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Quasi-Sure Exponential Stability of Stochastic Differential Delay Systems Driven by *G*-Brownian Motion

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Abstract: This paper focuses on the quasi-sure exponential stability of the stochastic differential delay equation driven by *G*-Brownian motion (SDDE-GBM): $d\xi(t) = f(t, \xi(t - \kappa_1(t)))dt + g(t, \xi(t - \kappa_2(t)))d\mathcal{Z}(t)$, where $\kappa_1(\cdot), \kappa_2(\cdot) : \mathbb{R}_+ \to [0, \tau]$ denote variable delays, and $\mathcal{Z}(t)$ denotes scalar *G*-Brownian motion, which has a symmetry distribution. It is shown that the SDDE-GBM is quasi-surely exponentially stable for each $\tau > 0$ bounded by τ^*, where the corresponding (non-delay) stochastic differential equation driven by *G*-Bronwian motion (SDE-GBM), $d\eta(t) = f(t, \eta(t))dt + g(t, \eta(t))d\mathcal{Z}(t)$, is quasi-surely exponentially stable. Moreover, by solving the non-linear equation on τ, we can obtain the implicit lower bound τ^*. Finally, illustrating examples are provided.

Keywords: SDDE-GBM; quasi-sure exponential stability; *G*-Brownian motion; delay bound; Borel–Cantelli's lemma

MSC: 60H10; 60J65

1. Introduction

As a crucial topic in stochastic systems theory and applications, in recent decades, the stability analyses of stochastic differential (delay) equations have been investigated by numerous researchers (see, e.g., [1–6] and the references therein). The existence, uniqueness, and related properties of solutions to stochastic differential equations driven by *G*-Brownian motion have been studied by a number of scholars (e.g., [7–12]). Meanwhile, the stability of stochastic differential equations driven by *G*-Brownian motion is explored in, e.g., references [13–20]. In particular, the topic of almost-sure exponential stability has significant meaning in the stability analysis of stochastic differential equations. Plenty of scholars have conducted research investigations on this topic, linking it to the almost-sure stability of stochastic differential systems (see, e.g., [21–29] and the references therein). Recently, in the *G*-Lévy framework, the Caratheodory approximation scheme for stochastic differential equations was investigated in [30], whereas the quasi-sure exponential stability for stochastic delay differential equations has not yet been explored.

The study of the (non-delay) SDEs has not been sufficiently explored as the systems depend on past states. Thus, it is necessary to discuss the stochastic differential delay equations (SDDEs). The related investigation is referred to in the references (e.g., [22,31]). Observably, Guo et al. [22] explored the almost-sure exponential stability of SDEs with multi-dimensional variable delays of the following form:

$$d\xi(t) = f(t, \xi(t - \kappa_1(t)))dt + g(t, \xi(t - \kappa_2(t)))dZ(t), \quad (1)$$

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where $\kappa_1(\cdot), \kappa_2(\cdot) : \mathbb{R}_+ \to [0, \tau]$ represent variable delays; and the process $Z(t)$ is a standard Brownian motion. The authors show that the SDDE (1) still preserves the almost-sure exponential stability for a sufficiently small delay if the corresponding SDE without delay is stable.

On the other hand, since Knightian uncertainty cannot be ignored in reality, the non-linear expectation theory is found by Peng [32], where G-Brownian-motion-driven stochastic differential equations are investigated. Subsequently, the theory of stability in stochastic differential equations is discussed by many scholars, such as [33–36] and the references therein. Therefore, only using the system modeled by SDDE (1) to describe some real-world systems may not be enough; we have to observe a stochastic system with both time-delays and G-Brownian motion, as follows:

$$d\xi(t) = f(t, \xi(t - \kappa_1(t)))dt + g(t, \xi(t - \kappa_2(t)))d\mathcal{Z}(t), \tag{2}$$

where $\mathcal{Z}(t)$ is a scalar G-Brownian motion.

The corresponding SDE-GBM (3) of the SDDE-GBM (2),

$$d\eta(t) = f(t, \eta(t))dt + g(t, \eta(t))d\mathcal{Z}(t), \tag{3}$$

plays the vital role in the analysis of stabilities.

Compared with SDE-GBM (3), what we concentrate on is the influence of time delay on the stability of the system (2) above. The following questions come out naturally.

(a) If the SDE-GBM (3) is quasi-surely exponentially stable, does the SDDE-GBM (2) have the same stability when τ is sufficiently small?
(b) If the claim in (a) holds, can an upper bound τ^* be obtained such that the SDDE-GBM (2) is quasi-surely exponentially stable for each $\tau < \tau^*$?
(c) Can we obtain the same conclusion as the case in SDDE (1)? Compared with the SDDE (1), what kind of effect does an ambiguity volatility have in the SDDE-GBM (2) system?

Symmetry is immune from a possible change; that is, we have symmetry when it is possible to perform some change in the situation that nevertheless leaves some aspect of the situation unaffected; thus, we have symmetry under the change in this respect. If some aspect of the situation is not immune from the change, then the situation is asymmetric under change. Through the symmetry of stochastic differential delay systems perturbed by G-Brownian motion, we will, stepwise, provide answers to the above three questions in the rest of this paper. The statements above clearly emphasize the differences between our work and the existing research.

Our work has the following innovative qualities:

(i) At first, the quasi-sure exponential stability of SDDEs is investigated in the G-Brownian motion framework.
(ii) Sublinear expectation theory is employed to solve the estimation and quasi-sure exponential stability of the solution to SDDE driven by G-Brownian motion.
(iii) By comparing the solution of SDDE-GBM with one of the corresponding SDE-GBM, we obtain the implicit lower bound τ^*. The methodology needs to overcome the difficulty incurred by nonlinear probability.

This paper is arranged as follows. Section 2 provides the preliminaries. In Section 3, we put forward several lemmas and present their proofs in order to verify the key theorem. We apply our main result to linear cases in Section 4, where the corollaries are given to illustrate our main result. Finally, Section 5 concludes the paper. A useful lemma is placed in Appendix A.

2. Notations and Preliminaries

Let $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ denote sublinear expectation space, where Ω is a given state set, and a linear space of real valued functions $\mathcal{H}$ defined on Ω can be considered as the space of random variables. The following concept of sublinear expectation comes from Peng [32].

Definition 1. *A functional $\hat{\mathbb{E}}$: $\mathcal{H} \to \mathbb{R}$ is called sublinear expectation if it fulfills the following criteria:*

(i) Monotonicity: For $X \geq Y$, $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$;
(ii) Constant preserving: For each constant c, $\hat{\mathbb{E}}[c] = c$;
(iii) Sub-additivity: $\hat{\mathbb{E}}[X+Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$ for any $X, Y \in \mathcal{H}$;
(iv) Positivity homogeneity: For each constant $\lambda \geq 0$, $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$.

Let $\mathcal{E}[X] := -\hat{\mathbb{E}}[-X]$ be a sublinear lower expectation for a random variable X. For a scalar G-Brownian motion $\mathcal{Z}(t)$, define the function $G(\alpha) := \frac{1}{2}\hat{\mathbb{E}}[\alpha \mathcal{Z}^2(1)] = \frac{1}{2}(\overline{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-)$, where $\mathcal{E}[\mathcal{Z}^2(1)] = -\hat{\mathbb{E}}[-\mathcal{Z}^2(1)] = \underline{\sigma}^2, \hat{\mathbb{E}}[\mathcal{Z}^2(1)] = \overline{\sigma}^2$, $0 < \underline{\sigma} \leq \overline{\sigma} < \infty$. And $\overline{\sigma}$ ($\underline{\sigma}$) is called maximum (minimun) ambiguous volatility.

