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Symmetric Circle Configurations from Regular Skeletal Polyhedra

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Abstract: The paper studies finite and infinite periodic point-circle configurations in ordinary Euclidean 3-space associated with regular skeletal polyhedra or related structures. The configurations preserve all the symmetries of the underlying polyhedron and, in most cases, are point-circle transitive. Illustrations of many highly symmetric point-circle configurations are provided.

Keywords: configuration; circle configuration; skeletal polyhedron; regular polyhedron; point-circle transitive

MSC: 51M20; 52B15

1. Introduction

Configurations of points and lines or other geometric objects have been studied extensively in the literature (see Grünbaum [1] and Pisanski & Servatius [2]). Because of their conceptual simplicity, the configurations occur in many different mathematical contexts and have been investigated from many different points of view—geometric, combinatorial, algebraic, topological, and others. Most work deals with the classical geometric case of point-line configurations in the plane or in space, or with combinatorial or topological configurations being viewed as incidence structures; for example, see Berman [3], Berman, Bokowski, Grünbaum and Pisanski [4], and Coxeter [5].

In this article, we explore point-circle configurations in ordinary Euclidean 3-space $\mathbb{E}^3$ associated with regular skeletal polyhedra or related discrete structures. The study of skeletal polyhedra was initiated in Grünbaum [6], although the term "skeletal polyhedron" was not used at that time. These skeletal polyhedra (or more generally, skeletal complexes) are finite or infinite periodic geometric graphs, which are equipped with additional polyhedral super-structure imposed by its polygon faces and are allowed to be finite (planar or skew) or infinite (linear, zigzag, or helical) polygons. A polyhedron is regular if its geometric symmetry group is transitive on the flags (incident vertex–edge–face triples). There are 48 regular skeletal polyhedra comprising 18 finite polyhedra (the five Platonic solids and four Kepler–Poinsot polyhedra, and their Petrie duals), and 30 infinite apeirohedra; for the full classification, see McMullen and Schulte [7] (Ch. 7E) or Grünbaum [6], Dress [8], McMullen [9]. These relatively new regular figures exhibit fascinating geometric, combinatorial, and algebraic properties and have applications in crystal chemistry [10].

The regular skeletal polyhedra and complexes give rise to highly symmetric point-circle configurations in $\mathbb{E}^3$. In this paper, we investigate the geometric properties of point-circle configurations, which are derived from a polyhedron by what we call the E-construction. Our approach and construction methods were inspired by the work in

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Gévay and Pisanski [11] and Gévay, Bašić, Kovič, and Pisanski [12], specifically by the V-construction of [11]. As we shall see, when applied to regular skeletal polyhedra with planar vertex figures, the V-construction also leads to highly symmetric point-circle configurations. The point-circle configurations obtained by our methods inherit the symmetry of the underlying polyhedron or complex and, in most cases, are point-circle transitive under their symmetry group.

The paper is organized as follows. In Section 2, we begin by reviewing the basic notions for configurations. In Section 3, E-construction is introduced in both combinatorial and geometric contexts. The skeletal regular polyhedra and their basic properties and classification are described in Section 4. In Section 5, we discuss the point-circle configurations associated with these regular polyhedra and investigate in detail the configurations derived via the E-construction from the polyhedra in the four possible classes: finite, planar, blended, and pure polyhedra. We provide attractive illustrations (generated using Mathematica [13]). In Section 5.4, we explain why in spite of their extremely high symmetry, chiral polyhedra will never lead to point-circle configurations via the E-construction or the V-construction. Section 6 concerns V-constructed circle configurations from regular polyhedra. The final Section 7 summarizes concluding remarks and proposes two open research problems.

2. Basic Notions

In this section, we briefly review the basics of combinatorial and geometrical configurations and introduce notation used later in the text.

Although most combinatorial configurations we encounter can be described in terms of a set of distinct "points" and a set of "blocks" in which the blocks can be identified uniquely with subsets of the set of points, it is useful to introduce terminology that allows us to deal with more general situations.

Following [2], an *incidence structure* is a triple $(\mathcal{P}, \mathcal{B}, \mathcal{I})$ where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets whose elements are called *points* or *blocks*, respectively, and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$ is an incidence relation. The *degree* of a point $p \in \mathcal{P}$ is defined to be the number of blocks incident to p, and the *degree* of a block $b \in \mathcal{B}$ is defined to be the number of points incident to b. We allow these numbers to be infinite if $\mathcal{P}$ or $\mathcal{B}$ are infinite sets. An incidence structure naturally determines a bipartite graph with vertex set $\mathcal{P} \cup \mathcal{B}$ and parts $\mathcal{P}$ and $\mathcal{B}$, such that $p \in \mathcal{P}$ and $b \in \mathcal{B}$ are joined by an edge if and only if p is incident to b. This graph is called the *Levi graph* of the incidence structure [2].

A *combinatorial configuration* $(\mathcal{P}, \mathcal{B})$ is an incidence structure, $(\mathcal{P}, \mathcal{B}, \mathcal{I})$, for which $\mathcal{B}$ is a collection of non-empty subsets of $\mathcal{P}$ and the incidence relation $\mathcal{I}$ is given by standard inclusion. Thus, for point $p \in \mathcal{P}$ and block $b \in \mathcal{B}$, point p and block b are "incident" if and only if $p \in b$. If $|\mathcal{P}| = m$, $|\mathcal{B}| = n$, then $(\mathcal{P}, \mathcal{B})$ is called an (m_r, n_s)-*configuration* if every point of P lies in exactly r blocks and every block in B contains exactly s points of P. We allow the numbers m, n, r, s to be infinite if $\mathcal{P}$ is infinite.

An *automorphism* of a combinatorial configuration $(\mathcal{P}, \mathcal{B})$ is a bijection $\gamma : \mathcal{P} \cup \mathcal{B} \to \mathcal{P} \cup \mathcal{B}$ such $\gamma(\mathcal{P}) = \mathcal{P}$, $\gamma(\mathcal{B}) = \mathcal{B}$, and for all $p \in P$, $b \in B$ we have $p \in b$ if and only if $\gamma(p) \in \gamma(b)$. The *automorphism group* of $(\mathcal{P}, \mathcal{B})$ consists of all of its automorphisms.

Geometric configurations are essentially geometrically realized combinatorial configurations. Here, "realization" will mean that the elements of $\mathcal{P}$ are represented by actual points in a Euclidean space, in our case of dimension 3, while the blocks are represented by lines, planes, circles, ellipses, spheres, ellipsoids, etc., along with standard geometric incidence (inclusion) as underlying incidence structure. We are primarily interested in representations of the blocks by circles. While our terminology largely aligns with that in [2], there will be instances where we use different terms.

A *point-circle configuration* $\mathcal{C} = (P, C)$ in $\mathbb{E}^3$ consists of a set of distinct points P in $\mathbb{E}^3$ and a set of distinct circles C in $\mathbb{E}^3$ such that every point of P lies on a circle of C, and every circle of C contains a point of P. As for combinatorial configurations, if $|P| = m$, $|C| = n$, then $\mathcal{C}$ is said to be an (m_r, n_s)-*configuration* if every point of P lies on r circles and every circle of C contains s points of P. Again m, n, r, s are allowed to be infinite.

Most research on circle configurations concerns planar configurations. For example, the Miquel configuration in the Euclidean plane has eight points and six circles such that each point lies on three circles and each circle contains four points; its Levi graph is the skeleton of both the rhombic dodecahedron and the Bilinski dodecahedron [2]. Our configurations will usually not be planar. The point-circle configuration in Figure 1 lies in 3-space and is associated with the dodecahedron (via the E-constructed).

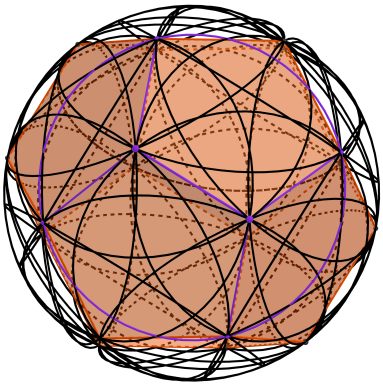


Figure 1. A circle configuration in $\mathbb{E}^3$ associated with the regular dodecahedron.

Most of our configurations are highly symmetric. A *symmetry* g of a point-circle configuration $\mathcal{C} = (P, C)$ in $\mathbb{E}^3$ is an isometry of $\mathbb{E}^3$ which preserves $\mathcal{C}$ (that is, $g(P) = P$ and $g(C) = C$). The *symmetry group* $G(\mathcal{C})$ consists of all symmetries of $\mathcal{C}$.

A point-circle configuration $\mathcal{C} = (P, C)$ is said to be *point transitive* or *circle transitive* if $G(\mathcal{C})$ acts transitively on the set of points P or the set of circles C, respectively. We call $\mathcal{C}$ *point-circle transitive* if $G(\mathcal{C})$ acts transitively on the set of all incident pairs consisting of a point and a circle.

A *point-circle realization* of a combinatorial configuration $(\mathcal{P}, \mathcal{B})$ in $\mathbb{E}^3$ is a point-circle configuration $\mathcal{C} = (P, C)$ in $\mathbb{E}^3$ together with bijections $\pi : \mathcal{P} \to P$ and $\beta : \mathcal{B} \to C$ such that $p \in b$ holds within $(\mathcal{P}, \mathcal{B})$ if and only if $\pi(p)$ lies on $\beta(b)$. Thus, informally speaking, the elements in $\mathcal{P}$ are realized by actual points in $\mathbb{E}^3$, the blocks in $\mathcal{B}$ are realized by circles in $\mathbb{E}^3$, and a point-block pair is incident in $(\mathcal{P}, \mathcal{B})$ if and only if the corresponding point-circle pair is incident in $\mathbb{E}^3$.

