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The Space of Continuous Linear Functionals on ℓ_1 Approximated by Weakly Symmetric Continuous Linear Functionals

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Abstract: We study the space of continuous linear functionals approximated by weakly symmetric continuous linear functionals on the complex Banach space ℓ_1 of all absolutely summing complex sequences. We construct the sequence of groups of symmetries on ℓ_1, obtain the structure of corresponding weakly symmetric continuous linear functionals, find the completion of the space, and construct for it a Schauder basis.

Keywords: symmetric function; weakly symmetric function; symmetric linear functional; Banach space of absolutely summing sequences

MSC: 46G20

1. Introduction

Algebras of analytic functions on infinite dimensional Banach spaces are usually nonseparable. For most interesting cases, their spectral structures and homomorphisms can be described only implicitly (see, e.g., [1–3]). On the other hand, a subalgebra of analytic functions, generated by a countable number of polynomials has the property that any continuous homomorphism is completely defined by its values on the sequence of polynomials [4,5]. Countably generated algebras of polynomials naturally appear as algebras of symmetric polynomials with respect to a group or semigroup of operators on a given Banach space (see for details [6,7] and references therein). The main idea of this paper is to consider a chain of groups of operators and so-called weakly symmetric functions that are symmetric with respect to operators in at least one of the groups. Clearly, in this way, we can obtain larger algebras of analytic functions, which can still be separable.

This work is devoted to the study of the space of continuous linear functionals approximated by weakly symmetric continuous linear functionals on the complex Banach space ℓ_1 of all absolutely summing complex sequences.

In general, a function on a vector space is called symmetric with respect to some fixed group of operators on this space if the function is invariant under the action on its argument of elements of the group. A function on a vector space is called weakly symmetric with respect to some fixed descending by inclusion sequence of groups of operators on this space if this function is symmetric with respect to at least one of the groups that belong to the sequence. Spaces of symmetric continuous polynomials and, in particular, spaces of symmetric continuous linear functionals, on Banach spaces are complete with respect to the norm of the uniform convergence on the closed unit ball, which is one of the most commonly used norms on these spaces. In contrast, spaces of weakly symmetric continuous polynomials on Banach spaces with respect to the above-mentioned norm are not necessarily complete. Therefore, completions of these spaces can contain functions

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that do not satisfy any conditions of symmetry. Consequently, such functions can be approximated by weakly symmetric functions each of which, by definition, is symmetric with respect to one of the above-mentioned groups. This fact makes it possible to apply to spaces of, in general, non-symmetric functions the technique developed for spaces of symmetric functions.

Symmetric polynomials with respect to the group of permutations of basis vectors of $\ell_p, 1 \leq p < \infty$ were considered first in [8,9]. Algebras of symmetric analytic functions for a general group of symmetry on Banach spaces were investigated in [10]. The spectrum of the algebra of symmetric analytic functions of bounded type on ℓ_1 was completely described in [11]. Problems related to extensions of symmetric analytic functions to larger spaces were studied in [12].

There are many ways for generalizations of symmetric polynomials (see, e.g., [13] for a survey on classical invariant theory and possible generalization for symmetric polynomials on free associative algebras). So-called block-symmetric or MacMahon polynomials are natural generalizations of symmetric polynomials (see, e.g., [14]) for the group of simultaneous permutations of basis vectors on finite Cartesian products of linear spaces. Block-symmetric polynomials on ℓ_p, were investigated in [6,15,16]. Next, it was observed that block-symmetric polynomials can be constructed on a single space ℓ_p modifying the action of the group of permutations. It allows the use of a sequence of groups for increasing dimensions of blocks and introduces the concept of weakly symmetric polynomials (see, [17] and references therein). Note that symmetric polynomial and their generalizations have many applications in different branches of mathematics and physics. In particular, applications of algebraic bases of symmetric polynomials to statistical quantum physics were proposed in [18–22].

In this work, we construct the sequence of groups of symmetries on the space ℓ_1. We obtain the structure of weakly symmetric, with respect to this sequence, continuous linear functionals on this space. Also, we find the Banach space, which is the completion of the space of all such functionals and construct its Schauder basis.

In Section 2, we give necessary definitions; in Section 3, we introduce weakly symmetric linear functionals on ℓ_1. In Section 4, we describe the Banach space Y, which is the closure of the space of continuous weakly symmetric linear functionals on ℓ_1. In Section 5, we construct a Schauder basis of the Banach space Y.

We refer the reader for details on polynomials and analytic functions on Banach spaces to [23,24] and on Schauder bases of Banach spaces to [25].

