symmetry

MDPI

Article

Asymmetric Finite-Range Persistence in Time Series Generated by the Modified Discrete Langevin Model

Zbigniew Czechowski

Institute of Geophysics, Polish Academy of Sciences, 01-452 Warsaw, Poland; zczech@igf.edu.pl

Abstract: The concept of asymmetric persistence in time series was proposed and an appropriate stochastic Langevin-type model was presented. The influence of this particular form of memory on the behavior of the generated time series was examined. It has been shown that asymmetry causes a significant distortion of the effect of drift forces and has a weaker impact on stochastic diffusion forces. Due to this, the current known methods for reconstructing the Langevin-type model fail. The results of this work may help in deriving a new reconstruction method.

Keywords: asymmetric persistence; correlations; time series; Langevin equation; model reconstruction procedure

check for updates

Academic Editors: Quanxin Zhu and Vladimir García-Morales

Received: 16 December 2024
Revised: 30 January 2025
Accepted: 10 February 2025
Published: 13 February 2025

Citation: Czechowski, Z. Asymmetric Finite-Range Persistence in Time Series Generated by the Modified Discrete Langevin Model. *Symmetry* **2025**, *17*, 287. https://doi.org/10.3390/sym17020287

Copyright: © 2025 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

The concept of asymmetric persistence appeared over 20 years ago in the context of economic time series [1]. A number of papers provided evidence that financial returns exhibit asymmetric dependence (e.g., Refs. [2–7])—there is often a stronger dependence between financial objects when the market is going down than when it is going up. On the other hand, another asymmetry is the skewness in the distribution of individual asset returns (e.g., Ref. [8]). Several methods for studying asymmetry were tested; however, later, two new measures of dependence were proposed: the local Gaussian correlation measure [9] and the modified Detrended Fluctuation Analysis (mDFA) procedure [10]. The first one was applied mainly to analyzing asymmetric correlations between different financial objects or to detecting financial contagions, while the latter was utilized for assessing asymmetric persistence in a time series.

Some mechanisms leading to asymmetric correlations may also be present in various physical, geophysical, or biological processes. In general, the processes governing an upward trend may be different from those governing a downward trend. For example, applying the mDFA procedure to northern hemisphere temperature and sunspot time series allowed for the detection of asymmetric correlations and the measurement of its level [10]. A simple scatterer-based mechanism in 1D random walk was recently proposed in Ref. [11].

In this work, we focus more on the modeling time series with asymmetric persistence than on methods of measuring this type of correlation. To this aim, a stochastic model of time series with persistence, based on the discrete Langevin equation, is proposed. The idea of modifying this equation first appeared in Ref. [12] (for a periodical case); then, it was adapted to the description of time series with persistence of order 1 [13], and further developed for a general case of persistence of order p [14]. The mechanism of p-order persistence means that the next increment is associated with a sequence of p signs (increase (+) or decrease (−)) of p previous increments. The persistence is a special case of

memory, which generally involves the influence of previous states of a process on the next state. In order to take into account the whole history of previous states, the well-known generalizations of the Langevin equation were derived by introducing the integral term [15] or by using a fractional derivative [16]. These models can describe persistent processes with long memory (understood as the power-like decay of the autocovariance function) very well, but the persistence is symmetrical.

A number of models have been proposed to describe asymmetric correlations in econometric data (see Ref. [17] for a comprehensive review). These were mainly appropriate generalizations of nonlinear models: TAR, the Markov switching model, and GARCH. In the first two, nonlinearities are introduced into the drift and in the third into the diffusion term, while in the Langevin equation, nonlinearities can occur in both terms simultaneously.

In Ref. [14], the analysis of the model and the procedure of its reconstruction from p-order persistent time series mainly concerned cases of symmetric persistence. However, this model offers the possibility of describing time series with asymmetric persistence; therefore, here, it will be used as a generator of this type of time series.

The main task of the present work is to analyze the impact of asymmetric persistence on the behavior of the generated time series. For this purpose, we use standard methods: Detrended Fluctuation Analysis (DFA) [18,19] to estimate the Hurst exponent, and the procedure of reconstruction of the Langevin equation [20]. Even though DFA does not recognize cases with asymmetric persistence, and the reconstruction procedure was dedicated to the Markov series only, the obtained results are valuable and will be useful in creating a new procedure for reconstructing the initial model with built-in p-order persistence [14] from asymmetric time series. Note that we do not use the modified DFA [10] in this work, because it is not suitable for stationary time series (while standard DFA works well for both non-stationary and stationary series).

Section 2 presents the model and its comparison with fractional Brownian motion (fBm). The analysis of p-order asymmetric and antisymmetric persistent walk is included in Section 3. The main section of this paper is Section 4, which examines the behavior of time series in the potential well, assuming various forms of the potential, diffusion functions, and persistence parameters. Section 5 presents the problems encountered by the modified reconstruction procedure [14] of the model in the case of asymmetric persistence. A summary and discussion of the results are provided in Section 6. In Appendix A, we have included a list of patterns (symbolizing seven previous increments) with persistence parameters assigned to them.

2. The Model

2.1. Persistence

We define a special form of correlation in time series $y(k)$, $k = 1, \ldots, N$. The next state $y(k+1)$ will depend on the current state $y(k)$ and on the sequence of p signs of the previous increments: $\Delta(k-p-1),\ldots,\Delta(k)$, where $\Delta(k) = y(k) - y(k-1)$. This relationship is related to the assumed probability of the next increase or decrease. We use a different notation for the probabilities (i.e., p $p(i)$) when the last increment was positive and a different notation (i.e., q $q(i)$) for those when it was negative (see Figure 1), where $i = 1, \ldots, 2^{p-1}$. This distinction will facilitate the introduction of the concepts of symmetric, asymmetric, and antisymmetric persistence in time series. Figure 1 shows the case when the last increment is positive (Figure 1a) or negative Figure 1b); then, the probability that the next increment will follow the trend is related to the probability given by the parameter $p(k)$ ($q(k)$).

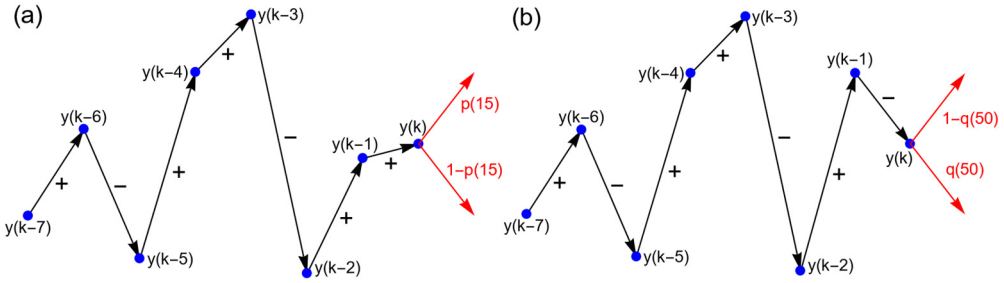


Figure 1. An illustration of the p-order persistence concept (for $p = 7$); $y(k-7), \ldots, y(k-1)$—values of the seven states of the time series preceding the current state $y(k)$; symbols + and − denote the signs of the seven previous increments: red arrows—next possible increment. (**a**) A case where the last increment has the positive sign; then, the sign of the next stochastic increment will also be positive with probability $p(15)$ and negative with probability $1 - p(15)$; (**b**) A case where the last increment has the negative sign; then, the sign of the next stochastic increment will also be negative with probability $q(50)$ and positive with probability $1 - q(50)$. The values of $p(15)$ and $q(50)$ depend on the sequence of seven signs of previous increments that are listed in Appendix A.

