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Lie Saturate and Controllability

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Abstract

We study the controllability of right-invariant bilinear systems on the complex and quaternionic special linear groups $\mathrm{Sl}(n,\mathbb{C})$ and $\mathrm{Sl}(n,\mathbb{H})$. The analysis relies on the *Lie saturate* $\mathrm{LS}(\Gamma)$, which characterizes controllability through convexity and closure properties of attainable sets, avoiding explicit Lie algebra computations. For $\mathrm{Sl}(n,\mathbb{C})$ with a strongly regular diagonal control matrix, we show that controllability is equivalent to the irreducibility of the drift matrix A, a property verified by the strong connectivity of its associated directed graph. For $\mathrm{Sl}(n,\mathbb{H})$, we derive controllability criteria based on quaternionic entries and the convexity of $\mathbb{T}^2$-orbits, which provide efficient sufficient conditions for general n and exact ones in the 2×2 case. These results link algebraic and geometric viewpoints within a unified framework and connect to recent graph-theoretic controllability analyses for bilinear systems on Lie groups. The proposed approach yields constructive and scalable controllability tests for complex and quaternionic systems.

Keywords: control system; controllability; Lie saturate; Lie group

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1. Introduction

The study of controllability for right-invariant control systems on Lie groups has been a subject of sustained interest in geometric control theory since the pioneering works of Jurdjevic and Sussmann [1,2]. Such systems, whose dynamics are generated by right-invariant vector fields, naturally arise in various areas of applied mathematics and engineering, including rigid body dynamics [3], quantum control [4], robotics and motion planning [5], and mechanical systems with symmetry [6]. The interplay between the algebraic structure of the Lie group and the geometry of attainable sets provides a fertile ground for understanding controllability and stabilization phenomena.

A central tool in this framework is the *Lie saturate* $\mathrm{LS}(\Gamma)$ of a family of vector fields $\Gamma \subset \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of a connected Lie group G. Introduced by Jurdjevic and Sussmann [7], the Lie saturate characterizes controllability in geometric terms: a right-invariant system is controllable if and only if $\mathrm{LS}(\Gamma) = \mathfrak{g}$. This concept refines the classical Lie algebra rank condition by incorporating convexity and closure properties of the attainable set, providing a powerful alternative to explicit Lie algebra computations.

However, verifying that $\mathrm{LS}(\Gamma) = \mathfrak{g}$ remains highly nontrivial in general. For systems on matrix Lie groups, structural properties of the drift and control matrices often enable a more combinatorial approach. In particular, the notion of *irreducibility* of matrices, together

with its graph-theoretic interpretation via strong connectivity of an associated directed graph [8,9], plays a key role in assessing whether the system's components are sufficiently coupled to ensure controllability. This connection between algebraic structure and network topology has been fruitfully exploited in recent studies of bilinear systems on graphs and Lie groups [10,11].

In this paper, we integrate the Lie saturate framework with graph-theoretic irreducibility criteria to establish controllability results for bilinear control systems on two significant classes of simple Lie groups: the complex special linear group $\mathrm{Sl}(n,\mathbb{C})$ and the quaternionic special linear group $\mathrm{Sl}(n,\mathbb{H})$. The former plays a central role in the control of multi-level quantum systems and complex representations [4,12], while the latter appears in quaternionic formulations of mechanical and quantum dynamics and relates closely to real forms of complex semisimple Lie algebras [13,14]. The general methodology aligns with classical approaches on $\mathrm{Sl}(n,\mathbb{R})$ developed by Gauthier and Bornard [15], but extends them to more intricate complex and quaternionic settings.

For $\mathrm{Sl}(n,\mathbb{C})$, under a *strong regularity* assumption on the control matrix B, originally introduced in [15], we show that controllability reduces to the irreducibility of the drift matrix A, equivalently, to the strong connectivity of its associated directed graph. This provides an explicit and purely combinatorial controllability criterion. For $\mathrm{Sl}(n,\mathbb{H})$, we generalize this construction by exploiting the real structure of $\mathfrak{sl}(n,\mathbb{H})$ and the convexity of torus orbits acting on quaternionic entries. In the 2×2 case, our conditions are both necessary and sufficient, extending and refining the quaternionic controllability results of [16].

Recent developments in the analysis of controllability on networks and bilinear Lie-group systems [10,11] further emphasize the relevance of combining algebraic and geometric reasoning. Our contribution complements these advances by providing constructive and scalable controllability criteria within the Lie saturate framework.

An important advantage of our approach is that the criteria we establish avoid a full computation of $\mathrm{Lie}(\Gamma)$, which is generally a difficult task even with symbolic computation software. Instead, by combining algebraic closure properties of the Lie saturate [17,18] with elementary manipulations based on graph connectivity, we derive controllability conditions that can be verified by straightforward algebraic or combinatorial checks.

The structure of the paper is as follows. In Section 2, we recall the definition and main properties of the Lie saturate, following [3,7], and review the graph-theoretic characterization of irreducible matrices [8]. Section 3 addresses the case of $\mathrm{Sl}(n,\mathbb{C})$, proving our main controllability theorem under strong regularity conditions. Section 4 extends the analysis to $\mathrm{Sl}(n,\mathbb{H})$, making explicit the quaternionic computations needed to establish the result. We conclude with remarks on possible generalizations to other semisimple Lie groups and future research directions.

2. The Lie Saturate and Irreducible Matrices

If G is a connected Lie group with Lie algebra $\mathfrak{g}$, we will denote by $\mathrm{Lie}(\Gamma)$ the Lie algebra generated by $\Gamma\subset\mathfrak{g}$, that is, the smallest Lie subalgebra of $\mathfrak{g}$ containing Γ.