For $q > 0$, let $M_G^{q,0}(0,T)$ be the family of all processes ψ with the norm

$$\|\psi\|_{M_G^q(0,T)} := \left\{ \hat{\mathbb{E}} \int_0^T |\psi(t)|^q dt \right\}^{1/q} < \infty.$$

In what follows, let P_0 be a Wiener measure on a given canonical probability space $(\Omega, \mathcal{B}(\Omega), P_0)$, $P^{\sigma_\cdot}(A) = P_0(Y^{\sigma_\cdot} \in A)$ for any event $A \in \mathcal{B}(\Omega)$, and $Y^{\sigma_\cdot}(t) = \int_0^t \sigma_s dw(s)$ P_0-a.s., where $(w(t))_{t\geq 0}$ is a standard Brownian motion under $(\Omega, \mathcal{B}(\Omega), P_0)$, and $\Omega = C_0(\mathbb{R}_+)$ is the space of all $\mathbb{R}$-valued continuous paths $(\omega(t))_{t\geq 0}$ with $\omega(0) = 0$ and $\Omega_t := \sigma\{\omega(\cdot \wedge t); \omega \in \Omega\}$. Obviously,

$$\begin{aligned}\mathcal{Z}(\cdot) :=& \{Y^{\sigma_\cdot};\ (\sigma_t)_{t\geq 0} \text{ is an } (\Omega_t)_{t\geq 0}\text{-progressively measurable process taking in} \\ & [\underline{\sigma}, \overline{\sigma}],\ P_0 \text{ a.s.}\}\end{aligned}$$

refers to G-Brownian motion under the probability measure family $\mathcal{P}$, where the multi-priors probability measure family $\mathcal{P}$, related to sublinear expectation $\hat{\mathbb{E}}$, is defined as follows (see Fei et al. [35]):

$$\begin{aligned}\mathcal{P} :=& \Big\{ P^{\sigma_\cdot} = P_0 \circ (Y^{\sigma_\cdot})^{-1}; (\sigma_t)_{t\geq 0} \\ & \text{is an } (\Omega_t)_{t\geq 0}\text{-progressively measurable process taking in } [\underline{\sigma}, \overline{\sigma}], P_0 \text{ a.s.} \Big\}, \end{aligned} \tag{4}$$

Thus, $d\mathcal{Z}(t) = \sigma_t dw(t)$, P_0-a.s.

For $\mathcal{P}$ related to sublinear expectation $\hat{\mathbb{E}}$, we define G-upper capacity $\mathbb{V}(\cdot)$ and G-lower capacity $\mathcal{V}(\cdot)$, associated to $\hat{\mathbb{E}}$ by

$$\begin{aligned}\mathbb{V}(A) &= \sup_{P\in\mathcal{P}} P(A),\ \forall\ A \in \mathcal{B}(\Omega), \\ \mathcal{V}(A) &= \inf_{P\in\mathcal{P}} P(A),\ \forall\ A \in \mathcal{B}(\Omega).\end{aligned}$$

Thus, a property is said to hold quasi-surely (q.s.) if there exists a polar set D with $\mathbb{V}(D) = 0$ such that it holds for each $\omega \in D^c$.

The following the Burkholder–Davis–Gundy inequality provides the explicit bounds (see, e.g., Fei et al. [35] (Lemma 2.4)).

Lemma 1 (Burkholder–Davis–Gundy inequality). *Let $r > 0$ and $\zeta = (\zeta(s))_{s\in[0,\mathcal{T}]} \in M_G^r(0,\mathcal{T})$. Then, for all $t \in [0,\mathcal{T}]$,*

$$\begin{aligned}
&\hat{\mathbb{E}} \sup_{s\le u\le t} \Big| \int_s^u \zeta(v) d\langle \mathcal{Z}\rangle(v)\Big|^r \\
&\quad \le (t-s)^{r-1}\overline{\sigma}^{2r}\hat{\mathbb{E}}\Big(\int_s^t |\zeta(v)|^2 dv\Big)^{r/2}, \text{ for } r \ge 1, \\
&\hat{\mathbb{E}} \sup_{s\le u\le t} \Big| \int_s^u \zeta(v) d\mathcal{Z}(v)\Big|^r \le C_r\overline{\sigma}^{r}\hat{\mathbb{E}}\Big(\int_s^t |\zeta(v)|^2 dv\Big)^{r/2}, \text{ for } r > 0,
\end{aligned}$$

where the constant C_r is defined as follows

$$\begin{aligned}
C_r &= \Big(\frac{32}{r}\Big)^{r/2}, \quad \text{if } 0 < r < 2; \\
C_r &= \Big(\frac{r^{r+1}}{2(r-1)^{r-1}}\Big)^{r/2}, \quad \text{if } r \ge 2.
\end{aligned}$$

Throughout this paper, we will use the following notations unless otherwise specified. For $z \in \mathbb{R}^n$, $|z|$ denotes its Euclidean norm. For a matrix D, let the superscript $D^\top$ represent its transpose, $|D| = \sqrt{\text{trace}(D^\top D)}$ its trace norm and $\|D\| = \max\{|Dz| : |z| = 1\}$ the operator norm. A filtered sublinear expectation space $(\Omega, \mathcal{H}, \{\Omega_t\}_{t\ge 0}, \hat{\mathbb{E}})$ carries with a filtration $\{\Omega_t\}_{t\ge 0}$. For $\tau > 0$, $C([-\tau,0];\mathbb{R}^n)$ represents the set of continuous functions ϑ from $[-\tau,0] \to \mathbb{R}^n$ with the norm $\|\vartheta\| = \sup_{-\tau\le v\le 0}|\vartheta(v)|$. For $t \ge t_0$, $L^2_{\Omega_t}(\Omega,\mathbb{R}^n)$ denotes the set of all Ω_t-measurable $\mathbb{R}^n$-valued random variables ς with $\mathbb{E}|\varsigma|^2 < \infty$, and $L^2_{\Omega_t}(\Omega, C([-\tau,0];\mathbb{R}^n))$ the set of Ω_t-measurable $C([-\tau,0];\mathbb{R}^n)$-valued random variables ϑ with $\mathbb{E}\|\vartheta\|^2 < \infty$. The functions $\kappa_1(\cdot), \kappa_2(\cdot) : \mathbb{R}_+ \to [0,\tau]$ are assumed to be Borel measurable functions.

In this paper, we will investigate the SDDE-GBM (2) with initial data $\xi(t_0+v) = \vartheta(v)$ for $v \in [-\tau,0]$, where $\vartheta \in L^2_{\Omega_{t_0}}(\Omega; C([-\tau,0];\mathbb{R}^n))$ for $t_0 \ge 0$, and $f, g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$.

Next, the quasi-sure exponential stability is defined as follows.

Definition 2. (i) *The trivial solution of SDDE-GBM (2) is said to be quasi-surely exponentially stable if* $\limsup_{t\to\infty} \frac{1}{t}|\log|\xi(t;\vartheta)| < 0$ q.s. *for all ϑ in (2).*

For the investigation on the stability of the system, we carry out the standing hypotheses on f and g.

Assumption 1. *For Borel measurable functions f and g, there exist constants $b_1, b_2 > 0$ such that*

$$\begin{aligned}
|f(t,u) - f(t,v)| &\le b_1|u-v|, \\
|g(t,u) - g(t,v)| &\le b_2|u-v|,
\end{aligned}$$

for all $u, v \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$.

Furthermore, we set the initial value $f(t,0) = g(t,0) = 0$ for each $t \ge 0$.

Obviously, this global Lipschitz condition induces the linear growth condition

$$|f(t,z)| \le b_1|z|, \qquad |g(t,z)| \le b_2|z| \tag{5}$$

for all $(z,t) \in \mathbb{R}^n \times \mathbb{R}_+$.

Similar to the discussion in Fei [36], under these assumptions, the SDDE-GBM (2) with the initial data ϑ has a unique solution $\xi(t)$ on $t \geq t_0 - \tau$, which has the property that $\hat{\mathbb{E}}|\xi(t)|^2 < \infty$.