2. Definitions and Preliminaries

Let ℓ_1 be the Banach space of all absolutely summing sequences of complex numbers with the norm

$$\|x\|_1 = \sum_{m=1}^{\infty} |x_m|,$$

where $x = (x_1, x_2, \ldots) \in \ell_1$. Let

$$e_m = (\underbrace{0, \ldots, 0}_{m-1}, 1, 0, \ldots)$$

for $m \in \mathbb{N}$. It is well known that the set $\{e_m\}_{m=1}^{\infty}$ is a Schauder basis in ℓ_1. Also, it is well known that the mapping

$$f \in \ell_1' \mapsto (f(e_1), f(e_2), \ldots) \in \ell_\infty \tag{1}$$

is an isometrical isomorphism, where ℓ_1' is the Banach space of all continuous linear functionals on ℓ_1 and ℓ_∞ is the Banach space of all bounded sequences of complex numbers with the norm

$$\|x\|_\infty = \sup_{m\in\mathbb{N}} |x_m|,$$

where $x = (x_1, x_2, \ldots) \in \ell_\infty$.

The case ℓ_1 is important for us because in contrast with ℓ_p, $p > 1$, [9], the space ℓ_1 admits a one-dimensional space of symmetric linear functionals spanned on the functional

$$F_1(x) = \sum_{m=1}^{\infty} x_m.$$

Using this fact, in the next section, we construct an infinite-dimensional space of weakly symmetric linear functionals on ℓ_1.

3. Weakly Symmetric Continuous Linear Functionals on ℓ_1

For $n \in \mathbb{N}$, let G_n be the group of all operators on ℓ_1 that permute coordinates of elements $x = (x_1, x_2, \ldots) \in \ell_1$ by blocks of the length n of the form $(x_1, x_2, \ldots, x_n), (x_{n+1}, x_{n+2}, \ldots, x_{2n}), \ldots$. For example, the operator

$$\ell_1 \ni (x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_{2n}, x_{2n+1}, x_{2n+2}, \ldots) \mapsto (x_{n+1}, x_{n+2}, \ldots, x_{2n}, x_1, x_2, \ldots, x_n, x_{2n+1}, x_{2n+2}, \ldots) \in \ell_1$$

belongs to G_n. Continuous G_n-symmetric polynomials on spaces of summable sequences were studied in [17]. The following fact is a particular result of ([17], Theorem 4 and Lemma 7).

Theorem 1. *Let $\mathbb{C}$ be the field of all complex numbers. For $n \in \mathbb{N}$, a continuous linear functional on ℓ_1 is G_n-symmetric if and only if it is a linear combination of functionals $f_s : \ell_1 \to \mathbb{C}$, defined by*

$$f_s((x_1, x_2, \ldots)) = x_s + x_{s+n} + x_{s+2n} + \ldots, \tag{2}$$

where $s \in \{1, \ldots, n\}$.

For every $s \in \{1, \ldots, n\}$, the image of the functional f_s, defined by (2), with respect to the isomorphism (1) is the sequence

$$(\underbrace{0, \ldots, 0}_{s-1}, 1, \underbrace{0, \ldots, 0}_{n-s}, \underbrace{0, \ldots, 0}_{s-1}, 1, \underbrace{0, \ldots, 0}_{n-s}, \ldots),$$

which is periodic with the period n. Consequently, taking into account that a linear combination of periodic sequences with the same period is also a periodic sequence with this period, Theorem 1 implies the following result.

Corollary 1. *For $n \in \mathbb{N}$, a continuous linear functional f on ℓ_1 is G_n-symmetric if and only if the image of f with respect to the isomorphism (1) is a periodic with the period n element of ℓ_∞.*

So, the space of all G_n-symmetric continuous linear functionals on ℓ_1 can be identified with the subspace of all periodic with the period n elements of ℓ_∞. We will denote this subspace by Y_n.

Consider the sequence of groups $\mathcal{G} = \left(G_1, G_2, G_{2^2}, \ldots, G_{2^n}, \ldots\right)$. Note that $G_1 \supset G_2 \supset G_{2^2} \supset \ldots \supset G_{2^n} \supset \ldots$. Consider weakly symmetric with respect to $\mathcal{G}$ continuous linear functionals on ℓ_1. The space of all such functionals can be identified with the union

$$Y_{\mathcal{G}} = \bigcup_{n=0}^{\infty} Y_{2^n}. \tag{3}$$

Note that $Y_{\mathcal{G}}$ is the subspace of ℓ_∞, elements of which have the following property: for every $y \in Y_{\mathcal{G}}$, there exists $n \in \mathbb{N} \cup \{0\}$ such that y is periodical with the period 2^n.

Thus, we have constructed the space of all continuous weakly symmetric linear functionals on ℓ_1. This approach cannot be applied to the general case of ℓ_p, because if $p > 1$,, then ℓ_p does not admit any symmetric (and so weakly symmetric) linear functional. In the next section, we consider continuous linear functionals on ℓ_1 that can be approximated by weakly symmetric continuous linear functionals. The set of all such functionals can be identified with the closure of $Y_{\mathcal{G}}$ in ℓ_∞.