Given a right-invariant control system Γ, let $\mathcal{A}_\Gamma$ denote its attainable set from the identity.

Definition 1. *The **Lie saturate** of* Γ, *written as* $\mathrm{LS}(\Gamma)$, *is the set*

$$\mathrm{LS}(\Gamma)=\{A\in\mathrm{Lie}(\Gamma)\mid \exp(tA)\in\mathrm{cl}(\mathcal{A}_\Gamma)\ \ \forall t\geq 0\}.$$

Some useful properties of the Lie saturate are listed in the next proposition.

Proposition 1 (cf. [7]). *Let G be a connected Lie group and $\Gamma\subset\mathfrak{g}$ a right-invariant control system. Then*

1. $\mathrm{LS}(\Gamma)$ *is topologically closed.*
2. $\mathrm{LS}(\Gamma)$ *is convex:*

$$X, Y \in \mathrm{LS}(\Gamma) \;\Rightarrow\; \alpha X + (1-\alpha)Y \in \mathrm{LS}(\Gamma) \;\; \forall \alpha \in [0,1].$$

3. $\mathrm{LS}(\Gamma)$ *is a positive cone:*

$$X \in \mathrm{LS}(\Gamma) \;\Rightarrow\; \alpha X \in \mathrm{LS}(\Gamma) \;\; \forall \alpha \geq 0.$$

Thus,

$$X, Y \in \mathrm{LS}(\Gamma) \;\Rightarrow\; \alpha X + \beta Y \in \mathrm{LS}(\Gamma) \;\; \forall \alpha, \beta \geq 0.$$

4. *For any* $\pm X, Y \in \mathrm{LS}(\Gamma)$ *and any* $t \in \mathbb{R}$,

$$\exp(t\,\mathrm{ad}X)Y = \exp(tX)\,Y\exp(-tX) \in \mathrm{LS}(\Gamma).$$

5. $\pm X, \pm Y \in \mathrm{LS}(\Gamma)$ *implies that* $\pm[X, Y] \in \mathrm{LS}(\Gamma)$.

The next theorem provides a central criterion that will be instrumental in proving controllability in the specific settings considered here.

Theorem 1. *A right-invariant control system* $\Gamma \subset \mathfrak{g}$ *is controllable on a connected Lie group G if and only if* $\mathrm{LS}(\Gamma) = \mathfrak{g}$.

Proof. See [1], Proposition 2. □

A graph-theoretic characterization of matrix irreducibility will play a key role in our analysis, as it provides a simple and intuitive way to translate algebraic coupling properties into combinatorial connectivity conditions. In what follows, we briefly recall this characterization. The matrices under consideration may have real, complex, or even quaternionic entries; however, for the purpose of irreducibility, the nature of the entries is immaterial. What matters is solely the 0 or non-0 pattern, which determines the presence or absence of arcs in the associated directed graph.

Definition 2. *An* $n \times n$ *matrix* A $(n \geq 2)$*, whose entries are real, complex, or quaternionic, is called* ***reducible*** *if there exists an* $n \times n$ *permutation matrix* P *such that*

$$PAP^t = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} *is an* $r \times r$ *block matrix, with* $1 \leq r < n$*. Otherwise, A is said to be* ***irreducible****. If* $n = 1$*, then A is reducible if its single entry is* 0*, and irreducible otherwise.*

Definition 3. *A* ***finite directed graph*** *associated to the* $n \times n$ *matrix* $A = (a_{ij})$ *is a set of n points in the plane, say* $P_1, \cdots, P_n$ *(called* ***nodes****), together with directed arcs connecting these points in such a way that point* P_i *is joined to point* P_j *by means of a line directed from* P_i *to* P_j *if and only if* $a_{ij} \neq 0$.

A directed path from node P_i to node P_j in the graph is simply a collection of concatenated directed arcs starting at P_i and ending at P_j, that is,

$$\overrightarrow{P_i P_{l_1}}, \;\; \overrightarrow{P_{l_1} P_{l_2}}, \cdots, \overrightarrow{P_{l_{r-1}} P_{l_r}}, \;\; \text{with } l_r = j.$$

Definition 4. *A directed graph is said to be* ***strongly connected*** *if for every ordered pair of nodes P_i and P_j there exists a directed path connecting P_i to P_j.*

The following simple result characterizes completely the irreducibility of a given square matrix A via the strongly connectedness of its directed graph. For completeness we prove it here, although a more detailed approach and further applications can be found in [8].

Theorem 2 (cf. [8], Theorem 1.6)**.** *A square matrix A of order n is irreducible if and only if its directed graph $g(A)$ is strongly connected.*

Proof. Suppose that A is reducible; then there exist $\tilde{A}$ and a permutation matrix P such that

$$A = P\tilde{A}P^t = P\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}P^t,$$

where A_{11} is of order r and A_{22} is of order $(n-r)$, for some $1 \leq r < n$. Set $v_1, \ldots, v_n$ for the nodes of $g(A)$ and $\tilde{v}_1, \ldots, \tilde{v}_n$ for the nodes of $g(\tilde{A})$. In $g(\tilde{A})$, observe that there does not exist a directed path from $\tilde{v}_i$ to $\tilde{v}_j$ if $r < i \leq n$ and $1 \leq j \leq r$. Hence $g(\tilde{A})$ is not strongly connected. As the directed graph of $\tilde{A}$ is obtained from that of A just by renumbering the nodes and this operation does not affect the connectedness of the directed graph, it follows that $g(A)$ is not strongly connected.