If the underlying set of points is clear from the context, we often simply refer to a point-circle configuration as a circle configuration. A point-circle configuration is said to be *isometric* if all its circles have the same radius. Every circle-transitive point-circle configuration is isometric, but vice versa, isometric point-circle configurations may not be circle-transitive.

3. The E-Construction

Many geometric structures give rise to interesting circle configurations. For example, if Q is a 3-dimensional convex polyhedron (with all its vertices) inscribed in a sphere in $\mathbb{E}^3$ (see [14]), then each face of Q is inscribed in a circle lying in this sphere, and the family of all such circles for faces of Q forms a circle configuration with point set given by the vertex set of Q. In this configuration, a point lies on a circle if and only if the point is the vertex of the corresponding face of Q. Thus, the combinatorics of the configuration is entirely

determined by Q. Every vertex-transitive convex polyhedron is naturally inscribed in a sphere and gives a circle configuration of this kind. Examples are the Platonic solids and Archimedean solids.

Similarly, if all faces of a convex polyhedron in $\mathbb{E}^3$ are regular polygons, then the family of in-circles of the faces of Q form a circle configuration whose point set is given by the edge midpoints of Q. In this configuration, only two circles meet at a point.

A more interesting construction involves the vertex figures of convex polytopes or other geometric figures. This is a particular instance of the V-construction discussed in Gévay and Pisanski [11] and Gévay [15]. Let Q be a convex polyhedron such that the vertices of Q adjacent to any given vertex v of Q lie in a plane and thus span a convex polygon called the vertex figure of Q at v. Suppose the vertex figure at each vertex of Q is inscribed in a circle. Then the family of all such circles for vertices of Q forms a circle configuration whose underlying point set is the vertex set of Q. For example, every simple convex polyhedron (with only 3-valent vertices) gives rise to such a configuration.

In Section 6, we revisit the V-construction in the context of regular skeletal polyhedra and derive highly symmetric finite or infinite circle configurations.

This article primarily investigates point-circle configurations obtained by what we call the *E-construction*. Here, "E" stands for "edge". At the combinatorial level, the E-construction proceeds from a combinatorial object such as a convex or abstract polyhedron, a map on a surface, or just a graph (with certain properties) and derives from it a combinatorial configuration whose point set is the vertex set of the original object and whose blocks are determined by the edges of the original object and consist of the vertices in "edge-neighborhoods". This combinatorial configuration may or may not be realizable by a circle configuration.

The E-construction works in fairly general contexts, and variations of the construction are possible as well. We begin by explaining the E-construction for convex polyhedra, and we explore other kinds of polyhedra after they have been introduced below.

Suppose Q is a convex polyhedron in $\mathbb{E}^3$ with vertex set $V(Q)$ and edge set $E(Q)$. For an edge e of Q, denote by $N(e)$ the set of vertices of Q which are adjacent to endpoints of e and lie in a face of Q containing e, but are not vertices of e. We call $N(e)$ the *edge neighborhood* of e, and let $B(Q)$ denote the family of these edge neighborhoods,

$$B(Q) := \{N(e) | e \in E(Q)\}. \tag{1}$$

In Figure 2, the two faces F_1 and F_2 of Q meet at edge e, and $N(e) = \{v_1, \cdots, v_4\}$. As every edge of Q lies in two faces of Q, each set $N(e)$ contains 2, 3, or 4 vertices, according to whether both faces at e are triangles, one face at e is a triangle, or no face at e is a triangle. There are instances of polytopes in which two edges e and e' can share the same edge neighborhood, that is, $N(e') = N(e)$; for example, each pair of antipodal edges of the 3-cube has this property. However, in almost all cases we are interested in, the edge neighborhoods $N(e)$ contain 3 or 4 vertices and uniquely determine the respective edges, e. In these cases $(V(Q), B(Q))$ gives a combinatorial configuration referred to as the *combinatorial E-configuration* of Q.

If we adopted a broader definition of combinatorial configuration and allowed blocks to be counted with multiplicity, then $(V(Q), B(Q))$ would also be a combinatorial configuration in the cases where edge neighborhoods do not uniquely determine the respective edge.

We are particularly interested in the case when no two triangular faces of Q share an edge (so that each edge neighborhood has at least three points), and the points in $N(e)$ are

coplanar and lie on a circle $c(e)$ for each edge e of Q. Then, up to allowing a multiplicity of circles, this gives a point-circle configuration $\mathcal{C}(Q) := (V(Q), C(Q))$, with

$$C(Q) := \{c(e) | e \in E(Q)\}. \tag{2}$$

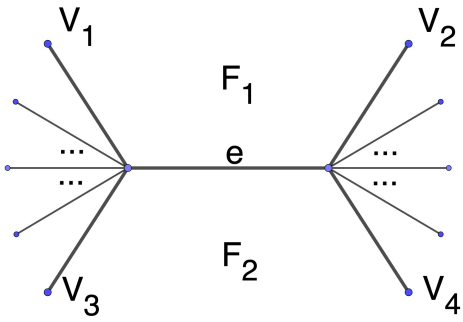


Figure 2. The edge neighborhood $N(e)$ of edge e, which consists of $v_1, \ldots, v_4$.

It is straightforward to generalize the E-construction from convex polyhedra to abstract polyhedra (see [7]). An *abstract polyhedron* is a (finite or infinite) partially ordered set Q with elements of *rank* j for $j = -1$, 0, 1, 2, or 3, called *j-faces* of Q; for $j = 0$, 1, or 2, the j-faces are also called vertices, edges, and faces (2-faces), respectively. The faces of ranks -1 and 3 are unique and are minimal and maximal, respectively; that is, if these faces are F_{-1} and F_3, respectively, then $F_{-1} \leq F \leq F_3$ for all F in Q. The faces F_{-1} and F_3 are called the *improper* faces of Q. (For convex polyhedra, the (-1)-face F_{-1} is the empty set, $\emptyset$, and the 3-face F_3 is the polyhedron itself. In discussing polyhedra, we will usually suppress the improper faces of ranks -1 and 3). A *flag* of Q is an incident triple consisting of a vertex, an edge, and a face. Further, Q is *strongly flag-connected*, meaning that any two flags Φ and Ψ of Q can be joined by a sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$, where Φ_{i-1} and Φ_i are *adjacent* (differ by one face), and $\Phi \cap \Psi \subseteq \Phi_i$ for each i. Finally, if F and G are a $(j-1)$-face and a $(j+1)$-face with $F < G$ and $0 \leq j \leq 2$, then there are exactly *two* j-faces H such that $F < H < G$. Thus, edges have two vertices, and edges lie in two faces. We usually identify a face F of Q with $\{G \mid G \leq F\}$ and note that this partially ordered set is isomorphic to (the face lattice of) a convex polygon or the apeirogon, again ignoring improper faces. Similarly, if F is a vertex, then $\{G \mid F < G\}$, called the *vertex figure* at F, is also isomorphic to a convex polygon or the apeirogon. Note that an abstract polyhedron with finite faces and vertex figures gives rise to a map on a surface, and it is often convenient to identify the polyhedron with this map.

The E-construction for abstract polyhedra works in much the same way as for convex polytopes. Suppose Q is an abstract polyhedron with vertex set $V(Q)$ and edge set $E(Q)$, and no two triangular faces of Q share an edge. For an edge e of Q, we let $N(e)$ denote the *edge neighborhood* of e, that is, the set of vertices of Q which are adjacent to a vertex of e and lie in a face of Q at edge e, but are not vertices of e; see again Figure 2. As before, we let $B(Q)$ denote the family of the edge neighborhoods,

$$B(Q) := \{N(e) \mid e \in E(Q)\}.$$

Then, with the same reservations with regards to multiplicity of blocks as in the case of convex polyhedra, $(V(Q), B(Q))$ is a combinatorial configuration referred to as the *combinatorial E-configuration* of Q.

Note that if the point v of the configuration $(V(Q), B(Q))$ is a vertex of Q surrounded by q 2-faces of Q, where none of which is a triangle, then v lies in $2q$ blocks $N(e)$. In fact, the $2q$ edges e occurring here are those edges of Q two steps away from v, in either direction, along the boundary of a 2-face of Q at v. If triangle faces occur at v, then each such face contributes only one edge to the count. For example, if q is even and triangle and non-triangle faces of Q alternate around v, then the point v lies in $3q/2$ blocks $N(e)$. Bear in mind that we are excluding the possibility that two triangle faces meet at an edge.

We are particularly interested in highly symmetric polyhedra and their respective E-configurations. In this case, the configurations usually inherit high symmetry.

An abstract polyhedron Q is *regular* if its (*combinatorial*) *automorphism group* $\Gamma(Q)$ is transitive on the flags. In this case, the automorphism group $\Gamma(Q)$ is generated by three involutions ρ_0, ρ_1, ρ_2 uniquely determined by the way in which they act on a chosen *base* flag. More specifically, ρ_0, ρ_1, and ρ_2 map this base flag to the flag differing from the base flag in its vertex, edge, or face, respectively; thus, ρ_i map the base flag to its "i-adjacent" flag (differing from the original flag in its i-face). If Q is of Schläfli type $\{p, q\}$ (that is, the faces are p-gons and the vertex figures are q-gons), then these generators satisfy the standard relations

$$\rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^p = (\rho_1\rho_2)^q = (\rho_0\rho_2)^2 = 1,$$

but in general, other independent relations too. Observe that, in a natural way, the group of the base face of Q is $\langle \rho_0, \rho_1 \rangle$, while that of the (base) vertex figure is $\langle \rho_1, \rho_2 \rangle$.

For an abstract regular polyhedron Q, the stabilizer of the base edge, e (say), leaves $N(e)$ invariant and is given by the subgroup $\langle \rho_0, \rho_2 \rangle$ isomorphic to $C_2 \times C_2$. In the notation of Figure 3, the generators ρ_0 and ρ_2 act on $N(e)$ like the permutations

$$(v_1\, v_2)(v_3\, v_4),\ (v_1\, v_3)(v_2\, v_4),$$

respectively. In particular, $\langle \rho_0, \rho_2 \rangle$ acts transitively on the four points $v_1, \ldots, v_4$.