4. Functionals That Can Be Approximated by Weakly Symmetric Functionals

Let Y be the subset of all $y = (y_1, y_2, \ldots) \in \ell_\infty$ that have the following property:

$$\text{for every } \varepsilon > 0 \text{ there exists } n \in \mathbb{N} \text{ such that } |y_j - y_{j+k2^n}| < \varepsilon \text{ for every } j \in \{1, \ldots, 2^n\} \text{ and } k \in \mathbb{N}. \tag{4}$$

Some elements of Y can be constructed as in the following example:

Example 1. *For $n \in \mathbb{N}$ and $c \in \mathbb{C}$, let us define the mapping $M_{n,c} : \mathbb{C}^{2^n} \to \mathbb{C}^{2^{n+1}}$ by*

$$M_{n,c}\left(\left(z_1, \ldots, z_{2^n}\right)\right) = \left(\left(z_1, \ldots, z_{2^n}, z_1 + c, \ldots, z_{2^n} + c\right),$$

where $\left(z_1, \ldots, z_{2^n}\right) \in \mathbb{C}^{2^n}$. Let a sequence $\{b_j\}_{j=0}^{\infty}$ of non-negative real numbers be such that the series $\sum_{j=0}^{\infty} b_j$ is convergent. Let $m \in \mathbb{N}$ and $z \in \mathbb{C}^{2^m}$. Let us define the sequence $y = (y_1, y_2, \ldots)$. For $j \in \mathbb{N}$, let y_j be equal to the jth coordinate of the vector

$$\left(M_{m+k,b_k} \circ M_{m+k-1,b_{k-1}} \circ \ldots \circ M_{m,b_0}\right)(z),$$

where k is an arbitrary non-negative integer such that $2^{m+k} \geq j$. It can be easily verified that y_j does not depend on the choice of k. Also, it can be verified that $y \in Y$.

Lemma 1. *Let $y = (y_1, y_2, \ldots) \in Y$. For every $n \in \mathbb{N} \cup \{0\}$,*

$$\left\|y - (y_1, \ldots, y_{2^n}, y_1, \ldots, y_{2^n}, \ldots)\right\|_\infty = \max_{s \in \{1, \ldots, 2^n\}} \sup_{k \in \mathbb{N}} |y_{s+k2^n} - y_s|.$$

Proof. Let $z = (z_1, z_2, \ldots)$ be defined by

$$z = y - (y_1, \ldots, y_{2^n}, y_1, \ldots, y_{2^n}, \ldots).$$

Let us calculate z_j for a given $j \in \mathbb{N}$. There exist unique $s \in \{1, \ldots, 2^n\}$ and $k \in \mathbb{N} \cup \{0\}$ such that

$$j = s + k2^n. \tag{5}$$

Then,

$$z_j = y_j - y_s.$$

Consequently, taking into account (5),

$$\|z\|_\infty = \max_{s\in\{1,\dots,2^n\}} \sup_{k\in\mathbb{N}\cup\{0\}} |y_{s+k2^n} - y_s|.$$

Note that, in the case $k = 0$, we have $y_{s+k2^n} - y_s = 0$ for every $s \in \{1,\dots,2^n\}$. Therefore,

$$\max_{s\in\{1,\dots,2^n\}} \sup_{k\in\mathbb{N}\cup\{0\}} |y_{s+k2^n} - y_s| = \max_{s\in\{1,\dots,2^n\}} \sup_{k\in\mathbb{N}} |y_{s+k2^n} - y_s|.$$

Thus,

$$\|z\|_\infty = \max_{s\in\{1,\dots,2^n\}} \sup_{k\in\mathbb{N}} |y_{s+k2^n} - y_s|.$$

This completes the proof. □

Proposition 1. *The space $Y_\mathcal{G}$, defined by (3), is dense in Y.*

Proof. Let $y = (y_1, y_2, \dots) \in Y$ and $\varepsilon > 0$. Let us construct $t \in Y_\mathcal{G}$ such that $\|y - t\|_\infty < \varepsilon$. By (4), there exists $n \in \mathbb{N}$ such that $|y_j - y_{j+k2^n}| < \varepsilon/2$ for every $j \in \{1,\dots,2^n\}$ and $k \in \mathbb{N}$. Consequently,

$$\max_{s\in\{1,\dots,2^n\}} \sup_{k\in\mathbb{N}} |y_{s+k2^n} - y_s| \le \varepsilon/2. \tag{6}$$

Let $t = (y_1, \dots, y_{2^n}, y_1, \dots, y_{2^n}, \dots)$. Then, $t \in Y_\mathcal{G}$. By Lemma 1, taking into account (6),

$$\|y - t\|_\infty \le \varepsilon/2 < \varepsilon.$$

So, $Y_\mathcal{G}$ is dense in Y. □

Proposition 2. *The set Y is closed in ℓ_∞.*

Proof. Let us show that Y contains every one of its adherent point. Let $x \in \ell_\infty$ be an adherent point of Y. Let us show that $x \in Y$. Let $\varepsilon > 0$. Since x is an adherent point of Y, there exists $y \in Y$ such that

$$\|x - y\|_\infty < \varepsilon/3. \tag{7}$$

Since $y \in Y$, there exists $n \in \mathbb{N}$ such that

$$|y_j - y_{j+k2^n}| < \varepsilon/3 \tag{8}$$

for every $j \in \{1,\dots,2^n\}$ and $k \in \mathbb{N}$. By (7) and (8),

$$\begin{aligned} |x_j - x_{j+k2^n}| =& |x_j - y_j + y_j - y_{j+k2^n} + y_{j+k2^n} - x_{j+k2^n}| \\ \le& |x_j - y_j| + |y_j - y_{j+k2^n}| + |y_{j+k2^n} - x_{j+k2^n}| \\ <& \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

for every $j \in \{1,\dots,2^n\}$ and $k \in \mathbb{N}$. So, $x \in Y$. This completes the proof. □

Propositions 1 and 2 imply the following result.