Conversely, assume that the graph $g(A)$ is not strongly connected. Then there exist nonempty subsets of vertices S_1 and S_2 in $g(A)$ such that no directed path connects $v_i \in S_2$ to $v_j \in S_1$. Denote $|S_1| = r$ and $|S_2| = n - r$. Relabel the vertices of $g(A)$ as $\tilde{v}_1, \ldots, \tilde{v}_n$ in such a way that $\tilde{v}_1, \ldots, \tilde{v}_r$ belong to S_1, and $\tilde{v}_{r+1}, \ldots, \tilde{v}_n$ belong to S_2. Apply the same permutation to the rows and columns of A to obtain a new matrix $\tilde{A}$. The graph associated with $\tilde{A}$ is exactly the relabeled version of $g(A)$. Since there are no directed paths in $g(\tilde{A})$ leading from vertices of S_2 to those of S_1, the matrix $\tilde{A}$ necessarily contains a zero submatrix of dimension $(n-r) \times r$ in its lower-left corner. Hence, $\tilde{A}$ is reducible. Because A and $\tilde{A}$ are related by a permutation similarity, it follows that A is also reducible. □

3. Controllability on $\mathbf{Sl}(n, \mathbb{C})$

In [19], the authors refine the main theorem of [1] by establishing new sufficient conditions for controllability on connected, simple, complex Lie groups. Their results improve earlier criteria by relaxing certain structural assumptions, thereby broadening the range of systems to which the theory applies. In the present section, we specialize our analysis to control systems evolving on the complex special linear group $\mathrm{Sl}(n, \mathbb{C})$. For this setting, we identify natural spectral and structural conditions under which it is possible to formulate *necessary and sufficient* controllability criteria for bilinear systems of the form $\Gamma = A + uB$. This represents a stronger result than the purely sufficient conditions obtained in [19], as it fully characterizes controllability in the specified framework. Our proof strategy is inspired by, and closely follows, the methodology developed in [15], where strong regularity properties of the drift term and graph-theoretic irreducibility conditions on the control term are combined with Lie saturate techniques to establish controllability. In adapting these ideas to $\mathrm{Sl}(n, \mathbb{C})$, we not only recover the essence of the approach in [15] but also extend it to yield a complete characterization of controllability that is verifiable through purely algebraic and combinatorial checks.

For the canonical basis of $(n, \mathbb{C})$ as a $\mathbb{C}$-vector space we take the set $\mathcal{B}$ formed by the matrices E_{ij}, $i \neq j$, $1 \leq i, j \leq n$, and $E_{ii} - E_{11}$, $1 < i \leq n$. Here, E_{ij} stands for the matrix which has 1 in the ij-entry and 0 elsewhere. Given $A \in (n, \mathbb{C})$ we denote by ${}^jA^i$ the matrix

which the only nonzero column is the j-th, and this column is exactly the i-th column of A. In a similar way, ${}_jA_i$ stands for the matrix having all lines zero, except for the j-th line, with this line being precisely the i-th line of A.

It is easy to see that $E_{ij}A = {}_iA_j$, $AE_{ij} = {}^jA^i$ and $[E_{ij}, A] = {}_iA_j - {}^jA^i$.

Definition 5 (cf. [1]). *A matrix $B \in (n, \mathbb{C})$ is said to be **strongly regular** if its eigenvalues $\lambda_1, \ldots, \lambda_n$ are all distinct and the real parts of the eigenvalues satisfy*

$$\mathrm{Re}(\lambda_p) - \mathrm{Re}(\lambda_q) \neq \mathrm{Re}(\lambda_s) - \mathrm{Re}(\lambda_t), \ \ \forall\ p, q, s, t \text{ such that } \{p, q\} \neq \{s, t\}.$$

Through section $B = \mathrm{diag}(b_1, b_2, \cdots, b_n)$ will indicate a fixed strongly regular element where $b_i \in \mathbb{C}$ for $i = 1, 2, \ldots, n$, and $\mathrm{Re}(b_p) - \mathrm{Re}(b_q) \neq \mathrm{Re}(b_s) - \mathrm{Re}(b_t)$, for all p, q, s, t such that $\{p, q\} \neq \{s, t\}$. We should propose the following additional hypotheses on B:

$$\mathrm{Im}(b_i) - \mathrm{Im}(b_j) \neq 0 \ \text{ whenever } i \neq j$$

and

$$\mathrm{Re}(b_1) > \mathrm{Re}(b_2) > \cdots > \mathrm{Re}(b_n).$$

Lemma 1. *If $B \in \mathrm{LS}(\Gamma)$, $\pm E_{ij} \in \mathrm{LS}(\Gamma)$ and $E_{jm} \in \mathrm{LS}(\Gamma)$, $i \neq j$, $j \neq m$, $i \neq m$, then*

$$\mathrm{span}_{\mathbb{C}}(E_{im}) \subset \mathrm{LS}(\Gamma).$$

Proof. We have that

$$\exp(-tE_{ij})E_{jm}\exp(tE_{ij}) = E_{jm} - tE_{im} \in \mathrm{LS}(\Gamma), \qquad \forall t \in \mathbb{R},$$

and this implies that

$$\lim_{t \to \pm\infty} \frac{1}{|t|}(E_{jm} - tE_{im}) = \pm E_{im} \in \mathrm{LS}(\Gamma).$$

Now,

$$\exp(tB)(\pm E_{im})\exp(-tB) = e^{t(b_i - b_m)}(\pm E_{im}) \in \mathrm{LS}(\Gamma), \qquad \forall t \in \mathbb{R}.$$

Choosing $t \in \mathbb{R}$ in a way that $e^{t(b_i - b_m)} \in \mathbb{R}$, we see that $\mathrm{span}_{\mathbb{R}}(E_{im}) \subset \mathrm{LS}(\Gamma)$. Similarly, taking $t \in \mathbb{R}$ such that $e^{t(b_i - b_m)}$ is purely imaginary, we get $\mathrm{span}_{\mathbb{R}}(\mathbf{i}E_{im})$. Putting these together, we conclude that $\mathrm{span}_{\mathbb{C}}(E_{im}) \subset \mathrm{LS}(\Gamma)$.