We shall denote the solution by $\xi(t; t_0, \vartheta)$, though we usually use $\xi(t)$ for simplicity. Next, we define $\xi_t = \{\xi(t+v) : v \in [-\tau, 0]\}$ for $t \geq t_0$ with $\xi_t \in L^2_{\Omega_t}(\Omega, C([-\tau, 0]; \mathbb{R}^n))$. Moreover, we have the solution of the SDDE-GBM (2) with the following form

$$\xi(t) = \xi(t; s, \xi_s),\ t_0 \leq s \leq t < \infty,$$

which reveals clearly that we only need the information on the given initial data ξ_s at time s to confirm $\xi(t)$ by solving the SDDE-GBM (2). The key technique used here is to compare the SDDE-GBM (2) with the corresponding SDE-GBM (3) with the initial value $\eta(t_0) = \eta_0$ and $\eta_0 \in L^2_{\Omega_{t_0}}(\Omega; \mathbb{R}^n)$. For Assumption 1, the unique solution of SDE-GBM (3) $\eta(t)$ on $t \geq t_0$ (see, Fei et al. [36]) satisfies $\hat{\mathbb{E}}|\eta(t)|^2 < \infty$ for any $t \geq t_0$. In order to emphasize the initial data η_0 at time t_0, we use the notation to represent the solution $\eta(t; t_0, \eta_0)$ instead of $\eta(t)$.

Obviously, we need to suppose that the SDE-GBM (3) is quasi-surely exponentially stable. We will prove the quasi-sure exponential stability of the SDE-GBM (3) in a theorem below. Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ represent the family of non-negative Lyapunov functions $V(z,t)$ defined on $\mathbb{R}^n \times \mathbb{R}_+$ which are once continuously differentiable in t and twice continuously differentiable in z. For each $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ and f, g in (2), we define the operator $\mathcal{L}V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ as follows:

$$\begin{aligned} \mathcal{L}V(z, y_1, y_2, t) &= V_t(z,t) + V_z(z,t) f(t, y_1) \\ &\quad + G(g^T(t, y_2) V_{zz}(z,t) g(t, y_2)), \end{aligned} \tag{6}$$

where

$$V_t(z,t) := \frac{\partial V}{\partial t}(z,t),\ V_z(z,t) := \frac{\partial V}{\partial z}(z,t),\ V_{zz}(z,t) := \Big(\frac{\partial^2 V}{\partial z_i z_j}(z,t)\Big)_{n \times n}.$$

Now, let us impose another assumption.

Assumption 2. *Let the function $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ verify that for all $(z,t) \in \mathbb{R}^n \times \mathbb{R}_+$,*

$$\begin{aligned} &a_1|z|^p \leq V(z,t) \leq \bar{a}_1|z|^p, \quad |V_z(z,t)g(t,z)|^2 \geq a_3(V(z,t))^2, \\ &\mathcal{L}V(z,t) \leq a_2 V(z,t), \quad 0.5\underline{\sigma}^2(1-\phi)a_3 > a_2, \end{aligned}$$

where constants $\phi \in (0,1)$, $\bar{a}_1 \geq a_1 > 0$, $a_2 \in \mathbb{R}$, $p > 0$, $a_3 \geq 0$ and $\varrho := \phi p < 1$.

The proposition deduced below shows that the SDE-GBM (3) is quasi-surely exponentially stable under Assumption 2. Here, our objective is to answer questions (a)–(c) listed above under these assumptions.

3. Main Results of Stability and Their Proofs

In this section, we provide several lemmas, which are necessary in order to provide positive answers to our desired questions. First of all, we cite the following lemma, which can be easily deduced due to Peng [32].

Lemma 2. *Let $F \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$, and $\xi(\cdot)$ be the solution of SDDE-GBM (2). Then,*

$$dF(\xi(t),t) \le \mathcal{L}F(\xi(t),\xi(t-\kappa_1(t)),\xi(t-\kappa_2(t)),t)dt + dM(t),\ t \ge 0,$$

where operation $\mathcal{L}$ is defined by (6), and

$$M(t) = F_z(\xi(t),t)g(t,\xi(t-\kappa_2(t)))d\mathcal{Z}(t)$$

is a continuous G-martingale with $M(0) = 0$.

Lemma 3. *Let Assumptions 1 and 2 hold. Then, the solution $\eta(t;t_0,\eta_0)$ of SDE-GBM (3) verifies*

$$\hat{\mathbb{E}}|\eta(t;t_0,\eta_0)|^\varrho \le Le^{-\gamma(t-t_0)}\hat{\mathbb{E}}|\eta_0|^\varrho, \tag{7}$$

for any $t \ge t_0$ and any $\eta_0 \in L^2_{\Omega_{t_0}}(\Omega;\mathbb{R}^n)$. Here,

$$L = (\bar{a}_1/a_1)^\phi, \quad \gamma = \phi(0.5\underline{\sigma}^2(1-\phi)a_3 - a_2). \tag{8}$$

Proof. Fix any t_0 and $\eta_0 \in \mathbb{R}^n$, with η_0 being deterministic. Let us first prove this case with the form $\eta(t;t_0,\eta_0) = \eta(t)$ for convenience. Assertion (7) holds when $\eta_0 = 0$, so we just need to consider whether the assertion holds or not as $\eta_0 \ne 0$. By Lemma A1, we show that if $\eta_0 \ne 0$, then $\eta(t) \ne 0$ for all $t \ge t_0$, quasi-surely. Set $\Upsilon(z,t) = (V(z,t))^\phi$. By Itô formula (see Peng [32]) and Lemma 2, we derive

$$\begin{aligned}
d\Big(e^{\gamma t}\Upsilon(\eta(t),t)\Big) &= \gamma e^{\gamma t}\Upsilon(\eta(t),t)dt + e^{\gamma t}\phi V^{\phi-1}(V_t(\eta(t),t) + V_z(\eta(t),t)f(t,\eta(t)))dt \\
&\quad + \frac{1}{2}e^{\gamma t}\phi V^{\phi-1}(\eta(t),t)g^\top(t,\eta(t))V_{zz}(\eta(t),t)g(t,\eta(t))d\langle\mathcal{Z}\rangle_t \\
&\quad - \frac{1}{2}e^{\gamma t}\phi(1-\phi)V^{\phi-2}(\eta(t),t)|V_z(\eta(t),t)g(t,\eta(t))|^2 d\langle\mathcal{Z}\rangle_t \\
&\quad + e^{\gamma t}\phi V^{\phi-1}(\eta(t),t)V_z(\eta(t),t)g(t,\eta(t))\Big)d\mathcal{Z}(t) \\
&\le \gamma e^{\gamma t}\Upsilon(\eta(t),t) + e^{\gamma t}\mathbb{L}\Upsilon(\eta(t),t) + d\mathbb{M}(t),
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
\mathbb{M}(t) &= \phi\int_{t_0}^t e^{\gamma s}V^{\phi-1}(\eta(s),s)V_z(\eta(s),s)g(s,\eta(s))d\mathcal{Z}(s), \\
\mathbb{L}\Upsilon(z,t) &= \phi(V(z,t))^{\phi-1}\mathcal{L}V(z,t) - 0.5\underline{\sigma}^2\phi(1-\phi)(V(z,t))^{\phi-2}|V_z(z,t)g(t,z)|^2.
\end{aligned}$$

Thus, from (9), we obtain

$$e^{\gamma t}\hat{\mathbb{E}}\Upsilon(\eta(t),t) \le e^{\gamma t_0}\Upsilon(\eta_0,t_0) + \hat{\mathbb{E}}\int_{t_0}^t [\gamma e^{\gamma s}\Upsilon(\eta(s),s) + e^{\gamma s}\mathbb{L}\Upsilon(\eta(s),s)]ds$$

for $t \ge t_0$.

