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Inclusive Subclasses of Bi-Univalent Functions Defined by Error Functions Subordinate to Horadam Polynomials

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Abstract: In this paper, by utilizing error functions subordinate to Horadam polynomials, we introduce the inclusive subclasses $A(a,\varsigma,r,u,\eta,\rho,\sigma)$, $B(a,\varsigma,r,u,\tau,\theta)$ and $C(a,\varsigma,r,u,\tau,\theta)$ of bi-univalent functions in the symmetric unit disk U. For functions in these subclasses, we derive estimations for the Maclaurin coefficients $|k_2|$ and $|k_3|$, as well as the Fekete–Szegö functional. Additionally, some related results are also obtained.

Keywords: analytic; univalent; bi-univalent functions; error functions; Horadam polynomials; Fekete–Szegö

1. Introduction and Preliminaries

Error functions in complex analysis expand the concepts of measuring discrepancies and quantifying deviations into the realm of complex numbers. These functions are essential for understanding the behavior of analytic and non-analytic functions, modeling complex variable physical processes, statistics, probability science, and solving differential equations (see [1,2]).

One of the most well-known examples is the complex error function, which generalizes the Gaussian error function to complex inputs and is also known as the Faddeeva function. This function plays a crucial role in statistical physics, quantum mechanics, and wave propagation, as it provides insights into oscillatory and exponential behaviors in the complex plane.

Error functions are particularly useful for evaluating singularities, estimating growth and decay rates in systems influenced by complex dynamics, and modeling intricate scenarios.

Alzer [3] and Coman [4] explored various traits and inequalities of error functions, while Elbert et al. [5] investigated the properties of complementary error functions.

The error function, denoted by erf, is defined by (see [6], p. 297)

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{s=0}^{\infty} \frac{(-1)^s z^{2s+1}}{(2s+1)s!}, \qquad (z \in \mathbb{C}). \tag{1}$$

Since $erf(z)$ is an odd function, it is symmetric with regard to the origin.

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Further, the imaginary error function, denoted by $erfi$, is defined by

$$erfi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{t^2} dt = \frac{2}{\sqrt{\pi}} \sum_{s=0}^{\infty} \frac{z^{2s+1}}{(2s+1)s!}, \quad (z \in \mathbb{C}). \tag{2}$$

The generalized error function of (1) is given by (see [6], p. 297)

$$erf_{\beta}(z) = \frac{\beta!}{\sqrt{\pi}} \int_0^z e^{t^{\beta}} dt = \frac{\beta!}{\sqrt{\pi}} \sum_{s=0}^{\infty} \frac{z^{\beta s+1}}{(\beta s+1)s!}, \quad (\beta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \mathbb{C}). \tag{3}$$

Also , the imaginary error function given by (2) can be generalized as follows:

$$erfi_{\beta}(z) = \frac{\beta!}{\sqrt{\pi}} \int_0^z e^{-t^{\beta}} dt = \frac{\beta!}{\sqrt{\pi}} \sum_{s=0}^{\infty} \frac{(-1)^s z^{\beta s+1}}{(\beta s+1)s!}, \quad (\beta \in \mathbb{N}_0,\ z \in \mathbb{C}). \tag{4}$$

Let Π denote the class of analytic and univalent functions D in the symmetric unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $D(0) = D'(0) - 1 = 0$. So, every function $D \in \Pi$ has the form (see [7])

$$D(z) = z + \sum_{s=2}^{\infty} k_s z^s, \quad (z \in U). \tag{5}$$

Thus, every function $D \in \Pi$ has an inverse D^{-1}, defined by

$$D^{-1}(D(z)) = z \qquad (z \in U)$$

and

$$D(D^{-1}(\varpi)) = \varpi \qquad (|\varpi| < r_0(D); r_0(D) \geq \frac{1}{4}),$$

where

$$G(\varpi) \equiv D^{-1}(\varpi) = \varpi - k_2\varpi^2 + (2k_2^2 - k_3)\varpi^3 - (5k_2^3 - 5k_2k_3 + k_4)\varpi^4 + \cdots. \tag{6}$$

Now, $D_1 \prec D_2$ or $D_1(z) \prec D_2(z)$ (the subordination of analytic functions D_1 and D_2) if, for all $z \in U$, there exists a function Φ with $\Phi(0) = 0$ and $|\Phi(z)| < 1$, such that

$$D_1(z) = D_2(\Phi(z)).$$

Further, if D_2 is univalent in U, then (see [8])

$$D_1(0) = D_2(0) \text{ and } D_1(U) \subset D_2(U) \Leftrightarrow D_1(z) \prec D_2(z).$$

A function D, given by (5), belongs to the subclass $\digamma$ (where $\digamma$ is the class of bi-univalent functions in U) if both $D(z)$ and $D^{-1}(\varpi)$ are univalent in U. For more details about the subclass $\digamma$, see [9–12].

The property that a function remains invariant when its variables are replaced with an equal or balanced number is known as symmetry, particularly in complex analysis and geometric function theory [7]. A complex function $D(z)$ is symmetric if and only if $D(z) = D(-z)$, $z \in U$. Otherwise, the values of the function at z and $-z$ are identical.

Clearly, the functions $erf_{\beta}(z)$ and $erfi_{\beta}(z)$ do not belong to the class Π. Therefore, it is natural to consider the following normalizations for these function, as proposed by Frasin in [13]

$$\mathcal{E}_{\beta}^{s}(z) = \frac{\sqrt{\pi}}{\beta!} z^{\left(1-\frac{1}{\beta}\right)} erf_{\beta}\left(z^{1/\beta}\right) = z + \sum_{s=2}^{\infty} \frac{(-1)^{s-1}}{((s-1)\beta+1)(s-1)!} z^s, \ (\beta \in \mathbb{N}, z \in U), \tag{7}$$

and

$$E^s_\beta(z) = \frac{\sqrt{\pi}}{\beta!} z^{\left(1-\frac{1}{\beta}\right)} erfi_\beta\left(z^{1/\beta}\right) = z + \sum_{s=2}^{\infty} \frac{1}{((s-1)\beta+1)(s-1)!} z^s, \qquad (\beta \in \mathbb{N}). \tag{8}$$

For functions $D(z) = z + \sum_{s=2}^{\infty} k_s z^s$ and $R(z) = z + \sum_{s=2}^{\infty} c_s z^s$, we define the convolution of D and R by

$$(D * R)(z) = z + \sum_{s=2}^{\infty} k_s c_s z^s, \ \ (z \in U).$$

Using the convolution, we define the following function:

$$ED^s_\beta(z) = D(z) * E^s_\beta(z) = z + \sum_{s=2}^{\infty} \frac{1}{((s-1)\beta+1)(s-1)!} k_s z^s, \ (\beta \in \mathbb{N}).$$

Note that, the normalization for Ramachandran et al. [14] is obtained for $\beta = 2$ in (7). The normalization for Mohammed et al. [15] is obtained for $\beta = 2$ in (8).