The number of all states for the p signs of the previous increments is 2^p. In Appendix A, we present a list of all states with assigned probabilities $p(k)$ ($q(k)$) for the case $p = 7$. The order determined by index k results from three adopted rules: the number of signs 'minus' ('plus') in the sequence; the moving of signs 'minus' ('plus') to the right; the last increment has a positive (negative) sign. Parameter p determines the order of persistence in our model.

2.2. Modified Discrete Langevin Model

The discrete version of the classical Langevin equation [21,22]

$$y(k+1) = y(k) + a(y(k))\Delta + \sqrt{b(y(k))\Delta}\,\xi_k \tag{1}$$

can describe an evolution of Markov (of order 1) series only. The deterministic drift term is given by the drift function $a(y)$, the stochastic diffusion term by the diffusion function $b(y)$ and the independent random variable ξ_k with normal density; Δ is the time step. In order to generalize the equation for finite-range correlated cases, an additional factor is introduced to the diffusion term:

$$y(k+1) = y(k) + a(y(k))\Delta + c(\boldsymbol{s_k}, r_k, \boldsymbol{D})\sqrt{b(y(k))\Delta}\,|\xi_k| \tag{2}$$

where the new function $c(\ldots)$ determines the sign of the term; therefore, the noise is replaced by the absolute value of the noise $|\xi_k|$. Function $c(\ldots)$ depends on vector $\boldsymbol{s_k} = [s_{k1}, s_{k2}, \ldots, s_{kp}]$, the random scalar variable r_k with uniform distribution in (0, 1) and the deterministic vector parameter $\boldsymbol{D}$. Vector $\boldsymbol{s_k}$ represents a pattern of signs of p previous increments, i.e., $\text{sign}(\Delta(k-p+1)), \ldots, \text{sign}(\Delta(k))$, before state $y(k)$, (e.g., [+ − + + − − +] for $p = 7$) whereas $\boldsymbol{D} = [p(1), p(2), \ldots p(2^{p-1}), q(1), q(2), \ldots, q(2^{p-1})]$ contains 2^p persistence parameters. Function $c(\ldots)$ takes only two values, 1 or −1. In each step, the value of r_k is drawn from the uniform distribution and ξ_k from the normal distribution. Let the pattern before $y(k)$ be, e.g., + − + + − − +], and the assumed probability of the pattern is $p(40)$ (see the list of patterns in Appendix A); then, when $r_k < p(40)$ function $c(\ldots) = 1$, otherwise $c(\ldots) = -1$. For a persistent series of order p, the function involves maintaining the tendency to increase (decrease) in the next step according to the assumed vector parameter $\boldsymbol{D}$. When all parameters $p(i)$ and $q(i)$ are greater than 0.5 (lower than 0.5) the time series is persistent (antipersistent) and is Markovian for $p(i) = q(i) = 0.5$. If, simultaneously, there appear components of $\boldsymbol{D}$ greater and smaller than 0.5, the series has a mixed persistence. The case

when $p(i) = q(i)$ ($p(i) = 1 - q(i)$) for all i is called symmetric (antisymmetric) persistence. In other cases, we have asymmetric persistence.

Obviously, when all components of $\boldsymbol{D}$ are the same, the p-persistent model is reduced to a 1-persistent model [13] (let us note that the persistence parameter d in that paper is equal to $1 - D$).

2.3. The Hurst Exponent

The range of memory in our model is limited (in each step) to signs of p previous increments. The purpose of this subsection is to show how this property is represented by the Detrended Fluctuation Analysis, which usually is used to estimate the Hurst exponent.

Let us briefly describe the DFA procedure [18]. By integration of time series $y(i)$, the profile (or double profile, when the integration is repeated) $u(k) = \sum_{i=1}^{k}[y(i) - \langle y \rangle]$ is determined (where $\langle y \rangle$ is the mean). The new signal $u(k)$ is divided into boxes of length n and, in each box, a trend of $u(k)$ is approximated by the polynomial $u_n(k)$ of order j. Then, the root-mean-square fluctuation is calculated

$$F(n) = \sqrt{\frac{1}{N_{max}} \sum_{k=1}^{N_{max}} [u(k) - u_n(k)]^2} \tag{3}$$

and the procedure is repeated for a broad range of box sizes n. For a correlated series, there is the scaling relation $F(n) \sim n^{\alpha}$, where α is a good approximation of the Hurst exponent H (for the double profile, $H \approx \alpha - 1$). The Hurst exponent is used as a measure of the long-term memory of the time series. It relates to the autocorrelations $\rho(k)$ of the time series, and the rate at which these decrease as the lag k between pairs of values increases:

$$\rho(k) \to Ck^{2H-2} \quad as \quad k \to \infty \tag{4}$$

In this work, we use the double profile (so the procedure also works well for strongly anticorrelated signals [19]) and polynomials of order 6.

Three time series were generated using Equation (2) with: $a(y) = 0$, $b(y) = 1$ (representing an analog of the Brownian walk), persistence of order $p = 1$, $p = 4$, $p = 7$, and symmetric parameters $p(k) = q(k)$, defined by the following formulas: $p(1) = 0.8$ for the case $p = 1$; $p(k) = 0.8 - 0.3((k-1)/15)^3$ for $p = 4$; and $p(k) = 0.8 - 0.3((k-1)/63)^3$ for $p = 7$. In the second and third cases, values of $p(k)$ start from a strong persistence of 0.8 and decrease with k to the Markovian value 0.5.

Figure 2a–c show the fluctuation functions obtained for these cases. The black line is a power law with exponent $\alpha = 1 + H$, where $H = 0.8$.

For larger values of n, the slope of the fluctuation function changes to a value corresponding to $H = 0.5$ (green line). We can observe that the crossover n_c between the two scaling regimes (the long-range correlation regime and the uncorrelated behavior) increases with order of persistence p, from $n_c = 100$ for $p = 1$ to $n_c = 500$ for $p = 7$. For comparison, Figure 2d shows $F(n)$ for the case of fractional Brownian motion with $H = 0.8$. Here, the long-range correlation regime has a size comparable to the length of the generated time series.

