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Comprehensive Subfamilies of Bi-Univalent Functions Defined by Error Function Subordinate to Euler Polynomials

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Abstract: Recently, several researchers have estimated the Maclaurin coefficients, namely $|q_2|$ and $|q_3|$, and the Fekete–Szegö functional problem of functions belonging to some special subfamilies of analytic functions related to certain polynomials, such as Lucas polynomials, Legendrae polynomials, Chebyshev polynomials, and others. This study obtains the bounds of coefficients $|q_2|$ and $|q_3|$, and the Fekete–Szegö functional problem for functions belonging to the comprehensive subfamilies $T(\zeta, \epsilon, \delta)$ and $J(\varphi, \delta)$ of analytic functions in a symmetric domain $\mathbb{U}$, using the imaginary error function subordinate to Euler polynomials. After specializing the parameters used in our main results, a number of new special cases are also obtained.

Keywords: analytic; univalent; bi-univalent; symmetric domain; error functions; Euler polynomials; Fekete–Szegö problem

MSC: 30C45

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Academic Editors: Junesang Choi and Ioan Rașa

Received: 28 November 2024
Revised: 3 January 2025
Accepted: 6 February 2025
Published: 8 February 2025

Citation: Al-Hawary, T.; Frasin, B.; Salah, J. Comprehensive Subfamilies of Bi-Univalent Functions Defined by Error Function Subordinate to Euler Polynomials. *Symmetry* **2025**, *17*, 256. https://doi.org/10.3390/sym17020256

1. Introduction

The study of bi-univalent functions using error functions combines sophisticated mathematical methods for error estimates and approximation with complex analysis, especially function theory. Subfamilies of univalent functions that are analytic in a particular domain are called bi-univalent functions. Using error functions to explore bi-univalent functions is motivated by a combination of classical function theory, numerical analysis, and applications to engineering and physics. We can improve our comprehension of bi-univalent functions by using error functions, which offer more accurate characterizations, sharper bounds, and better approximations.

Also, there are numerous uses for the error function in probability science, statistics, applied mathematics, and partial differential equation physics. The error function in quantum mechanics is crucial for estimating the likelihood of seeing a particle in a given area. Alzer [1] and Coman [2] provided a variety of error function properties and inequalities, whereas Elbert et al. [3] investigated the properties of complementary error functions.

Let AUF symbolize the family of analytic and univalent functions Q in the symmetric domain $\mathbb{U} = \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$ and satisfy $Q(0) = Q'(0) - 1 = 0$ of the form

$$Q(\varsigma) = \varsigma + \sum_{k=2}^{\infty} q_k \varsigma^k, \quad (\varsigma \in \mathbb{U}). \tag{1}$$

Every function $Q \in AUF$ has an inverse Q^{-1}, defined by

$$Q^{-1}(Q(\varsigma)) = \varsigma \text{ and } \varpi = Q(Q^{-1}(\varpi)) \ \ (\varsigma \in \mathbb{U}, \ |\varpi| < r_0(Q) \geq \frac{1}{4}),$$

where (see [4])

$$Q^{-1}(\varpi) = H(\varpi) = \varpi - q_2\varpi^2 + (2q_2^2 - q_3)\varpi^3 - (q_4 + 5q_2^3 - 5q_3q_2)\varpi^4 + \cdots . \quad (2)$$

Let Π be the family of bi-univalent functions in $\mathbb{U}$ given by (1) (Q is bi-univalent in $\mathbb{U}$ if Q and Q^{-1} are univalent in $\mathbb{U}$) (see [5]).

The function Q is subordinate to H, symbolized by $Q \prec H$, if there exists the function $\varpi \in AUF$, and the functions Q and H are analytic in $\mathbb{U}$, such that

$$\varpi(0) = 0 \text{ and } |\varpi(\varsigma)| < 1, \qquad (\varsigma \in \mathbb{U})$$

such that

$$Q(\varsigma) = H(\varpi(\varsigma)).$$

Also, if H is univalent in $\mathbb{U}$, then

$$Q(\varsigma) \prec H(\varsigma) \quad \text{if and only if} \quad Q(0) = H(0) \text{ and } Q(\mathbb{U}) \subset H(\mathbb{U}).$$

Abramowitz and Stegun [6] defined the following error function

$$\operatorname{erf}(\varsigma) = \frac{2}{\sqrt{\pi}} \int_0^{\varsigma} e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \varsigma^{2k+1}}{(2k+1)k!}, \qquad (\varsigma \in \mathbb{C}). \quad (3)$$

Further, defines the following error function, whereas erfi denotes the imaginary error function

$$\operatorname{erf} i(\varsigma) = \frac{2}{\sqrt{\pi}} \int_0^{\varsigma} e^{t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\varsigma^{2k+1}}{(2k+1)k!}, \qquad (\varsigma \in \mathbb{C}). \quad (4)$$

Since the error function is odd (i.e., $\operatorname{erf}(-\varsigma) = -\operatorname{erf}(\varsigma)$), it is symmetric with respect to the origin.