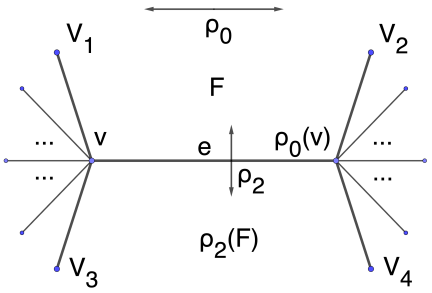


Figure 3. The action of $\langle \rho_0, \rho_2 \rangle$ on $N(e)$.

Moreover, as Q is regular, the configuration $(V(Q), B(Q))$ is point-block transitive under the automorphism group of Q. In fact, in this case $\Gamma(Q)$ acts transitively on the edges of Q and thus on the blocks in $B(Q)$, and the stabilizer of an edge in Q also stabilizes the corresponding block in $B(Q)$ and acts transitively on the four points of this block. If Q is of type $\{p, q\}$ (with $p \neq 3$), then each point of $(V(Q), B(Q))$ lies in $2q$ blocks.

An abstract polyhedron Q is called (combinatorially) *chiral* if $\Gamma(Q)$ has two orbits on the flags, such that adjacent flags are in distinct orbits. If Q is chiral, then each point also lies in $2q$ blocks if Q has type $\{p, q\}$ (with $p \neq 3$), but now an edge stabilizer in $\Gamma(Q)$ has two orbits on the points of the corresponding block.

A *Petrie polygon* of an abstract regular polyhedron Q is a path along edges such that any two, but no three, successive edges belong to a 2-face of Q. Any two Petrie polygons of an abstract regular polyhedron Q are equivalent under $\Gamma(Q)$. The *Petrie dual* of an abstract regular polyhedron Q has the same vertices and edges as Q, but has as its 2-faces the Petrie polygons of Q. This is again a polyhedron, denoted Q^π, if and only if a Petrie polygon visits any vertex of Q at most once (see [7] (Lemma 7B3)). In this case Q^π is an abstract regular polyhedron.

The following lemma follows directly from the definition of the Petrie dual.

Lemma 1. *An abstract regular polyhedron Q and its Petrie-dual Q^π (if this is a polyhedron) have isomorphic combinatorial E-configurations.*

This lemma will apply in particular to the skeletal regular polyhedra introduced in the next section. In this case, the E-configurations of a polyhedron and its Petrie-dual will be identical.

We remark that circle configurations associated with maps (or polyhedra) on surfaces have also been studied by Izquierdo and Stokes [16]. For interesting combinatorial configurations with cyclic or polycyclic symmetries, see also [17].

4. Skeletal Polyhedra

In this section, we briefly review the regular and chiral skeletal polyhedra in $\mathbb{E}^3$ (see [7] (Ch. 7E) and [18,19]).

Informally, a skeletal polyhedron in $\mathbb{E}^3$ is a collection of vertices, edges, and finite or infinite polygons, all assembled in a manner typical for geometric polyhedra. By a *finite polygon*, or an *n-gon*, we mean a figure formed by distinct points $v_1, \ldots, v_n$, called the *vertices* of the polygon, along with the line segments (v_i, v_{i+1}), for each i, and the line segment (v_n, v_1), called the *edges* of the polygon. Similarly, an *infinite polygon* is a figure consisting of infinitely distinct points, $\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots$, called the *vertices* of the polygon, and of line segments (v_i, v_{i+1}) for each i, called the *edges* of the polygon, such that each compact subset intersects only finitely many line segments.

A *skeletal polyhedron*, or a *geometric polyhedron*, Q in $\mathbb{E}^3$ similarly has three kinds of elements: vertices, edges, and faces. More precisely, Q consists of a set V of points, called the *vertices*, a set E of line segments, called the *edges*, and a set F of polygons, called the *faces*, with the following properties. The graph (V, E) determined by V and E is connected; this graph is called the *edge graph* of Q. The vertex figure at the vertices of Q are cycles. Recall that the *vertex figure* at a vertex v is a graph whose vertex set consists of the neighbors of v in Q and whose edge set is given by the line segments (u, w) where u, w are neighbors of v such that (u, v) and (v, w) are edges of a common face of Q. Further, each edge of Q lies in exactly two faces of Q. Finally, Q is *discrete*, in the sense that each compact set meets only finitely many faces of Q.

Given a polyhedron Q, the *symmetry group* $G = G(Q)$ of Q consists of all isometries of the ambient space, $\mathbb{E}^3$, or the affine hull of Q if Q is planar, which leave Q invariant. By a *flag* of Q we mean an incident triple consisting of a vertex, an edge, and a face of Q. Two flags of Q are called *0-adjacent*, *1-adjacent*, or *2-adjacent*, respectively, if they differ in their their vertices, edges, or faces. Two flags of Q are said to be *adjacent* if they are j-adjacent for some $j = 0, 1, 2$.

A skeletal polyhedron Q is called (geometrically) *regular* if $G(Q)$ is transitive on the flags of Q. A skeletal polyhedron Q is said to be (geometrically) *chiral* if $G(Q)$ acts with two orbits on the flags of Q such that any two adjacent flags are in different orbits.

The regular skeletal polyhedra have been completely classified and are also known as the *Grünbaum–Dress polyhedra* (see Grünbaum [6], Dress [8], McMullen & Schulte [7,20] (Ch. 7E), and McMullen [9]. Using the classification scheme of [7] (Ch. 7E), the altogether 48 regular polyhedra naturally fall into the following classes. There are eighteen finite polyhedra, namely the five Platonic solids and the four Kepler–Poinsot polyhedra (see [21]), which are also referred to as the *classical regular polyhedra*, and their Petrie duals, as well as six planar polyhedra, namely the regular tessellations by triangles, squares, and hexagons, and their Petrie-duals. There also are twelve "blended" polyhedra, denoted $P\#L$, obtained by blending the six planar polyhedra P with either a line segment or a linear apeirogon L positioned on a line perpendicular to the plane of the planar polyhedron. Strictly speaking, the geometry of the blended polyhedra depends on the ratio of the edge lengths of the planar and the linear components, but if these ratios are discounted, there are twelve polyhedra. Finally, there are twelve "pure" regular polyhedra. The geometry of some regular skeletal polyhedra is somewhat more involved.

For a regular polyhedron Q in $\mathbb{E}^3$, the symmetry group $G(Q)$ is generated by three geometric reflections r_0, r_1, r_2 in points, lines, or planes. Suppose $\{v, e, f\}$ is a fixed flag of Q, called the *base* flag of Q. Then the generators of $G(Q)$ can be chosen in such a way that r_0, r_1, and r_2 map the base flag to the flag differing in v, e, or f, respectively; thus, the generator r_j of $G(Q)$ maps the base flag to its j-adjacent flag. The nine classical regular polyhedra (Platonic solids and Kepler–Poinsot polyhedra) are the only regular polyhedra for which all three generators r_0, r_1, r_2 are plane reflections. If Q has Schläfli symbol $\{p, q\}$ (recall that this means that the faces of Q are p-gons and the vertex figures are q-gons), then the symmetry group $G(P)$ satisfies the Coxeter type relations

$$r_0^2 = r_1^2 = r_2^2 = (r_0 r_1)^p = (r_1 r_2)^q = (r_0 r_2)^2 = 1, \tag{3}$$

but usually other independent relations too. It is convenient to describe regular skeletal polyhedra by padded Schläfli symbols obtained from the basic Schläfli symbol $\{p, q\}$ by adding symbols that signify specific types of information (such as extra defining relations for the symmetry group). The *mirror vector* (or *dimension vector*) of a regular polyhedron Q records the dimensions of the three reflection mirrors (fixed point sets) and is denoted as $(\dim r_0, \dim r_1, \dim r_2)$.

The chiral skeletal polyhedra in $\mathbb{E}^3$ have also been completely classified (see [18]). Chiral polyhedra do not occur in the classical theory, and no finite, planar, or "blended" examples exist. There are six infinite families of chiral polyhedra (apeirohedra) in $\mathbb{E}^3$, each with two or one (integral or real) parameters depending on whether the classification scheme is by congruence or similarity of polyhedra, respectively. In three families, the polyhedra have finite skew faces; in the other three, the polyhedra have helical faces. For certain exceptional parameter values, the respective polyhedra become regular and, in some cases, finite. Each family has two exceptional parameter values, and it is convenient to include the two polyhedra in the family. Then, the six resulting larger families will contain exactly two regular polyhedra, while all other polyhedra are chiral.

For a chiral polyhedron Q in $\mathbb{E}^3$, the symmetry group $G(Q)$ is generated by two isometries s_1, s_2, which, as in the above, are determined by their action on a base flag $\{v, e, f\}$ of Q. Suppose v' denotes the vertex of e distinct from v, and f' the face at e distinct from f. Then the generator s_1 fixes the base face f while permuting the vertices of f cyclically so that $s_1(v) = v'$; and s_2 stabilizes v while permuting the vertices in the vertex figure at v cyclically so that $s_2(f') = f$. These generators s_1, s_2 are non-involutory and satisfy the relations

$$s_1^p = s_2^q = (s_1 s_2)^2 = 1,$$

among others, where, as in the above, $\{p, q\}$ is the basic Schläfli type of Q.

Just as a geometric polyhedron Q can be viewed as a realization of its underlying abstract polyhedron, Q' (say), there is a geometric configuration that realizes the combinatorial E-configuration $(V(Q'), B(Q'))$ of Q' on Q itself. Its point set is the vertex set $V(Q)$ of Q, and its block set $B(Q)$ consists of the subsets of $V(Q)$ corresponding to blocks in $B(Q)$. Thus $(V(Q), B(Q))$ realizes $(V(Q'), B(Q'))$ on Q itself. We say that Q satisfies the *coplanarity condition* if the points of each block in $B(Q)$ are coplanar.