Several well-known families of orthogonal polynomials include the Legendre, Jacobi, Laguerre, Hermite, and Chebyshev families [16–18]. Each family has its own weight function and interval, as well as unique properties and applications.

Orthogonal polynomials are widely used in mathematical modeling to solve ordinary differential equations that meet specific model requirements. In addition to their significance in contemporary mathematics, orthogonal polynomials have numerous applications in physics and engineering. They are particularly important in problems related to approximation theory. Furthermore, approximation theory, probability theory, interpolation, differential equations, quantum physics, and mathematical statistics all make extensive use of these polynomials (see [19–23]).

The Horadam polynomials are a class of polynomials that generalize other families such as the Fibonacci, Chebyshev, Pell, Pell-Lucas, and Lucas polynomials based on recurrence relations. These polynomials are named after Australian mathematician Murray S. Klamkin Horadam, who introduced them in 1978.

Horadam polynomials exhibit many fascinating properties and have connections to various areas of mathematics, including number theory, algebraic geometry, and combinatorics.

In 1965, Horadam [24,25] defined the following linear recurrence relation:

$$h_{s+2} = r h_{s+1} + u h_s, \ h_0 = a, \ h_1 = \varsigma, \ a, r, u, \varsigma \in \mathbb{R}, \ s \in \{0, 1, 2, \cdots\}.$$

For $s \in \{3, 4, \cdots\}$, Horadam polynomial $h_s(x)$ is defined by the following recurrence relation:

$$h_s(x) = r x h_{s-1}(x) + u h_{s-2}(x), \tag{9}$$

with

$$h_1(x) = a, h_2(x) = \varsigma x \text{ and } h_3(x) = r\varsigma x^2 + au, \ a, r, u, \varsigma \in \mathbb{R}. \tag{10}$$

The generating function of the Horadam polynomial $h_s(x)$ is obtained as

$$F(x, z) = \sum_{s=1}^{\infty} h_s(x) z^{n-1} = \frac{a + (\varsigma - ar)xz}{1 - rxz - uz^2}. \tag{11}$$

Remark 1. *For specific values of a, ς, r and u, we obtain various polynomials from the Horadam polynomials $h_s(x)$ (see [24,25]). Below are some examples:*

1. *If $a = \varsigma = r = u = 1$, we obtain the Fibonacci polynomials $FI_s(x)$;*
2. *If $a = 2$ and $\varsigma = r = u = 1$, we obtain the Lucas polynomials $LU_s(x)$;*
3. *If $a = \varsigma = 1$, $r = 2$ and $u = -1$, we obtain the first kind of Chebyshev polynomials $CT_s(x)$;*
4. *If $a = 1$, $\varsigma = r = 2$ and $u = -1$, we obtain the second kind of Chebyshev polynomials $CU_s(x)$;*
5. *If $a = u = 1$ and $\varsigma = r = 2$, we obtain the Pell polynomials $P_s(x)$;*
6. *If $a = \varsigma = r = 2$ and $u = 1$, we obtain the first kind of Pell–Lucas polynomials $PL_s(x)$.*

Many researchers have studied bi-univalent functions related to orthogonal polynomials (see [26–29]).

Using error functions and subordinates into Horadam polynomials, we introduce the inclusive subclasses $A(a,\varsigma,r,u,\eta,\rho,\sigma)$, $B(a,\varsigma,r,u,\tau,\theta)$, and $C(a,\varsigma,r,u,\tau,\theta)$. For these subclasses, we estimate the upper bounds of the coefficients $|k_2|$, $|k_3|$ and $|k_3 - \epsilon k_2^2|$.

2. Coefficient Bounds for the Subclasses $A(a,\varsigma,r,u,\eta,\rho,\sigma)$, $B(a,\varsigma,r,u,\tau,\theta)$, and $C(a,\varsigma,r,u,\tau,\theta)$

The definitions of the new comprehensive subclasses $A(a,\varsigma,r,u,\eta,\rho,\sigma)$, $B(a,\varsigma,r,u,\tau,\theta)$, and $C(a,\varsigma,r,u,\tau,\theta)$ using error functions and Horadam polynomials are given first in this section.

Definition 1. *Let $\eta \geq 1$, $\rho,\sigma \geq 0$, $z,\varpi \in \mathbb{C}$, and let the function $F(x,z)$ be given by (11). A function $D \in \digamma$, defined by (5), is said to belong to the subclass $A(a,\varsigma,r,u,\eta,\rho,\sigma)$ if it satisfies the following subordinations:*

$$(1-\eta)\left(\frac{ED^s_\beta(z)}{z}\right)^\rho + \eta\left(\left(ED^s_\beta(z)\right)'\right)^{1-\rho} + \sigma z\left(ED^s_\beta(z)\right)'' \prec F(x,z) + 1 - a \tag{12}$$

and

$$(1-\eta)\left(\frac{EG^s_\beta(\varpi)}{\varpi}\right)^\rho + \eta\left(\left(EG^s_\beta(\varpi)\right)'\right)^{1-\rho} + \sigma\varpi\left(EG^s_\beta(\varpi)\right)'' \prec F(x,\varpi) + 1 - a. \tag{13}$$

Definition 2. *Let $-\pi < \theta \leq \pi$, $\tau \geq 1$, $z,\varpi \in \mathbb{C}$, and let the function $F(x,z)$ be given by (11). A function $D \in \digamma$, defined by (5), is said to belong to the subclass $B(a,\varsigma,r,u,\tau,\theta)$ if it satisfies the following subordinations:*

$$\left(\frac{ED^s_\beta(z)}{z}\right)^\tau + \frac{1+e^{i\theta}}{2}\left(\frac{z\left(ED^s_\beta(z)\right)''}{\left(ED^s_\beta(z)\right)'}\right) \prec F(x,z) + 1 - a \tag{14}$$

and

$$\left(\frac{EG^s_\beta(\varpi)}{\varpi}\right)^\tau + \frac{1+e^{i\theta}}{2}\left(\frac{\varpi\left(EG^s_\beta(\varpi)\right)''}{\left(EG^s_\beta(\varpi)\right)'}\right) \prec F(x,\varpi) + 1 - a. \tag{15}$$