□

As in the previous lemma, if A and $\pm E_{ij}$ belong to $\mathrm{LS}(\Gamma)$, $i \neq j$, then $\pm({}^jA^i - {}_iA_j) \in \mathrm{LS}(\Gamma)$. A simple calculation with the curve $\exp(-tE_{ij})A\exp(tE_{ij})$, $t \in \mathbb{R}$, shows this fact.

Lemma 2. *If A, $\pm B$ and $\pm E_{ij}$ (with $i \neq j$) are elements in $\mathrm{LS}(\Gamma)$ and if c_{km} (with $k \neq m$) is a nonzero and nondiagonal entry in ${}^jA^i$ or in ${}_iA_j$, then $\mathrm{span}_{\mathbb{C}}(E_{km}) \subset \mathrm{LS}(\Gamma)$.*

Proof. According to the previous paragraph, we have $\pm({}^jA^i - {}_iA_j) \in \mathrm{LS}(\Gamma)$. Then, the curves

$$\pm\exp(tB)({}^jA^i - {}_iA_j)\exp(-tB)$$

are entirely contained ($t \in \mathbb{R}$) in the Lie saturate $\mathrm{LS}(\Gamma)$. The km-entries of these curves are given by

$$\pm({}^jA^i - {}_iA_j)_{km}e^{(b_k - b_m)t},$$

where $({}^jA^i - {}_iA_j)_{km}$ indicates the km-entry of the matrix ${}^jA^i - {}_iA_j$. Let k_0m_0 be the index corresponding to the maximum of the differences $\mathrm{Re}(b_k) - \mathrm{Re}(b_m)$ such that $({}^jA^i - {}_iA_j)_{km} \neq 0$. We can choose a sequence $t_N \to \infty$ in a way that $e^{t_N(b_{k_0} - b_{m_0})} \in \mathbb{R}$. Moreover, taking $t_N \in \mathbb{R}$ such that

$$\cos(t_N(\mathrm{Im}(b_{k_0}) - \mathrm{Im}(b_{m_0}))) = 1 \quad \text{and} \quad \sin(t_N(\mathrm{Im}(b_{k_0}) - \mathrm{Im}(b_{m_0}))) = 0,$$

we get

$$e^{t_N(b_{k_0} - b_{m_0})} = e^{t_N(\mathrm{Re}(b_{k_0}) - \mathrm{Re}(b_{m_0}))} \in \mathbb{R}.$$

This choice implies that

$$\pm \frac{1}{e^{t_N(b_{k_0} - b_{m_0})}} \exp(tB)({}^jA^i - {}_iA_j)\exp(-tB) \in \mathrm{LS}(\Gamma), \quad \forall N \in \mathbb{N}.$$

The nonzero entries of these two curves have the form

$$\pm c_{km} \frac{e^{t_N(b_k - b_m)}}{e^{t_N(b_{k_0} - b_{m_0})}} = \pm c_{km} \frac{e^{t_N(b_k - b_m)}}{e^{t_N(\mathrm{Re}(b_{k_0}) - \mathrm{Re}(b_{m_0}))}}.$$

Taking limits as $t_N \to \infty$, all these entries converge to 0, except the one corresponding to the k_0m_0-entry, which is exactly $c_{k_0m_0}$. In other words,

$$\lim_{t_N \to \infty} \frac{1}{e^{t_N(b_{k_0} - b_{m_0})}} \exp(tB)({}^jA^i - {}_iA_j)\exp(-tB) = c_{k_0m_0}E_{k_0m_0} \in \mathrm{LS}(\Gamma).$$

This implies that $\exp(tB)(c_{k_0m_0}E_{k_0m_0})\exp(-tB) \in \mathrm{LS}(\Gamma)$, for all $t \in \mathbb{R}$. But

$$\exp(tB)(c_{k_0m_0}E_{k_0m_0})\exp(-tB) = e^{t(b_{k_0} - b_{m_0})}c_{k_0m_0}E_{k_0m_0}.$$

We can make four different choices of $t \in \mathbb{R}$ to get

$$\cos(t_N(\mathrm{Im}(b_{k_0}) - \mathrm{Im}(b_{m_0}))) = \pm 1 \ \text{ and } \ \mathbf{i}\sin(t_N(\mathrm{Im}(b_{k_0}) - \mathrm{Im}(b_{m_0}))) = \pm \mathbf{i},$$

and this shows that $\mathrm{span}_{\mathbb{R}}(c_{k_0m_0}E_{k_0m_0}) \subset \mathrm{LS}(\Gamma)$ and $\mathrm{span}_{\mathbb{R}}(\mathbf{i}c_{k_0m_0}E_{k_0m_0}) \subset \mathrm{LS}(\Gamma)$, that is,

$$\mathrm{span}_{\mathbb{C}}(c_{k_0m_0}E_{k_0m_0}) = \mathrm{span}_{\mathbb{C}}(E_{k_0m_0}) \subset \mathrm{LS}(\Gamma).$$

We can then consider the element $({}^jA^i - {}_iA_j) - c_{k_0m_0}E_{k_0m_0}$ and iterate the preceding procedure. At this point the maximum over the numbers $\mathrm{Re}(b_k) - \mathrm{Re}(b_m)$ will have decreased and this iteration will successively show that all the needed inclusions are true. □

We can now state a controllability theorem on $(n, \mathbb{C})$.