Theorem 2. *The set Y is the closure of $Y_\mathcal{G}$ with respect to the norm of ℓ_∞.*

It is well known that the closure of a vector subspace in a normed space is a vector subspace too. Consequently, since $Y_\mathcal{G}$ is a vector subspace of ℓ_∞, Theorem 2 implies the following result.

Corollary 2. *The set Y is a vector subspace of ℓ_∞.*

Proposition 2 and Corollary 2 imply the following result.

Theorem 3. *The set Y with operations and topology, inherited from ℓ_∞, is a Banach space.*

We can see that the space Y is a proper closed subset in ℓ_∞. Note that Y is separable. Moreover, in the next section, we construct a Schauder basis of Y.

5. Construction of a Schauder Basis of Y

For a given $n \in \mathbb{N} \cup \{0\}$ and $j \in \{1, \dots, 2^n\}$, let

$$A_{n,j} = \{j + k2^n : \ k \in \mathbb{N} \cup \{0\}\}. \tag{9}$$

It can be checked that

$$A_{n,j} = A_{n+1,j} \sqcup A_{n+1,j+2^n} \tag{10}$$

for every $n \in \mathbb{N} \cup \{0\}$ and $j \in \{1, \dots, 2^n\}$.

For $A \subset \mathbb{N}$, let 1_A be the sequence $(x_1, x_2, \dots)$ such that

$$x_j = \begin{cases} 1, & \text{if } j \in A, \\ 0, & \text{otherwise} \end{cases}$$

for $j \in \mathbb{N}$. By (10),

$$1_{A_{n,j}} = 1_{A_{n+1,j}} + 1_{A_{n+1,j+2^n}} \tag{11}$$

for every $n \in \mathbb{N} \cup \{0\}$ and $j \in \{1, \dots, 2^n\}$.

Let the mapping $\varkappa : \mathbb{N} \to \mathbb{N} \cup \{0\}$ be defined by

$$\varkappa(j) = \lceil \log_2 j \rceil, \tag{12}$$

where $j \in \mathbb{N}$. In other words, $\varkappa(j)$ is the minimal number $m \in \mathbb{N} \cup \{0\}$ such that $j \leq 2^m$.

For $j \in \mathbb{N}$, let

$$b_j = 1_{A_{\varkappa(j),j}}, \tag{13}$$

where $\varkappa$ is defined by (12).

For $j \in \mathbb{N}$, let $\alpha_j : Y \to \mathbb{C}$ be defined by

$$\alpha_j(y) = \begin{cases} y_1, & \text{if } j = 1, \\ y_j - y_{j-2^{\varkappa(j)-1}}, & \text{if } j \geq 2, \end{cases} \tag{14}$$

where $y = (y_1, y_2, \dots) \in Y$.

Proposition 3. *Let $y = (y_1, y_2, \dots) \in Y$. For every $n \in \mathbb{N} \cup \{0\}$,*

$$\sum_{j=1}^{2^n} \alpha_j(y) b_j = (y_1, \dots, y_{2^n}, y_1, \dots, y_{2^n}, \dots), \tag{15}$$

where b_j and α_j are defined by (13) and (14), respectively.

Proof. We proceed by induction on n. Consider the case $n = 0$. In this case, taking into account (9), (13) and (14),

$$\sum_{j=1}^{2^n} \alpha_j(y)b_j = \alpha_1(y)b_1 = y_1 1_{A_{\varkappa(1),1}} = y_1 1_{A_{0,1}} = y_1 1_{A_{\{1+k2^0:\, k\in\mathbb{N}\cup\{0\}\}}} = y_1 1_{A_{\mathbb{N}}}$$
$$= y_1(1,1,\ldots) = (y_1, y_1, \ldots).$$

So, in this case, the equality (15) holds.

Suppose the equality (15) holds for $n = m$, where m is some element of $\mathbb{N}\cup\{0\}$. Let us show that this equality holds for $n = m+1$. Note that

$$\sum_{j=1}^{2^{m+1}} \alpha_j(y)b_j = \sum_{j=1}^{2^m} \alpha_j(y)b_j + \sum_{j=2^m+1}^{2^{m+1}} \alpha_j(y)b_j. \tag{16}$$

By the induction hypothesis,

$$\sum_{j=1}^{2^m} \alpha_j(y)b_j = (y_1, \ldots, y_{2^m}, y_1, \ldots, y_{2^m}, \ldots). \tag{17}$$

Taking into account (9),

$$(y_1, \ldots, y_{2^m}, y_1, \ldots, y_{2^m}, \ldots) = \sum_{j=1}^{2^m} y_j 1_{A_{m,j}}. \tag{18}$$