The error function (3) is generalized as follows:

$$\begin{aligned} \operatorname{erf}_\mu(\varsigma) &= \frac{\mu!}{\sqrt{\pi}} \int_0^{\varsigma} e^{-t^\mu} dt, \ \ \mu \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \\ &= \frac{\mu!}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \varsigma^{\mu k+1}}{(\mu n+1)k!}, \qquad (\varsigma \in \mathbb{C}). \end{aligned} \quad (5)$$

From (5), we have $\operatorname{erf}_0(\varsigma) = \frac{\varsigma}{e\sqrt{\pi}}$, $\operatorname{erf}_1(\varsigma) = \frac{1-e^{\varsigma}}{\sqrt{\pi}}$, $\operatorname{erf}_2(\varsigma) = \operatorname{erf}(\varsigma)$.

Clearly, the function $\operatorname{erf}_\mu(\varsigma)$ does not belong in the family AUF. Thus, it is natural to consider the following function:

$$\mathcal{E}_\mu(\varsigma) = \frac{\sqrt{\pi}}{\mu!} \varsigma^{\left(1-\frac{1}{\mu}\right)} \operatorname{erf}_\mu\left(\varsigma^{1/\mu}\right) = \varsigma + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{((k-1)\mu+1)(k-1)!} \varsigma^k, \ (\mu \in \mathbb{N}, \varsigma \in \mathbb{U}). \quad (6)$$

Also, the imaginary error function (4) is generalized as follows:

$$\operatorname{erf} i_\mu(\varsigma) = \frac{\mu!}{\sqrt{\pi}} \int_0^{\varsigma} e^{t^\mu} dt \ = \frac{\mu!}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\varsigma^{\mu k+1}}{(\mu k+1)k!}, \quad (\mu \in \mathbb{N}_0, \varsigma \in \mathbb{C}). \quad (7)$$

Further, the normalization of the generalized imaginary error function $\mathrm{erfi}_{\mu}(\varsigma)$ is given by

$$E_{\mu}(\varsigma) = \frac{\sqrt{\pi}}{\mu!}\varsigma^{\left(1-\frac{1}{\mu}\right)}\,\mathrm{erf}\,i_{\mu}\left(\varsigma^{1/\mu}\right)$$
$$= \varsigma + \sum_{k=2}^{\infty} \frac{1}{((k-1)\mu+1)(k-1)!}\varsigma^{k}, \quad (\mu \in \mathbb{N}, \varsigma \in \mathbb{U}). \tag{8}$$

Making use of the convolution, we construct the linear operator $EQ_{\mu} : AUF \to AUF$ to be given as

$$EQ_{\mu}(\varsigma) = Q(\varsigma) * E_{\mu}(\varsigma) = \varsigma + \sum_{k=2}^{\infty} \frac{1}{((k-1)\mu+1)(k-1)!} q_k \varsigma^{k}, \ (\mu \in \mathbb{N}, \varsigma \in \mathbb{U}). \tag{9}$$

Remark 1. *If we take $\mu = 2$ in (6), we obtain the normalization for Ramachandran et al. [7], and if we take $\mu = 2$ in (8), we obtain the normalization for Mohammed et al. [8].*

Understanding complex functions and their geometric features requires an understanding of Euler polynomials, which originated in the studies of Leonhard Euler in the seventeenth century. In geometric function theory, they are essential for describing conformal mappings that locally preserve angles. They are also extensively used in many branches of geometric function theory, such as Riemann surface theory, Schwarz–Christoffel mappings, and the study of univalent functions. The complex connection between geometric transformations and analytic functions made possible by Euler polynomials is clarified by these applications.

Euler polynomials $G_k(\delta)$ are defined using the generating function (see, e.g., [9,10]):

$$\mathcal{K}(\delta, H) = \frac{2e^{H\delta}}{e^{H}+1} = \sum_{k=0}^{\infty} G_k(\delta)\frac{H^k}{k!}, \left(\frac{1}{2} < \delta \le 1,\ |H| < \pi\right).$$

A precise formula for $G_k(\delta)$ is given by

$$G_j(\delta) = \sum_{i=0}^{j} \frac{1}{2^i} \sum_{u=0}^{i} (-1)^u \binom{i}{u} (\delta+u)^j. \tag{10}$$

From (10), the function $G_i(\delta)$ in terms of G_u is obtained as follows:

$$G_i(\delta) = \sum_{u=0}^{i} \frac{G_u}{2^u}\binom{i}{u}(\delta - \frac{1}{2})^{i-u}.$$

The initial Euler polynomial values are as follows:

$$\begin{aligned} G_0(\delta) &= 1; \\ G_1(\delta) &= \frac{2\delta-1}{2}; \\ G_2(\delta) &= \delta^2 - \delta; \\ G_3(\delta) &= \frac{4\delta^3 - 6\delta^2 + 1}{4}; \\ G_4(\delta) &= \delta^4 - 2\delta^3 + \delta. \end{aligned} \tag{11}$$

In 2010, Srivastava et al. [5] found bounds for the coefficients $|q_2|$ and $|q_3|$ of functions in two interesting subfamilies of the function family Π. Motivated by this work, many researchers have studied new subfamilies of Π to obtain new bounds for the coefficients $|q_2|$ and $|q_3|$, like Amourah et al. [11], Deniz [12], Tang et al. [13], Yousef et al. [14], and others.