Theorem 3. *Let $B = \mathrm{diag}(b_1, b_2, \cdots, b_n) \in (n, \mathbb{C})$ be a strongly regular element satisfying*

$$\mathrm{Re}(b_1) > \mathrm{Re}(b_2) > \cdots > \mathrm{Re}(b_n) \ \text{ and } \ \mathrm{Im}(b_i) - \mathrm{Im}(b_j) \neq 0 \ \text{ whenever } i \neq j.$$

If $A \in (n, \mathbb{C})$ is such that $a_{1n} \neq 0 \neq a_{n1}$, then the invariant control system $\Gamma = A + uB$ is controllable on $\mathrm{Sl}(n, \mathbb{C})$ if and only if A is irreducible.

Proof. At first we prove that the irreducibility of A is a necessary condition for controllability. To see this we suppose that A is reducible and show that the system Γ is not controllable. If the matrix A is reducible, then there exists a permutation matrix P such that

$$PAP^t = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}. \tag{1}$$

Being B a diagonal matrix, it is easy to see that any permutation matrix leaves B in the diagonal form (by conjugation as above), and this means that both A and B are conjugate to matrices of the form (1). The set formed by the matrices having the form (1) is a Lie subalgebra of $(n, \mathbb{C})$. Consequently, $\mathrm{Lie}(\Gamma) \subsetneq (n, \mathbb{C})$ and this implies that $\mathrm{LS}(\Gamma) \neq (n, \mathbb{C})$.

Now we prove the sufficiency; in other words, we show that the system Γ under the conditions imposed on A and B is controllable.

In this way, our first aim is to show that $\pm E_{1n} \in \mathrm{LS}(\Gamma)$ and $\pm E_{n1} \in \mathrm{LS}(\Gamma)$.

As $\mathrm{LS}(\Gamma)$ is a closed convex positive cone, we have $\exp(t\,B)A\exp(t\,B) \in \mathrm{LS}(\Gamma)$ for all $t \in \mathbb{R}$. Taking $(t_N)_{N\in\mathbb{N}} \subset \mathbb{R}$ as a sequence such that $e^{\pm t_N(b_n - b_1)} \in \mathbb{R}$, simple calculations together with limits show us that

$$e^{t_N(b_n-b_1)} \in \mathbb{R}\exp(t\,B)A\exp(t\,B) \in \mathrm{LS}(\Gamma) \longrightarrow a_{1n}E_{n1} = A_{n1} \in \mathrm{LS}(\Gamma)$$

and

$$e^{-t_N(b_n-b_1)} \in \mathbb{R}\exp(t\,B)A\exp(t\,B) \in \mathrm{LS}(\Gamma) \longrightarrow a_{1n}E_{1n} = A_{1n} \in \mathrm{LS}(\Gamma).$$

Knowing that $A_{1n}, A_{n1} \in \mathrm{LS}(\Gamma)$, we have

$$\exp(t\,\mathrm{ad}(B))A_{1n} \in \mathrm{LS}(\Gamma) \quad \text{and} \quad \exp(t\,\mathrm{ad}(B))A_{n1} \in \mathrm{LS}(\Gamma),$$

for all $t \in \mathbb{R}$. But it is easy to see that

$$\exp(t\,\mathrm{ad}(B))A_{1n} = \exp(t\,B)A_{1n}\exp(-t\,B) = e^{t(b_1-b_n)}A_{1n}$$

and

$$\exp(t\,\mathrm{ad}(B))A_{n1} = \exp(t\,B)A_{n1}\exp(-t\,B) = e^{t(b_n-b_1)}A_{n1}.$$

Calling $M = \mathrm{Im}(b_1) - \mathrm{Im}(b_n)$ and letting $t = \frac{\pi}{M}$, we have

$$e^{t(b_1-b_n)} = e^{\frac{\pi(b_1-b_n)}{M}} = e^{\pi\frac{\mathrm{Re}(b_1-b_n)}{\mathrm{Im}(b_1-b_n)}}(\cos(\pi) + \mathbf{i}\sin(\pi)) = -e^{\pi\frac{\mathrm{Re}(b_1-b_n)}{\mathrm{Im}(b_1-b_n)}}$$

and

$$e^{t(b_n-b_1)} = e^{\frac{\pi(b_n-b_1)}{M}} = e^{\pi\frac{\mathrm{Re}(b_n-b_1)}{\mathrm{Im}(b_n-b_1)}}(\cos(\pi) + \mathbf{i}\sin(\pi)) = -e^{\pi\frac{\mathrm{Re}(b_n-b_1)}{\mathrm{Im}(b_n-b_1)}}.$$

This choice together with $t = 0$ shows us that $\mathrm{span}_{\mathbb{R}}\{A_{1n}, A_{n1}\} \subset \mathrm{LS}(\Gamma)$, but this is equivalent to

$$\mathrm{span}_{\mathbb{R}}\{E_{1n}, E_{n1}\} \subset \mathrm{LS}(\Gamma).$$

In the same fashion, choosing $t \in \mathbb{R}$ such that $\cos(t) = 0$ and $\sin(t) = \pm 1$, we get $\mathrm{span}_{\mathbb{R}}\{\mathbf{i}A_{1n}, \mathbf{i}A_{n1}\} \subset \mathrm{LS}(\Gamma)$. And these considerations allow us to conclude that

$$\mathrm{span}_{\mathbb{C}}\{E_{1n}\} \subset \mathrm{LS}(\Gamma) \quad \text{and} \quad \mathrm{span}_{\mathbb{C}}\{E_{n1}\} \subset \mathrm{LS}(\Gamma).$$