By (17) and (18),

$$\sum_{j=1}^{2^m} \alpha_j(y)b_j = \sum_{j=1}^{2^m} y_j 1_{A_{m,j}}. \tag{19}$$

Taking into account (13) and (14),

$$\sum_{j=2^m+1}^{2^{m+1}} \alpha_j(y)b_j = \sum_{j=2^m+1}^{2^{m+1}} (y_j - y_{j-2^{\varkappa(j)-1}}) 1_{A_{\varkappa(j),j}}. \tag{20}$$

Note that $\varkappa(j) = m+1$ for every $j \in \{2^m+1, \ldots, 2^{m+1}\}$. Consequently,

$$\sum_{j=2^m+1}^{2^{m+1}} (y_j - y_{j-2^{\varkappa(j)-1}}) 1_{A_{\varkappa(j),j}} = \sum_{j=2^m+1}^{2^{m+1}} (y_j - y_{j-2^m}) 1_{A_{m+1,j}}. \tag{21}$$

By (20) and (21),

$$\begin{aligned} \sum_{j=2^m+1}^{2^{m+1}} \alpha_j(y)b_j &= \sum_{j=2^m+1}^{2^{m+1}} (y_j - y_{j-2^m}) 1_{A_{m+1,j}} \\ &= \sum_{j=2^m+1}^{2^{m+1}} y_j 1_{A_{m+1,j}} - \sum_{j=2^m+1}^{2^{m+1}} y_{j-2^m} 1_{A_{m+1,j}}. \end{aligned} \tag{22}$$

Note that

$$\sum_{j=2^m+1}^{2^{m+1}} y_{j-2^m} 1_{A_{m+1,j}} = \sum_{s=1}^{2^m} y_s 1_{A_{m+1,s+2^m}}, \tag{23}$$

where $s = j - 2^m$. By (22) and (23),

$$\sum_{j=2^m+1}^{2^{m+1}} \alpha_j(y) b_j = \sum_{j=2^m+1}^{2^{m+1}} y_j 1_{A_{m+1,j}} - \sum_{j=1}^{2^m} y_j 1_{A_{m+1,j+2^m}}.$$

Consequently, taking into account (16) and (19),

$$\sum_{j=1}^{2^{m+1}} \alpha_j(y) b_j = \sum_{j=1}^{2^m} y_j 1_{A_{m,j}} + \sum_{j=2^m+1}^{2^{m+1}} y_j 1_{A_{m+1,j}} - \sum_{j=1}^{2^m} y_j 1_{A_{m+1,j+2^m}}. \tag{24}$$

Taking into account (11),

$$\begin{aligned} \sum_{j=1}^{2^m} y_j 1_{A_{m,j}} - \sum_{j=1}^{2^m} y_j 1_{A_{m+1,j+2^m}} &= \sum_{j=1}^{2^m} y_j \left(1_{A_{m,j}} - 1_{A_{m+1,j+2^m}}\right) \\ &= \sum_{j=1}^{2^m} y_j \left(1_{A_{m+1,j}} + 1_{A_{m+1,j+2^m}} - 1_{A_{m+1,j+2^m}}\right) \\ &= \sum_{j=1}^{2^m} y_j 1_{A_{m+1,j}}. \end{aligned} \tag{25}$$

By (24) and (25), taking into account (9),

$$\begin{aligned} \sum_{j=1}^{2^{m+1}} \alpha_j(y) b_j &= \sum_{j=1}^{2^m} y_j 1_{A_{m+1,j}} + \sum_{j=2^m+1}^{2^{m+1}} y_j 1_{A_{m+1,j}} \\ &= \sum_{j=1}^{2^{m+1}} y_j 1_{A_{m+1,j}} \\ &= (y_1, \ldots, y_{2^{m+1}}, y_1, \ldots, y_{2^{m+1}}, \ldots). \end{aligned}$$

So, for the case $n = m + 1$, the equality (15) holds. This completes the proof. □

Lemma 1 and Proposition 3 imply the following corollary.

Corollary 3. *Let $y = (y_1, y_2, \ldots) \in Y$. For every $n \in \mathbb{N} \cup \{0\}$,*

$$\left\| y - \sum_{j=1}^{2^n} \alpha_j(y) b_j \right\|_\infty = \max_{s \in \{1,\ldots,2^n\}} \sup_{k \in \mathbb{N}} |y_{s+k2^n} - y_s|.$$

Proposition 4. *For every $y = (y_1, y_2, \ldots) \in Y$ and $r \in \mathbb{N}$,*

$$\sum_{j=1}^{r} \alpha_j(y) b_j = \left(y_{\nu(1)}, y_{\nu(2)}, \ldots, y_{\nu(2^{\varkappa(r)})}, y_{\nu(1)}, y_{\nu(2)}, \ldots, y_{\nu(2^{\varkappa(r)})}, \ldots\right), \tag{26}$$

where $\varkappa$, b_j and α_j are defined by (12), (13) and (14), respectively, and the mapping $\nu : \{1, \ldots, 2^{\varkappa(r)}\} \to \mathbb{N}$ is defined by

$$\nu(j) = \begin{cases} j, & \text{if } j \in \{1, \ldots, r\}, \\ j - 2^{\varkappa(r)-1}, & \text{if } j \in \{r+1, \ldots, 2^{\varkappa(r)}\}. \end{cases} \tag{27}$$

Proof. Consider the case $r = 2^{\varkappa(r)}$, i.e., r is a power of 2. In this case, ν is the identity map. Therefore, the right-hand side of the equality (26) is equal to $(y_1, y_2, \ldots, y_r, y_1, y_2, \ldots, y_r, \ldots)$,

which is equal to the left-hand side of the equality (26) by Proposition 3. So, the equality (26) holds.