Then, through Assumption 2, we obtain

$$\mathbb{L}\Upsilon(z,t) \le -\gamma\Upsilon(z,t), \tag{10}$$

which shows

$$e^{\gamma t}\hat{\mathbb{E}}\Upsilon(\eta(t),t) \le e^{\gamma t_0}\Upsilon(\eta_0,t_0),\ \forall t \ge t_0.$$

Due to $a_1^{\phi}|z|^{\varrho} \leq \Upsilon(z,t) \leq \bar{a}_1^{\phi}|z|^{\varrho}$, we obtain

$$\hat{\mathbb{E}}|\eta(t)|^{\varrho} \leq L|\eta_0|^{\varrho}e^{-\gamma(t-t_0)}, \ \forall t \geq t_0.$$

Thus, the assertion is confirmed for $\eta_0 \in \mathbb{R}^n$.

On the other hand, for each fixed $\eta_0 \in L^2_{\Omega_{t_0}}(\Omega;\mathbb{R}^n)$, thanks to the property of the conditional expectation under sublinear expectation, the assertion

$$\hat{\mathbb{E}}|\eta(t)|^{\varrho} = \hat{\mathbb{E}}\big(\hat{\mathbb{E}}(|\eta(t)|^{\varrho} \mid \Omega_{t_0})\big) \leq \hat{\mathbb{E}}\big(L|\eta_0|^{\varrho}e^{-\gamma(t-t_0)}\big) = Le^{-\gamma(t-t_0)}\hat{\mathbb{E}}|\eta_0|^{\varrho} \tag{11}$$

preserves for $t \geq t_0$ as well. Thus, the proof is complete. □

Now, we can give the proof of the quasi-sure exponential stability of the SDE-GBM (3) via Lemma 3.

Proposition 1. *With the assertion (7) proved in Lemma 3, we can determine that for any initial data $\eta_0 \in L^2_{\Omega_{t_0}}(\Omega;\mathbb{R}^n)$, the solution of SDE-GBM (3) fulfills*

$$\limsup_{t\to\infty} \frac{1}{t}\log(|\eta(t;t_0,\eta_0)|) < 0 \quad \text{q.s.},$$

which means the SDE-GBM (3) is quasi-surely exponentially stable.

Proof. First, we set $\Upsilon(\eta(t),t) = (V(\eta(t),t))^{\phi}$ as Lemma 3. Due to (9), we then have

$$e^{\gamma t}\Upsilon(\eta(t),t) \leq e^{\gamma t_0}\Upsilon(\eta_0,t_0) + \int_{t_0}^{t}[\gamma e^{\gamma s}\Upsilon(\eta(s),s) + e^{\gamma s}\mathbb{L}\Upsilon(\eta(s),s)]ds + \mathbb{M}(t).$$

Obviously, $\mathbb{M}(t)$ is a G-martingale. Moreover, through (10) and Assumption 2, we obtain

$$0 \leq e^{\gamma t}\Upsilon(\eta(t),t) \leq e^{\gamma t_0}\Upsilon(\eta_0,t_0) + \mathbb{M}(t) \leq e^{\gamma t_0}\bar{a}_1^{\phi}|\eta_0|^{\varrho} + \mathbb{M}(t).$$

Therefore, by applying G-semimartingale convergence theorem (see, e.g., Peng et al. [37] (Theorem 3.3)), there exists a finite random variable $\zeta \geq 0$ such that

$$\mathbb{M}(t) \leq \zeta, \quad \text{q.s.}$$

Given above result, it follows that

$$|\eta(t)|^{\varrho} \leq a_1^{-\phi}e^{-\gamma(t-t_0)}(\bar{a}_1^{\phi}|\eta_0|^{\varrho} + \zeta e^{-\gamma t_0}) \quad \text{q.s.},$$

which shows

$$\lim_{t\to\infty}\frac{1}{t}\log|\eta(t)| \leq -\frac{\gamma}{\varrho} \quad \text{q.s.}$$

Therefore, the SDE-GBM (3) is quasi-surely exponentially stable. Thus, the proof is complete. □

What follows is a lemma about the properties related to the p-th moment of the solution to the SDDE-GBM (2).

Lemma 4. *Let Assumption 1 and Assumption 2 hold. Write $\xi(t;t_0,\vartheta)=\xi(t)$ for any $\vartheta\in L^2_{\Omega_{t_0}}(\Omega;C([-\tau,0];\mathbb{R}^n))$. Then, for any $t\geq t_0$,*

$$\hat{\mathbb{E}}|\xi(t)|^{\varrho}\leq\left(2e^{(2b_1+\overline{\sigma}^2b_2^2)(t-t_0)}\right)^{\varrho/2}\hat{\mathbb{E}}\|\vartheta\|^{\varrho}, \tag{12}$$

$$\hat{\mathbb{E}}\Big(\sup_{0\leq u\leq\tau}|\xi(t+u)-\xi(t)|^{\varrho}\Big)\leq\mathbb{Q}_1(\tau,\varrho,t-t_0)\hat{\mathbb{E}}\|\vartheta\|^{\varrho}, \tag{13}$$

and

$$\hat{\mathbb{E}}\Big(\sup_{t_0\leq u\leq t}|\xi(u)|^{\varrho}\Big)\leq\Big(4+\frac{8(t-t_0)b_1^2+32\overline{\sigma}^2b_2^2}{2b_1+\overline{\sigma}^2b_2^2}\exp\{(2b_1+\overline{\sigma}^2b_2^2)(t-t_0)\}\Big)^{\varrho/2}\hat{\mathbb{E}}\|\vartheta\|^{\varrho}, \tag{14}$$

where, for $T\geq 0$,

$$\mathbb{Q}_1(\tau,\varrho,T)=\left(4\tau(\tau b_1^2+4\overline{\sigma}^2b_2^2)e^{(2b_1+\overline{\sigma}^2b_2^2)(T+\tau)}\right)^{\varrho/2}. \tag{15}$$

Proof. We only have to consider the deterministic initial case here for $\vartheta\in C([-\tau,0];\mathbb{R}^n)$ since the conditional expectation can be proved in a similar manner to (11). Applying Lemma 2, along with Assumption 1, it is not difficult to determine that, for any $t\geq t_0$,

$$\begin{aligned}\hat{\mathbb{E}}|\xi(t)|^2&\leq|\xi(t_0)|^2+\hat{\mathbb{E}}\int_{t_0}^{t}\left[2\xi(s)^Tf(s,\xi(s-\kappa_1(s)))+\overline{\sigma}^2|g(s,\xi(s-\kappa_2(s)))|^2\right]ds\\&\leq\hat{\mathbb{E}}|\xi(t_0)|^2+\left[2b_1+\overline{\sigma}^2b_2^2\right]\int_{t_0}^{t}\Big(\sup_{t_0-\tau\leq v\leq s}\hat{\mathbb{E}}|\xi(v)|^2\Big)ds.\end{aligned}$$

Obviously, the above inequality remains correct when we turn the left-hand-side term to its supremum:

$$\hat{\mathbb{E}}\sup_{t_0\leq u\leq t}|\xi(u)|^2\leq|\xi(t_0)|^2+\left[2b_1+\overline{\sigma}^2b_2^2\right]\int_{t_0}^{t}\Big(\sup_{t_0-\tau\leq v\leq s}\hat{\mathbb{E}}|\xi(v)|^2\Big)ds,$$

which shows

$$\begin{aligned}\sup_{t_0-\tau\leq u\leq t}\hat{\mathbb{E}}|\xi(u)|^2&\leq\|\vartheta\|^2+\sup_{t_0\leq u\leq t}\hat{\mathbb{E}}|\xi(u)|^2\\&\leq 2\|\vartheta\|^2+\left[2b_1+\overline{\sigma}^2b_2^2\right]\int_{t_0}^{t}\Big(\sup_{t_0-\tau\leq v\leq s}\hat{\mathbb{E}}|\xi(v)|^2\Big)ds.\end{aligned}$$

By Gronwall's inequality, it follows that

$$\sup_{t_0-\tau\leq u\leq t}\hat{\mathbb{E}}|\xi(u)|^2\leq 2\|\vartheta\|^2e^{(2b_1+\overline{\sigma}^2b_2^2)(t-t_0)}. \tag{16}$$