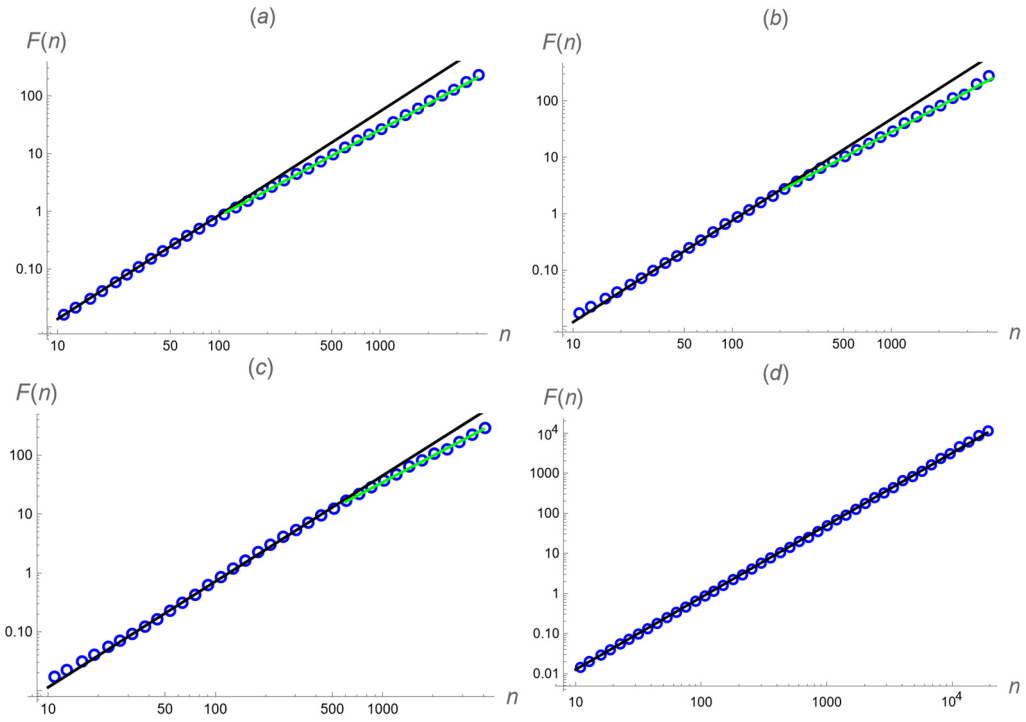


Figure 2. The fluctuation functions $F(n)$ (blue circles) obtained from the p-order persistent time series generated by Equation (2) with $a(y) = 0$, $b(y) = 1$ and symmetric parameters, $p(k) = q(k)$. (**a**) Case $p = 1$; (**b**) $p = 4$; (**c**) $p = 7$; (**d**) $F(n)$ is obtained from fBm with $H = 0.8$. The black line is a power law with exponent $\alpha = 1 + H$ where $H = 0.8$, while for green line $H = 0.5$.

2.4. Comparison with Fractional Brownian Motion

Fractional Brownian motion [23] was introduced as a generalization of Brownian motion for the case of long-range correlations. For this purpose, fractional integral was used, which involved taking into account the entire history of the process. Unlike our p-persistent model in which each state remembers only p previous increments, in fBm the memory includes all previous values of the process. Positive and negative correlations are symmetric in fBm, which corresponds to symmetric persistence in our model. Let us treat the series generated by fBm with $H = 0.7$ as if it were generated by our 7-order persistent model; then, the reconstruction of vector parameter $\boldsymbol{D}$ leads to values $p(i)$ and $q(i)$, shown in Figure 3. Since $p(i) = q(i)$, $i = 1,\ldots, 64$, this confirms the symmetry of persistence in fBm.

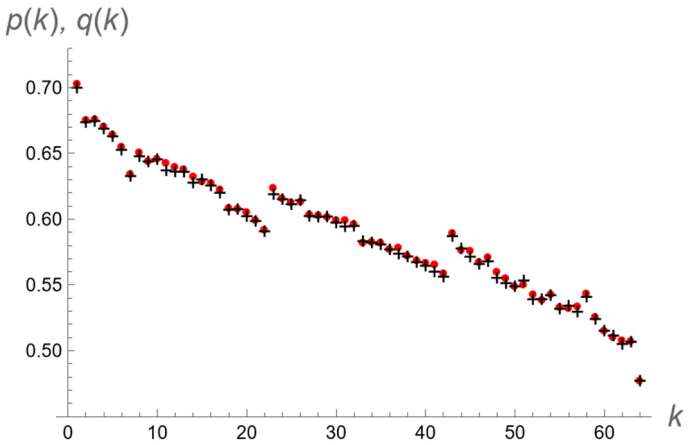


Figure 3. Persistence parameters $p(k)$ (red points) and $q(k)$ (black +) reconstructed from fBm with $H = 0.7$.

Moreover, Figure 3 shows how the memory built into fBm is expressed by the behavior of persistence parameters. In general, their values decrease with increasing k, but we noticed a few jumps. There may be two reasons for the jumps. The first leads to the conclusion that the arrangement of $p(k)$ given in Appendix A is not the proper one and requires a slight renumbering. The second reason may be due to the fact that correlations in fBm are not the same as the persistence in our model. The memory in our model refers to the values of the current state and the signs of the seven previous increments, while fBm remembers values of the previous states. Therefore, the concept of persistence ('trend') must be distinguished from the more general concept of memory.

3. Asymmetric Persistence

Asymmetric persistence means that all or only some of the parameters $p(i)$ differ from the corresponding parameters $q(i)$. Then, the upward trend in time series differs from the downward trend. In this section, we present three examples illustrating the impact of asymmetric persistence on certain features of the generated time series.

In Example 1, we generate a time series of length $N = 10^6$ using Equation (2) with $a(y) = 0$, $b(y) = 1$, $p = 7$, time step $\Delta = 0.01$, and persistence parameters in the following forms:

$$p(k) = 0.9 - 1.6{\cdot}10^{-6}{\cdot}\Big(k-1\Big)^3, \tag{5}$$

$$q(k) = 0.6 - 4.0{\cdot}10^{-7}{\cdot}\Big(k-1\Big)^3. \tag{6}$$

Values of the parameters start from 0.9 (0.6) and decrease with k to 0.5. Equations (5) and (6) are specific examples of equations:

$$p(k) = p(1) - (p(1) - 0.5)\left(\frac{k-1}{63}\right)^3, \tag{7}$$

$$q(k) = q(1) - (q(1) - 0.5)\left(\frac{k-1}{63}\right)^3, \tag{8}$$

that will model the values of persistence parameters in this work. By adopting a functional form of the parameters, we avoid the tedious task of determining the value of each of the 128 persistence parameters. Assuming the selected forms (7) and (8), we follow the quite natural principle that correlations leading to the persistent behavior of a time series weaken over time. Therefore, the probability of another positive increment will be higher when all seven previous increments were positive than when only six previous increments were positive and one, the most distant one (k − 6), was negative, etc. Using the third power instead of a linear relationship in the second term in Equation (7) causes a slow decline in $p(k)$ for small values of k, which only accelerates for large k. This means that, e.g., longer sequences of previous increases have a greater tendency to maintain this upward trend than if a linear relationship were used.

After applying the DFA procedure, we obtain the fluctuation function $F(n)$ shown in Figure 4a. In the first scaling regime (black line) the Hurst exponent $H = 0.686$, and then drops to $H = 0.5$ (green line).

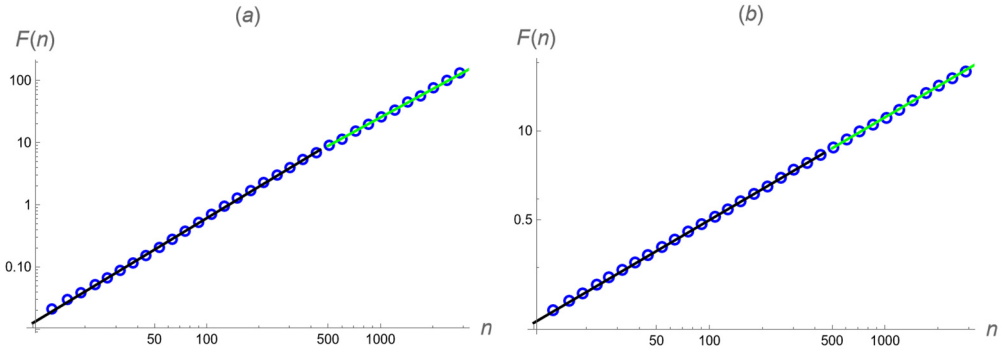


Figure 4. Fluctuation functions $F(n)$ (blue circles) obtained from p-order persistent time series generated by Equation (2) with $a(y) = 0$, $b(y) = 1$. (**a**) The case with asymmetric parameters given by Equations (5) and (6). The black line is a power law with exponent $\alpha = 1 + H$ where $H = 0.686$, while for green line $H = 0.5$. (**b**) The case with antisymmetric parameters given by Equations (9) and (10). The black and green lines are a power law with exponent $\alpha = 1 + H$, where $H = 0.5$.

In Example 2, we only change the values of persistence parameters:

$$p(k) = 0.2 + 1.2 \times 10^{-6} \cdot \Big(k - 1\Big)^3, \tag{9}$$

$$q(k) = 0.8 - 1.2 \times 10^{-6} \cdot \Big(k - 1\Big)^3. \tag{10}$$

In this antisymmetric case, the fluctuation function gives the Hurst exponent $H = 0.5$ (see Figure 4b, black and green line), which suggests that the time series is uncorrelated. However, according to the assumed values of the parameters, the time series is strongly persistent in the downward trend and strongly antipersistent in the upward trend.