For a univalent function Q, Fekete and Szegö [15] derived a sharp constraint of the functional $\left|q_3 - \vartheta q_2^2\right|$ with real $\vartheta(0 \leq \vartheta < 1)$. Since then, the classical Fekete–Szegö problem or inequality has been defined as the problem of determining the sharp bounds for this functional of family functions AUF with any complex ϑ.

The novelty of this work is evident in that many authors have used several special functions in their articles; they have never used error functions in subfamilies of bi-univalent functions.

In this work, we construct two new and extensive subfamilies of bi-univalent functions using a particular special function, the imaginary error function and Euler polynomial, denoted by $T(\zeta, \epsilon, \delta)$ and $J(\varphi, \delta)$, and find initial bounds for the coefficients $|q_2|$ and $|q_3|$, as well as the Fekete–Szegö inequality. Also, a number of new corollaries are displayed.

2. Bounds of the Subfamilies $T(\zeta, \epsilon, \delta)$ and $J(\varphi, \delta)$

At the beginning of this section, we should define the comprehensive subfamilies $T(\zeta, \epsilon, \delta)$ and $J(\varphi, \delta)$ using an error function subordinate to Euler polynomials.

Definition 1. *For $Q \in T(\zeta, \epsilon, \delta)$, assume the following subordinations are satisfied:*

$$(1-\zeta)\frac{EQ_\mu(\varsigma)}{\varsigma} + \zeta\left(EQ_\mu(\varsigma)\right)' + \epsilon\varsigma\left(EQ_\mu(\varsigma)\right)'' \prec \mathcal{K}(\delta, \varsigma) = \sum_{k=0}^{\infty} G_k(\delta)\frac{\varsigma^k}{k!} \tag{12}$$

and

$$(1-\zeta)\frac{EH_\mu(\varpi)}{\varpi} + \zeta\left(EH_\mu(\varpi)\right)' + \epsilon\varpi\left(EH_\mu(\varpi)\right)'' \prec \mathcal{K}(\delta, \varpi) = \sum_{k=0}^{\infty} G_k(\delta)\frac{\varpi^k}{k!}, \tag{13}$$

where $\zeta \geq 1, \epsilon \geq 0, \frac{1}{2} < \delta \leq 1, \varsigma, \varpi \in \mathbb{U}$ and $H = Q^{-1}$.

Definition 2. *For $Q \in J(\varphi, \delta)$, assume the following subordinations are satisfied:*

$$\left(EQ_\mu(\varsigma)\right)' + \varsigma\frac{e^{i\varphi}+1}{2}\left(EQ_\mu(\varsigma)\right)'' \prec \mathcal{K}(\delta, \varsigma) = \sum_{k=0}^{\infty} G_k(\delta)\frac{\varsigma^k}{k!} \tag{14}$$

and

$$\left(EH_\mu(\varpi)\right)' + \varpi\frac{e^{i\varphi}+1}{2}\left(EH_\mu(\varpi)\right)'' \prec \mathcal{K}(\delta, \varpi) = \sum_{k=0}^{\infty} G_k(\delta)\frac{\varpi^k}{k!}, \tag{15}$$

where $-\pi < \varphi \leq \pi, \frac{1}{2} < \delta \leq 1, \varsigma, \varpi \in \mathbb{U}$ and $H = Q^{-1}$.

Example 1. *If $\zeta = 1$ in Definition 1, we obtain the subfamily $T(1, \epsilon, \delta)$, which satisfies the following requirements:*

$$\left(EQ_\mu(\varsigma)\right)' + \epsilon\varsigma\left(EQ_\mu(\varsigma)\right)'' \prec \mathcal{K}(\delta, \varsigma) = \sum_{k=0}^{\infty} G_k(\delta)\frac{\varsigma^k}{k!}$$

and

$$\left(EH_\mu(\varpi)\right)' + \epsilon\varpi\left(EH_\mu(\varpi)\right)'' \prec \mathcal{K}(\delta, \varpi) = \sum_{k=0}^{\infty} G_k(\delta)\frac{\varpi^k}{k!},$$

where $\epsilon \geq 0, \frac{1}{2} < \delta \leq 1, \varsigma, \varpi \in \mathbb{U}$ and $H = Q^{-1}$.

Example 2. *If $\epsilon = 0$ in Definition 1, we obtain the subfamily $T(\zeta, 0, \delta)$, which satisfies the following requirements:*

$$(1-\zeta)\frac{EQ_{\mu}(\varsigma)}{\varsigma}+\zeta\big(EQ_{\mu}(\varsigma)\big)'\prec\mathcal{K}(\delta,\varsigma)=\sum_{k=0}^{\infty}G_k(\delta)\frac{\varsigma^k}{k!}$$

and

$$(1-\zeta)\frac{EH_{\mu}(\varpi)}{\varpi}+\zeta\big(EH_{\mu}(\varpi)\big)'\prec\mathcal{K}(\delta,\varpi)=\sum_{k=0}^{\infty}G_k(\delta)\frac{\varpi^k}{k!},$$

where $\zeta\geq 1$, $\frac{1}{2}<\delta\leq 1$, $\varsigma,\varpi\ \in\ \mathbb{U}$ *and* $H=Q^{-1}$.