The well-known Hölder inequality ($\varrho\in(0,1)$) deduces

$$\hat{\mathbb{E}}|\xi(t)|^{\varrho}\leq(\hat{\mathbb{E}}|\xi(t)|^2)^{\varrho/2}\leq\left(2e^{(2b_1+\overline{\sigma}^2b_2^2)(t-t_0)}\right)^{\varrho/2}\|\vartheta\|^{\varrho}.$$

Therefore, the first statement (12) is right. Via the Hölder inequality, Lemma 1, and Assumption 1, we further derive the following:

$$\begin{aligned}
&\hat{\mathbb{E}}\Big(\sup_{0\le u\le\tau}|\xi(t+u)-\xi(t)|^2\Big)\\
&\quad\le 2\tau\hat{\mathbb{E}}\int_t^{t+\tau}|f(s,\xi(s-\kappa_1(s)))|^2ds+8\overline{\sigma}^2\hat{\mathbb{E}}\int_t^{t+\tau}|g(s,\xi(s-\kappa_2(s)))|^2ds\\
&\quad\le 2\tau b_1^2\int_t^{t+\tau}\hat{\mathbb{E}}|\xi(s-\kappa_1(s))|^2ds+8\overline{\sigma}^2b_2^2\int_t^{t+\tau}\hat{\mathbb{E}}|\xi(s-\kappa_2(s))|^2ds\\
&\quad\le 4\tau(\tau b_1^2+4\overline{\sigma}^2b_2^2)e^{(2b_1+\overline{\sigma}^2b_2^2)(t+\tau-t_0)}\|\vartheta\|^2.
\end{aligned}$$

Applying the Hölder inequality again, we obtain

$$\hat{\mathbb{E}}\Big(\sup_{0\le u\le\tau}|\xi(t+u)-\xi(t)|^\varrho\Big)\le\mathbb{Q}_1(\tau,\varrho,t-t_0)\mathbb{E}\|\vartheta\|^\varrho,$$

where $\mathbb{Q}_1(\tau,\varrho,t-t_0)$ has been defined in (15). Thereby, we verify the second assertion (13). We can likewise determine via the Hölder inequality and Lemma 1 that

$$\begin{aligned}
\hat{\mathbb{E}}(\sup_{t_0\le u\le t}|\xi(u)|^2)&\le 3|\xi(t_0)|^2+3(t-t_0)b_1^2\int_{t_0}^t\hat{\mathbb{E}}|\xi(s-\kappa_1(s))|^2ds\\
&\quad+12\overline{\sigma}^2b_2^2\int_{t_0}^t\hat{\mathbb{E}}|\xi(s-\kappa_2(s))|^2ds\\
&\le\Big(3+\frac{6(t-t_0)b_1^2+24\overline{\sigma}^2b_2^2}{2b_1+\overline{\sigma}^2b_2^2}\exp\{(2b_1+\overline{\sigma}^2b_2^2)(t-t_0)\}\Big)\|\vartheta\|^2.
\end{aligned}$$

Thus, it is easy to show that

$$\begin{aligned}
\hat{\mathbb{E}}(\sup_{t_0\le u\le t}|\xi(u)|^\varrho)\le\Big(3+&\frac{6(t-t_0)b_1^2+24\overline{\sigma}^2b_2^2}{2b_1+\overline{\sigma}^2b_2^2}\\
&\times\exp\{(2b_1+\overline{\sigma}^2b_2^2)(t-t_0)\}\Big)^{\varrho/2}\|\vartheta\|^\varrho,
\end{aligned}$$

which shows (14). Thus, the proofs of the assertions have all been completed. □

The below lemma associates the solution of the SDDE-GBM (2) to that of the SDE-GBM (3).

Lemma 5. *Suppose that Assumption 1 and Assumption 2 hold. Let* $\vartheta\in L^2_{\Omega_{t_0}}(\Omega;C([-\tau,0];\mathbb{R}^n))$ *and denote* $\xi(t;t_0,\vartheta)=\xi(t)$*. Then, for any* $t\ge t_0+\tau$*, we obtain*

$$\hat{\mathbb{E}}|\eta(t;t_0+\tau,\xi(t_0+\tau))-\xi(t)|^\varrho\le\mathbb{Q}_2(\tau,\varrho,t-t_0)\hat{\mathbb{E}}\|\vartheta\|^\varrho,\tag{17}$$

where (for $T\ge\tau$*)*

$$\begin{aligned}
\mathbb{Q}_2(\tau,\varrho,T)=\Big(&\frac{4\tau(\tau b_1+\overline{\sigma}^2b_2^2)(b_1+2\overline{\sigma}^2b_2^2)}{2b_1+\overline{\sigma}^2b_2^2}\\
&\times\exp\{(3b_1+2\overline{\sigma}^2b_2^2)(T-\tau)\}\\
&\times[\exp\{(2b_1+\overline{\sigma}^2b_2^2)T\}-\exp\{(2b_1+\overline{\sigma}^2b_2^2)\tau\}]\Big)^{\varrho/2}.
\end{aligned}\tag{18}$$

Proof. For the random variable ϑ, as in the proof in (11), we use the definition of the conditional expectation once again, which means we just have to give the proof for the deterministic initial data $\vartheta\in C([-\tau,0];\mathbb{R}^n)$. We denote $\eta(t;t_0+\tau,\xi(t_0+\tau))=\eta(t)$ like before. Through Lemma 2 and Assumption 1, we obtain for $t\ge t_0+\tau$,

$$
\begin{aligned}
&\hat{\mathbb{E}}|\xi(t)-\eta(t)|^2\\
&\quad\le \hat{\mathbb{E}}\Big[\int_{t_0+\tau}^{t} 2(\xi(s)-\eta(s))^T(f(s,\xi(s-\kappa_1(s)))-f(s,\eta(s)))ds\\
&\quad+\overline{\sigma}^2\int_{t_0+\tau}^{t}|g(s,\xi(s-\kappa_2(s)))-g(s,\eta(s))|^2ds\\
&\quad\le \hat{\mathbb{E}}\int_{t_0+\tau}^{t}\Big[2b_1|\xi(s)-\eta(s)||\xi(s-\kappa_1(s))-\eta(s)|+\overline{\sigma}^2b_2^2|\xi(s-\kappa_2(s))-\eta(s)|^2\Big]ds\\
&\quad\le (3b_1+2\overline{\sigma}^2b_2^2)\int_{t_0+\tau}^{t}|\xi(s)-\eta(s)|^2ds+b_1\int_{t_0+\tau}^{t}\hat{\mathbb{E}}|\xi(s)-\xi(s-\kappa_1(s))|^2ds\\
&\quad+2\overline{\sigma}^2b_2^2\int_{t_0+\tau}^{t}\hat{\mathbb{E}}|\xi(s)-\xi(s-\kappa_2(s))|^2ds.
\end{aligned}
$$

Through Gronwall's inequality, we then obtain

$$
\begin{aligned}
\hat{\mathbb{E}}|\eta(t)-\xi(t)|^2 &\le \exp\{(3b_1+2\overline{\sigma}^2b_2^2)(t-t_0-\tau)\}\\
&\quad\times\Big(b_1\int_{t_0+\tau}^{t}\hat{\mathbb{E}}|\xi(s)-\xi(s-\kappa_1(s))|^2ds\\
&\quad+2\overline{\sigma}^2b_2^2\int_{t_0+\tau}^{t}\hat{\mathbb{E}}|\xi(s)-\xi(s-\kappa_2(s))|^2ds\Big),
\end{aligned}
\tag{19}
$$

whereas through (16), we can show that for $s\in[t_0+\tau,t]$,

$$
\begin{aligned}
&\hat{\mathbb{E}}|\xi(s)-\xi(s-\kappa_i(s))|^2\\
&\quad\le 2\tau b_1^2\int_{s-\kappa_i(s)}^{s}\hat{\mathbb{E}}|\xi(u-\kappa_1(u))|^2du\\
&\quad+2\overline{\sigma}^2b_2^2\int_{s-\kappa_i(s)}^{s}\hat{\mathbb{E}}|\xi(u-\kappa_2(u))|^2du\\
&\quad\le 4\tau(\tau b_1^2+\overline{\sigma}^2b_2^2)\|\vartheta\|^2e^{(2b_1+\overline{\sigma}^2b_2^2)(s-t_0)},
\end{aligned}
$$

for $i=1$ or 2. Plugging these into (19) yields

$$
\hat{\mathbb{E}}|\eta(t)-\xi(t)|^2\le(\mathbb{Q}_2(\tau,\varrho,t-t_0))^{2/\varrho}\|\vartheta\|^2,
$$

where $\mathbb{Q}_2(\tau,\varrho,t-t_0)$ has been provided in (18). By applying Hölder's inequality, we obtain

$$
\hat{\mathbb{E}}|\eta(t)-\xi(t)|^{\varrho}\le\mathbb{Q}_2(\tau,\varrho,t-t_0)\|\vartheta\|^{\varrho},
$$

which is our required claim (17). Thus, we complete the proof. □

On the basis of all these lemmas, we can now provide the main results of this paper, as well as their proof.