Definition 3. *Let $-\pi < \theta \leq \pi$, $\tau \geq 1$, $z, \varpi \in \mathbb{C}$, and let the function $F(x,z)$ be given by (11). A function $D \in \digamma$, defined by (5), is said to belong to the subclass $C(a, \varsigma, r, u, \tau, \theta)$ if it satisfies the following subordinations:*

$$\left(\left(ED_{\beta}^{s}(z)\right)'\right)^{\tau} + \frac{1+e^{i\theta}}{2}\left(z\left(ED_{\beta}^{s}(z)\right)''\right) \prec F(x,z) + 1 - a \tag{16}$$

and

$$\left(\left(EG_{\beta}^{s}(\varpi)\right)'\right)^{\tau} + \frac{1+e^{i\theta}}{2}\left(\varpi\left(EG_{\beta}^{s}(\varpi)\right)''\right) \prec F(x,\varpi) + 1 - a. \tag{17}$$

Lemma 1 ([30]). *Let $E_1, E_2 \in \mathbb{R}$, and $\tau_1, \tau_2 \in \mathbb{C}$. If $|\tau_1| < \hbar$ and $|\tau_2| < \hbar$, then*

$$|(E_1 + E_2)\tau_1 + (E_1 - E_2)\tau_2| \leq \begin{cases} 2|E_1|\hbar \ \ for \ \ |E_1| \geq |E_2|, \\ 2|E_2|\hbar \ \ for \ \ |E_1| \leq |E_2|. \end{cases}$$

In the following theorem, we estimate the initial coefficients $|k_2|$ and $|k_3|$, as well as the Fekete–Szegö functional for the subclass $A(a, \varsigma, r, u, \eta, \rho, \sigma)$.

Theorem 1. *Let $D \in \digamma$, defined by (5), belong to the subclass $A(a, \varsigma, r, u, \eta, \rho, \sigma)$. Then,*

$$|k_2| \leq \min\left\{\frac{|\varsigma x|(\beta+1)}{|\rho(1-3\eta)+2(\sigma+\eta)|}, \frac{|\varsigma x|\sqrt{2|\varsigma x|}}{\sqrt{\left|\varsigma^2 x^2\left(\frac{\rho(\rho-1)(3\eta+1)}{(\beta+1)^2} + \frac{\rho(1-4\eta)+3(2\sigma+\eta)}{2\beta+1}\right) - 2(r\varsigma x^2 + au)\left(\frac{\rho(1-3\eta)+2(\sigma+\eta)}{\beta+1}\right)^2\right|}}\right\},$$

$$|k_3| \leq \min\left\{\frac{2|\varsigma x|(2\beta+1)}{|\rho(1-4\eta)+3(2\sigma+\eta)|} + \frac{\varsigma^2 x^2(\beta+1)^2}{|\rho(1-3\eta)+2(\sigma+\eta)|^2}, \frac{2|\varsigma x|(2\beta+1)}{|\rho(1-4\eta)+3(2\sigma+\eta)|} + \frac{|\varsigma x|\sqrt{2|\varsigma x|}}{\left|\varsigma^2 x^2\left(\frac{\rho(\rho-1)(3\eta+1)}{(\beta+1)^2} + \frac{\rho(1-4\eta)+3(2\sigma+\eta)}{2\beta+1}\right) - 2(r\varsigma x^2 + au)\left(\frac{\rho(1-3\eta)+2(\sigma+\eta)}{\beta+1}\right)^2\right|}\right\}$$

and

$$\left|k_3 - \epsilon k_2^2\right| \leq \begin{cases} \frac{2|\varsigma x|(2\beta+1)}{|\rho(1-4\eta)+3(2\sigma+\eta)|} & |\Psi(\epsilon)| < \frac{2\beta+1}{\rho(1-4\eta)+3(2\sigma+\eta)}, \\ 2|\varsigma x||\Psi(\epsilon)| & |\Psi(\epsilon)| \geq \frac{2\beta+1}{\rho(1-4\eta)+3(2\sigma+\eta)}, \end{cases}$$

where

$$\Psi(\epsilon) = \frac{(1-\epsilon)\varsigma^2 x^2}{\varsigma^2 x^2\left(\frac{\rho(\rho-1)(3\eta+1)}{(\beta+1)^2} + \frac{\rho(1-4\eta)+3(2\sigma+\eta)}{2\beta+1}\right) - 2(r\varsigma x^2 + au)\left(\frac{\rho(1-3\eta)+2(\sigma+\eta)}{\beta+1}\right)^2}.$$

Proof. Let $D \in A(a, \varsigma, r, u, \eta, \rho, \sigma)$. From the subordinations (12) and (13), there exist two analytic functions t_1 and t_2 such that $t_1(0) = t_2(0) = 0$ and $|t_1(z)| < 1, |t_2(\varpi)| < 1$, satisfying the following conditions:

$$(1-\eta)\left(\frac{ED_\beta^s(z)}{z}\right)^\rho + \eta\left(\left(ED_\beta^s(z)\right)'\right)^{1-\rho} + \sigma z\left(ED_\beta^s(z)\right)'' = F(x, \alpha(z)) + 1 - a, \ \ z \in U \tag{18}$$

and

$$(1-\eta)\left(\frac{EG_\beta^s(\varpi)}{\varpi}\right)^\rho + \eta\left(\left(EG_\beta^s(\varpi)\right)'\right)^{1-\rho} + \sigma\varpi\left(EG_\beta^s(\varpi)\right)'' = F(x, \gamma(\varpi)) + 1 - a, \ \varpi \in U, \tag{19}$$

where α and γ are analytic of the form

$$\alpha(z) = t_1 z + t_2 z^2 + t_3 z^3 + \cdots,$$

and

$$\gamma(z) = b_1\varpi + b_2\varpi^2 + b_3\varpi^3 + \cdots,$$

such that $\alpha(0) = \gamma(0) = 0$ and $|\alpha(z)| < 1, |\gamma(z)| < 1$ for $z, \varpi \in U$.