Example 3. *If* $\varphi=\pi$ *in Definition 2, we obtain the subfamily* $J(\pi,\delta)$*, which satisfies the following requirements:*

$$\big(EQ_{\mu}(\varsigma)\big)'\prec\mathcal{K}(\delta,\varsigma)=\sum_{k=0}^{\infty}G_k(\delta)\frac{\varsigma^k}{k!}$$

and

$$\big(EH_{\mu}(\varpi)\big)'\prec\mathcal{K}(\delta,\varpi)=\sum_{k=0}^{\infty}G_k(\delta)\frac{\varpi^k}{k!},$$

where $\frac{1}{2}<\delta\leq 1$, $\varsigma,\varpi\ \in\ \mathbb{U}$ *and* $H=Q^{-1}$.

Example 4. *If* $\varphi=0$ *in Definition 2, we obtain the subfamily* $J(0,\delta)$*, which satisfies the following requirements:*

$$\big(EQ_{\mu}(\varsigma)\big)'+\varsigma\big(EQ_{\mu}(\varsigma)\big)''\prec\mathcal{K}(\delta,\varsigma)=\sum_{k=0}^{\infty}G_k(\delta)\frac{\varsigma^k}{k!}$$

and

$$\big(EH_{\mu}(\varpi)\big)'+\varpi\big(EH_{\mu}(\varpi)\big)''\prec\mathcal{K}(\delta,\varpi)=\sum_{k=0}^{\infty}G_k(\delta)\frac{\varpi^k}{k!},$$

where $\frac{1}{2}<\delta\leq 1$, $\varsigma,\varpi\ \in\ \mathbb{U}$ *and* $H=Q^{-1}$.

Remark 2. *All the previous subfamilies mentioned are inspired by subfamilies used by many researchers when* $Re(Q'(\varsigma))\ >\ \alpha$. *From this, we can determine that* $Re(Q'(\varsigma))\ >\ 0$, *which is the condition for the function* Q *to be univalent in the open disk* $\mathbb{U}$. *For instance, the family* $Re\Big((1-\zeta)\frac{Q(\varsigma)}{\varsigma}+\zeta Q'(\varsigma)\ +\epsilon\varsigma Q''(\varsigma)\Big)\ >\ \alpha$ *was studied by Frasin et al. [16],* $Re(Q'(\varsigma)+\epsilon\varsigma Q''(\varsigma))>\alpha$ *was studied by Ponnusamy [17], and* $Re(Q'(\varsigma))>\alpha$ *was studied by Ezrohi [18].*

Lemma 1 ([19])**.** *Let* $X(\varsigma)\in\digamma$ *be given by*

$$X(\varsigma)=1+m_1\varsigma+m_2^2\varsigma+\cdots,\ \ (Re(X(\varsigma))>0,\ \ \varsigma\in\mathbb{U}),$$

$|m_n|\leq 2$ *for each* $n\in\mathbb{N}$.

In the next Theorems, we estimate the initial coefficient $|q_2|$, $|q_3|$ and solve the Fekete–Szegö problem for the subfamilies $T(\zeta,\epsilon,\delta)$ and $J(\varphi,\delta)$, respectively.

Theorem 1. *Let* $Q\in\Pi$ *be given by (1) in the subfamily* $T(\zeta,\epsilon,\delta)$*, where* $\zeta\geq 1$, $\epsilon\geq 0$, $\frac{1}{2}<\delta\leq 1$, $\varsigma,\varpi\ \in\ \mathbb{U}$ *and* $H=Q^{-1}$*. Then,*

$$|q_2|\leq\sqrt{D(\epsilon,\zeta,\delta)},$$

$$|q_3|\leq\frac{(2\delta-1)^2(\mu+1)^2}{4(2\epsilon+\zeta+1)^2}+\frac{(2\delta-1)(2\mu+1)}{6\epsilon+2\zeta+1}$$

and

$$\left|q_3-\vartheta q_2^2\right| \leq \begin{cases} \frac{2(2\delta-1)(2\mu+1)}{(6\epsilon+2\zeta+1)} \text{ if } \ 0 \leq |1-\vartheta|D(\epsilon,\zeta,\delta) < \frac{(2\delta-1)(2\mu+1)}{6\epsilon+2\zeta+1}, \\ \\ 2|1-\vartheta|D(\epsilon,\zeta,\delta) \text{ if } |1-\vartheta|D(\epsilon,\zeta,\delta) \geq \frac{(2\delta-1)(2\mu+1)}{6\epsilon+2\zeta+1}, \end{cases}$$

where

$$D(\epsilon,\zeta,\delta) = \frac{(2\delta-1)^3(2\mu+1)(\mu+1)^2}{\left|(6\epsilon+2\zeta+1)(\mu+1)^2(2\delta-1)^2-4(2\epsilon+\zeta+1)^2(2\mu+1)(\delta^2-3\delta+1)\right|}.$$