Theorem 1. *Suppose that Assumptions 1 and 2 hold. Then, for any initial data,*

$$
\vartheta\in L^2_{\Omega_{t_0}}(\Omega,C([-\tau,0];\mathbb{R}^n)),
$$

there exists a positive number τ^* such that the solution of SDDE-GBM (2) verifies

$$
\limsup_{t\to\infty}\frac{1}{t}\log(|\xi(t;t_0,\vartheta)|)<0\quad\text{q.s.},
\tag{20}
$$

provided $\tau < \tau^*$. Indeed, we can select a constant $\varepsilon \in (0,1)$ and set

$$T = \frac{1}{\gamma}\log\Big(\frac{2^{0.5\varrho}L}{\varepsilon}\Big); \tag{21}$$

finally, let $\tau^* > 0$ be the unique root to the following equation (in τ)

$$\varepsilon e^{(b_1+0.5\overline{\sigma}^2b_2^2)\varrho\tau} + \mathbb{Q}_1(\tau,\varrho,\tau+T) + \mathbb{Q}_2(\tau,\varrho,\tau+T) = 1, \tag{22}$$

where L, γ will be given by (8), while $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are provided by (15) and (18), respectively.

Proof. We prove the theorem in a progressive manner for better understanding.
Step 1. First, we choose the constants satisfied Assumption 2. After that, we fix the free parameter $\varepsilon \in (0,1)$ and T is defined by (21); thus, we have

$$2^{0.5\varrho}Le^{-\gamma T} = \varepsilon. \tag{23}$$

We discover that ϱ and T can be determined once the free parameters ϕ and ε are chosen. Based on the expressions of the quantities, the left-hand-side term of (22) is continuously increasing and equal to ε, as $\tau = 0$. Hence, Equation (22) must have a unique root, denoted by $\tau^* \geq 0$.

Fix $\tau \in (0,\tau^*)$ and $\vartheta \in L^2_{\Omega_{t_0}}(\Omega; C([-\tau,0];\mathbb{R}^n))$ arbitrarily. For the sake of simplicity, we use $\xi(t)$, $\eta(t_0+\tau+T)$ to substitute $\xi(t;t_0,\vartheta)$ for $t \geq t_0$ and $\eta(t_0+\tau+T;t_0+\tau,\xi(t_0+\tau))$, respectively. In terms of Lemmas 3 and 4, we obtain

$$\begin{aligned}\hat{\mathbb{E}}|\eta(t_0+\tau+T)|^\varrho &\leq L\hat{\mathbb{E}}|\xi(t_0+\tau)|^\varrho e^{-\gamma T}\\ &\leq Le^{-\gamma T}\Big(2e^{(2b_1+\overline{\sigma}^2b_2^2)\tau}\Big)^{\varrho/2}\hat{\mathbb{E}}\|\vartheta\|^\varrho.\end{aligned} \tag{24}$$

Due to the elementary inequality $(a+b)^\varrho \leq a^\varrho + b^\varrho$ for $0 < \varrho < 1$, we obtain

$$\hat{\mathbb{E}}|\xi(t_0+\tau+T)|^\varrho \leq \hat{\mathbb{E}}|\eta(t_0+\tau+T)|^\varrho + \hat{\mathbb{E}}|\xi(t_0+\tau+T)-\eta(t_0+\tau+T)|^\varrho.$$

By employing (24), as well as (17), we derive

$$\hat{\mathbb{E}}|\xi(t_0+2\tau+T)|^\varrho \leq \big(Le^{-\gamma T}(2e^{(2b_1+\overline{\sigma}^2b_2^2)\tau})^{\varrho/2} + \mathbb{Q}_2(\tau,\varrho,\tau+T)\big)\hat{\mathbb{E}}\|\vartheta\|^\varrho. \tag{25}$$

On the other hand, by employing (13), we obtain

$$\begin{aligned}&\hat{\mathbb{E}}\|\xi_{t_0+2\tau+T}\|^\varrho\\ &\quad\leq \hat{\mathbb{E}}|\xi(t_0+\tau+T)|^\varrho + \hat{\mathbb{E}}\Big(\sup_{0\leq u\leq\tau}|\xi(t_0+\tau+T)-\xi(t_0+\tau+T+u)|^\varrho\Big)\\ &\quad\leq \hat{\mathbb{E}}|\xi(t_0+\tau+T)|^\varrho + \mathbb{Q}_1(\tau,\varrho,\tau+T)\hat{\mathbb{E}}\|\vartheta\|^\varrho.\end{aligned} \tag{26}$$

Applying (23) and (25) to (26), we finally obtain

$$\begin{aligned}&\hat{\mathbb{E}}\|\xi_{t_0+2\tau+T}\|^\varrho\\ &\quad\leq \Big(Le^{-\gamma T}(2e^{(2b_1+\overline{\sigma}^2b_2^2)\tau})^{\varrho/2}\\ &\quad\quad+\mathbb{Q}_1(\tau,\varrho,\tau+T)+\mathbb{Q}_2(\tau,\varrho,\tau+T)\Big)\hat{\mathbb{E}}\|\vartheta\|^\varrho\\ &\quad= \Big(\varepsilon e^{(b_1+0.5\overline{\sigma}^2b_2^2)\varrho\tau}+\mathbb{Q}_1(\tau,\varrho,\tau+T)+\mathbb{Q}_2(\tau,\varrho,\tau+T)\Big)\hat{\mathbb{E}}\|\vartheta\|^\varrho.\end{aligned} \tag{27}$$

For any $\tau < \tau^*$, we use a form similar to (22):

$$\varepsilon e^{(b_1+0.5\overline{\sigma}^2 b_2^2)\varrho\tau} + \mathbb{Q}_1(\tau, \varrho, \tau+T) + \mathbb{Q}_2(\tau, \varrho, \tau+T) < 1.$$

We may obtain some $\mu > 0$ such that

$$\varepsilon e^{(b_1+0.5\overline{\sigma}^2 b_2^2)\varrho\tau} + \mathbb{Q}_1(\tau, \varrho, \tau+T) + \mathbb{Q}_2(\tau, \varrho, \tau+T) = e^{-\mu(2\tau+T)}.$$

Moreover (27) can be simplified to the following form:

$$\hat{\mathbb{E}}\|\xi_{t_0+2\tau+T}\|^\varrho \le e^{-\mu(2\tau+T)}\hat{\mathbb{E}}\|\vartheta\|^\varrho. \tag{28}$$

Step 2. Next, we investigate the solution of the SDDE-GBM (2) with the initial data $\xi_{t_0+2\tau+T}$ at $t = t_0 + 2\tau + T$. From Step 1, we deduce

$$\hat{\mathbb{E}}\|\xi_{t_0+2(2\tau+T)}\|^\varrho \le e^{-\mu(2\tau+T)}\hat{\mathbb{E}}\|\xi_{t_0+2\tau+T}\|^\varrho,$$

which, along with (28), means

$$\hat{\mathbb{E}}\|\xi_{t_0+2(2\tau+T)}\|^\varrho \le e^{-2\mu(2\tau+T)}\hat{\mathbb{E}}\|\vartheta\|^\varrho.$$