From the equalities (18) and (19), we have

$$\begin{aligned}&(1-\eta)\left(\frac{ED_\beta^s(z)}{z}\right)^\rho + \eta\left(\left(ED_\beta^s(z)\right)'\right)^{1-\rho} + \sigma z\left(ED_\beta^s(z)\right)''\\ &= 1 + h_2(x)t_1 z + \left(h_2(x)t_2 + h_3(x)t_1^2\right)z^2 + \cdots, \ z \in U,\end{aligned} \tag{20}$$

and

$$\begin{aligned}&(1-\eta)\left(\frac{EG_\beta^s(\varpi)}{\varpi}\right)^\rho + \eta\left(\left(EG_\beta^s(\varpi)\right)'\right)^{1-\rho} + \sigma\varpi\left(EG_\beta^s(\varpi)\right)''\\ &= 1 + h_2(x)b_1\varpi + \left(h_2(x)b_2 + h_3(x)b_1^2\right)\varpi^2 + \cdots, \ \varpi \in U.\end{aligned} \tag{21}$$

Itis common knowledge that if

$$|\alpha(z)| = \left|t_1 z + t_2 z^2 + t_3 z^3 + \cdots\right| < 1, \ z \in U$$

and

$$|\gamma(z)| = \left|b_1\varpi + b_2\varpi^2 + b_3\varpi^3 + \cdots\right| < 1, \ \varpi \in U,$$

then

$$|t_i| \le 1 \text{ and } |b_i| \le 1, \ z \in U. \tag{22}$$

Fromequating the coefficients for Equations (20) and (21), we obtain

$$\frac{\rho(1-3\eta) + 2(\sigma+\eta)}{\beta+1}k_2 = h_2(x)t_1, \tag{23}$$

$$\frac{\rho(\rho-1)(3\eta+1)}{2(\beta+1)^2}k_2^2 + \frac{\rho(1-4\eta) + 3(2\sigma+\eta)}{2(2\beta+1)}k_3 = h_2(x)t_2 + h_3(x)t_1^2, \tag{24}$$

$$-\frac{\rho(1-3\eta) + 2(\sigma+\eta)}{\beta+1}k_2 = h_2(x)b_1, \tag{25}$$

and

$$\left[\frac{\rho(\rho-1)(3\eta+1)}{2(\beta+1)^2}+\frac{\rho(1-4\eta)+3(2\sigma+\eta)}{2\beta+1}\right]k_2^2-\frac{[\rho(1-4\eta)+3(2\sigma+\eta)]}{2(2\beta+1)}k_3 = h_2(x)b_2+h_3(x)b_1^2. \tag{26}$$

From (23) and (25), it follows that

$$t_1=-b_1 \tag{27}$$

and

$$2\left[\frac{\rho(1-3\eta)+2(\sigma+\eta)}{\beta+1}\right]^2k_2^2=(h_2(x))^2\left(t_1^2+b_1^2\right). \tag{28}$$

If we add (24) to (26), we have

$$\left[\frac{\rho(\rho-1)(3\eta+1)}{(\beta+1)^2}+\frac{\rho(1-4\eta)+3(2\sigma+\eta)}{2\beta+1}\right]k_2^2=h_2(x)(t_2+b_2)+h_3(x)\left(t_1^2+b_1^2\right). \tag{29}$$

Substituting the value of $t_1^2+d_1^2$ from (28) in (29), we obtain

$$\left[\frac{\rho(\rho-1)(3\eta+1)}{(\beta+1)^2}+\frac{\rho(1-4\eta)+3(2\sigma+\eta)}{2\beta+1}-\frac{2h_3(x)}{(h_2(x))^2}\left(\frac{\rho(1-3\eta)+2(\sigma+\eta)}{\beta+1}\right)^2\right]k_2^2 = h_2(x)(t_2+b_2). \tag{30}$$

Using (10) and (22) for the relations (23) and (30), we have, respectively,

$$|k_2|\leq\frac{|\varsigma x|(\beta+1)}{|\rho(1-3\eta)+2(\sigma+\eta)|}$$

and

$$|k_2|\leq\frac{|\varsigma x|\sqrt{2|\varsigma x|}}{\sqrt{\left|\varsigma^2x^2\left(\frac{\rho(\rho-1)(3\eta+1)}{(\beta+1)^2}+\frac{\rho(1-4\eta)+3(2\sigma+\eta)}{2\beta+1}\right)-2(r\varsigma x^2+au)\left(\frac{\rho(1-3\eta)+2(\sigma+\eta)}{\beta+1}\right)^2\right|}}.$$

Also, if we subtract (26) from (24), we have

$$\frac{\rho(1-4\eta)+3(2\sigma+\eta)}{2\beta+1}\left(k_3-k_2^2\right)=h_2(x)(t_2-b_2)+h_3(x)\left(t_1^2-b_1^2\right). \tag{31}$$

Then, in view of (27) and (28), Equation (31) becomes

$$k_3=\frac{h_2(x)(t_2-b_2)(2\beta+1)}{\rho(1-4\eta)+3(2\sigma+\eta)}+k_2^2. \tag{32}$$

With (23), Equation (32) becomes

$$k_3=\frac{h_2(x)(t_2-b_2)(2\beta+1)}{\rho(1-4\eta)+3(2\sigma+\eta)}+\frac{(h_2(x))^2t_1^2(\beta+1)^2}{[\rho(1-3\eta)+2(\sigma+\eta)]^2}, \tag{33}$$

and using (30) in (32),

$$k_3 = \frac{h_2(x)(t_2 - b_2)(2\beta+1)}{\rho(1-4\eta) + 3(2\sigma+\eta)} + \frac{h_2(x)(t_2 + b_2)}{\left[\frac{\rho(\rho-1)(3\eta+1)}{(\beta+1)^2} + \frac{\rho(1-4\eta)+3(2\sigma+\eta)}{2\beta+1} - \frac{2h_3(x)}{(h_2(x))^2}\left(\frac{\rho(1-3\eta)+2(\sigma+\eta)}{\beta+1}\right)^2\right]}. \quad (34)$$

Using (10) and (22) for the relations (33) and (34), we have, respectively,

$$|k_3| \le \frac{2|\varsigma x|(2\beta+1)}{|\rho(1-4\eta)+3(2\sigma+\eta)|} + \frac{\varsigma^2 x^2(\beta+1)^2}{|\rho(1-3\eta)+2(\sigma+\eta)|^2}$$

and

$$|k_3| \le \frac{2r\varsigma(2\beta+1)}{|\rho(1-4\eta)+3(2\sigma+\eta)|} + \frac{|\varsigma x|\sqrt{2|\varsigma x|}}{\left|\varsigma^2 x^2\left(\frac{\rho(\rho-1)(3\eta+1)}{(\beta+1)^2} + \frac{\rho(1-4\eta)+3(2\sigma+\eta)}{2\beta+1}\right) - 2(r\varsigma x^2 + au)\left(\frac{\rho(1-3\eta)+2(\sigma+\eta)}{\beta+1}\right)^2\right|}.$$