We proceed to prove that $\mathrm{LS}(\Gamma) = (n, \mathbb{C})$. With A being irreducible, Theorem 2 tells us that for any $m \neq n$ there exists a directed path on the graph of A from node P_n to node P_m. We can choose this path free of loops, that is, a path

$$\overrightarrow{P_n P_{i_1}},\ \overrightarrow{P_{i_1} P_{i_2}}, \cdots, \overrightarrow{P_{i_p} P_m}\ .$$

without repetition of index. In particular,

$$P_{i_k} \neq P_n, \quad P_{i_k} \neq P_m \quad \forall k = 1, 2, \ldots, p.$$

Further, we are interested in the case which this path does not pass through P_1. The only two obvious possibilities are:

- This path in fact does not pass through P_1 or
- $P_{i_k} = P_1$ for some k.

  If the second case occurs, taking the greatest k such that $P_{i_k} = P_1$, we get the path

$$\overrightarrow{P_1 P_{i_{k+1}}},\ \overrightarrow{P_{i_{k+1}} P_{i_{k+2}}}, \cdots, \overrightarrow{P_{i_p} P_m},$$

such that $P_{i_{k+h}} \neq P_1$, $P_{i_{k+h}} \neq P_m$ and $P_{i_{k+h}} \neq P_n$ for every h. In every case we get either

1. a path from node P_n to node P_m not passing through P_1
   or
2. a path from node P_1 to node P_m not passing through P_n.

Note that the former situation can be obtained from the latter just by interchanging 1 and n, and this would not affect the generality of our reasoning. Therefore we can assume that the path is from P_n to P_m not passing through P_1, and the arcs of this path correspond, respectively, to $a_{ni_1}, a_{i_1 i_2}, \ldots, a_{i_p m}$ (nonzero entries of A).

As $\pm E_{1n} \in \mathrm{LS}(\Gamma)$ and $\pm(a_{ni_1} E_{1i_1})_{1i_1}$ is a nonzero and non diagonal (because $i_k \neq 1$) entry in $\pm_1 A_n$, an application of Lemma 2 shows that $\mathrm{span}_{\mathbb{C}}\{E_{1i_1}\} \subset \mathrm{LS}(\Gamma)$. A simple iteration of this reasoning shows also that

$$\mathrm{span}_{\mathbb{C}}\{E_{1i_2}, \ldots, E_{1m}\} \subset \mathrm{LS}(\Gamma).$$

With $\pm E_{n1}$ being elements of $\mathrm{LS}(\Gamma)$, as $E_{1m} \in \mathrm{LS}(\Gamma)$, Lemma 1 shows us that $\mathrm{span}_{\mathbb{C}}\{E_{nm}\} \subset \mathrm{LS}(\Gamma)$.

We now take a path from node P_m to node P_n with corresponding nonzero and nondiagonal entries $a_{mj_1}, a_{j_1 j_2}, \ldots, a_{j_p n}$. We follow a procedure like the previous one but in the opposite sense; that is, starting by the last arc on the path, which corresponds to the entry $a_{j_p n}(E_{j_p 1})_{j_p 1}$ of ${}^1A^n$, we conclude that $\mathrm{span}_{\mathbb{C}}\{E_{j_p 1}\} \subset \mathrm{LS}(\Gamma)$. Then we follow with the iterations until the first arc on the path. At this point we have that $a_{mj_1}(E_{m1})_{m1}$ is a nonzero and nondiagonal entry on ${}^1A^{j_1}$; thus Lemma 2 implies that $\mathrm{span}_{\mathbb{C}}\{E_{m1}\} \subset \mathrm{LS}(\Gamma)$.

Since $\mathrm{span}_{\mathbb{C}}\{E_{m1}\} \subset \mathrm{LS}(\Gamma)$ and $\mathrm{span}_{\mathbb{C}}\{E_{1n}\} \subset \mathrm{LS}(\Gamma)$, Lemma 1 ensures that $\mathrm{span}_{\mathbb{C}}\{E_{mn}\} \subset \mathrm{LS}(\Gamma)$.

Let $k \neq 1, n, m$. Applying the same reasoning as before, we get

$$\mathrm{span}_{\mathbb{C}}\{E_{km}\} \subset \mathrm{LS}(\Gamma) \quad \text{and} \quad \mathrm{span}_{\mathbb{C}}\{E_{mk}\} \subset \mathrm{LS}(\Gamma).$$

This means that $\mathrm{span}_{\mathbb{C}}\{E_{km}\} \subset \mathrm{LS}(\Gamma)$ for all pairs of indices (k, m) with $k \neq m$, but these subspaces generate $(n, \mathbb{C})$ as a Lie algebra. Consequently, $\mathrm{LS}(\Gamma) = (n, \mathbb{C})$, and this implies that Γ is controllable on $\mathrm{Sl}(n, \mathbb{C})$. □

4. Controllability on $\mathbf{Sl}(n,\mathbb{H})$

The Lie algebra $\mathfrak{sl}(2,\mathbb{H})$ is the real simple Lie algebra formed by matrices having trace with zero real part and it complexifies to a complex Lie algebra isomorphic to $\mathfrak{sl}(4,\mathbb{C})$. It is a fifteen-dimensional Lie algebra whose basis vectors will be denoted as follows.