Proof. Since $Q(\varsigma)=\varsigma+\sum_{k=2}^{\infty} q_k\varsigma^k \in T(\zeta,\sigma,\delta)$, from (12) and (13), we have

$$(1-\zeta)\frac{EQ_\mu(\varsigma)}{\varsigma}+\zeta\big(EQ_\mu(\varsigma)\big)'+\epsilon\varsigma\big(EQ_\mu(\varsigma)\big)'' \prec \mathcal{K}(\delta,\varsigma) \tag{16}$$

and

$$(1-\zeta)\frac{EH_\mu(\varpi)}{\varpi}+\zeta\big(EH_\mu(\varpi)\big)'+\epsilon\varpi\big(EH_\mu(\varpi)\big)'' \prec \mathcal{K}(\delta,\varpi). \tag{17}$$

We define the functions $s_1, s_2 : \mathbb{U} \to \mathbb{U}$, with $s_1(0)=s_2(0)=0$ and $|s_1(\varsigma)|<1$, $|s_2(\varpi)|<1$ for all $\varsigma,\varpi \in \mathbb{U}$. So, we can define $\rho,\sigma \in \digamma$ as

$$\rho(\varsigma)=\frac{s_1(\varsigma)+1}{1-s_1(\varsigma)}=1+\rho_1\varsigma+\rho_2\varsigma^2+\rho_3\varsigma^3+\cdots, \ |\rho_k|\leq 2, \ \varsigma\in\mathbb{U}.$$

$$\Rightarrow s_1(\varsigma)=\frac{\rho(\varsigma)-1}{\rho(\varsigma)+1}=\frac{\rho_1}{2}\varsigma+\left(\frac{\rho_2}{2}-\frac{\rho_1^2}{4}\right)\varsigma^2+\frac{1}{2}\left(\rho_3-\rho_1\rho_2+\frac{\rho_1^3}{4}\right)\varsigma^3+\cdots \tag{18}$$

and

$$\sigma(\varpi)=\frac{s_2(\varpi)+1}{1-s_2(\varpi)}=1+\sigma_1\varpi+\sigma_2\varpi^2+\sigma_3\varpi^3+\cdots, \ |\sigma_k|\leq 2, \ \varpi\in\mathbb{U}.$$

$$\Rightarrow s_2(\varpi)=\frac{\sigma(\varpi)-1}{\sigma(\varpi)+1}=\frac{\sigma_1}{2}\varpi+\left(\frac{\sigma_2}{2}-\frac{\sigma_1^2}{4}\right)\varpi^2+\frac{1}{2}\left(\sigma_3-\sigma_1\sigma_2+\frac{\sigma_1^3}{4}\right)\varpi^3+\cdots. \tag{19}$$

Using (18) and (19), we obtain

$$\begin{aligned}\mathcal{K}(\delta,s_1(\varsigma)) &= G_0(\delta)+\frac{G_1(\delta)}{2}\rho_1\varsigma+\left(\frac{G_1(\delta)}{2}\left(\rho_2-\frac{\rho_1^2}{2}\right)+\frac{G_2(\delta)}{8}\rho_1^2\right)\varsigma^2 \\ &+\left(\frac{G_1(\delta)}{2}\left(\rho_3-\rho_1\rho_2+\frac{\rho_1^3}{4}\right)+\frac{G_2(\delta)}{4}\left(\rho_1\rho_2-\frac{\rho_1^3}{2}\right)+\frac{G_3(\delta)}{48}\rho_1^3\right)\varsigma^3+\cdots\end{aligned} \tag{20}$$

and

$$\begin{aligned}\mathcal{K}(\delta,s_2(\varpi)) &= G_0(\delta)+\frac{G_1(\delta)}{2}\sigma_1\varpi+\left(\frac{G_1(\delta)}{2}\left(\sigma_2-\frac{\sigma_1^2}{2}\right)+\frac{G_2(\delta)}{8}\sigma_1^2\right)\varpi^2 \\ &+\left(\frac{G_1(\delta)}{2}\left(\sigma_3-\sigma_1\sigma_2+\frac{\sigma_1^3}{4}\right)+\frac{G_2(\delta)}{4}\left(\sigma_1\sigma_2-\frac{\sigma_1^3}{2}\right)+\frac{G_3(\delta)}{48}\sigma_1^3\right)\varpi^3+\cdots.\end{aligned} \tag{21}$$

From (16), (17) and (20), (21), we obtain

$$\frac{2\epsilon+\zeta+1}{\mu+1}q_2=\frac{G_1(\delta)}{2}\rho_1, \tag{22}$$

$$\frac{6\epsilon+2\zeta+1}{2(2\mu+1)}q_3=\frac{G_1(\delta)}{2}\left(\rho_2-\frac{\rho_1^2}{2}\right)+\frac{G_2(\delta)}{8}\rho_1^2, \tag{23}$$

$$-\frac{2\epsilon+\zeta+1}{\mu+1}q_2=\frac{G_1(\delta)}{2}\sigma_1, \tag{24}$$

and

$$\frac{6\epsilon+2\zeta+1}{2(2\mu+1)}\left(2q_2^2-q_3\right)=\frac{G_1(\delta)}{2}\left(\sigma_2-\frac{\sigma_1^2}{2}\right)+\frac{G_2(\delta)}{8}\sigma_1^2. \tag{25}$$