Also, from (32) and (30), we obtain

$$\begin{aligned} k_3 - \epsilon k_2^2 &= \frac{h_2(x)(t_2 - b_2)(2\beta+1)}{\rho(1-4\eta)+3(2\sigma+\eta)} + (1-\epsilon)k_2^2 \\ &= \frac{h_2(x)(t_2 - b_2)(2\beta+1)}{\rho(1-4\eta)+3(2\sigma+\eta)} \\ &+ \frac{(1-\epsilon)(t_2+d_2)(h_2(x))^3}{(h_2(x))^2\left(\frac{\rho(\rho-1)(3\eta+1)}{(\beta+1)^2} + \frac{\rho(1-4\eta)+3(2\sigma+\eta)}{2\beta+1}\right) - 2h_3(x)\left(\frac{\rho(1-3\eta)+2(\sigma+\eta)}{\beta+1}\right)^2} \\ &= h_2(x)\left[\left(\Psi(\epsilon) + \frac{2\beta+1}{\rho(1-4\eta)+3(2\sigma+\eta)}\right)t_2 + \left(\Psi(\epsilon) - \frac{2\beta+1}{\rho(1-4\eta)+3(2\sigma+\eta)}\right)b_2\right], \end{aligned}$$

where

$$\Theta(\epsilon) = \frac{(1-\epsilon)(h_2(x))^2}{(h_2(x))^2\left(\frac{\rho(\rho-1)(3\eta+1)}{(\beta+1)^2} + \frac{\rho(1-4\eta)+3(2\sigma+\eta)}{2\beta+1}\right) - 2h_3(x)\left(\frac{\rho(1-3\eta)+2(\sigma+\eta)}{\beta+1}\right)^2}.$$

Then, in view of (22) for $|t_2|$ and $|d_2|$, and Lemma 1, we obtain

$$\begin{aligned} \left|k_3 - \epsilon k_2^2\right| &\le \begin{cases} \frac{2|h_2(x)|(2\beta+1)}{|\rho(1-4\eta)+3(2\sigma+\eta)|} & |\Psi(\epsilon)| < \frac{2\beta+1}{\rho(1-4\eta)+3(2\sigma+\eta)}, \\ 2|h_2(x)||\Psi(\epsilon)| & |\Psi(\epsilon)| \ge \frac{2\beta+1}{\rho(1-4\eta)+3(2\sigma+\eta)}. \end{cases} \\ &\equiv \begin{cases} \frac{2|\varsigma x|(2\beta+1)}{|\rho(1-4\eta)+3(2\sigma+\eta)|} & |\Psi(\epsilon)| < \frac{2\beta+1}{\rho(1-4\eta)+3(2\sigma+\eta)}, \\ 2|\varsigma x||\Psi(\epsilon)| & |\Psi(\epsilon)| \ge \frac{2\beta+1}{\rho(1-4\eta)+3(2\sigma+\eta)}, \end{cases} \end{aligned}$$

where

$$\Psi(\epsilon) = \frac{(1-\epsilon)\varsigma^2 x^2}{\varsigma^2 x^2\left(\frac{\rho(\rho-1)(3\eta+1)}{(\beta+1)^2} + \frac{\rho(1-4\eta)+3(2\sigma+\eta)}{2\beta+1}\right) - 2(r\varsigma x^2 + au)\left(\frac{\rho(1-3\eta)+2(\sigma+\eta)}{\beta+1}\right)^2},$$

which completes the proof. □

We need the following lemma to prove the next Theorems.

Lemma 2 ([31]). *If* $C(z) = 1 + k_1 z + k_2 z^2 + \cdots \in \Upsilon$, $z \in U$, *then there exist some* ρ, δ *with* $|\rho| \leq 1$, $|\delta| \leq 1$, *such that*

$$2k_2 = k_1^2 + \rho(4 - k_1^2) \text{ and } 4k_3 = k_1^3 + 2k_1\rho(4 - k_1^2) - (4 - k_1^2)k_1\rho^2 + 2(4 - k_1^2)(1 - |\rho|^2)\delta. \tag{35}$$

In the next theorem, we estimate the Fekete–Szegö functional and the initial coefficients $|k_2|$ and $|k_3|$ for the subclass $B(a, \varsigma, r, u, \tau, \theta)$.

Theorem 2. *Let* $D \in \digamma$, *defined by (5), belong to the subclass* $B(a, \varsigma, r, u, \tau, \theta)$. *Then,*

$$|k_2| \leq \min\left\{ \frac{|\varsigma x|(\beta+1)}{|e^{i\theta} + \tau + 1|}, \frac{|\varsigma x|\sqrt{2|\varsigma x|}}{\sqrt{\left|\varsigma^2 x^2 \left(\frac{\tau(\tau-1) - 4(e^{i\theta}+1)}{(\beta+1)^2} + \frac{3\left(e^{i\theta}+1\right)+\tau}{2\beta+1} \right) - \frac{2(r\varsigma x^2 + au)\left(e^{i\theta}+\tau+1\right)^2}{(\beta+1)^2}\right|}} \right\},$$

$$|k_3| \leq \min\left\{ \frac{\varsigma^2 x^2 (\beta+1)^2}{\left|e^{i\theta} + \tau + 1\right|^2} + \frac{2|\varsigma x|(2\beta+1)}{|3(e^{i\theta}+1) + \tau|}, \right.$$
$$\left. \frac{2|\varsigma x|^3}{\left|\varsigma^2 x^2 \left(\frac{\tau(\tau-1) - 4(e^{i\theta}+1)}{(\beta+1)^2} + \frac{3\left(e^{i\theta}+1\right)+\tau}{2\beta+1} \right) - \frac{2(r\varsigma x^2 + au)\left(e^{i\theta}+\tau+1\right)^2}{(\beta+1)^2}\right|} + \frac{2|\varsigma x|(2\beta+1)}{|3(e^{i\theta}+1) + \tau|} \right\}$$

and

$$\left|k_3 - \varrho k_2^2\right| \leq \begin{cases} \frac{4|\varsigma x|(2\beta+1)}{|3(e^{i\theta}+1)+\tau|} & |1-\varrho| < \frac{(2\beta+1)\left|e^{i\theta}+\tau+1\right|^2}{\varsigma x(\beta+1)^2 |3(e^{i\theta}+1)+\tau|}, \\ \frac{4\varsigma^2 x^2 (\beta+1)^2 |1-\varrho|}{\left|e^{i\theta}+\tau+1\right|^2} & |1-\varrho| \geq \frac{(2\beta+1)\left|e^{i\theta}+\tau+1\right|^2}{\varsigma x(\beta+1)^2 |3(e^{i\theta}+1)+\tau|}. \end{cases} \tag{36}$$