5. E-Constructed Point-Circle Configurations

In this section we study circle configurations derived from regular skeletal polyhedra Q in $\mathbb{E}^3$ via the E-construction. In each case, we proceed from the configuration $(V(Q), B(Q))$ of Q and attempt to represent the blocks by circles, each passing through the points of the block.

First recall from Lemma 1 that it suffices to consider only one polyhedron from each pair of Petrie duals. Leaving aside the pairs involving a polyhedron with triangle faces, which are excluded here, we find that for all but four pairs of Petrie duals, the point sets of blocks are coplanar and lie on a circle, and thus the pair gives a highly symmetric circle configuration. Further, as Q is edge-transitive, the coplanarity of the point sets of blocks can be verified for a single edge.

The following lemma helps verify the coplanarity condition for most regular polyhedra.

Lemma 2. *Let Q be a regular polyhedron in $\mathbb{E}^3$ of type $\{p, q\}$, $p \neq 3$, with symmetry group $G(Q) = \langle r_0, r_1, r_2 \rangle$.*
(a) The points of each block in $B(Q)$ lie on a sphere centered at the midpoint of the respective edge of Q, of radius independent of the respective edge.
(b) If r_0 or r_2 are plane reflections, then Q satisfies the coplanarity condition, and the points of each block in $B(Q)$ lie on a circle of radius independent of the respective edge.

Proof. Let e be the base of Q. Then the stabilizer of e in $G(Q)$ is given by $\langle r_0, r_2 \rangle$, and in the notation of Figure 3, the generators r_0 and r_2 act on $N(e) = \{v_1, \ldots, v_4\}$ like the permutations $(v_1\, v_2)(v_3\, v_4)$ and $(v_1\, v_3)(v_2\, v_4)$, respectively. In particular, $\langle r_0, r_2 \rangle$ fixes the midpoint of e and acts simply transitively on $N(e)$. Thus, the points of $N(e)$ are equidistant from the midpoint of e and lie on a sphere centered at this point.

If r_0 is a plane reflection, its reflection plane is the perpendicular bisector of e and the lines $\overline{v_1 v_3}$ and $\overline{v_2 v_4}$ are mapped onto each other under r_0. Thus, $v_1, \ldots, v_4$ are coplanar. Similarly, if r_2 is a plane reflection, then e lies in the reflection plane and r_2 maps the lines $\overline{v_1 v_2}$ and $\overline{v_3 v_4}$ onto each other, which proves that $v_1, \ldots, v_4$ is coplanar. In either case, as the points $v_1, \ldots, v_4$ lie on a sphere about the midpoint of e, they also lie on a circle.

The remaining statements follow from the edge transitivity of $G(Q)$ and the fact that a symmetry of Q taking e to an edge e' also takes $N(e)$ to $N(e')$. □

Note that Lemma 2(a) shows that, regardless of coplanarity, each regular polyhedron Q gives rise to an E-constructed point-sphere configuration (or point-circle configuration if Q is planar), with congruent spheres (circles) centered at the midpoints of the edges of Q.

Theorem 1. *Let Q be a regular polyhedron in $\mathbb{E}^3$ of type $\{p, q\}$. $p \neq 3$, with symmetry group $G(Q) = \langle r_0, r_1, r_2 \rangle$, and suppose Q satisfies the coplanarity condition.*
(a) Then the E-construction applied to Q yields a circle configuration $\mathcal{C}(Q) = (V(Q), C(Q))$ on which $G(P)$ acts point-circle transitively.

(b) If Q has the property that different edges determine different blocks $N(e)$, then $G(P)$ acts simply point-circle transitively on $\mathcal{C}(Q)$ (that is, with trivial point-circle stabilizers).

Proof. Let e denote the base edge of Q, and let $c(e)$ denote the circle determined by e. By the edge-transitivity of $G(Q)$ on Q, the circle configuration is circle transitive under $G(Q)$. Now consider the stabilizer of $c(e)$ in $G(Q)$. Clearly, the stabilizer $\langle r_0, r_2 \rangle$ of e in $G(Q)$ also stabilizes $c(e)$, and this group is isomorphic to $C_2 \times C_2$ and acts simply transitively on the four points on $c(e)$. This establishes the first part.

For the second part, as the circles are uniquely determined by their edges in Q, any element of $G(Q)$ stabilizing the circle $c(e)$ will also fix the base edge e and thus belong to $\langle r_0, r_2 \rangle$. It follows that the circle stabilizer coincides with the edge stabilizer. Thus $G(Q)$ acts simply point-circle transitively on the circle configuration. □

The cube is an example where two (opposite) edges determine the same circle (see Figure 4). In this case, $G(Q)$ still acts point-circle transitively, but the circle stabilizers also contain the reflection in the plane of the circle and have order 8.

In Figure 4 and the figures below, colors have been added for visual clarity and better distinction of geometric components.

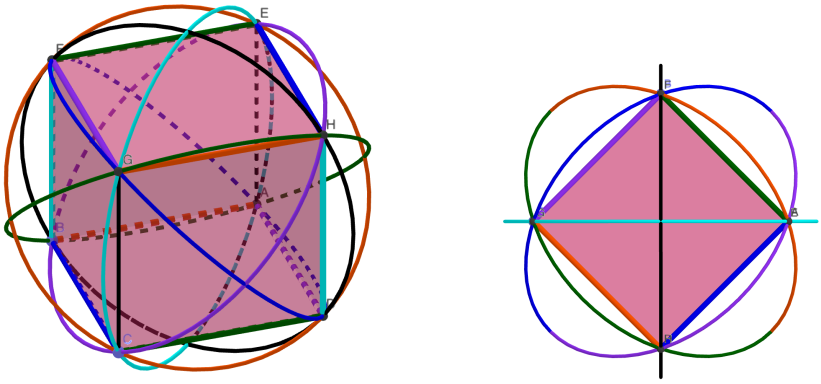


Figure 4. Circle configuration for the cube. The colors of the circles and of their corresponding edges of the cube match. A view from the "top" is shown on the right.

As described in Section 4, the regular skeletal polyhedra fall into four classes consisting of finite, planar, blended, or pure polyhedra, respectively. We discuss each class separately. In each case the polyhedron Q can be constructed from its symmetry group $G(Q) = \langle r_0, r_1, r_2 \rangle$ by Wythoff's construction. This proceeds from a common fixed point v of the generators r_1 and r_2, which is taken as the base vertex v of Q, and constructs the base edge e as the line segment $(v, r_0(v))$ and the base face f as $\{s(e) \mid s \in \langle r_0, r_1 \rangle\}$. The vertices, edges, and faces of the polyhedron are then the images of the base vertex, base edge, and base face under $G(Q)$. The construction of the circle configuration $C(Q)$ for Q can proceed similarly from an incident pair consisting of a *base point* and a *base circle*: the vertex $r_1 r_0(v)$ of Q lies on $c(e)$ and can serve as the base point of the configuration, and the circle $c(e)$ corresponding to the base edge e of Q can serve as the *base circle*. The entire configuration $C(Q)$ then consists of the images of the base point and base circle under $G(Q)$. Alternatively, one could find the points and circles of $C(Q)$ as the images of v and $c(e)$, respectively (but note that v does not lie on $c(e)$).

5.1. Configurations from Finite Regular Polyhedra

The eighteen finite regular polyhedra are the five Platonic solids, the four Kepler–Poinsot polyhedra, and their Petrie duals. In each case, the generator r_2 is a plane reflection,

so, in particular, Lemma 2 (b) applies. In fact, for the classical finite regular polyhedra, all three generators r_0, r_1, r_2 are plane reflections, and for their Petrie duals, the generators transition to r_0r_2, r_1, r_2, leaving r_2 unchanged.

The Petrie dual pairs of finite polyhedra can be found in [7] (p. 218). Discounting the pairs that contain a polyhedron with triangle faces, only the following five pairs can contribute a circle configuration, with π indicating Petrie duality:

$$\begin{array}{ccc} \{4,3\} & \overset{\pi}{\longleftrightarrow} & \{6,3\}_4 \\ \{5,3\} & \overset{\pi}{\longleftrightarrow} & \{10,3\} \\ \{\frac{5}{2},3\} & \overset{\pi}{\longleftrightarrow} & \{\frac{10}{3},3\} \\ \{5,\frac{5}{2}\} & \overset{\pi}{\longleftrightarrow} & \{6,\frac{5}{2}\} \\ \{\frac{5}{2},5\} & \overset{\pi}{\longleftrightarrow} & \{6,5\}. \end{array} \tag{4}$$

The case of the cube $\{4,3\}$ (and its Petrie dual $\{6,3\}_4$) is special in that the circle configuration has fewer circles than the polyhedron has edges. In this case, opposite edges of the cube determine the same circle, with the same four points on it, so in a sense, the circle occurs twice. The configuration is shown in Figure 4.

The polyhedra $\{5,3\}$ and $\{\frac{5}{2},3\}$ in the second and third rows of (4) are geometrically distinct but combinatorially isomorphic. They share the same vertex sets. Still, these two geometrically distinct polyhedra share the same circle configuration, with 20 points and 30 circles. The edge sets of either polyhedron determine the sets of blocks $N(e)$, and thus, the circle configurations are the same. The configuration for $\{\frac{5}{2},3\}$ is depicted in Figure 5 and is the same as the configuration shown in Figure 1 relative to $\{5,3\}$. Again, colors have been added for visual clarity.

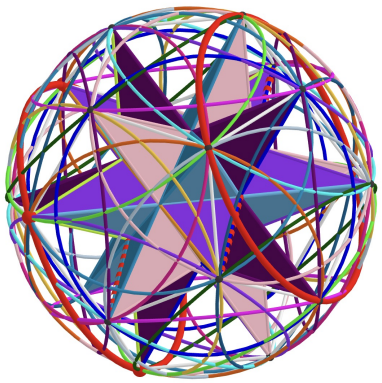


Figure 5. Circle configuration for the great stellated dodecahedron $\{\frac{5}{2},3\}$. Each red circle is associated with a dotted red edge of the polyhedron.