Upon adding the Equation (22) to (24) and performing some calculations, we obtain

$$\rho_1=-\sigma_1 \tag{26}$$

and

$$\frac{8(2\epsilon+\zeta+1)^2}{(\mu+1)^2}q_2^2=G_1^2(\delta)(\rho_1^2+\sigma_1^2). \tag{27}$$

$$\Rightarrow q_2^2=\frac{G_1^2(\delta)(\rho_1^2+\sigma_1^2)(\mu+1)^2}{8(2\epsilon+\zeta+1)^2}. \tag{28}$$

Adding the Equation (23) to (25) gives

$$\frac{4(6\epsilon+2\zeta+1)}{2\mu+1}q_2^2=2G_1(\delta)(\rho_2+\sigma_2)+(\rho_1^2+\sigma_1^2)\left(\frac{1}{2}G_2(\delta)-G_1(\delta)\right).$$

According to (26), we obtain

$$\frac{4(6\epsilon+2\zeta+1)}{2\mu+1}q_2^2=2G_1(\delta)(\rho_2+\sigma_2)+\rho_1^2(G_2(\delta)-2G_1(\delta)). \tag{29}$$

From Equations (26) and (27), we obtain

$$\rho_1^2=\frac{4(2\epsilon+\zeta+1)^2q_2^2}{G_1^2(\delta)(\mu+1)^2}. \tag{30}$$

By replacing ρ_1^2 in Equation (29), we obtain

$$q_2^2=\frac{G_1^3(\delta)(\rho_2+\sigma_2)(2\mu+1)(\mu+1)^2}{4\left[(6\epsilon+2\zeta+1)(\mu+1)^2G_1^2(\delta)-(2\epsilon+\zeta+1)^2(2\mu+1)(G_2(\delta)-2G_1(\delta))\right]}. \tag{31}$$

Applying (11) and Lemma 1, we obtain

$$\begin{aligned}|q_2|&\le\sqrt{\frac{(2\delta-1)^3(2\mu+1)(\mu+1)^2}{\left|(6\epsilon+2\zeta+1)(\mu+1)^2(2\delta-1)^2-4(2\epsilon+\zeta+1)^2(2\mu+1)(\delta^2-3\delta+1)\right|}}\\&=\sqrt{D(\epsilon,\zeta,\delta)},\end{aligned}$$

where

$$D(\epsilon,\zeta,\delta)=\frac{(2\delta-1)^3(2\mu+1)(\mu+1)^2}{\left|(6\epsilon+2\zeta+1)(\mu+1)^2(2\delta-1)^2-4(2\epsilon+\zeta+1)^2(2\mu+1)(\delta^2-3\delta+1)\right|}.$$

Subtracting (25) from (23), and in view of (26), we obtain

$$q_3 = q_2^2 + \frac{G_1(\delta)(\rho_2 - \sigma_2)(2\mu+1)}{2(6\epsilon + 2\zeta + 1)}. \tag{32}$$

Substituting the value of q_2^2 from (28) and using (26), we have

$$q_3 = \frac{G_1^2(\delta)\rho_1^2(\mu+1)^2}{4(2\epsilon+\zeta+1)^2} + \frac{G_1(\delta)(\rho_2 - \sigma_2)(2\mu+1)}{2(6\epsilon + 2\zeta + 1)}. \tag{33}$$

Applying (11) and Lemma 1, we obtain

$$\begin{aligned} |q_3| &\leq \frac{G_1^2(\delta)|\rho_1|^2(\mu+1)^2}{4(2\epsilon+\zeta+1)^2} + \frac{G_1(\delta)(|\rho_2| + |\sigma_2|)(2\mu+1)}{2(6\epsilon + 2\zeta + 1)} \\ &\leq \frac{(2\delta-1)^2(\mu+1)^2}{4(2\epsilon+\zeta+1)^2} + \frac{(2\delta-1)(2\mu+1)}{6\epsilon + 2\zeta + 1}. \end{aligned}$$

From (32), we have

$$q_3 - \vartheta q_2^2 = \frac{G_1(\delta)(\rho_2 - \sigma_2)(2\mu+1)}{2(6\epsilon + 2\zeta + 1)} + (1-\vartheta)q_2^2.$$

Using (11) after the triangular inequality, we arrive at

$$\begin{aligned} \left|q_3 - \vartheta q_2^2\right| &\leq \frac{G_1(\delta)(|\rho_2| + |\sigma_2|)(2\mu+1)}{2(6\epsilon + 2\zeta + 1)} + |1-\vartheta||q_2|^2 \\ &\leq \frac{(2\delta-1)(2\mu+1)}{(6\epsilon + 2\zeta + 1)} + |1-\vartheta|D(\epsilon,\zeta,\delta). \end{aligned}$$