It should be emphasized here that the values $H = 0.686$ in Example 1 and $H = 0.5$ in Example 2 are equal to the average values of $p(k) + q(k)$, i.e.,

$$\langle p(k) + q(k) \rangle = \frac{1}{2^p} \sum_{k=1}^{2^{p-1}} (p(k) + q(k)), \tag{11}$$

where $p(k)$ and $q(k)$ are given by Formulas (5) and (6) or (9) and (10). This is not a coincidence. We performed many simulations assuming different forms and values of persistence parameters and different forms of drift and diffusion functions. In all cases, H estimated by DFA was equal to $\langle p(k) + q(k) \rangle$. This means that DFA incorrectly recognizes the level of correlation in time series where asymmetric correlations exist because it actually only finds their average effect.

In both of the above examples, the absence of a drift term (i.e., $a(y) = 0$) was assumed (like in the free random walk), which means that there were no potential $V(y)$ of the drift force (where $V'(y) = -a(y)$), which would limit the dynamics of the process. In Example 3, we introduce a simple quadratic potential $V(y) = y^2/2$ (then $a(y) = -y$), but other assumptions are adopted from Example 2. The generated time series is stationary (unlike in Examples 1 and 2). Its values fluctuate around the average value $y = -3.56$ (see Figure 5a) instead of $y = 0$, even though the minimum of the potential is at zero. According to a simple physical interpretation, the Langevin equation describes the fluctuation of a particle in the potential well (see Figure 5b). The fluctuations are governed by the stochastic diffusion forces that interact with drift forces (potential). For the case of symmetric persistence, the stochastic forces are amplified (weakened) by the action of persistent (antipersistent) trends, but the values of time series are still clustered around the minimum of the potential (see green distribution function $f(y)$ in Figure 5b). Asymmetric persistence (e.g., the one given by Equations (9) and (10)) leads to an asymmetry of stochastic forces and, as a result, it will cause a shift in the distribution function (see the red distribution in Figure 5b).

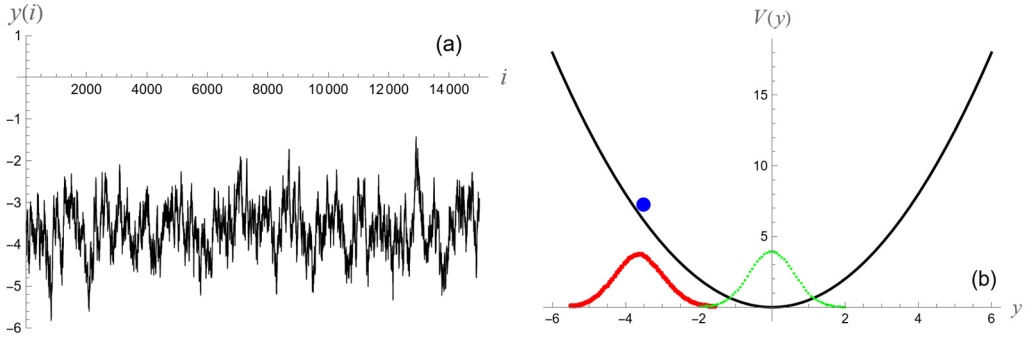


Figure 5. (**a**) Time series generated by Equation (2) with $a(y) = -y$, $b(y) = 1$ and antisymmetric parameters, Equations (9) and (10). (**b**) Potential $V(y)$ (black graph) and distribution function $f(y)$ (red graph) of the time series shown in plot (**a**). For a comparison, the green graph represents the distribution function for a symmetric case. The blue dot indicates a particle in the potential field.

4. Reconstruction of the Drift and Diffusion Functions from Time Series

For the case of the classical Langevin equation, the procedure of reconstruction of the drift and diffusion functions (referred to here as the standard procedure) was proposed in Ref. [20] and then developed in many papers [24–27]. However, for generalized non-Markov models, the method fails. In papers [13,14], it was shown that for a Langevin-type model with symmetric persistence, the standard procedure leads to a wrong reconstruction of the drift function but to a good estimation of the diffusion function; therefore, the new modified procedure was proposed. Unfortunately, for general asymmetric cases, this procedure also fails.

In this chapter, we will show the influence of various forms of persistence on the deformations of drift and diffusion functions reconstructed using the standard procedure. To this aim, we assume three different cases of the functions:

$$(M1) \quad a(y) = -y; \quad b(y) = 1; \tag{12}$$

$$(M2) \quad a(y) = -y^2 + \tfrac{1}{2}, \quad b(y) = y; \tag{13}$$

$$(M3) \quad a(y) = -y^3 + y, \quad b(y) = y^2. \tag{14}$$

however, these three examples have a common feature—the classical Langevin equation with these forms of functions generates stationary time series characterized by a Gaussian (or half-Gaussian) distribution function. By maintaining this type of uniformity, we avoid the influence of the form of the distribution function on our analysis.

4.1. Symmetric Persistence

We generate three sets of time series using Equation (2) where drift and diffusion functions are given by three cases, M1–M3. Each set consists of two time series; in one, we assume the following symmetric form of persistence parameters:

$$p(k) = q(k) = 0.9 - 1.6{\cdot}10^{-6}{\cdot}(k-1)^3, \tag{15}$$

and in the other, the form

$$p(k) = q(k) = 0.7 - 8.0{\cdot}10^{-7}{\cdot}(k-1)^3. \tag{16}$$

The order of persistence is $p = 7$, the time step is $\Delta = 0.01$, and the length of series is $N = 2 \times 10^6$. Applying the standard procedure [20] to the generated series, we obtain reconstructions of the drift functions, which differ significantly from the initial assumed

forms in M1–M3 (see Figure 6). However, the reconstruction of the diffusion function is correct (in accordance with the conclusions of Ref. [14]). Figure 6a suggests that the obtained deviations may be the result of rotation of the initial drift function, but in nonlinear cases we are dealing with a combination of rotation with additional slight deformation. The angle of 'rotation' of the drift function graph increases as the average value of $p(k) + q(k)$ increases (which corresponds to the value of H estimated by DFA).

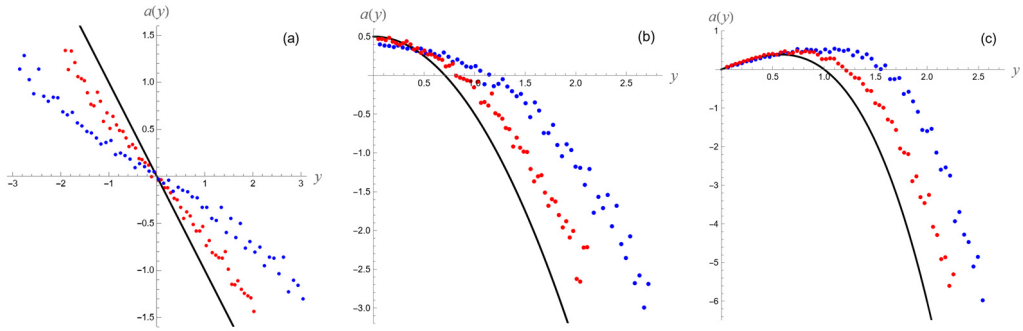


Figure 6. Standard procedure reconstructions of drift functions from time series generated by Equation (2) for two symmetric forms of persistence parameters given by Equation (15) (blue points) and Equation (16) (red points). (**a**) Case M1; (**b**) case M2; (**c**) case M3. Black graphs represent the initial forms of $a(y)$.