Proof. Let $D \in B(a, \varsigma, r, u, \tau, \theta)$. From the subordinations (14) and (15), we can write

$$\left(\frac{ED_\beta^s(z)}{z} \right)^\tau + \frac{1 + e^{i\theta}}{2} \left(\frac{z\left(ED_\beta^s(z)\right)''}{\left(ED_\beta^s(z)\right)'} \right) = F(x, \alpha(z)) + 1 - a, \quad z \in U$$

and

$$\left(\frac{EG_\beta^s(\varpi)}{\varpi} \right)^\tau + \frac{1 + e^{i\theta}}{2} \left(\frac{\varpi\left(EG_\beta^s(\varpi)\right)''}{\left(EG_\beta^s(\varpi)\right)'} \right) = F(x, \gamma(\varpi)) + 1 - a, \quad \varpi \in U.$$

Thus, we have

$$\left(\frac{ED_\beta^s(z)}{z} \right)^\tau + \frac{1 + e^{i\theta}}{2} \left(\frac{z\left(ED_\beta^s(z)\right)''}{\left(ED_\beta^s(z)\right)'} \right)$$
$$= 1 + h_2(x)t_1 z + \left(h_2(x)t_2 + h_3(x)t_1^2\right)z^2 + \cdots, \ z \in U. \tag{37}$$

and

$$\left(\frac{EG^s_\beta(\varpi)}{\varpi}\right)^\tau+\frac{1+e^{i\theta}}{2}\left(\frac{\varpi\left(EG^s_\beta(\varpi)\right)''}{\left(EG^s_\beta(\varpi)\right)'}\right)$$
$$=1+h_2(x)b_1\varpi+\left(h_2(x)b_2+h_3(x)b_1^2\right)\varpi^2+\cdots,\ \varpi\in U. \tag{38}$$

From Equations (37) and (38), we have

$$\frac{e^{i\theta}+\tau+1}{\beta+1}k_2=h_2(x)t_1, \tag{39}$$

$$\frac{\frac{1}{2}\tau(\tau-1)-2(e^{i\theta}+1)}{(\beta+1)^2}k_2^2+\frac{3\left(e^{i\theta}+1\right)+\tau}{2(2\beta+1)}k_3=h_2(x)t_2+h_3(x)t_1^2, \tag{40}$$

$$-\frac{e^{i\theta}+\tau+1}{\beta+1}k_2=h_2(x)b_1, \tag{41}$$

and

$$\left[\frac{3\left(e^{i\theta}+1\right)+\tau}{2\beta+1}+\frac{\frac{1}{2}\tau(\tau-1)-2(e^{i\theta}+1)}{(\beta+1)^2}\right]k_2^2-\frac{3\left(e^{i\theta}+1\right)+\tau}{2(2\beta+1)}k_3=h_2(x)b_2+h_3(x)b_1^2. \tag{42}$$

From (39) and (41), it follows that

$$t_1=-b_1 \tag{43}$$

and

$$\frac{2\left(e^{i\theta}+\tau+1\right)^2}{(\beta+1)^2}k_2^2=(h_2(x))^2\left(t_1^2+b_1^2\right). \tag{44}$$

Substituting the value of $t_1^2+b_1^2$ from (44) after we add Equations (40) and (42), we have

$$\left(\frac{\tau(\tau-1)-4(e^{i\theta}+1)}{(\beta+1)^2}+\frac{3\left(e^{i\theta}+1\right)+\tau}{2\beta+1}-\frac{2h_3(x)\left(e^{i\theta}+\tau+1\right)^2}{(h_2(x))^2(\beta+1)^2}\right)k_2^2=h_2(x)(t_2+b_2). \tag{45}$$

Utilizing (10) and (22) after using the triangle inequality for Equations (39) and (45), we obtain, respectively,

$$|k_2|\le\frac{|\varsigma x|(\beta+1)}{|e^{i\theta}+\tau+1|}\ \text{ and }\ |k_2|\le\frac{|\varsigma x|\sqrt{2|\varsigma x|}}{\sqrt{\left|\varsigma^2x^2\left(\frac{\tau(\tau-1)-4(e^{i\theta}+1)}{(\beta+1)^2}+\frac{3\left(e^{i\theta}+1\right)+\tau}{2\beta+1}\right)-\frac{2(r\varsigma x^2+au)\left(e^{i\theta}+\tau+1\right)^2}{(\beta+1)^2}\right|}}.$$

Also, if we subtract (42) from (40), we obtain

$$\frac{3(e^{i\theta}+1)+\tau}{2\beta+1}\left(k_3-k_2^2\right)=h_2(x)(t_2-b_2)+h_3(x)\left(t_1^2-b_1^2\right). \tag{46}$$

In view of (43), Equation (46) becomes

$$k_3=k_2^2+\frac{h_2(x)(t_2-b_2)(2\beta+1)}{3(e^{i\theta}+1)+\tau}. \tag{47}$$

Equation (47) with (43) and (44) becomes

$$k_3=\frac{(h_2(x))^2(\beta+1)^2t_1^2}{\left(e^{i\theta}+\tau+1\right)^2}+\frac{h_2(x)(t_2-b_2)(2\beta+1)}{3(e^{i\theta}+1)+\tau}. \tag{48}$$

Utilizing (10) and (22) after using the triangle inequality for Equation (48), we obtain

$$|k_3| \leq \frac{\varsigma^2 x^2(\beta+1)^2}{|e^{i\theta}+\tau+1|^2} + \frac{2|\varsigma x|(2\beta+1)}{|3(e^{i\theta}+1)+\tau|}.$$

Similarly, using (45) in relation to (47), we obtain

$$k_3 = \frac{(h_2(x))^3(t_2+b_2)}{(h_2(x))^2\left(\frac{\tau(\tau-1)-4(e^{i\theta}+1)}{(\beta+1)^2} + \frac{3(e^{i\theta}+1)+\tau}{2\beta+1}\right) - \frac{2h_3(x)\left(e^{i\theta}+\tau+1\right)^2}{(\beta+1)^2}} + \frac{h_2(x)(t_2-b_2)(2\beta+1)}{3(e^{i\theta}+1)+\tau} \tag{49}$$