If

$$|1-\vartheta|D(\epsilon,\zeta,\delta) \leq \frac{(2\delta-1)(2\mu+1)}{(6\epsilon + 2\zeta + 1)}$$

we obtain

$$\left|q_3 - \vartheta q_2^2\right| \leq \frac{2(2\delta-1)(2\mu+1)}{6\epsilon + 2\zeta + 1},$$

and if

$$|1-\vartheta|D(\epsilon,\zeta,\delta) \geq \frac{(2\delta-1)(2\mu+1)}{6\epsilon + 2\zeta + 1}$$

we obtain

$$\left|q_3 - \vartheta q_2^2\right| \leq 2|1-\vartheta|D(\epsilon,\zeta,\delta).$$

which are the Theorem 1 assertions. □

Theorem 2. *Let $Q \in \Pi$ of the form (1) in the subfamily $J(\varphi,\delta)$, where $-\pi < \varphi \leq \pi$, $\frac{1}{2} < \delta \leq 1$, $\varsigma, \varpi \in \mathbb{U}$ and $H = Q^{-1}$. Then,*

$$|q_2| \leq \sqrt{Y(\varphi,\delta)},$$

$$|q_3| \leq \frac{(2\delta-1)^2(\mu+1)^2}{4(e^{i\varphi}+3)^2} + \frac{(2\delta-1)(2\mu+1)}{3(e^{i\varphi}+2)}$$

and

$$\left|q_3 - \vartheta q_2^2\right| \leq \begin{cases} \frac{2(2\delta-1)(2\mu+1)}{3(e^{i\varphi}+2)} & \text{if } \; 0 \leq |1-\vartheta|Y(\varphi,\delta) < \frac{(2\delta-1)(2\mu+1)}{3(e^{i\varphi}+2)}, \\ 2|1-\vartheta|Y(\varphi,\delta) & \text{if } \; |1-\vartheta|Y(\varphi,\delta) \geq \frac{(2\delta-1)(2\mu+1)}{3(e^{i\varphi}+2)}, \end{cases}$$

where

$$\Upsilon(\varphi,\delta)=\frac{2(2\delta-1)^3(2\mu+1)(\mu+1)^2}{2\left|3\left(e^{i\varphi}+2\right)(\mu+1)^2(2\delta-1)^2-4\left(e^{i\varphi}+3\right)^2(2\mu+1)\left(\delta^2-3\delta+1\right)\right|}.$$

Proof. Since $Q(\varsigma)=\varsigma+\sum_{k=2}^{\infty}q_k\varsigma^k\in J(\varphi,\delta)$, from Equations (14), (15), (20), and (21), we can write

$$\left(EQ_\mu(\varsigma)\right)'+\varsigma\frac{e^{i\varphi}+1}{2}\left(EQ_\mu(\varsigma)\right)''\prec\mathcal{K}(\delta,\varsigma) \tag{34}$$

and

$$\left(EH_\mu(\varpi)\right)'+\varpi\frac{e^{i\varphi}+1}{2}\left(EH_\mu(\varpi)\right)''\prec\mathcal{K}(\delta,\varpi). \tag{35}$$

From Equations (34) and (35), and the functions $\mathcal{K}(\delta,\varsigma)$ and $\mathcal{K}(\delta,\varpi)$, respectively, which are given by (20) and (21), we have

$$\frac{e^{i\varphi}+3}{\mu+1}q_2=\frac{G_1(\delta)}{2}\rho_1, \tag{36}$$

$$\frac{3\left(e^{i\varphi}+2\right)}{2(2\mu+1)}q_3=\frac{G_1(\delta)}{2}\left(\rho_2-\frac{\rho_1^2}{2}\right)+\frac{G_2(\delta)}{8}\rho_1^2, \tag{37}$$

$$-\frac{e^{i\varphi}+3}{\mu+1}q_2=\frac{G_1(\delta)}{2}\sigma_1, \tag{38}$$

and

$$\frac{3\left(e^{i\varphi}+2\right)}{2(2\mu+1)}\left(2q_2^2-q_3\right)=\frac{G_1(\delta)}{2}\left(\sigma_2-\frac{\sigma_1^2}{2}\right)+\frac{G_2(\delta)}{8}\sigma_1^2. \tag{39}$$

We obtain the findings provided by Theorem 2 using the same method used to prove Theorem 1. □

3. Some Corollaries

By specializing the parameters in our main results for the previous section, we obtain some corollaries, for example:

Corollary 1. *Let $Q\in T(1,\epsilon,\delta)$, where $\epsilon\geq 0$, $\frac{1}{2}<\delta\leq 1$, $\varsigma,\varpi\in\mathbb{U}$. Then,*

$$|q_2|\leq\sqrt{D(\epsilon,1,\delta)},$$

$$|q_3|\leq\frac{(2\delta-1)^2(\mu+1)^2}{16(\epsilon+1)^2}+\frac{(2\delta-1)(2\mu+1)}{3(2\epsilon+1)}$$

and

$$\left|q_3-\vartheta q_2^2\right|\leq\begin{cases}\frac{2(2\delta-1)(2\mu+1)}{3(2\epsilon+1)} \ \text{if}\ 0\leq|1-\vartheta|D(\epsilon,1,\delta)<\frac{(2\delta-1)(2\mu+1)}{3(2\epsilon+1)},\\[2ex] 2|1-\vartheta|D(\epsilon,1,\delta)\ \text{if}\ |1-\vartheta|D(\epsilon,1,\delta)\geq\frac{(2\delta-1)(2\mu+1)}{3(2\epsilon+1)},\end{cases}$$

where

$$D(\epsilon,1,\delta)=\frac{(2\delta-1)^3(2\mu+1)(\mu+1)^2}{\left|3(2\epsilon+1)(\mu+1)^2(2\delta-1)^2-16(\epsilon+1)^2(2\mu+1)\left(\delta^2-3\delta+1\right)\right|}.$$