4.2. Antisymmetric Persistence

Here, we also apply three cases, M1–M3, but each set consists of three time series because the following three antisymmetric forms of persistence parameters were assumed for them:

$$p(k) = 1 - q(k) = 0.8 - 1.2 \times 10^{-6} \cdot (k-1)^3, \tag{17}$$

$$p(k) = 1 - q(k) = 0.7 - 8.0 \times 10^{-7} \cdot (k-1)^3, \tag{18}$$

$$p(k) = 1 - q(k) = 0.6 - 4.0 \times 10^{-7} \cdot (k-1)^3. \tag{19}$$

Figure 7 shows the reconstructed drift functions. With increasing values of $p(1)$ in Equations (17)–(19), the graphs of drift functions are increasingly shifted from the graph of the initial function; however, the transformation is not a pure translation. Moreover, the diffusion functions $b(y)$ are also incorrectly reconstructed in antisymmetric case, although the deviations are less significant (see Figure 8). Particularly unsatisfactory, in the context of reconstruction, is the result of the DFA procedure, which gives an estimate of the Hurst exponent $H \approx 0.5$ for all cases with antisymmetric persistence parameters (as shown in Section 3, DFA gives the average value of $p(k) + q(k)$). Thus, DFA treats antisymmetric cases as Markovian regardless of the level of persistence.

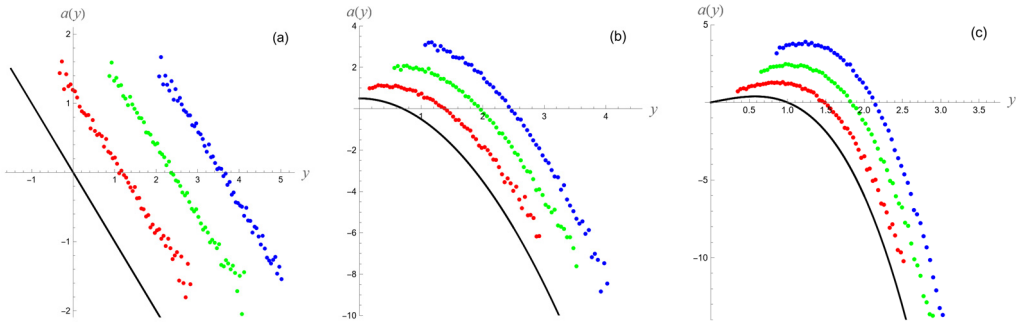


Figure 7. Standard procedure reconstructions of drift functions from time series generated by Equation (2) for three antisymmetric forms of persistence parameters given by Equation (17) (blue points), Equation (18) (green points) and Equation (19) (red points). (**a**) Case M1; (**b**) case M2; (**c**) case M3. Black graphs represent the initial forms of $a(y)$.

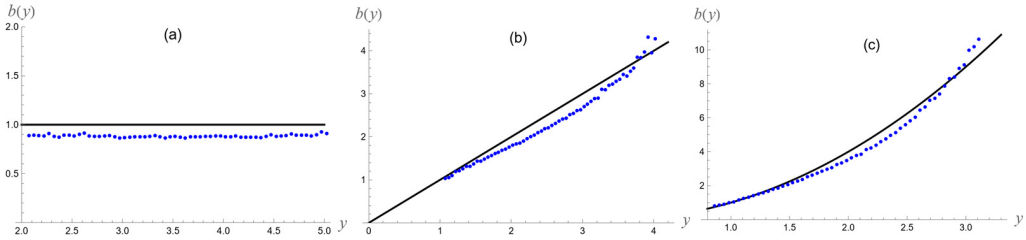


Figure 8. Standard procedure reconstructions of diffusion functions from time series generated by Equation (2) for the antisymmetric form of persistence parameters given by Equation (17) (blue points). (**a**) Case M1; (**b**) case M2; (**c**) case M3. Black graphs represent the initial forms of $b(y)$.

4.3. Asymmetric Persistence

In this subsection, we assume three forms of persistence parameters that are neither symmetric nor antisymmetric. Two of them are regular:

$$p(k) = 0.9 - 1.6 \times 10^{-6} \cdot (k-1)^3, \tag{20}$$

$$q(k) = 0.6 - 4.0 \times 10^{-7} \cdot (k-1)^3, \tag{21}$$

and:

$$p(k) = 0.9 - 1.6 \times 10^{-6} \cdot (k-1)^3, \tag{22}$$

$$q(k) = 0.3 + 8.0 \times 10^{-7} \cdot (k-1)^3. \tag{23}$$

Parameters given by Equations (20) and (21) mean strong upward persistence and weak downward persistence, while Equation (23) indicates a downward antipersistent trend. The reconstructed drift functions (Figure 9) are significantly deviated from initial forms. The deviations are combinations of transformations mentioned in the previous two subsections: 'rotation' and 'translation'. Similarly to the antisymmetric case, the reconstructed diffusion functions differ from the initial functions, although the deviations are not large (see Figure 10).

In the third example, in which only the case M1 is analyzed, the form of $p(k)$ ($q(k)$) is random (see Figure 11a) with a decreasing trend (without a trend). Also, in this case, the estimation of the Hurst exponent by DFA (see Figure 11b) is consistent with the average $\langle p(k) + q(k) \rangle$, and the reconstruction of the drift function (see Figure 11c) undergoes similar displacements ('translation', 'rotation') as in the previous examples.

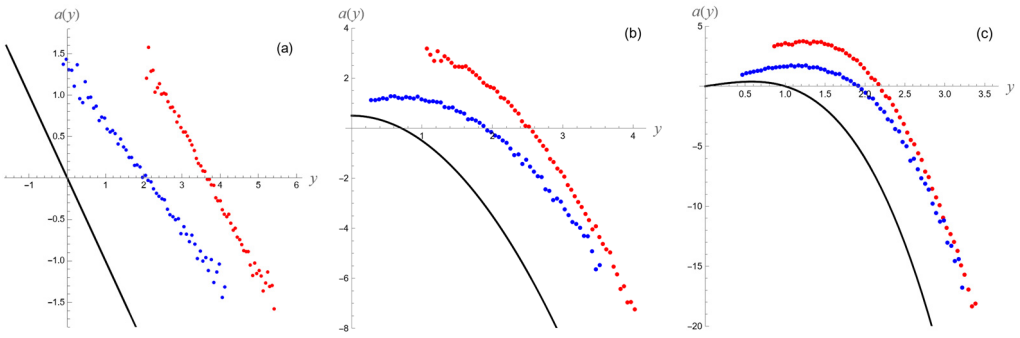


Figure 9. Standard procedure reconstructions of drift functions from time series generated by Equation (2) for two asymmetric forms of persistence parameters given by Equations (20) and (21) (blue points) and Equations (22) and (23) (red points). (**a**) Case M1; (**b**) case M2; (**c**) case M3. Black graphs represent the initial forms of $a(y)$.

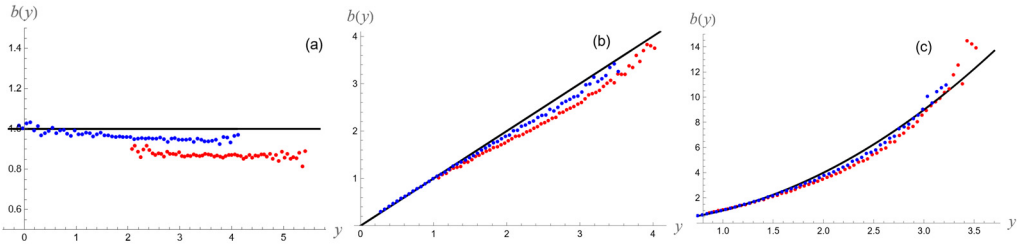


Figure 10. Standard procedure reconstructions of diffusion functions from time series generated by Equation (2) for two asymmetric forms of persistence parameters given by Equations (20) and (21) (blue points) and Equations (22) and (23) (red points). (**a**) Case M1; (**b**) case M2; (**c**) case M3. Black graphs represent the initial forms of $b(y)$.