The circle configurations for the pair of dual polyhedra $\{5,\frac{5}{2}\}$ and $\{\frac{5}{2},5\}$ in the fourth and fifth rows of (4) also are the same, for the same reason reason as for the previous pair of polyhedra. Each has 12 points and 15 circles, each with multiplicity 2. They are shown side by side in Figure 6. The circles in each case are the equatorial circles in the planes perpendicular to the two-fold symmetry axes (through the midpoints of pairs of antipodal edges), and since the two polyhedra share the same symmetry groups and same vertex sets, the collections of circles must coincide. This provides an alternative explanation of why the two circle configurations are identical for this pair of polyhedra.

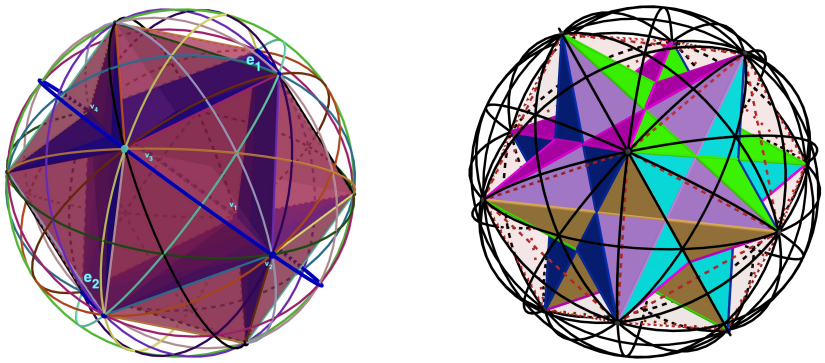


Figure 6. Circle configuration for the pair of polyhedra $\{5, \frac{5}{2}\}$, $\{\frac{5}{2}, 5\}$, shown on the left relative to the great dodecahedron $\{5, \frac{5}{2}\}$ and on the right relative to the small stellated dodecahedron $\{\frac{5}{2}, 5\}$. The circles are the equatorial circles in planes perpendicular to the axis of two-fold symmetry for the polyhedra and have been colored with different colors. In the figure on the left, the edges labelled e_1 and e_2 determine the same circle, colored blue, lying in the equatorial plane halfway between e_1 and e_2.

5.2. Configurations from Planar or Blended Regular Polyhedra

The circle configurations for the planar regular polyhedra are straightforward. These polyhedra are the three regular tessellations $\{4,4\}$, $\{3,6\}$, and $\{6,3\}$, and their Petrie duals Lemma 2. The triangular tessellation $\{3,6\}$ and its Petrie dual will not give a circle configuration, but the other pairs of Petrie duals will. The two corresponding circle configurations are shown in Figure 7.

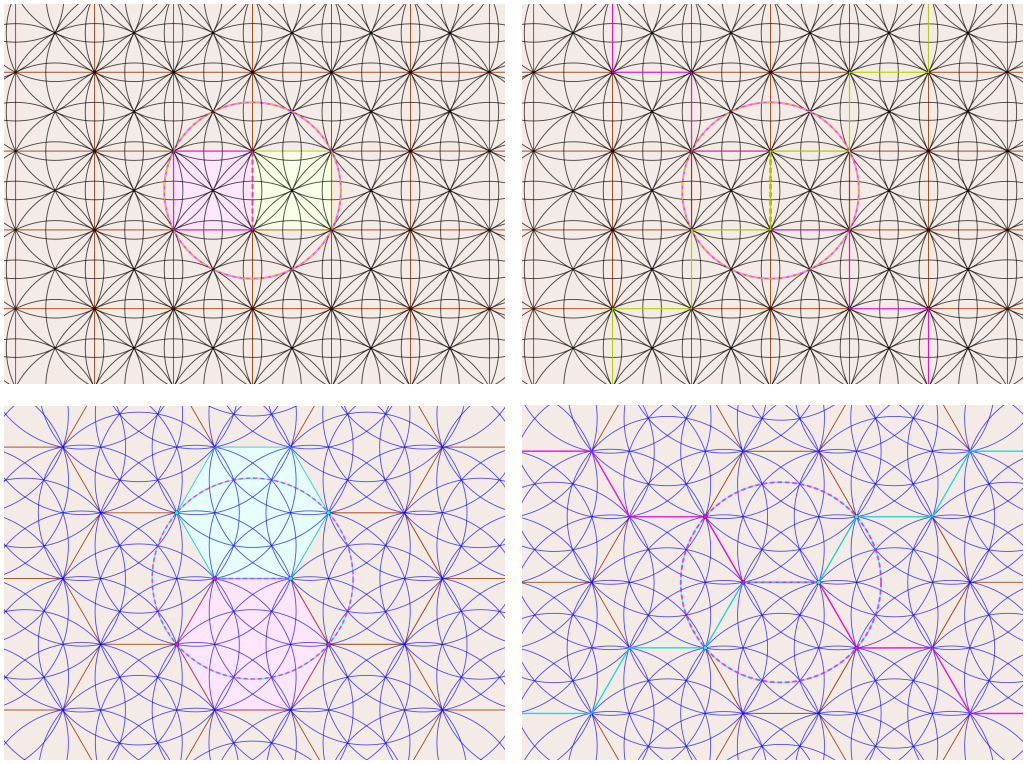


Figure 7. Circle configurations for the tessellations $\{4,4\}$ and $\{6,3\}$, each shown in two ways: relative to the tessellation itself and relative to its Petrie dual. The figures on the left highlight a pair of adjacent faces of the tessellation, along with their common edge and the circle corresponding to this edge.

The twelve blended regular polyhedra $P\#L$ are orthogonal blends of two components: a planar regular polyhedron P and a linear regular polygon L (a line segment $\{\ \}$ or an apeirogon $\{\infty\}$). In each case, the generator r_2 is a plane reflection. In fact, for a planar regular polyhedron, the generator r_2 is a reflection in a line so that, under orthogonal blending with either a line segment or an apeirogon, the generator r_2 becomes a plane reflection symmetry for the blended polyhedron. Thus, Lemma 2 (b) applies and shows that the blended polyhedra contributes six circle configurations.

More explicitly, suppose P lies in $\mathbb{E}^2$ and has symmetry group $G(P) = \langle r_0', r_1', r_2' \rangle$, and L lies on the line $\mathbb{E}^1$ and has symmetry group $G(L) = \langle r_0'' \rangle$ or $G(L) = \langle r_0'', r_1'' \rangle$ if $L = \{\ \}$ or $L = \{\infty\}$, respectively. If 3-space is taken in the form $\mathbb{E}^3 = \mathbb{E}^2 \perp \mathbb{E}^1$ with coordinates x, y, z (and with $((x, y), z)$ identified with (x, y, z)), then the generators r_0, r_1, r_2 of the symmetry group of $G(P\#L)$ are given by

$$\begin{aligned} r_0(x,y,z) &= (r_0'(x,y), r_0''(z)), \\ r_1(x,y,z) &= \begin{cases} (r_1'(x,y), y, z) \text{ if } L = \{\ \} \\ (r_1'(x,y), r_1''(x,y), z) \text{ if } L = \{\infty\}, \end{cases} \\ r_2(x,y,z) &= (r_2'(x,y), z). \end{aligned}$$

The polyhedron $P\#L$ itself is obtained from these generators of the group by applying Wythoff's construction with the initial (base) vertex $v := (x_0, y_0, z_0)$, where (x_0, y_0) and z_0 are the initial vertices of P and L, respectively.

Note that taking Petrie duals commutes with blending. In other words, $(P\#L)^\pi = P^\pi\#L$; here, the linear component remains unchanged. It follows that the six blends with line segments and the six blends with apeirogons each give three circle configurations, up to relative scaling of the components of the blend.

The three blended polyhedra $\{4,4\}\#\{\ \}$, $\{6,3\}\#\{\ \}$, and $\{3,6\}\#\{\ \}$ all lie in a slab of $\mathbb{E}^3$ bounded by a pair of parallel planes H_1 and H_2 (say), and are obtained from a pair of tessellations $\{4,4\}$, $\{6,3\}$, or $\{3,6\}$ in H_1 and H_2, respectively, positioned so that one tessellation is a translate of the other under a translation perpendicular to the planes. A typical face of the blend then is a polygon lying in the boundary of a right prism whose base and top are square, triangle, or hexagon tiles of these tessellations.

As a consequence, the three circle configurations for the blends with line segments also all lie between two parallel planes. The configurations for the polyhedra $\{4,4\}\#\{\ \}$, $\{6,3\}\#\{\ \}$, and $\{3,6\}\#\{\ \}$ are shown in Figures 8, 9 and 10, respectively.

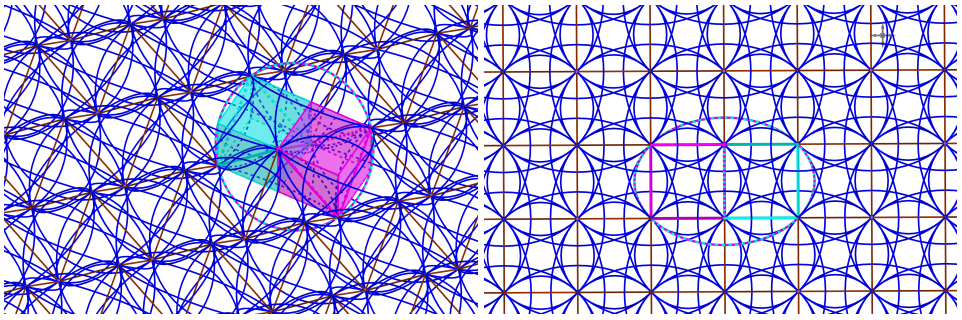


Figure 8. Circle configurations for $\{4,4\}\#\{\ \}$, the blend of a square tessellation with a line segment. The polyhedron lies in a slab between parallel planes equipped with square tessellations, each perpendicularly positioned to the other relative to the slab. Each face is a tetragon (skew square) lying in the boundary of a right prism inside the slab whose top and bottom faces are squares of the tessellations. The figure on the left highlights a pair of adjacent tetragon faces of the polyhedron, along with their common (dotted) edge and their corresponding pair of adjacent right prisms. The dotted circle corresponds to the dotted edge. The figure on the right shows a top view of the configuration.