Corollary 2. *Let $Q \in T(\zeta, 0, \delta)$, where $\zeta \geq 1$, $\frac{1}{2} < \delta \leq 1$, $\varsigma, \varpi \in \mathbb{U}$. Then,*

$$|q_2| \leq \sqrt{D(0, \zeta, \delta)},$$

$$|q_3| \leq \frac{(2\delta - 1)^2 (\mu + 1)^2}{4(\zeta + 1)^2} + \frac{(2\delta - 1)(2\mu + 1)}{2\zeta + 1}$$

and

$$\left|q_3 - \vartheta q_2^2\right| \leq \begin{cases} \frac{2(2\delta-1)(2\mu+1)}{(2\zeta+1)} \ \textit{if}\ 0 \leq |1 - \vartheta| D(0, \zeta, \delta) < \frac{(2\delta-1)(2\mu+1)}{2\zeta+1}, \\ \\ 2|1 - \vartheta| D(0, \zeta, \delta) \ \textit{if}\ |1 - \vartheta| D(0, \zeta, \delta) \geq \frac{(2\delta-1)(2\mu+1)}{2\zeta+1}, \end{cases}$$

where

$$D(0, \zeta, \delta) = \frac{(2\delta - 1)^3 (2\mu + 1)(\mu + 1)^2}{\left|(2\zeta + 1)(\mu + 1)^2 (2\delta - 1)^2 - 4(\zeta + 1)^2 (2\mu + 1)(\delta^2 - 3\delta + 1)\right|}.$$

Corollary 3. *Let $Q \in J(\pi, \delta)$ where $\frac{1}{2} < \delta \leq 1$, $\varsigma, \varpi \in \mathbb{U}$. Then,*

$$|q_2| \leq \sqrt{Y(\pi, \delta)},$$

$$|q_3| \leq \frac{(2\delta - 1)^2 (\mu + 1)^2}{16} + \frac{(2\delta - 1)(2\mu + 1)}{3}$$

and

$$\left|q_3 - \vartheta q_2^2\right| \leq \begin{cases} \frac{2(2\delta-1)(2\mu+1)}{3} \ \textit{if}\ 0 \leq |1 - \vartheta| Y(\pi, \delta) < \frac{(2\delta-1)(2\mu+1)}{3}, \\ \\ 2|1 - \vartheta| Y(\pi, \delta) \ \textit{if}\ |1 - \vartheta| Y(\pi, \delta) \geq \frac{(2\delta-1)(2\mu+1)}{3}, \end{cases}$$

where

$$Y(\pi, \delta) = \frac{(2\delta - 1)^3 (2\mu + 1)(\mu + 1)^2}{\left|3(\mu + 1)^2 (2\delta - 1)^2 - 16(2\mu + 1)(\delta^2 - 3\delta + 1)\right|}.$$

Corollary 4. *Let $Q \in J(0, \delta)$ where $\frac{1}{2} < \delta \leq 1$, $\varsigma, \varpi \in \mathbb{U}$. Then,*

$$|q_2| \leq \sqrt{Y(0, \delta)},$$

$$|q_3| \leq \frac{(2\delta - 1)^2 (\mu + 1)^2}{64} + \frac{(2\delta - 1)(2\mu + 1)}{9}$$

and

$$\left|q_3 - \vartheta q_2^2\right| \leq \begin{cases} \frac{2(2\delta-1)(2\mu+1)}{9} \ \textit{if}\ 0 \leq |1 - \vartheta| Y(0, \delta) < \frac{(2\delta-1)(2\mu+1)}{9}, \\ \\ 2|1 - \vartheta| Y(0, \delta) \ \textit{if}\ |1 - \vartheta| Y(0, \delta) \geq \frac{(2\delta-1)(2\mu+1)}{9}, \end{cases}$$

where

$$Y(0, \delta) = \frac{2(2\delta - 1)^3 (2\mu + 1)(\mu + 1)^2}{2\left|9(\mu + 1)^2 (2\delta - 1)^2 - 64(2\mu + 1)(\delta^2 - 3\delta + 1)\right|}.$$

4. Conclusions

Numerous distinguished mathematicians have recently researched special functions since they are used in so many different mathematical and scientific fields. The aim of this study is to define new subfamilies of analytical functions using error functions subordinate to Euler polynomials. For functions in the subfamilies $T(\zeta, \epsilon, \delta)$ and $J(\varphi, \delta)$, we obtained the initial bounds for the coefficients $|q_2|$ and $|q_3|$, and the Fekete–Szegö inequality. The upper bounds for $|q_2|$, $|q_3|$ and $|q_3 - \vartheta q_2^2|$ are still an open problem for $|q_k|$, $k \geq 3$. Using the linear operator EQ_μ given in (9) could inspire researchers to find new bounds for the coefficients $|q_2|$ and $|q_3|$, and the Fekete–Szegö inequality for different subfamilies of normalized analytic functions with negative coefficients defined in the open unit disk $\mathbb{U}$.

Author Contributions: Conceptualization, B.F.; methodology, J.S.; validation and formal analysis, T.A.-H.; investigation and resources, J.S.; data curation, B.F. and T.A.-H.; writing—review and editing, B.F. and T.A.-H.; visualization and supervision, T.A.-H. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The data are contained within the article.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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