Utilizing(10) and (22) after using the triangle inequality for Equation (49), we obtain

$$|k_3| \leq \frac{2|\varsigma x|^3}{\left|\varsigma^2 x^2\left(\frac{\tau(\tau-1)-4(e^{i\theta}+1)}{(\beta+1)^2} + \frac{3(e^{i\theta}+1)+\tau}{2\beta+1}\right) - \frac{2(r\varsigma x^2+au)\left(e^{i\theta}+\tau+1\right)^2}{(\beta+1)^2}\right|} + \frac{2|\varsigma x|(2\beta+1)}{|3(e^{i\theta}+1)+\tau|}.$$

Also, using (43) and (44), we obtain $k_2^2 = \frac{(h_2(x))^2(\beta+1)^2 t_1^2}{\left(e^{i\theta}+\tau+1\right)^2}$. Thus, from (47), we obtain

$$\begin{aligned} k_3 - \varrho k_2^2 &= \frac{h_2(x)(t_2-b_2)(2\beta+1)}{3(e^{i\theta}+1)+\tau} + (1-\varrho)k_2^2 \\ &= \frac{h_2(x)(t_2-b_2)(2\beta+1)}{3(e^{i\theta}+1)+\tau} + (1-\varrho)\frac{(h_2(x))^2(\beta+1)^2 t_1^2}{\left(e^{i\theta}+\tau+1\right)^2}. \end{aligned}$$

From Lemma 2, we have $2t_2 = t_1^2 + \rho(4-t_1^2)$ and $2b_2 = b_1^2 + \zeta(4-b_1^2)$, $|\rho| \leq 1$, $|\zeta| \leq 1$, and using (43), we obtain

$$t_2 - b_2 = \frac{4-t_1^2}{2}(\rho-\zeta),$$

and thus,

$$k_3 - \varrho k_2^2 = \frac{h_2(x)(4-t_1^2)(2\beta+1)(\rho-\zeta)}{2\left(3(e^{i\theta}+1)+\tau\right)} + (1-\varrho)\frac{(h_2(x))^2(\beta+1)^2 t_1^2}{\left(e^{i\theta}+\tau+1\right)^2}.$$

Using the triangle inequality, taking $|\rho| = m$, $|\zeta| = v$, $m, v \in [0,1]$, and assuming that $t_1 = p \in [0,2]$, we obtain

$$\left|k_3 - \varrho k_2^2\right| \leq \frac{|h_2(x)|(4-p^2)(2\beta+1)(m+v)}{2|3(e^{i\theta}+1)+\tau|} + |1-\varrho|\frac{(h_2(x))^2(\beta+1)^2 p^2}{|e^{i\theta}+\tau+1|^2}. \tag{50}$$

Using (10) for Equation (50), we obtain

$$\left|k_3 - \varrho k_2^2\right| \leq \frac{|\varsigma x|(4-p^2)(2\beta+1)(m+v)}{2|3(e^{i\theta}+1)+\tau|} + |1-\varrho|\frac{\varsigma^2 x^2(\beta+1)^2 p^2}{|e^{i\theta}+\tau+1|^2}.$$

Assume that $B_1(p) = \frac{\varsigma^2 x^2|1-\varrho|(\beta+1)^2 p^2}{|e^{i\theta}+\tau+1|^2} \geq 0$ and $B_2(p) = \frac{|\varsigma x|(4-p^2)(2\beta+1)}{2|3(e^{i\theta}+1)+\tau|} \geq 0$, then the inequality (50) can be rewritten as

$$\left|k_3 - \varrho k_2^2\right| \leq B_1(p) + B_2(p)(m+v) =: J(m,v), \ m, v \in [0,1].$$

Therefore,

$$\max\{J(m,v):\ m,v\in[0,1]\}=J(1,1)=B_1(p)+2B_2(p)=:M(p),\ p\in[0,2],$$

where

$$M(p)=\frac{\varsigma^2x^2(\beta+1)^2}{\left|e^{i\theta}+\tau+1\right|^2}\left(|1-\varrho|-\frac{(2\beta+1)\left|e^{i\theta}+\tau+1\right|^2}{|\varsigma x|(\beta+1)^2\left|3(e^{i\theta}+1)+\tau\right|}\right)p^2+\frac{4|\varsigma x|(2\beta+1)}{\left|3(e^{i\theta}+1)+\tau\right|}.$$

Since

$$M'(p)=\frac{2\varsigma^2x^2(\beta+1)^2}{\left|e^{i\theta}+\tau+1\right|^2}\left(|1-\varrho|-\frac{(2\beta+1)\left|e^{i\theta}+\tau+1\right|^2}{|\varsigma x|(\beta+1)^2\left|3(e^{i\theta}+1)+\tau\right|}\right)p,$$

it is clear that $M'(p)\le 0$ iff $|1-\varrho|\le\frac{(2\beta+1)\left|e^{i\theta}+\tau+1\right|^2}{|\varsigma x|(\beta+1)^2\left|3(e^{i\theta}+1)+\tau\right|}$. Hence, the function L is decreasing on $[0,2]$; therefore,

$$\max\{M(p):\ p\in[0,2]\}=M(0)=\frac{4|\varsigma x|(2\beta+1)}{\left|3(e^{i\theta}+1)+\tau\right|}.$$

Also, $M'(p)\ge 0$ iff $|1-\varrho|\ge\frac{(2\beta+1)\left|e^{i\theta}+\tau+1\right|^2}{|\varsigma x|(\beta+1)^2\left|3(e^{i\theta}+1)+\tau\right|}$. So, M is an increasing function over $[0,2]$, so

$$\max\{M(p):\ p\in[0,2]\}=M(2)=\frac{4\varsigma^2x^2(\beta+1)^2}{\left|e^{i\theta}+\tau+1\right|^2}|1-\varrho|$$

and the estimation (36) has been validated. □

In the next theorem, we estimate the initial coefficients $|k_2|$ and $|k_3|$, as well as the Fekete–Szegö functional for the subclass $C(a,\varsigma,r,u,\tau,\theta)$.