Consider the case $r < 2^{\varkappa(r)}$. Note that

$$\sum_{j=1}^{r} \alpha_j(y) b_j = \sum_{j=1}^{2^{\varkappa(r)}} \alpha_j(y) b_j - \sum_{j=r+1}^{2^{\varkappa(r)}} \alpha_j(y) b_j. \tag{28}$$

By Proposition 3,

$$\sum_{j=1}^{2^{\varkappa(r)}} \alpha_j(y) b_j = (y_1, \ldots, y_{2^{\varkappa(r)}}, y_1, \ldots, y_{2^{\varkappa(r)}}, \ldots). \tag{29}$$

Taking into account (9),

$$(y_1, \ldots, y_{2^{\varkappa(r)}}, y_1, \ldots, y_{2^{\varkappa(r)}}, \ldots) = \sum_{j=1}^{2^{\varkappa(r)}} y_j 1_{A_{\varkappa(r),j}}. \tag{30}$$

By (29) and (30),

$$\sum_{j=1}^{2^{\varkappa(r)}} \alpha_j(y) b_j = \sum_{j=1}^{2^{\varkappa(r)}} y_j 1_{A_{\varkappa(r),j}}. \tag{31}$$

By (13) and (14),

$$\sum_{j=r+1}^{2^{\varkappa(r)}} \alpha_j(y) b_j = \sum_{j=r+1}^{2^{\varkappa(r)}} (y_j - y_{j-2^{\varkappa(j)-1}}) 1_{A_{\varkappa(j),j}}. \tag{32}$$

Note that $\varkappa(j) = \varkappa(r)$ for every $j \in \{r+1, \ldots, 2^{\varkappa(j)}\}$. Consequently

$$\sum_{j=r+1}^{2^{\varkappa(r)}} (y_j - y_{j-2^{\varkappa(j)-1}}) 1_{A_{\varkappa(j),j}} = \sum_{j=r+1}^{2^{\varkappa(r)}} (y_j - y_{j-2^{\varkappa(r)-1}}) 1_{A_{\varkappa(r),j}}. \tag{33}$$

By (32) and (33),

$$\sum_{j=r+1}^{2^{\varkappa(r)}} \alpha_j(y) b_j = \sum_{j=r+1}^{2^{\varkappa(r)}} (y_j - y_{j-2^{\varkappa(r)-1}}) 1_{A_{\varkappa(r),j}}. \tag{34}$$

Therefore, by (28), (31) and (34),

$$\begin{aligned}
\sum_{j=1}^{r} \alpha_j(y) b_j &= \sum_{j=1}^{2^{\varkappa(r)}} y_j 1_{A_{\varkappa(r),j}} - \sum_{j=r+1}^{2^{\varkappa(r)}} (y_j - y_{j-2^{\varkappa(r)-1}}) 1_{A_{\varkappa(r),j}} \\
&= \sum_{j=1}^{r} y_j 1_{A_{\varkappa(r),j}} + \sum_{j=r+1}^{2^{\varkappa(r)}} y_j 1_{A_{\varkappa(r),j}} - \sum_{j=r+1}^{2^{\varkappa(r)}} (y_j - y_{j-2^{\varkappa(r)-1}}) 1_{A_{\varkappa(r),j}} \\
&= \sum_{j=1}^{r} y_j 1_{A_{\varkappa(r),j}} + \sum_{j=r+1}^{2^{\varkappa(r)}} (y_j - y_j + y_{j-2^{\varkappa(r)-1}}) 1_{A_{\varkappa(r),j}} \\
&= \sum_{j=1}^{r} y_j 1_{A_{\varkappa(r),j}} + \sum_{j=r+1}^{2^{\varkappa(r)}} y_{j-2^{\varkappa(r)-1}} 1_{A_{\varkappa(r),j}} \\
&= \sum_{j=1}^{r} y_{\nu(j)} 1_{A_{\varkappa(r),j}} + \sum_{j=r+1}^{2^{\varkappa(r)}} y_{\nu(j)} 1_{A_{\varkappa(r),j}} \\
&= \sum_{j=1}^{2^{\varkappa(r)}} y_{\nu(j)} 1_{A_{\varkappa(r),j}} \\
&= (y_{\nu(1)}, y_{\nu(2)}, \ldots, y_{\nu(2^{\varkappa(r)})}, y_{\nu(1)}, y_{\nu(2)}, \ldots, y_{\nu(2^{\varkappa(r)})}, \ldots).
\end{aligned}$$