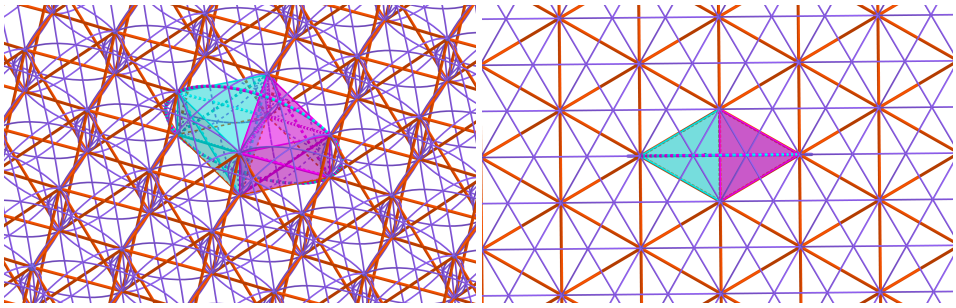


Figure 9. Circle configurations for $\{3,6\}\#\{\,\}$, the blend of a triangle tessellation with a line segment. The polyhedron lies in a slab between parallel planes equipped with triangular tessellations, each perpendicularly positioned to the other relative to the slab. Each face of the polyhedron is a skew hexagon lying in the boundary of a right triangular prism inside the slab whose top and bottom faces are triangles of the tessellations. The figure on the left highlights a pair of adjacent skew hexagon faces of the polyhedron, along with their common (dotted) edge and their corresponding pair of adjacent right triangular prisms. The dotted circle corresponds to the dotted edge. The figure on the right shows a top view of the configuration.

The three circle configurations for the blends with apeirogons, $Q := P\#\{\infty\}$, are more complicated. The facets of Q are regular helical polygons rising above the faces of the planar polyhedron P. In the E-constructed circle configuration, the circles similarly rise above the faces of P following the helical faces of Q. The three configurations for $\{4,4\}\#\{\infty\}$, $\{6,3\}\#\{\infty\}$, and $\{3,6\}\#\{\infty\}$ are shown in Figures 11–13.

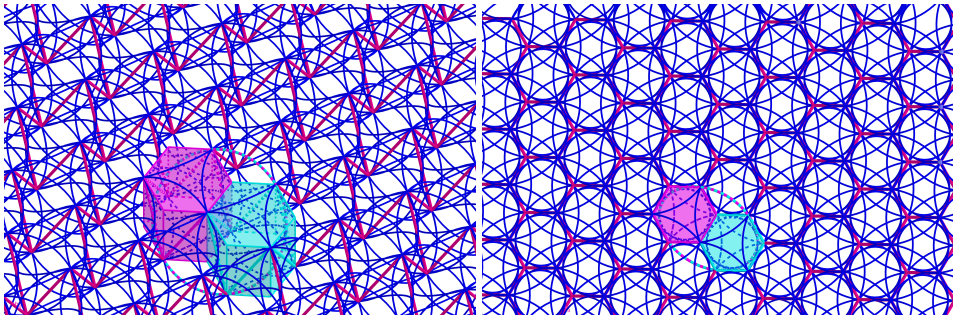


Figure 10. Circle configurations for $\{6,3\}\#\{\,\}$, the blend of a hexagonal tessellation with a line segment. The figure on the left highlights a pair of adjacent skew hexagon faces of the polyhedron and the corresponding pair of adjacent hexagonal prisms inside a slab that is formed by parallel planes equipped with suitable hexagonal tessellations. The dotted circle corresponds to the dotted edge shared by the two hexagon faces. The figure on the right shows a top view of the configuration.

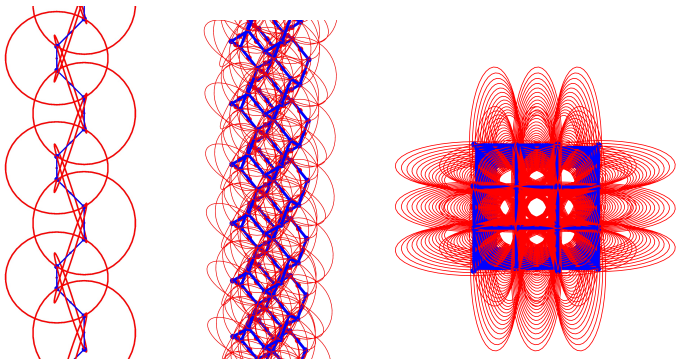


Figure 11. Circle configurations for $\{4,4\}\#\{\infty\}$, the blend of the square tessellation with an apeirogon. The figure on the left indicates how the circles associated with the edges of a single (square based) helical face follow this face as its edges are traversed. The figure in the middle shows the circles generated by a helical face and the eight neighboring helical faces; the figure on the right gives a top view of this family of circles.

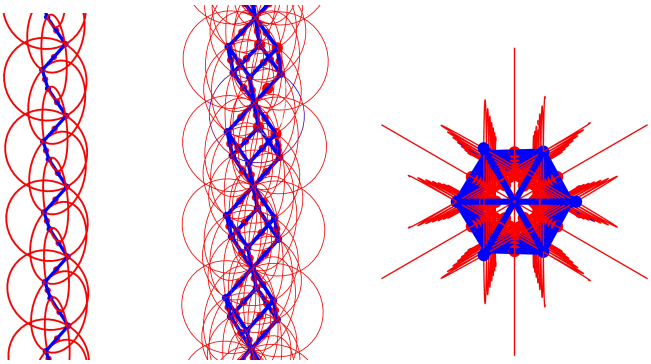


Figure 12. Circle configurations for $\{3,6\}\#\{\infty\}$, the blend of the triangular tessellation with an apeirogon. The figure on the left indicates how the circles associated with the edges of a single (triangle based) helical face follow this face as its edges are traversed. The figure in the middle shows the circles generated by six neighboring helical faces; the figure on the right gives a top view of this family of circles.

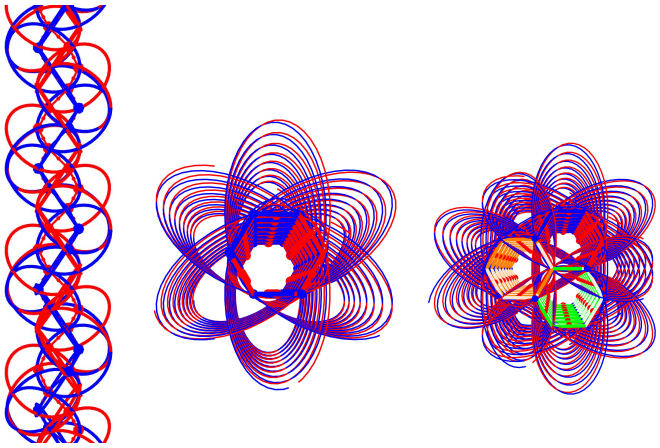


Figure 13. Circle configurations for $\{6,3\}\#\{\infty\}$, the blend of the hexagonal tessellation with an apeirogon. The helical faces of the polyhedron "rise above" the hexagons of the hexagonal plane tessellation $\{6,3\}$, with two helical faces per hexagon, and with these faces sharing every third vertex. The figure on the left indicates the circles associated with the edges of the two helical faces "over" the same hexagon, with red and blue circles arising from the red or blue helical face, respectively. The figure in the middle shows a top view of this arrangement of circles. The figure on the right shows a top view of the family of circles generated by the six helical faces sharing a vertex of the polyhedron.

5.3. Configurations from Pure Regular Polyhedra

The geometry of the twelve pure polyhedra is more involved. Their mirror vectors can be found in [7] (p. 225).

In the case of six polyhedra, the generator r_2 is a plane reflection, and we can appeal again to Lemma 2. These six polyhedra fall into three pairs of Petrie duals,

$$\begin{array}{ccc} \{4,6\,|\,4\} & \overset{\pi}{\longleftrightarrow} & \{\infty,6\}_{4,4} \\ \{6,4\,|\,4\} & \overset{\pi}{\longleftrightarrow} & \{\infty,4\}_{6,4} \\ \{6,6\,|\,3\} & \overset{\pi}{\longleftrightarrow} & \{\infty,6\}_{6,3}\,, \end{array} \tag{5}$$

each contributing a circle configuration. The polyhedra on the left in (5) are the three Petrie–Coxeter polyhedra, with convex faces and skew vertex figures. The configuration for $\{4,6\,|\,4\}$ is shown in Figure 14.

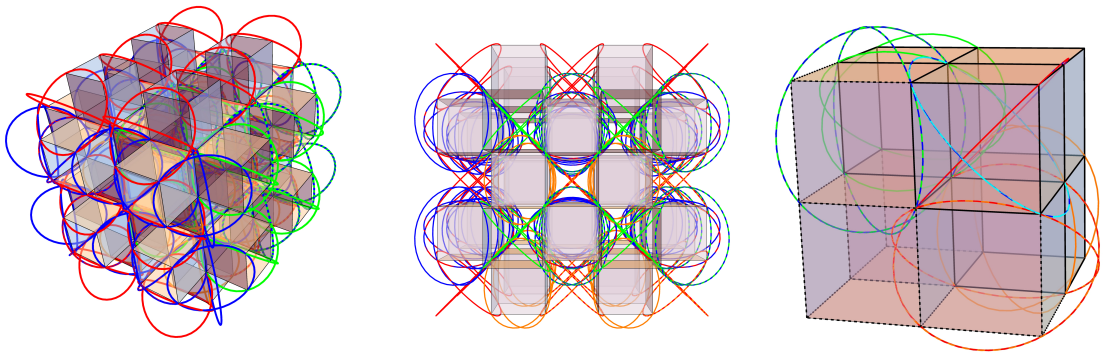


Figure 14. Two views of the circle configurations for $\{4,6\,|\,4\}$. The figure on the right shows how a $2\times2\times2$ block of cubes of the underlying cubical tessellation contributes to the circle configuration: six of the eight cubes contribute two circles per cube, and the other two do not contribute a circle.