Set E_{ij} to be the matrix whose (k,m) entry is given by

$$(E_{ij})_{km} = \delta_{ik}\cdot\delta_{jm}, \quad \text{where} \quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and let E_{ij}^{q} denote the 2×2 matrix whose (k,m) entry is given by

$$(E_{ij}^{q})_{km} = q\cdot\delta_{ik}\cdot\delta_{jm}, \quad \text{where} \quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Then, the set $\mathcal{B}$ formed by the matrices

$$E_{22}-E_{11}, E_{12}, E_{21}, E_{12}^{\mathbf{i}}, E_{12}^{\mathbf{j}}, E_{12}^{\mathbf{k}}, E_{21}^{\mathbf{i}}, E_{21}^{\mathbf{j}}, E_{21}^{\mathbf{k}}, E_{11}^{\mathbf{i}}, E_{11}^{\mathbf{j}}, E_{11}^{\mathbf{k}}, E_{22}^{\mathbf{i}}, E_{22}^{\mathbf{j}}, E_{22}^{\mathbf{k}}$$

will be considered as the canonical basis of $\mathfrak{sl}(2,\mathbb{H})$ as a vector space over $\mathbb{R}$.

For sake of convenience, we reproduce here Lemma 4.3 and its Corollary 4.4 proved in [20].

Lemma 3. *Consider the torus $\mathbb{T}^2$ acting on the quaternions $\mathbb{H}$ by*

$$\phi((t,s),q) = e^{\mathbf{i}t}qe^{-\mathbf{i}s}.$$

Write $q = a+b$, with $a = x_1+x_2\mathbf{i}\in\mathbb{H}_{\{\mathbf{1},\mathbf{i}\}}$ and $b = x_3\mathbf{j}+x_4\mathbf{k}\in\mathbb{H}_{\{\mathbf{j},\mathbf{k}\}}$. Suppose that $q\notin\mathbb{H}_{\{\mathbf{1},\mathbf{i}\}}\cup\mathbb{H}_{\{\mathbf{j},\mathbf{k}\}}$, that is, $a\neq 0\neq b$. Then, the orbit $\mathbb{T}^2q$ is a two-dimensional torus and $\mathbb{H}$ is the convex cone generated by $\mathbb{T}^2q$.

Corollary 1. *Consider $c_1,c_2\in\mathbb{R}$ with $c_1c_2\neq 0$ and c_1/c_2 irrational. Take $q\in\mathbb{H}$ with $q\notin\mathbb{H}_{\{1,\mathbf{i}\}}\cup\mathbb{H}_{\{\mathbf{j},\mathbf{k}\}}$. Then $\mathbb{H}$ is the closed convex cone generated by the curve $e^{\mathbf{i}tc_1}qe^{-\mathbf{i}tc_2}$.*

With these results and notations in mind, we are able to state a controllability theorem that slightly improves Theorem 4.1 in [20] for the Lie group $\mathrm{Sl}(2,\mathbb{H})$, in the sense that under certain conditions, we get a necessary and sufficient condition for controllability.

Theorem 4. *Let $A,B\in\mathfrak{sl}(2,\mathbb{H})$ be matrices such that $B = \mathrm{diag}\{a+\mathbf{i}b_1, -a+\mathbf{i}b_2\}$ with $b_1b_2\neq 0$ and b_1/b_2 irrational. Denote the $(1,2)$ and $(2,1)$ entries of the matrix A by $p\in\mathbb{H}$ and $q\in\mathbb{H}$, respectively. Let $\mathbb{H}_{1,\mathbf{i}}$ and $\mathbb{H}_{\mathbf{j},\mathbf{k}}$ be the (real) subspaces of $\mathbb{H}$ spanned by $\{1,\mathbf{i}\}$ and $\{\mathbf{j},\mathbf{k}\}$, respectively, and suppose that p and q do not belong to $\mathbb{H}_{1,\mathbf{i}}\cup\mathbb{H}_{\mathbf{j},\mathbf{k}}$. Then the right-invariant control system with unrestricted controls $\Gamma = A+\mathbb{R}B$ is controllable on $\mathrm{Sl}(2,\mathbb{H})$ if and only if p and q are both nonzero quaternions.*

Proof. We start by proving the necessary condition. Since we are dealing with matrices of order 2, the condition $p\neq 0\neq q$ is equivalent to A being an irreducible matrix. So, if we suppose that A is reducible, there would exist a permutation matrix P such that

$$PAP^{T} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}. \tag{2}$$

As B is diagonal, it is easy to see that any permutation matrix leaves B in the diagonal form (by conjugation as above), and this means that both A and B are conjugate to matrices of the form (2). The set formed by the matrices with the form (2) is a Lie subalgebra of $(2,\mathbb{H})$. Consequently, $\mathrm{Lie}(\Gamma) \subsetneq (2,\mathbb{H})$ and this implies that $\mathrm{LS}(\Gamma) \neq (2,\mathbb{H})$.

By the other hand, to prove that Γ is controllable it suffices to show that $\mathrm{LS}(\Gamma) = \mathfrak{sl}(2,\mathbb{H})$. First of all, since $\mathrm{LS}(\Gamma)$ is a closed convex positive cone, then $A + tB \in \mathrm{LS}(\Gamma)$ implies that $\pm B \in \mathrm{LS}(\Gamma)$ as a consequence of the following convergences

$$\frac{A+tB}{|t|} \longrightarrow \pm B \text{ as } t \to \pm\infty.$$

Thus, $A, \pm B \in \mathrm{LS}(\Gamma)$ and we can apply Proposition 1 item (4) with $X = B$ and $Y = A$ to conclude that $\exp(t\,\mathrm{ad}B)A \in \mathrm{LS}(\Gamma)$ for all $t \in \mathbb{R}$.