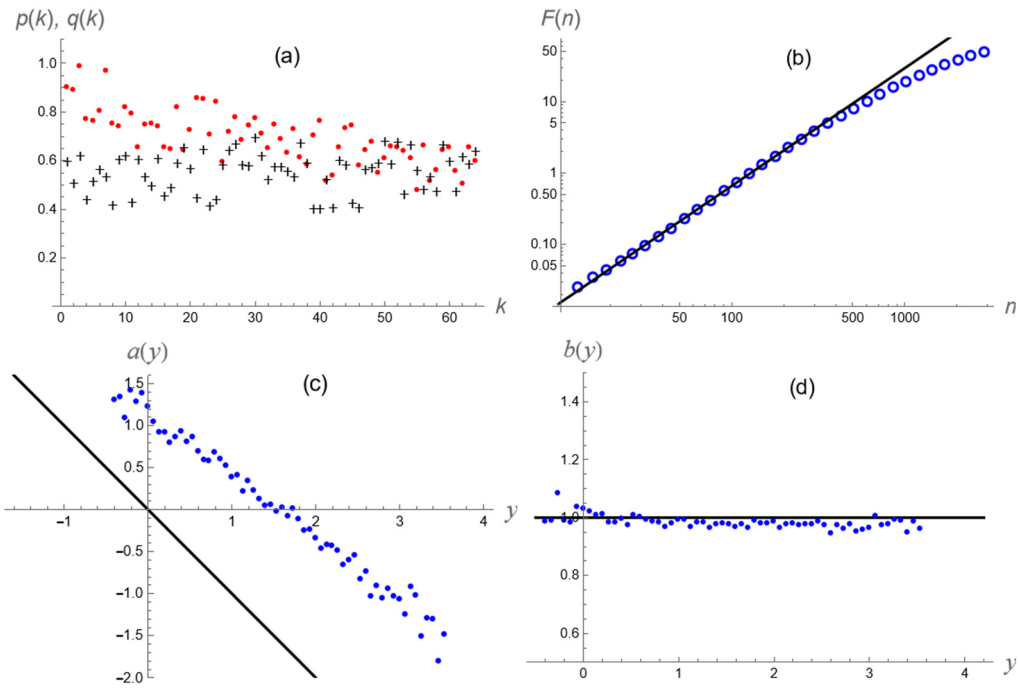


Figure 11. The case with random persistence parameters. (**a**) Illustration of $p(k)$ (red points) and $q(k)$ (black +); (**b**) fluctuation functions $F(n)$ (blue circles); the black line is a power law with exponent $\alpha = 1 + H$, where $H = 0.652$; (**c**) standard procedure reconstruction of drift function (blue points); black line represents the initial form, $a(y) = -y$; (**d**) standard procedure reconstruction of diffusion function (blue points); black line represents the initial form, $b(y) = 1$.

5. Problems with the Modified Reconstruction Procedure

The modified reconstruction procedure for the case of p-order persistent model was proposed and tested in Ref. [14]. It works well for symmetric persistence. The algorithm consists of five steps:

Step 1. First reconstruction $a_1(y)$ and $b_1(y)$ of the input $a(y)$ and $b(y)$ using the standard procedure [20];
Step 2. Estimation of persistence parameters $p(k)$ and $q(k)$ from histograms of

$$c(\boldsymbol{s_k}, r_k, \boldsymbol{D}) \approx (y(k+1) - y(k)) - a_1(y)\Delta; \tag{24}$$

Step 3. Second reconstruction $a_2(y)$ from the new time series generated by Equation (2) with $a_1(y)$, $b_1(y)$, and parameters $p(k)$ and $q(k)$ (estimated in Step 2) as the input;
Step 4. Third reconstruction $a_3(y)$ from new time series generated by Equation (2) with $a_2(y)$, $b_1(y)$, and parameters $p(k)$ and $q(k)$ (estimated in Step 2) as the input;
Step 5. The final reconstruction of the drift function is given by the formula

$$a_R = a_1(y) - [a_2(y) - a_1(y)]h(D), \tag{25}$$

where $h(D)$ is the factor of proportionality which depends on $a_1(y)$, $a_2(y)$, and $a_3(y)$ (see the appendix in Ref. [14]).

In this section, we compare how the modified reconstruction procedure works for the symmetric and antisymmetric persistence parameters. Two time series were generated by Equation (2) with $a(y) = -(y-1)$, $b(y) = 1$, and persistence parameters given by Equation (16) for the symmetric case and by Equation (18) for the antisymmetric case. The result of the first step is crucial, because the reconstruction of the drift function is used to estimate the persistence parameters in Step 2. Figure 12 shows that for the symmetric case, the parameters are estimated quite well, but for the antisymmetric case there is a big gap between the input values (black and red lines) and the estimated values (black and red points). This is due to the fact that $a_1(y)$ (the blue points in Figure 13) is much closer to the input drift function $a(y)$ (black dashed line) in the symmetric case than in the antisymmetric case where $a_1(y)$ is far-shifted. An incorrect reconstruction of persistence parameters leads to an incorrect final reconstruction of the drift function (the orange line in Figure 13) for the example with antisymmetric persistence.

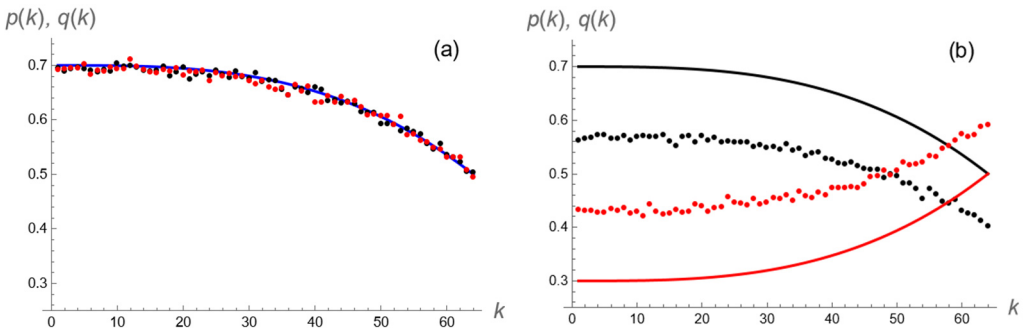


Figure 12. Persistence parameters $p(k)$ (red points) and $q(k)$ (black points) reconstructed from generated time series. (**a**) Symmetric case; (**b**) antisymmetric case. Continuous lines (blue, black and red) represent the input values of these parameters.

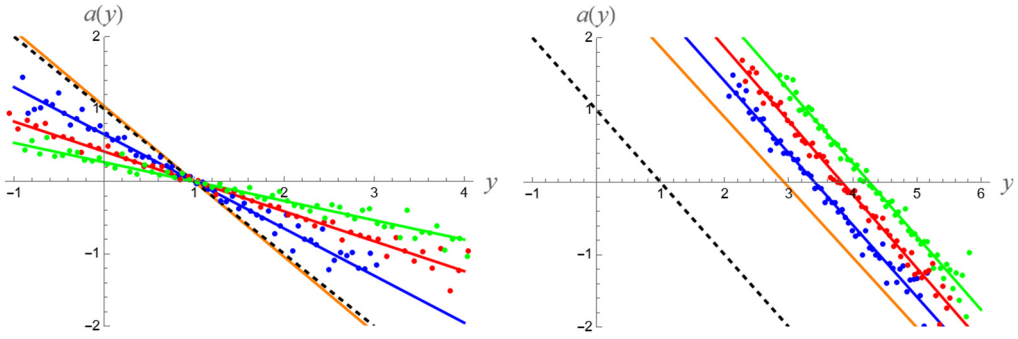


Figure 13. Visualizations of successive steps of the modified reconstruction procedure. Points represent the results of the standard procedure. Continuous lines are the least-square fits for $a_1(y)$ (blue), $a_2(y)$ (red), and $a_3(y)$ (green). The black dashed line shows the input $a(y)$; the orange line is the final reconstruction $a_R(y)$. Left plot—symmetric case; right plot—antisymmetric case.