This completes the proof. □

Corollary 4. *For every $y = (y_1, y_2, \ldots) \in Y$ and $r \in \mathbb{N}$,*

$$\left\| \sum_{j=1}^{2^{\varkappa(r)}} \alpha_j(y) b_j - \sum_{j=1}^{r} \alpha_j(y) b_j \right\|_\infty = \begin{cases} 0, & \text{if } r = 2^{\varkappa(r)}, \\ \max\limits_{s \in \{r+1, \ldots, 2^{\varkappa(r)}\}} |y_s - y_{s-2^{\varkappa(r)-1}}|, & \text{if } r < 2^{\varkappa(r)}. \end{cases}$$

Proof. The case $r = 2^{\varkappa(r)}$ is trivial. Consider the case $r < 2^{\varkappa(r)}$. Let $z = (z_1, z_2, \ldots)$ be defined by

$$z = \sum_{j=1}^{2^{\varkappa(r)}} \alpha_j(y) b_j - \sum_{j=1}^{r} \alpha_j(y) b_j.$$

Let $j \in \mathbb{N}$. Let us find z_j. Let $s \in \{1, \ldots, 2^{\varkappa(r)}\}$ and $k \in \mathbb{N} \cup \{0\}$ be such that $j = s + k2^{\varkappa(r)}$. By Propositions 3 and 4,

$$z_j = y_s - y_{\nu(s)},$$

where ν is defined by (27). By (27),

$$y_{\nu(s)} = \begin{cases} y_s, & \text{if } s \in \{1, \ldots, r\}, \\ y_{s-2^{\varkappa(r)-1}}, & \text{if } s \in \{r+1, \ldots, 2^{\varkappa(r)}\}. \end{cases}$$

Consequently,

$$z_j = \begin{cases} 0, & \text{if } s \in \{1, \ldots, r\}, \\ y_s - y_{s-2^{\varkappa(r)-1}}, & \text{if } s \in \{r+1, \ldots, 2^{\varkappa(r)}\}. \end{cases}$$

Therefore,

$$\|z\|_\infty = \max_{s \in \{r+1, \ldots, 2^{\varkappa(r)}\}} |y_s - y_{s-2^{\varkappa(r)-1}}|.$$

This completes the proof. □

Theorem 4. *For every $y \in Y$, the series $\sum_{j=1}^{\infty} \alpha_j(y) b_j$, where b_j and α_j are defined by (13) and (14), respectively, converges to y with respect to the norm $\|\cdot\|_\infty$. Consequently, $\{b_j\}_{j=1}^{\infty}$ is a Schauder basis of the space Y.*

Proof. Let $y = (y_1, y_2, \ldots) \in Y$. Let us show that the series $\sum_{j=1}^{\infty} \alpha_j(y) b_j$ converges to y with respect to the norm $\|\cdot\|_\infty$, that is,

$$\lim_{r \to \infty} \left\| y - \sum_{j=1}^{r} \alpha_j(y) b_j \right\|_\infty = 0. \tag{35}$$

Let $\varepsilon > 0$. Since $y \in Y$, there exists $n \in \mathbb{N}$ such that

$$|y_j - y_{j+k2^n}| < \varepsilon/4 \tag{36}$$

for every $j \in \{1, \ldots, 2^n\}$ and $k \in \mathbb{N} \cup \{0\}$. Let us show that

$$\left\| y - \sum_{j=1}^{r} \alpha_j(y) b_j \right\|_\infty < \varepsilon \tag{37}$$

for every $r > 2^n$. Let $r \in \mathbb{N}$ be such that $r > 2^n$. Let us prove (37). First, we will establish some auxiliary inequalities.

By (36),

$$\begin{aligned} |y_{j+k_1 2^n} - y_{j+k_2 2^n}| &= |y_{j+k_1 2^n} - y_j + y_j - y_{j+k_2 2^n}| \\ &\leq |y_{j+k_1 2^n} - y_j| + |y_j - y_{j+k_2 2^n}| \\ &< \varepsilon/4 + \varepsilon/4 = \varepsilon/2 \end{aligned} \tag{38}$$

for every $j \in \{1, \ldots, 2^n\}$ and $k_1, k_2 \in \mathbb{N} \cup \{0\}$.

Since $r > 2^n$, it follows that $\varkappa(r) > n$, where $\varkappa$ is defined by (12). Consequently, since both $\varkappa(r)$ and n are integers, it follows that

$$\varkappa(r) \geq n + 1. \tag{39}$$

Note that

$$y - \sum_{j=1}^{r} \alpha_j(y) b_j = y - \sum_{j=1}^{2^{\varkappa(r)}} \alpha_j(y) b_j + \sum_{j=1}^{2^{\varkappa(r)}} \alpha_j(y) b_j - \sum_{j=1}^{r} \alpha_j(y) b_j.$$

Consequently,

$$\left\| y - \sum_{j=1}^{r} \alpha_j(y) b_j \right\|_\infty \leq \left\| y - \sum_{j=1}^{2^{\varkappa(r)}} \alpha_j(y) b_j \right\|_\infty + \left\| \sum_{j=1}^{2^{\varkappa(r)}} \alpha_j(y) b_j - \sum_{j=1}^{r} \alpha_j(y) b_j \right\|_\infty. \tag{40}$$