Theorem 3. *Let $D\in\digamma$, defined by (5), belongs to the subclass $C(a,\varsigma,r,u,\tau,\theta)$. Then,*

$$|k_2|\le\min\left\{\frac{|\varsigma x|(\beta+1)}{\left|e^{i\theta}+2\tau+1\right|},\ \frac{|\varsigma x|\sqrt{2|\varsigma x|}}{\sqrt{\left|\varsigma^2x^2\left(\frac{4\tau(\tau-1)}{(\beta+1)^2}+\frac{3\left[e^{i\theta}+\tau+1\right]}{2\beta+1}\right)-\frac{2(r\varsigma x^2+au)\left(e^{i\theta}+2\tau+1\right)^2}{(\beta+1)^2}\right|}}\right\},$$

$$|k_3|\le\min\left\{\frac{\varsigma^2x^2(\beta+1)^2}{\left|e^{i\theta}+2\tau+1\right|^2}+\frac{2|\varsigma x|(2\beta+1)}{3\left|e^{i\theta}+\tau+1\right|},\right.$$
$$\left.\frac{\varsigma^2x^2(\beta+1)^2}{\left|e^{i\theta}+2\tau+1\right|^2}+\frac{2|\varsigma x|^3}{\left|\varsigma^2x^2\left(\frac{4\tau(\tau-1)}{(\beta+1)^2}+\frac{3\left[e^{i\theta}+\tau+1\right]}{2\beta+1}\right)-\frac{2(r\varsigma x^2+au)\left(e^{i\theta}+2\tau+1\right)^2}{(\beta+1)^2}\right|}\right\}$$

and

$$\left|k_3-\varrho k_2^2\right|\le\begin{cases}\frac{8|\varsigma x|(2\beta+1)}{3\left|e^{i\theta}+\tau+1\right|} & |1-\varrho|<\frac{2(2\beta+1)\left|e^{i\theta}+2\tau+1\right|^2}{3|\varsigma x|(\beta+1)^2\left|e^{i\theta}+\tau+1\right|},\\[2ex] \frac{4\varsigma^2x^2(\beta+1)^2|1-\varrho|}{\left|e^{i\theta}+2\tau+1\right|^2} & |1-\varrho|\ge\frac{2(2\beta+1)\left|e^{i\theta}+2\tau+1\right|^2}{3|\varsigma x|(\beta+1)^2\left|e^{i\theta}+\tau+1\right|}.\end{cases}$$

Proof. Let $D \in C(a, \varsigma, r, u, \tau, \theta)$. From the from subordinations (16) and (17), we can write

$$\left(\left(ED_\beta^s(z)\right)'\right)^\tau + \frac{1+e^{i\theta}}{2}\left(z\left(ED_\beta^s(z)\right)''\right) = F(x, \alpha(z)) + 1 - a, \ \ z \in U$$

and

$$\left(\left(EG_\beta^s(\varpi)\right)'\right)^\tau + \frac{1+e^{i\theta}}{2}\left(z\left(EG_\beta^s(\varpi)\right)''\right) = F(x, \gamma(\varpi)) + 1 - a, \ \ \varpi \in U.$$

Thus, we have

$$\begin{aligned}&\left(\left(ED_\beta^s(z)\right)'\right)^\tau + \frac{1+e^{i\theta}}{2}\left(z\left(ED_\beta^s(z)\right)''\right)\\&= 1 + \frac{t_1}{4}z + \frac{1}{48}\left(12t_2 - 7t_1^2\right)z^2 + \frac{1}{192}\left(17t_1^3 - 56t_1t_2 + 48t_3\right)z^3 + \cdots, \ z \in U.\end{aligned} \tag{51}$$

and

$$\begin{aligned}&\left(\left(EG_\beta^s(\varpi)\right)'\right)^\tau + \frac{1+e^{i\theta}}{2}\left(z\left(EG_\beta^s(\varpi)\right)''\right)\\&= 1 + \frac{b_1}{4}\varpi + \frac{1}{48}\left(12b_2 - 7b_1^2\right)\varpi^2 + \frac{1}{192}\left(17b_1^3 - 56b_1b_2 + 48b_3\right)\varpi^3 + \cdots, \ \varpi \in U.\end{aligned} \tag{52}$$

From Equations (51) and (52), we obtain

$$\frac{e^{i\theta} + 2\tau + 1}{\beta + 1}k_2 = h_2(x)t_1, \tag{53}$$

$$\frac{2\tau(\tau-1)}{(\beta+1)^2}k_2^2 + \frac{3\left[e^{i\theta} + \tau + 1\right]}{2(2\beta+1)}k_3 = h_2(x)t_2 + h_3(x)t_1^2, \tag{54}$$

$$-\frac{e^{i\theta} + 2\tau + 1}{\beta + 1}k_2 = h_2(x)b_1, \tag{55}$$

and

$$\left[\frac{3\left[e^{i\theta} + \tau + 1\right]}{2\beta+1} + \frac{2\tau(\tau-1)}{(\beta+1)^2}\right]k_2^2 - \frac{3\left[e^{i\theta} + \tau + 1\right]}{2(2\beta+1)}k_3 = h_2(x)b_2 + h_3(x)b_1^2. \tag{56}$$

Using the last four equations and by the same technique for proving Theorem 2, we obtain the conclusions of Theorem 3. □

Remark 2. *For the subclasses $A(a, \varsigma, r, u, \eta, \rho, \sigma)$, $B(a, \varsigma, r, u, \tau, \theta)$, and $C(a, \varsigma, r, u, \tau, \theta)$, we can derive numerous corollaries for specific values of $a, \varsigma, r, u, \eta, \rho, \sigma$ in Theorem 1, and $a, \varsigma, r, u, \tau, \theta$ in Theorems 2 and 3. In particular, in view of Remark 1, we can derive several results related to Fibonacci polynomials, Lucas polynomials, Chebyshev polynomials of the first kind, Chebyshev polynomials of the second kind, Pell polynomials, and Pell–Lucas polynomials.*

3. Conclusions

The error function, defined by (1), plays a significant role in mathematics and its related disciplines. It is particularly notable for its wide range of applications, including statistics, probability theory, partial differential equations, special functions, and physics. It is worth mentioning that the error function is also commonly referred to as the probability integral in the literature.

In this work, we introduced the inclusive subclasses $A(a, \varsigma, r, u, \eta, \rho, \sigma)$, $B(a, \varsigma, r, u, \tau, \theta)$, and $C(a, \varsigma, r, u, \tau, \theta)$, which are subclasses of bi-univalent functions defined using the error function and subordinated to Horadam polynomials. For functions belonging to these subclasses, we have derived estimations for the Maclaurin coefficients $|k_2|$ and $|k_3|$, as well as the Fekete–Szegö functional.

The findings of this investigation open avenues for further exploration, particularly due to the unique characterizations and proofs presented. These results not only enrich the theory of analytic and bi-univalent function subclasses but also pave the way for future research involving other special functions within these subclasses. The interplay between the error function, Horadam polynomials, and the introduced subclasses could inspire new directions in the study of complex functions and their applications.

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