In order to overcome this problem, it is necessary to find a method to estimate the shift in function $a_1(y)$ relative to $a(y)$ caused by the asymmetry. We cannot just shift the graph $a_1(y)$ to zero, because we do not know what the shift built into the input drift function was (in our case, the shift is 1), including the shift caused by (unknown) values of persistence parameters. Analytical results of similar studies (in our parallel work) on the 1-order asymmetric persistence model (with two persistence parameters only, $p(1)$ and $q(1)$) may provide some guidance. It has been shown that the shift depends on the difference $q(1) - p(1)$ and on the form of the diffusion function; however, in the case of higher-order persistence, this type of relationship would be much more complicated.

6. Discussion

The introduction of the concept of asymmetric persistence allows for a deeper analysis of time series and their more proper modeling. Hidden mechanisms in complex geophysical, biological, or economic processes may be of a different nature when they lead to an upward trend than those which lead to a downward trend. Standard models assuming symmetry not only oversimplify the examined process but also do not provide the opportunity to identify these mechanisms. However, like most generalizations, the assumption of asymmetric persistence also leads to additional difficulties and challenges.

The present work concerns a stochastic time-series model based on the Langevin equation with built-in finite-range persistence. Although the appropriate model was already derived in our paper [14], the procedure for its reconstruction from time series was tested for symmetric persistence. Unfortunately, this procedure does not work in asymmetric cases. In the present paper, we re-analyze this p-order persistent model in the context of its application to time series in which asymmetric persistence may operate. We increased the memory length from $p = 5$ (in Ref. [14]) to $p = 7$, which resulted in a fourfold increase in the number of parameters. The comparison of this model with the fractional Brownian motion has shown that the persistence in the fractional model is symmetric. To make our model more similar to fBm, we should slightly change the order of persistence parameters in the appendix.

The next part of the work concerned the analysis of the effectiveness of standard procedures for the analysis of time series with asymmetric persistence and the reconstruction of the model from these data. The time series were generated by Equation (2), assuming different forms of the drift and diffusion functions and persistence parameters.

The example of p-order persistent walk shows that the estimation of the Hurst exponent made by the DFA procedure may suggest misleading conclusions when we are dealing with asymmetric persistence. For example, for any antisymmetric persistence, it gives the

Markovian value $H = 0.5$. The explanation is quite simple—DFA gives the average result equal to $\langle p(k) + q(k) \rangle$.

By adding the potential, the freedom of the persistent trend is limited, and it complicates the dynamics of the process. The stochastic force associated with the trend competes with the drift force, and the process may find a different steady-state location. This causes a shift in the generated time series, which makes the reconstruction of the function $a(y)$ very difficult. In the symmetric case, there is no such shift, and then the reconstruction procedure proposed in Ref. [14] works well. However, for asymmetric cases, a new procedure must be developed.

From the examples analyzed in Section 4, we can conclude that the reconstructions of drift function $a(y)$ obtained by the standard procedure [20] are very different from the initial function. For symmetric persistence, the main transformation of the drift function is rotation; for antisymmetric, it is shift; and for more general asymmetric cases, it is a combination of rotation and shift. In addition, the functions experience another, much smaller shape deformation. It should be emphasized that asymmetric persistence also causes incorrect reconstructions of the diffusion function $b(y)$ obtained from the standard method. (the reconstruction of $b(y)$ is correct only in the symmetric case).

Referring to the reconstruction procedure of the p-persistent model [14], the basic problem caused by the asymmetry of persistence is the significant shift in the reconstructed drift function relative to the initial $a(y)$. When dealing with a real time series, we do not know whether the potential is shifted or whether the shift is a result of asymmetric persistence in the process (moreover, both of these factors may act simultaneously). In fact, this is the main reason for the failure of the reconstruction procedure proposed in Ref. [14]. Rotations do not cause changes as big as shifts, and the procedure was able to identify them. Therefore, developing a new procedure for the model with asymmetric persistence is a big challenge. The results of this work should help in this task, because they show problems that need to be solved. Perhaps another modification of mDFA [10] will enable the estimation of the shift in the drift function caused by asymmetric persistence in stationary series.

Funding: This research was funded in part by National Science Centre, Poland, Contract No. UMO-2022/45/B/ST10/01621.

Data Availability Statement: The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest: The author declares no conflicts of interest.

Appendix A

The basic feature of time series in this work is (asymmetric) persistence, i.e., a certain tendency to maintain a trend. Therefore, we adopted three rules that precisely determined the order of $p(k)$, i.e., assigning $p(k)$ to appropriate configurations of sign sequences:

(1) For $p(k)$, the last increment Δ_k is positive;
(2) We arrange the groups of sequences which have the same number of 'plus' signs in the following way one by one: the fewer 'plus' signs in the sequence, the lower their place in the order;
(3) The order of sequences of signs of p previous increments within each group depends on the 'minus' signs shifting to the right, i.e., the more 'minus' signs shifted closer to the right, the higher the value of k;

and similar rules for $q(k)$:

(1) For $q(k)$, the last increment Δ_k is negative;

(2) We arrange the groups of sequences which have the same number of signs 'minus' in the following way one by one: the fewer 'minus' signs in the sequence, the lower their place in the order;

(3) The order of sequences of signs of p previous increments within each group depends on the 'plus' signs shifting to the right, i.e., the more 'plus' signs shifted closer to the right, the higher the value of k.

Here, we present a list of patterns symbolizing the signs of seven previous increments with the parameters $p(k)$ assigned to them. The list for parameters $q(k)$ is analogous, so one only needs to replace: p by q, "+" by "−", and "−" by "+".

Δ_{k-6}	Δ_{k-5}	Δ_{k-4}	Δ_{k-3}	Δ_{k-2}	Δ_{k-1}	Δ_k
+	+	+	+	+	+	p_1
−	+	+	+	+	+	p_2
+	−	+	+	+	+	p_3
+	+	−	+	+	+	p_4
+	+	+	−	+	+	p_5
+	+	+	+	−	+	p_6
+	+	+	+	+	−	p_7
−	−	+	+	+	+	p_8
−	+	−	+	+	+	p_9
+	−	−	+	+	+	p_{10}
−	+	+	−	+	+	p_{11}
+	−	+	−	+	+	p_{12}
+	+	−	−	+	+	p_{13}
−	+	+	+	−	+	p_{14}
+	−	+	+	−	+	p_{15}
+	+	−	+	−	+	p_{16}
+	+	+	−	−	+	p_{17}
−	+	+	+	+	−	p_{18}
+	−	+	+	+	−	p_{19}
+	+	−	+	+	−	p_{20}
+	+	+	−	+	−	p_{21}
+	+	+	+	−	−	p_{22}
−	−	−	+	+	+	p_{23}
−	−	+	−	+	+	p_{24}
−	+	−	−	+	+	p_{25}
+	−	−	−	+	+	p_{26}
−	−	+	+	−	+	p_{27}
−	+	−	+	−	+	p_{28}
+	−	−	+	−	+	p_{29}
−	+	+	−	−	+	p_{30}
+	−	+	−	−	+	p_{31}
+	+	−	−	−	+	p_{32}
−	−	+	+	+	−	p_{33}
−	+	−	+	+	−	p_{34}
+	−	−	+	+	−	p_{35}
−	+	+	−	+	−	p_{36}
+	−	+	−	+	−	p_{37}
+	+	−	−	+	−	p_{38}
−	+	+	+	−	−	p_{39}
+	−	+	+	−	−	p_{40}
+	+	−	+	−	−	p_{41}
+	+	+	−	−	−	p_{42}