The six remaining pure polyhedra, as well as the three Petrie–Coxeter polyhedra in (5), are precisely the nine infinite regular polyhedra that occur in the extended families of chiral and regular polyhedra described in Section 4. The coplanarity condition for these polyhedra can be analyzed using the methods described in the next section, and we refer to these here; of course, one can also check the coplanarity condition directly.

Two of these six polyhedra, $\{6,4\}_6$ and $\{\infty,4\}_{\cdot,*3}$, are self-Petrie, meaning that their Petrie duals are congruent to the original polyhedron. Interestingly, these two polyhedra will not lead to circle configurations via the E-construction because they fail the coplanarity condition.

The other four polyhedra fall into two pairs of Petrie duals,

$$\begin{array}{ccc} \{4,6\}_6 & \overset{\pi}{\longleftrightarrow} & \{6,6\}_4 \\ \{\infty,3\}^{(a)} & \overset{\pi}{\longleftrightarrow} & \{\infty,3\}^{(b)}\,. \end{array} \tag{6}$$

The second pair also does not meet the coplanarity condition and will not give a circle configuration. On the other hand, the first pair does have the coplanarity property and yields a circle configuration via the E-construction.

5.4. Chiral Polyhedra and the E-Construction

Chiral polyhedra do not lend themselves to point-circle configurations via the E-construction as the coplanarity condition fails in each case. This is remarkable, as chiral polyhedra are almost regular, failing regularity just by a factor of 2. In a way, this seems to indicate that the highly symmetric circle configurations arising from regular skeletal polyhedra are somewhat exceptional among E-constructed configurations.

The chiral polyhedra belong to six two-parameter families of polyhedra, each of which also includes exactly two regular polyhedra parameterized in the same way as the chiral polyhedra. In three families, one regular polyhedron is finite; in the other three, both regular polyhedra are infinite. Thus, altogether, the six families contain nine infinite regular polyhedra, as mentioned in the previous section.

Checking coplanarity for these six families is not entirely straightforward as the two parameters are integral or real. By way of example, we explain the procedure for the family of polyhedra $\{P(a,b)\}_{a,b}$ of type $\{6,6\}$, with integral parameters a,b, relatively prime, in which the chiral polyhedra have skew hexagonal faces and vertex figures ([18] (Section 5)).

The symmetry group of a polyhedron $Q := P(a,b)$ is generated by

$$\begin{aligned} s_1&\colon (x_1,x_2,x_3)\mapsto(-x_2,x_3,x_1)+(0,-b,-a),\\ s_2&\colon (x_1,x_2,x_3)\mapsto-(x_3,x_1,x_2). \end{aligned}$$

Here, s_1 is a rotatory reflection (a rotation about an axis followed by a reflection in a plane perpendicular to the axis) that cyclically permutes the vertices of the hexagonal base face f. The generator s_2 is a rotatory reflection that, when applied to the vertex u in the base edge e distinct from the base vertex v, produces the hexagonal vertex figure at v; in the present case, $v = o$. The symmetry $t := s_1 s_2$ is given by

$$t\colon (x_1, x_2, x_3) \mapsto (-x_1, x_2, -x_3) + (a, 0, b)$$

and is a half-turn whose rotation axis passes through the midpoint of e and is perpendicular to e. The vertices, edges, and faces of the polyhedron are the images of v, e, and f under the group $G(Q)$.

Note that $e = \{o, u\}$ and $u = t(o) = (a, 0, b)$. The vertices of the vertex figure at o are given by $s_2^j(u)$ with $j = 0, \ldots, 5$; of these six vertices, only $s_2(u) = (-b, -a, 0)$ and $s_2^{-1}(u) = (0, -b, -a) = s_1(o)$ are points in the block $N(e)$. The two remaining points in $N(e)$ are the images of the latter two under t, namely, $(a + b, -a, b)$ and $(a, -b, a + b)$, respectively. Thus

$$N(e) = \{(-b, -a, 0), (0, -b, -a), (a + b, -a, b), (a, -b, a + b)\}.$$

For a point $x = (x_1, x_2, x_3)$ in $\mathbb{E}^3$ define the vector $\hat{x}$ in $\mathbb{E}^4$ by $\hat{x} := (x_1, x_2, x_3, 1)$. Then it is clear that four points $v_1, \ldots, v_4$ in $\mathbb{E}^3$ are affinely dependent (that is, coplanar) if and only if the respective vectors $\hat{v}_1, \ldots, \hat{v}_4$ are linearly dependent in $\mathbb{E}^4$ (that is, have vanishing determinant).

For the four points in $N(e)$ the determinant is given by

$$\det \begin{pmatrix} -b & 0 & a+b & a \\ -a & -b & -a & -b \\ 0 & -a & b & a+b \\ 1 & 1 & 1 & 1 \end{pmatrix} = 2a^3 + 2a^2 b - 2ab^2 - 2b^3 = 2(a+b)^2(a-b)\,.$$

Thus, the determinant is 0 if and only if $a = \pm b$. It follows that $P(a, b)$ satisfies the coplanarity condition if and only if $a = \pm b$. These parameter values characterize precisely the two regular polyhedra in the infinite family of polyhedra $\{P(a, b)\}_{a,b}$, namely $\{6, 6 \,|\, 3\}$ and $\{6, 6\}_4$. In other words, no chiral polyhedron in the family $\{P(a, b)\}_{a,b}$ satisfies the coplanarity condition, but the two regular polyhedra do.

The five other extended families of chiral and regular polyhedra can be dealt with in similar ways. In each case the chiral polyhedra fail the coplanarity condition. In the three extended families of polyhedra with finite faces, the two regular polyhedra satisfy the coplanarity condition by arguments similar to those above. In particular, this establishes the coplanarity for $\{4, 6 \,|\, 4\}$, $\{6, 4 \,|\, 4\}$, $\{6, 6 \,|\, 3\}$, $\{4, 6\}_6$, and $\{6, 4\}_6$, and $\{6, 6\}_4$. In the three remaining families, all polyhedra, but one, have helical faces and fail the coplanarity condition, including one regular polyhedron in each case; the one exception is a finite regular polyhedron and satisfies the coplanarity condition. The three infinite regular polyhedra failing the condition are $\{\infty, 3\}^{(a)}$, $\{\infty, 3\}^{(b)}$, and $\{\infty, 4\}_{\cdot,*3}$.

Note that the V-construction applied to chiral polyhedra also will not lead to point-circle configurations. In fact, all chiral polyhedra in $\mathbb{E}^3$ have skew vertex figures (and skew faces), and thus, no circle will pass through all the vertices in a vertex figure.

6. V-Constructed Point-Circle Configurations

In this section, we briefly discuss the V-construction for regular skeletal polyhedra in $\mathbb{E}^3$. By our discreteness assumption, each regular skeletal polyhedron has finite vertex

figures, which either are planar (convex or star-) regular polygons or non-planar (skew) regular polygons. The V-construction only applies in cases where the vertex figures are planar. Given a regular skeletal polyhedron with planar vertex figures, the V-construction yields a point-circle configuration whose points are the vertices of Q and whose circles are the circumcircles of the vertex figures at the vertices of Q. As a regular polyhedron Q and its Petrie dual Q^π share the same underlying edge graph, and the same vertex figures, the corresponding point-circle configurations for Q and Q^π are identical.

Theorem 2. *Let Q be a regular polyhedron in $\mathbb{E}^3$ with symmetry group $G(Q) = \langle r_0, r_1, r_2 \rangle$, and suppose Q has planar vertex figures.*
(a) Then the V-construction applied to Q yields a point-circle configuration $\mathcal{C}$ on which $G(Q)$ acts point-circle transitively, in all but two cases with a point-circle stabilizer of order 2.
(b) If C is the circumcircle of the vertex figure at the base vertex v of Q, and v' is the vertex in the base edge of Q distinct from v, then the stabilizer in $G(Q)$ of the point-circle pair (v', C) of $\mathcal{C}$ is given by $\langle r_2 \rangle$ except when Q is the octahedron or its Petrie dual. In the exceptional cases, the stabilizer is given by $\langle r_2, r_C \rangle \cong C_2 \times C_2$, where r_C denotes the reflection in the plane through C.

Proof. Let Q be of type $\{p, q\}$. By the classification of regular skeletal polyhedra, $q \leq 6$, and Q is finite if $q = 5$. Thus, the planar vertex figures of Q are convex q-gons, or possibly pentagrams if $q = 5$.

Let $\{v, e, f\}$ denote the base flag of Q. Each point-circle pair of the circle configuration is determined by an incident vertex–edge pair of Q, and this vertex–edge pair is unique except in the cases of the octahedron and its Petrie dual. As $G(Q)$ acts transitively on the incident vertex–edge pairs of Q, it follows that $G(Q)$ also acts point-circle transitively on the V-constructed point-circle configuration.

Suppose v' is the vertex of e distinct from v, and (v', C) is the point-circle pair of $\mathcal{C}$ associated with the vertex–edge pair (v, e) of Q. Then the stabilizer $\langle \rho_2 \rangle$ of the vertex–edge pair (v, e) in $G(Q)$ also stabilizes the point-circle pair (v', C). It remains to show that $\langle \rho_2 \rangle$ is the full stabilizer of (v', C) in $G(Q)$ except when Q is the octahedron or its Petrie dual.