Let us show that

$$\left\| y - \sum_{j=1}^{2^{\varkappa(r)}} \alpha_j(y) b_j \right\|_\infty < \varepsilon/2. \tag{41}$$

By Corollary 3,

$$\left\| y - \sum_{j=1}^{2^{\varkappa(r)}} \alpha_j(y) b_j \right\|_\infty = \max_{s \in \{1, \ldots, 2^{\varkappa(r)}\}} \sup_{k \in \mathbb{N}} |y_{s+k2^{\varkappa(r)}} - y_s|. \tag{42}$$

Let $s \in \{1, \ldots, 2^{\varkappa(r)}\}$ and $k \in \mathbb{N}$. Let us show that

$$|y_{s+k2^{\varkappa(r)}} - y_s| < \varepsilon/2. \tag{43}$$

Let $t \in \{1, \ldots, 2^n\}$ and $q_1 \in \mathbb{N} \cup \{0\}$ be such that

$$s = t + q_1 2^n.$$

Then,

$$\begin{aligned} s + k2^{\varkappa(r)} &= t + q_1 2^n + k2^{\varkappa(r)} \\ &= t + q_1 2^n + k2^{\varkappa(r)-n} 2^n \\ &= t + \left(q_1 + k2^{\varkappa(r)-n}\right) 2^n \\ &= t + q_2 2^n, \end{aligned}$$

where $q_2 = q_1 + k2^{\varkappa(r)-n}$. Taking into account (39), $\varkappa(r) - n$ is a positive integer and, consequently, $q_2 \in \mathbb{N} \cup \{0\}$. Therefore, by (38),

$$|y_{s+k2^{\varkappa(r)}} - y_s| = |y_{t+q_2 2^n} - y_{t+q_1 2^n}| < \varepsilon/2.$$

So, the inequality (43) holds. By (42) and (43), the inequality (41) holds.

Let us show that

$$\Big\|\sum_{j=1}^{2^{\varkappa(r)}} \alpha_j(y)b_j - \sum_{j=1}^{r} \alpha_j(y)b_j\Big\|_\infty < \varepsilon/2. \tag{44}$$

In the case $r = 2^{\varkappa(r)}$, the inequality (44) is trivial. Consider the case $r < 2^{\varkappa(r)}$. In this case, by Corollary 4,

$$\Big\|\sum_{j=1}^{2^{\varkappa(r)}} \alpha_j(y)b_j - \sum_{j=1}^{r} \alpha_j(y)b_j\Big\|_\infty = \max_{s\in\{r+1,\dots,2^{\varkappa(r)}\}} |y_s - y_{s-2^{\varkappa(r)-1}}|. \tag{45}$$

Let $s \in \{r+1,\dots,2^{\varkappa(r)}\}$. Let us show that

$$|y_s - y_{s-2^{\varkappa(r)-1}}| < \varepsilon/2. \tag{46}$$

Let $u \in \{1,\dots,2^n\}$ and $v_1 \in \mathbb{N}\cup\{0\}$ be such that

$$s - 2^{\varkappa(r)-1} = u + v_1 2^n.$$

Then,

$$\begin{aligned} s &= u + v_1 2^n + 2^{\varkappa(r)-1} \\ &= u + v_1 2^n + 2^{\varkappa(r)-1-n} 2^n \\ &= u + \big(v_1 + 2^{\varkappa(r)-1-n}\big)2^n \\ &= u + v_2 2^n, \end{aligned}$$

where $v_2 = (v_1 + 2^{\varkappa(r)-1-n})$. Taking into account (39), $\varkappa(r) - 1 - n$ is a non-negative integer and, consequently, $v_2 \in \mathbb{N}\cup\{0\}$. Therefore, by (38),

$$|y_s - y_{s-2^{\varkappa(r)-1}}| = |y_{u+v_2 2^n} - y_{u+v_1 2^n}| < \varepsilon/2.$$

So, the inequality (46) holds. By (45) and (46), the inequality (44) holds.

By (40), (41) and (44), the inequality (37) holds. Therefore, the equality (35) holds. Consequently, $\{b_j\}_{j=1}^\infty$ is a Schauder basis of the space Y. This completes the proof. □

6. Discussion and Conclusions

We constructed a new Banach space Y consisting of linear functionals on ℓ_1 that can be approximated by weakly symmetric linear functionals and found a Schauder basis of this space. For the next step, we will investigate weakly symmetric polynomials on ℓ_1 and describe algebras of polynomials and analytic functions on ℓ_1 consisting of elements that can be approximated by the weakly symmetric polynomials. It would allow us to spread methods of investigations of algebras of symmetric analytic functions to the largest case where functions already are not symmetric. As we mentioned above, spaces ℓ_p do not admit symmetric linear functionals if $p > 1$, but they admit symmetric n-homogeneous polynomials for $n \geq p$. Thus, further investigation of weakly symmetric polynomials can be extended for the general case of ℓ_p.

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