$$
\begin{array}{ccccccc}
- & - & - & - & + & + & + \quad p_{43} \\
- & - & - & + & - & + & + \quad p_{44} \\
- & - & + & - & - & + & + \quad p_{45} \\
- & + & - & - & - & + & + \quad p_{46} \\
+ & - & - & - & - & + & + \quad p_{47}
\end{array}
$$

$$
\begin{array}{ccccccc}
- & - & - & + & + & - & + \quad p_{48} \\
- & - & + & - & + & - & + \quad p_{49} \\
- & + & - & - & + & - & + \quad p_{50} \\
+ & - & - & - & + & - & + \quad p_{51} \\
- & - & + & + & - & - & + \quad p_{52} \\
- & + & - & + & - & - & + \quad p_{53} \\
+ & - & - & + & - & - & + \quad p_{54} \\
- & + & + & - & - & - & + \quad p_{55} \\
+ & - & + & - & - & - & + \quad p_{56} \\
+ & + & - & - & - & - & + \quad p_{57}
\end{array}
$$

$$
\begin{array}{ccccccc}
- & - & - & - & - & + & + \quad p_{58} \\
- & - & - & - & + & - & + \quad p_{59} \\
- & - & - & + & - & - & + \quad p_{60} \\
- & - & + & - & - & - & + \quad p_{61} \\
- & + & - & - & - & - & + \quad p_{62} \\
+ & - & - & - & - & - & + \quad p_{63}
\end{array}
$$

$$
\begin{array}{ccccccc}
- & - & - & - & - & - & + \quad p_{64}
\end{array}
$$

References

1. Franses, P.H.; Paap, R. Modelling Asymmetric Persistence over the Business Cycle. Econometric Institute Research Papers (No. EI 9852). 1998. Available online: http://hdl.handle.net/1765/1525 (accessed on 9 February 2025).
2. Silvapulle, P.; Granger, C.W.J. Large returns, conditional correlation and portfolio diversification: A value-at-risk approach. *Quant. Financ.* **2001**, *1*, 542–551. [CrossRef]
3. Okimoto, T. New evidence of asymmetric dependence structures in international equity Markets. *J. Financ. Quant. Anal.* **2008**, *43*, 787–816. [CrossRef]
4. Ang, A.; Chen, J. Asymmetric correlations of equity portfolios. *J. Financ. Econ.* **2002**, *63*, 443–494. [CrossRef]
5. Hong, Y.; Tu, J.; Zhou, G. Asymmetries in stock returns: Statistical tests and economic evaluation. *Rev. Financ. Stud.* **2007**, *20*, 1547–1581. [CrossRef]
6. Chollete, L.; Heinen, A.; Valdesogo, A. Modeling international financial returns with a multivariate regime switching copula. *J. Financ. Econom.* **2009**, *7*, 437–480. [CrossRef]
7. Garcia, R.; Tsafack, G. Dependence structure and extreme comovements in international equity and bond markets. *J. Bank. Financ.* **2011**, *35*, 1954–1970. [CrossRef]
8. Rydberg, T.H.; Shephard, N. A modelling framework for the prices and times of trades made on the New York stock exchange. In *Nonlinear and Nonstationary Signal Processing*; Fitzgerald, W.J., Smith, R.L., Walden, A.T., Young, P.C., Eds.; Cambridge University Press: London, UK, 2000; pp. 217–246.
9. Tjøstheim, D.; Hufthammer, K.O. Local Gaussian correlation: A new measure of dependence. *J. Econom.* **2013**, *172*, 33–48. [CrossRef]
10. Alvarez-Ramirez, J.; Rodriguez, E.; Echeverria, J.C. A DFA approach for assessing asymmetric correlations. *Phys. A* **2009**, *388*, 2263–2270. [CrossRef]
11. Rossetto, V. The one-dimensional asymmetric persistent random walk. *J. Stat. Mech.* **2018**, *2018*, 043204. [CrossRef]
12. Czechowski, Z.; Telesca, L. Construction of A Langevin model from time series with a periodical correlation function: Application to wind speed data. *Phys. A* **2013**, *392*, 5592–5603. [CrossRef]
13. Czechowski, Z. Reconstruction of the modified discrete Langevin equation from persistent time series. *Chaos* **2016**, *26*, 053109. [CrossRef] [PubMed]
14. Czechowski, Z. Discrete Langevin-type equation for p-order persistent time series and procedure of its reconstruction. *Chaos* **2021**, *31*, 063102. [CrossRef]
15. Kubo, R. The Fluctuation-Dissipation Theorem. *Rep. Prog. Phys.* **1966**, *29*, 255–284. [CrossRef]
16. Mainardi, F.; Mura, A.; Tampieri, F. Brownian motion and anomalous diffusion revisited via a fractional Langevin Equation. *Mod. Probl. State Phys.* **2009**, *8*, 3. Available online: http://arxiv.org/abs/1004.3505 (accessed on 20 April 2010).

17. Cao, L.; Sun, R.; Ma, T.; Liu, C. On Asymmetric Correlations and Their Applications in Financial Markets. *J. Risk Financ. Manag.* **2023**, *16*, 187. [CrossRef]
18. Kantelhardt, J.W.; Koscielny-Bunde, E.; Rego, H.H.A.; Havlin, S.; Bunde, A. Detecting long-range correlations with detrended fluctuation analysis. *Phys. A* **2001**, *295*, 441–454. [CrossRef]
19. Kantelhardt, J.W.; Zschiegner, S.A.; Koscielny-Bunde, E.; Havlin, S.; Bunde, A.; Stanley, H.E. Multifractal detrended fluctuation analysis of nonstationary time series. *Phys. A* **2002**, *316*, 87–114. [CrossRef]
20. Siegert, S.; Friedrich, R.; Peinke, J. Analysis of data sets of stochastic systems. *Phys. Lett. A* **1998**, *243*, 275–280. [CrossRef]
21. Grasman, J.; van Herwaarden, O.A. *Asymptotic Methods for the Fokker-Planck Equation and the Exit Problem in Applications*; Springer: Berlin/Heidelberg, Germany, 1999.
22. Oksendal, B. *Stochastic Differential Equations: An Introduction with Applications*; Springer: Berlin/Heidelberg, Germany, 1998.
23. Mandelbrot, B.B.; Van Ness, J.W. Fractional Brownian Motions, Fractional Noises and Applications. *SIAM Rev.* **1968**, *10*, 422–437. [CrossRef]
24. Sura, P.; Barsugli, J. A note on estimating drift and diffusion parameters from timeseries. *J. Phys. Lett. A* **2002**, *305*, 304–311. [CrossRef]
25. Gottschal, J.; Peinke, J. On the definition and handling of different drift and diffusion estimates. *New J. Phys.* **2008**, *10*, 083034. [CrossRef]
26. Anteneodo, C.; Riera, R. Arbitrary order corrections for finite-time drift and diffusion coefficients. *Phys. Rev. E* **2009**, *80*, 031103. [CrossRef] [PubMed]
27. Friedrich, R.; Peinke, J.; Sahimi, M.; Reza Rahimi Tabar, M. Approaching complexity by stochastic methods: From biological systems to turbulence. *Phys. Rep.* **2011**, *506*, 87–162. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.