By repeating this procedure, we obtain

$$\hat{\mathbb{E}}\|\xi_{t_0+\ell(2\tau+T)}\|^\varrho \le e^{-\ell\mu(2\tau+T)}\hat{\mathbb{E}}\|\vartheta\|^\varrho \tag{29}$$

for all $\ell = 1, 2, \ldots$. Since this holds for $\ell = 0$, inequality (29) holds for all $\ell = 0, 1, 2, \ldots$. From (14) in Lemma 4 and (29), we obtain

$$\hat{\mathbb{E}}\Big(\sup_{t_0+\ell(2\tau+T)\le t\le t_0+(\ell+1)(2\tau+T)} |\xi(t)|^\varrho\Big) \le C\hat{\mathbb{E}}\|\xi_{t_0+\ell(2\tau+T)}\|^\varrho \le Ce^{-\ell\mu(2\tau+T)}\hat{\mathbb{E}}\|\vartheta\|^\varrho \tag{30}$$

for all $\ell = 0, 1, 2, \ldots$, where

$$C = \Big(4 + \frac{8(2\tau+T)b_1^2 + 32\overline{\sigma}^2 b_2^2}{2b_1 + \overline{\sigma}^2 b_2^2} e^{(2b_1+\overline{\sigma}^2 b_2^2)(2\tau+T)}\Big)^{\varrho/2}.$$

Step 3. Note that

$$\sum_{\ell=0}^{\infty} e^{-\ell\mu(2\tau+T)} < \infty.$$

Since, through (30) and Chebyshev's inequality (see, e.g., Chen et al. [38] (Proposition 2.1)),

$$\sum_{\ell=0}^{\infty} \mathbb{V}\Big(\sup_{t_0+\ell(2\tau+T)\le t\le t_0+(\ell+1)(2\tau+T)} |\xi(t)|^\varrho \ge e^{-0.5\ell\mu(2\tau+T)}\Big) \le C\hat{\mathbb{E}}\|\vartheta\|^\varrho \sum_{\ell=0}^{\infty} e^{-0.5\ell\mu(2\tau+T)},$$

we obtain

$$\sum_{\ell=0}^{\infty} \mathbb{V}\Big(\sup_{t_0+\ell(2\tau+T)\le t\le t_0+(\ell+1)(2\tau+T)} |\xi(t)|^\varrho \ge e^{-0.5\ell\mu(2\tau+T)}\Big) < \infty.$$

The Borel–Cantelli lemma under non-linear expectations (see, e.g., Chen et al. [38] (Lemma 2.2)) indicated that for some set $\hat{\Omega} \subseteq \Omega$ with $\mathbb{V}(\Omega \setminus \hat{\Omega}) = 0$, there is an integer $\ell_0 = \ell_0(\omega)$ for each $\omega \in \hat{\Omega}$ such that

$$\sup_{t_0+\ell(2\tau+T)\le t\le t_0+(\ell+1)(2\tau+T)} |\xi(t)|^{\varrho} \le e^{-0.5\ell\mu(2\tau+T)}, \qquad \forall \ell \ge \ell_0(\omega),$$

which shows

$$\limsup_{t\to\infty} \frac{1}{t} \log(|\xi(t,\omega)|) \le -\frac{\mu}{2\varrho}$$

for all $\omega \in \hat{\Omega}$. Therefore, Equation (20) holds. Thus, we complete the proof. □

4. Applications to Specific Systems

Now, we begin to testify that our new Theorem 1 is useful. For the initial data $\xi_{t_0} = \vartheta \in L^2_{\Omega_{t_0}}(\Omega; C([-\tau, 0]; \mathbb{R}))$, we consider a scalar linear SDDE-GBM:

$$d\xi(t) = \alpha\xi(t-\tau)dt + \sigma\xi(t-\tau)d\mathcal{Z}(t) \tag{31}$$

on $t \ge t_0$. Here, $\xi(t) \in \mathbb{R}$, $\mathcal{Z}(t)$ is a scalar G-Brownian motion. Constants $\alpha \ge 0$ and τ, $\sigma > 0$ satisfy

$$2\alpha < \underline{\sigma}^2\sigma^2. \tag{32}$$

It is easy to verify that Assumption 1 holds with $b_1 = \alpha$, $b_2 = \sigma$. Now, we can go on to confirm Assumption 2 by considering the corresponding scalar SDE-GBM:

$$d\eta(t) = \alpha\eta(t)dt + \sigma\eta(t)d\mathcal{Z}(t).$$

Let a $C^{2,1}$-function $V(z,t) = z^2$ for $(z,t) \in \mathbb{R} \times \mathbb{R}_+$ and choose $\phi \in (0, 0.5)$ such that

$$(1-2\phi)\underline{\sigma}^2\sigma^2 > 2\alpha. \tag{33}$$

In this way, we can easily determine that

$$\mathcal{L}V(z,t) \le (2\alpha + \underline{\sigma}^2\sigma^2)z^2, \ |V_z(z,t)\sigma z|^2 = 4\sigma^2 z^4,$$

which shows that the parameters from Assumption 2 fulfill

$$p = 2, \bar{a}_1 = a_1 = 1, a_2 = 2\alpha + \underline{\sigma}^2\sigma^2, a_3 = 4\sigma^2. \tag{34}$$

Together with (33) and (34), Assumption 2 is satisfied. Due to Theorem 1, we have the following corollary.

Corollary 1. *Let conditions (32) and (33) hold. Then, for any initial data,*

$$\vartheta \in L^2_{\Omega_{t_0}}(\Omega; C([-\tau, 0]; \mathbb{R})),$$

there exists a positive number τ^* such that the solution of the SDDE-GBM (31) fulfils

$$\limsup_{t\to\infty} \frac{1}{t} \log(|\xi(t; t_0, \vartheta)|) < 0 \quad \text{q.s.},$$

provided $\tau < \tau^*$.

In order to apply our theory to obtain an estimate on τ^* in Corollary 1, we set $\gamma = \phi((1-2\phi)\underline{\sigma}^2\sigma^2 - 2\alpha)$. We select another constant $\varepsilon \in (0,1)$ and denote

$$T = \frac{1}{\gamma}\log\Big(\frac{2^\phi}{\varepsilon}\Big).$$

Let $\tau^* > 0$ be the unique root to the following equation (in τ):

$$\varepsilon e^{(2\alpha+\overline{\sigma}\sigma^2)\phi\tau} + \mathbb{Q}_3(\tau,\varrho,\tau+T) + \mathbb{Q}_4(\tau,\varrho,\tau+T) = 1,$$

where

$$\mathbb{Q}_3(\tau,\varrho,\tau+T) = \Big(4\tau(\tau\alpha^2 + 4\overline{\sigma}^2\sigma^2)e^{(2\alpha+\overline{\sigma}^2\sigma^2)(T+2\tau)}\Big)^{\varrho/2}$$

and

$$\begin{aligned}\mathbb{Q}_4(\tau,\varrho,\tau+T) =& \Big(\frac{4\tau(\alpha+2\overline{\sigma}^2\sigma^2)(\tau\alpha^2+\overline{\sigma}^2\sigma^2)}{2\alpha+\overline{\sigma}^2\sigma^2} \\ &\times e^{(3\alpha+2\overline{\sigma}^2\sigma^2)T}\big[e^{(2\alpha+\overline{\sigma}^2\sigma^2)(T+\tau)} - e^{(2\alpha+\overline{\sigma}^2\sigma^2)\tau}\big]\Big)^{\varrho/2}.\end{aligned}$$

Finally, a simpler SDDE-GBM is given as follows:

$$d\xi(t) = \sigma\xi(t-\tau)d\mathcal{Z}(t), \tag{35}$$

which is the special case of (31) when $\alpha = 0$. The following corollary is straightforward.