Suppose $s \in G(Q)$ stabilizes (v', C), that is, s fixes the circle C and the point v' on C. If we can show that s also fixes v, then s fixes the base vertex and base edge of Q and thus $s \in \langle r_2 \rangle$, as claimed.

To this end observe that s permutes the vertices of the vertex figure at v (that is, the q points on C) and thus fixes the center z of C. As the edges of Q connecting v to the points on C all have the same length, v must lie on the line l perpendicular, at z, to the plane containing C; as the polyhedron $\{6, 6\}_4$ shows, it is possible that $v = z$. Now, as s leaves the plane H through C invariant, s also keeps l invariant and thus maps v to itself or to the point $\hat{v} := 2z - v$ on l on the other side of H at the same distance from z as v. If $s(v) = v$, we are done since s fixes the base vertex; in particular, this settles the case when $v = z$.

Let $v \neq z$, and suppose that $s(v) = \hat{v}$. Then, $\hat{v}$ is a vertex of Q since $s \in G(Q)$. Also, as s permutes the q points of C and the vertex figures of Q are congruent planar regular q-gons, the vertices v and $\hat{v}$ must share the same vertex figure. By the vertex-transitivity of $G(Q)$, this "bipyramidal structure" must occur at every circle of the configuration, including the circle C' associated with v'. As s fixes v', it maps edge $e = \{v', v\}$ of Q to an edge, namely $\{v', \hat{v}\}$, and both edges connect v' to points in C'. Next, since s also fixes z, it must leave the line l' through v', z pointwise fixed and induce a plane isometry on the plane H' perpendicular to l' at z that maps v to $\hat{v}$. This plane isometry has an invariant line, namely $H \cap H'$, and thus must be a half-turn about a point or a reflection in the invariant line; in the latter case, the reflection line must be the perpendicular bisector of v and $\hat{v}$ in H'. It follows that s itself must be a half-turn about l' or the reflection in the plane H, the latter being the

perpendicular bisector of v and $\hat{v}$ in $\mathbb{E}^3$. It follows that H and H' are perpendicular, that H' contains C', and that z is the center of both C and C'. In particular, since v and $\hat{v}$ are opposite vertices in C', the number q must be even.

Thus $q = 4$ or $q = 6$. We show that the case $q = 4$ leads to the two exceptional cases mentioned in the theorem and that the case $q = 6$ can be eliminated.

Suppose $q = 4$. Then the two points of $C \cap C'$ are vertices of Q and thus contribute points to C and C'; they also are connected to v' by edges of Q. It follows that Q must contain the entire 1-skeleton of the octahedron spanned by the six points of C and C'; in fact, as these six vertices already account for the full vertex valency, 4 (=q), the 1-skeleton of Q must coincide with the 1-skeleton of the octahedron. This situation can only occur if Q is an octahedron or the Petrie dual of the octahedron.

In the case of the octahedron $\{3, 4\}$ and its Petrie dual $\{6, 4\}_3$, the above isometry s either is the reflection r_C in the plane H through C or the half-turn about l'; but the latter is just $r_C r_2$. Note that both r_C and $r_C r_2$ belong to $G(Q)$, and both are possible choices for s for the octahedron or its Petrie dual. In either case, $r_C s$ stabilizes the point-circle pair (v', C) and fixes the base vertex v and base edge e. It follows that $r_C s \in \langle r_2 \rangle$. Thus the stabilizer of (v', C) in $G(Q)$ is $\langle r_2, r_C \rangle$, which is isomorphic to $C_2 \times C_2$ as r_2 and r_C commute.

What remains is to reject the possibility that $q = 6$. No finite regular polyhedron has 6-valent vertices. Planar polyhedra and blends of planar polyhedra with a line segment cannot have bipyramidal structure of the above kind at the circles of the corresponding V-constructed point-circle configuration. Blends of planar polyhedra with an apeirogon all have skew vertex figures and can be discounted. This leaves the pure polyhedra of which only two have planar hexagonal vertex figures, namely $\{6, 6\}_4$ and its Petrie dual $\{4, 6\}_6$. In both cases, each vertex is the center of its vertex figure, and thus, we are in the case $v = z$ already discussed above. Recall here that polyhedron $\{6, 6\}_4$ can be derived from the Petrie–Coxeter polyhedron $\{4, 6 \,|\, 4\}$, by bi-coloring the vertices of $\{4, 6 \,|\, 4\}$ black and white and then taking the (skew) vertex figures of $\{4, 6 \,|\, 4\}$ at the white vertices as faces for $\{6, 6\}_4$; the vertex figures of $\{6, 6\}_4$ then are planar hexagons centered at the black vertices of $\{4, 6 \,|\, 4\}$.

In conclusion, in all but two cases, the stabilizer of the point-circle pair (v', C) of $\mathcal{C}$ is given by $\langle r_2 \rangle$, and in the two exceptional cases by $\langle r_2, r_C \rangle \cong C_2 \times C_2$. This completes the proof. □

7. Concluding Remarks and Open Problems

In this paper, we investigated the highly symmetric circle configurations obtained from the regular skeletal polyhedra in $\mathbb{E}^3$. These circle configurations inherit their symmetry from the polyhedron and are point-circle transitive. However, as the family of chiral polyhedra shows, the high symmetry of the underlying polyhedron alone is not a guarantee for the applicability of the E-construction as the coplanarity condition may fail.

If a convex or skeletal polyhedron satisfies the coplanarity condition and its symmetry group has k orbits on the edges, then the E-construction yields a circle configuration with at most k orbits on the circles. In particular, the edge transitivity of a polyhedron will imply the circle transitivity of the configuration. Among the Archimedean convex polyhedra, exactly four have an edge transitive symmetry group: the cuboctahedron and icosidodecahedron, and their duals (the rhombic dodecahedron and rhombic triacontahedron, respectively). See Figure 15 for the configuration derived from the cuboctahedron. These four polyhedra yield circle-transitive circle configurations via the E-construction. However, these configurations are not point-circle transitive. In fact, in each case, the symmetry group of the polyhedron is the full octahedral or icosahedral group, with edge stabilizers (for the 24 or 60 edges, respectively) of order 2, and must necessarily coincide with the full symmetry group of the

configuration since the full octahedral and icosahedral groups are not proper subgroups of any finite group of isometries in $\mathbb{E}^3$. Thus, as the edge stabilizers contribute only two symmetries, the stabilizer of a circle will not act transitively on the points of the circle.

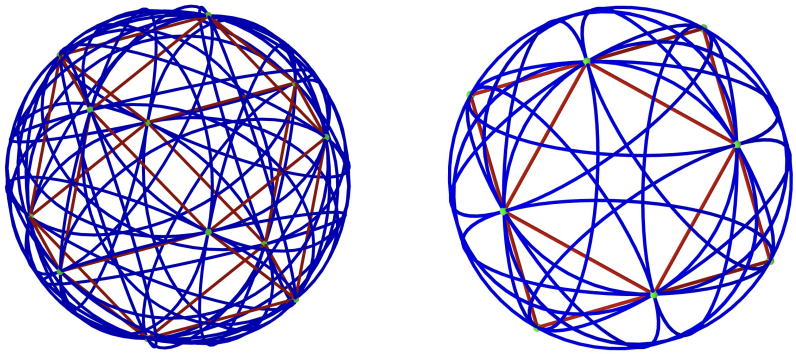


Figure 15. Circle configurations for the cuboctahedron, with a top view shown on the right.

For more general polyhedra-like structures such as the regular polygonal complexes K of [22], there are generalizations of the E-construction method. For example, if the faces at every edge e of K occur in cyclic order such that every pair of successive faces in this order satisfies the coplanarity condition relative to e, then a circle configuration can often be obtained in much the same way as before, namely by choosing the circles that pass through the four (or three) vertices determined by each pair of successive faces. The simplest such complex is the 2-skeleton of the cubical tessellation in $\mathbb{E}^3$, that is, the complex formed by the vertices, edges, and square faces of the tessellation. The resulting circular configuration is point-circle transitive and is shown in Figure 16.

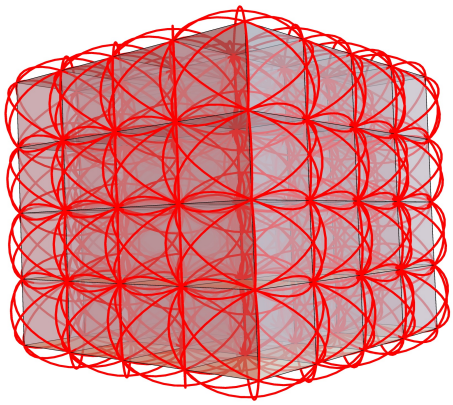


Figure 16. Circle configuration for the 2-skeleton of the cubical tessellation of $\mathbb{E}^3$. Each cube of the tessellation contributes six circles.

Tilings of space can also be an interesting source of circle configurations. Let $\mathcal{T}$ be a face-to-face tiling of $\mathbb{E}^3$ whose tiles are convex polyhedra, and suppose each tile of T permits the construction of a circle configuration $C(T)$ via the E-construction. Then, the union of all circle configurations $C(T)$, taken over all tiles T of $\mathcal{T}$, is again a circle configuration in $\mathbb{E}^3$. For example, the above circle configuration associated with the 2-skeleton of the cubical tessellation occurs as a special case when $\mathcal{T}$ is the cubical tessellation. The symmetry group of the tessellations acts point-circle transitively on the configuration, with a point stabilizer of order 48, a circle stabilizer of order 16 (the reflection in the plane of a circle fixes every point on the circle), and a point-circle stabilizer of order 2 (generated by this reflection).

The circle configurations presented in this paper suggest the following two attractive open research problems.

Problem 1. *Classify all finite or infinite discrete circle configurations in* $\mathbb{E}^3$*, which are point-circle transitive.*

Problem 2. *Classify all finite or infinite discrete circle configurations in* $\mathbb{E}^3$*, which are circle transitive.*

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