Corollary 2. *Let $\tau^* > 0$ be the unique root to the equation*

$$\begin{aligned}\varepsilon e^{\overline{\sigma}^2\sigma^2\phi\tau} &+ \Big(16\tau\overline{\sigma}^2\sigma^2 e^{\overline{\sigma}^2\sigma^2(T+2\tau)}\Big)^\phi \\ &+ \Big(8\tau\overline{\sigma}^2\sigma^2 e^{2\overline{\sigma}^2\sigma^2 T}\big[e^{\overline{\sigma}^2\sigma^2(T+\tau)} - e^{\overline{\sigma}^2\sigma^2\tau}\big]\Big)^\phi = 1,\end{aligned} \tag{36}$$

where $\varepsilon \in (0,1)$ and $\phi \in (0,0.5)$ are two free parameters and

$$\gamma = \phi(1-2\phi)\underline{\sigma}^2\sigma^2,\ T = \frac{1}{\gamma}\log\Big(\frac{2^\phi}{\varepsilon}\Big).$$

Then, the solution of the SDDE-GBM (35) satisfies

$$\limsup_{t\to\infty}\frac{1}{t}\log(|\xi(t;t_0,\vartheta)|) < 0 \quad \text{q.s.},$$

provided $\tau < \tau^*$, with any initial data $\vartheta \in L^2_{\Omega_{t_0}}(\Omega; C([-\tau,0];\mathbb{R}))$.

Proof. Let

$$\begin{aligned}F(\tau) := \varepsilon e^{\overline{\sigma}^2\sigma^2\phi\tau} &+ \Big(16\tau\overline{\sigma}^2\sigma^2 e^{\overline{\sigma}^2\sigma^2(T+2\tau)}\Big)^\phi \\ &+ \Big(8\tau\overline{\sigma}^2\sigma^2 e^{2\overline{\sigma}^2\sigma^2 T}\big[e^{\overline{\sigma}^2\sigma^2(T+\tau)} - e^{\overline{\sigma}^2\sigma^2\tau}\big]\Big)^\phi - 1,\end{aligned}$$

which derives, for $\tau \geq 0$,

$$\begin{aligned} F'(\tau) =& \varepsilon\bar{\sigma}^2\sigma^2\phi e^{\bar{\sigma}^2\sigma^2\phi\tau} \\ &+ \phi\tau^{\phi-1}(1+2\tau\bar{\sigma}^2\sigma^2)\left(16\bar{\sigma}^2\sigma^2 e^{\bar{\sigma}^2\sigma^2(T+2\tau)}\right)^{\phi} \\ &+ \phi\tau^{\phi-1}\left(8\bar{\sigma}^2\sigma^2 e^{2\bar{\sigma}^2\sigma^2 T}[e^{\bar{\sigma}^2\sigma^2(T+\tau)} - e^{\bar{\sigma}^2\sigma^2\tau}]\right)^{\phi}(1+\tau\bar{\sigma}^2\sigma^2) > 0. \end{aligned}$$

Therefore, $F(\tau)$ is increasing. Since $F(0) = \varepsilon - 1 < 0$ and $F(+\infty) = \infty$, Equation (36) has a unique solution on τ. Thus, the proof is complete. □

5. Conclusions

We first introduce the SDDE system with G-Brownian motion and discover that if the time lag in the ambiguity system follows $\tau < \tau^*$, then it retains the same stability as long as the corresponding (non-delay) system is stable. Moreover, the upper bound τ^* is given out to further confirm our result. In order to obtain our main result, we provide several lemmas and their proofs using stochastic calculus under non-linear expectations. Compared with the usual stochastic delay system driven only by classical Brownian motion, the ambiguity system makes all the parameters in the system different so as to ensure the stability of the system. It is shown that when a real system is disturbed via G-Brownian motion, we can no longer use the classical stochastic calculus to handle it. The following question is thus put forward: if the coefficients of SDDE-driven G-Brownian motion (2) are highly non-linear (see, e.g., [35]), can we obtain a bound τ^* such that the system (2) attains quasi-sure exponential stability for delay $\tau < \tau^*$ on the condition that the corresponding non-delay SDE driven by G-Brownian motion (3) has quasi-sure exponential stability? We will further explore this problem.

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Appendix A

In this appendix, we provide a useful lemma.

Lemma A1. *Let Assumptions 1 and 2 hold. Then, for the initial value $\eta_0 \neq 0$, the solution $\eta(t) = \eta(t; t_0, \eta_0)$ of SDE-GBM (3) satisfies*

$$\mathcal{V}\big(\eta(t; t_0, \eta_0) \neq 0 \quad on \quad t \geq t_0\big) = 1;$$

that is, the sample path of any solution starting from a non-zero state will never reach the origin quasi-surely.

Proof. If the assertion does not hold, then there would exist some $\eta_0 \neq 0$ such that $\mathcal{V}(v < \infty) > 0$, where v is the first time at which the solution of Equation (3) reaches zero, i.e.,

$$v = \inf\{t \geq t_0;\ \eta(t) = 0\}.$$

Thus, we can find a pair of constants, $T > t_0$ and $\varpi > 0$, sufficiently large for $\mathcal{V}(\Theta) > 0$, where $\Theta = \{\omega; v \le T,\ |\eta(t)| \le \varpi - 1,\ \forall t_0 \le t \le T\}$.

Set $C^{2,1}$-function $V(z,t) = |z|^{-1}$ for $z \neq 0$. Therefore, by Lemma 2, the operator has the form that

$$\begin{aligned}\mathcal{L}V(z,t) &= -|z|^{-3} z^T f(t,z) + \frac{\overline{\sigma}^2}{2}(-|z|^{-3}|g(t,z)|^2 + 3|z|^{-5}|z^T g(t,z)|^2) \\ &\le |z|^{-2}|f(t,z)| + \overline{\sigma}^2 |z|^{-3}|g(t,z)|^2.\end{aligned}$$

Thus, according to the Lipschitz condition for the functions f and g, we have, for $z \neq 0, t_0 \le t \le T$,

$$\mathcal{L}V(z,t) \le (b_1 + \overline{\sigma}^2 b_2^2)|z|^{-1} = A|z|^{-1},$$

where we have set $A = b_1 + \overline{\sigma}^2 b_2^2$. Therefore, we obtain

$$\mathcal{L}V(z,t) \le A|z|^{-1} = AV(z,t), \quad 0 < |z| \le \varpi \ \text{ and } t_0 \le t \le T.$$

Now, for any $\varepsilon \in (0, |z_0|)$, define the stopping time $v_\varepsilon = \inf\{t \ge t_0 :\ |\eta(t)| \notin (\varepsilon, \varpi)\}$. For $\omega \in \Theta$, $v_\varepsilon \le T$ and $|\eta(v_\varepsilon)| = \varepsilon$, we obtain, by Lemma 2,

$$\begin{aligned}&\hat{\mathbb{E}}[e^{-A(v_\varepsilon \wedge T - t_0)} V(\eta(v_\varepsilon \wedge T), v_\varepsilon \wedge T)] \\ &\le V(z_0, t_0) + \hat{\mathbb{E}} \int_{t_0}^{v_\varepsilon \wedge T} e^{-A(s-t_0)}[-AV(\eta(s),s) + \mathcal{L}V(\eta(s),s)]ds \\ &\le V(z_0,t_0) = |z_0|^{-1}.\end{aligned}$$

The above inequality shows that

$$\hat{\mathbb{E}}[e^{-A(T-t_0)}\varepsilon^{-1} I_\Theta] \le |z_0|^{-1}.$$

Hence, $\mathbb{V}(\Theta) \le \varepsilon |z_0|^{-1} e^{A(T-t_0)}$. Letting $\varepsilon \to 0$ leads to that conclusion that $\mathbb{V}(\Theta) = 0$, but this contradicts the definition of Θ. Thus, the proof is complete. □

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