Direct calculations shows us that

$$e^{-2at}\exp(t\,\mathrm{ad}B)A = \begin{bmatrix} e^{-2at}s & e^{\mathbf{i}t(b_1-b_2)}p \\ e^{-4at}e^{\mathbf{i}t(b_1-b_2)}q & e^{-2at}r \end{bmatrix}, \text{ where } A = \begin{bmatrix} s & p \\ q & r \end{bmatrix}$$

and the quaternions p and q are as in the statement of the theorem. It is clear that when $t \to \infty$, all the entries of this matrix converge to 0 except for the $(1,2)$-entry.

Choosing a sequence $t_k \to \infty$ such that $e^{\mathbf{i}t_k(b_1-b_2)} \to 1$, we conclude that

$$X = \begin{bmatrix} 0 & p \\ 0 & 0 \end{bmatrix} \in \mathrm{LS}(\Gamma).$$

Similarly, we have

$$e^{2at}\exp(t\,\mathrm{ad}B)A = \begin{bmatrix} e^{2at}s & e^{4at}e^{\mathbf{i}t(b_1-b_2)}p \\ e^{\mathbf{i}t(b_1-b_2)}q & e^{2at}r \end{bmatrix} \in \mathrm{LS}(\Gamma) \qquad \forall t \in \mathbb{R}$$

and taking limits as $t \to -\infty$, we get

$$Y = \begin{bmatrix} 0 & 0 \\ q & 0 \end{bmatrix} \in \mathrm{LS}(\Gamma).$$

Again by item (4) of Proposition 1, we have that

$$e^{-2at}\exp(tB)X\exp(tB) = \begin{bmatrix} 0 & e^{\mathbf{i}tb_1}pe^{-\mathbf{i}tb_2} \\ 0 & 0 \end{bmatrix} \in \mathrm{LS}(\Gamma)$$

and

$$e^{2at}\exp(tB)Y\exp(tB) = \begin{bmatrix} 0 & 0 \\ e^{\mathbf{i}tb_2}qe^{-\mathbf{i}tb_1} & 0 \end{bmatrix} \in \mathrm{LS}(\Gamma).$$

Applying Corollary 1 to these curves, we conclude that $\mathrm{span}_{\mathbb{H}}\{X\}$ and $\mathrm{span}_{\mathbb{H}}\{Y\}$ are contained in $\mathrm{LS}(\Gamma)$. In other words, this means that

$$\mathrm{span}_{\mathbb{R}}\{E_{12}, E_{21}, E_{12}^{\mathbf{i}}, E_{12}^{\mathbf{j}}, E_{12}^{\mathbf{k}}, E_{21}^{\mathbf{i}}, E_{21}^{\mathbf{j}}, E_{21}^{\mathbf{k}}\} \subset \mathrm{LS}(\Gamma).$$

Now, by item (5) of Proposition 1, we have that

$$\pm[E_{21}, E_{12}] = \pm E_{22} - E_{11} \in \mathrm{LS}(\Gamma).$$

Also,

$$\pm[E_{12}^{\mathbf{j}}, E_{21}^{\mathbf{k}}] = \pm \begin{bmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{bmatrix} \in \mathrm{LS}(\Gamma) \text{ and } \pm [E_{21}, E_{12}^{\mathbf{k}}] = \pm \begin{bmatrix} -\mathbf{i} & 0 \\ 0 & \mathbf{i} \end{bmatrix} \in \mathrm{LS}(\Gamma).$$

Adding these terms together will give us

$$\pm E_{11}^{\mathbf{i}} \in \mathrm{LS}(\Gamma) \text{ and } \pm E_{22}^{\mathbf{i}} \in \mathrm{LS}(\Gamma).$$

And we go on with similar computations:

$$\pm[E_{12}^{\mathbf{i}}, E_{21}^{\mathbf{k}}] = \pm \begin{bmatrix} -\mathbf{j} & 0 \\ 0 & -\mathbf{j} \end{bmatrix} \in \mathrm{LS}(\Gamma) \text{ and } \pm [E_{21}, E_{12}^{\mathbf{j}}] = \pm \begin{bmatrix} -\mathbf{j} & 0 \\ 0 & \mathbf{j} \end{bmatrix} \in \mathrm{LS}(\Gamma)$$

implies that

$$\pm E_{11}^{\mathbf{j}} \in \mathrm{LS}(\Gamma) \text{ and } \pm E_{22}^{\mathbf{j}} \in \mathrm{LS}(\Gamma).$$

Finally,

$$\pm[E_{12}^{\mathbf{i}}, E_{21}^{\mathbf{j}}] = \pm \begin{bmatrix} \mathbf{k} & 0 \\ 0 & \mathbf{k} \end{bmatrix} \in \mathrm{LS}(\Gamma) \text{ and } \pm [E_{21}, E_{12}^{\mathbf{k}}] = \pm \begin{bmatrix} -\mathbf{k} & 0 \\ 0 & \mathbf{k} \end{bmatrix} \in \mathrm{LS}(\Gamma),$$

which leads us to conclude that

$$\pm E_{11}^{\mathbf{k}} \in \mathrm{LS}(\Gamma) \text{ and } \pm E_{22}^{\mathbf{k}} \in \mathrm{LS}(\Gamma).$$

Consequently, $\pm\mathcal{B} \subset \mathrm{LS}(\Gamma)$, proving $\mathrm{LS}(\Gamma) = \mathfrak{sl}(2, \mathbb{H})$. □

Using Theorem 1, we also can state a similar theorem about the controllability of $\Gamma = A + \mathbb{R}B$, for specific $A, B \in (n, \mathbb{H})$. The reader should note that it is a weaker version of Theorem 4.1 in [20], because even if the Lie algebra rank condition is not asked, it follows immediately from the equality $\mathrm{LS}(\Gamma) = (n, \mathbb{H})$. However, if we keep in mind that the Lie algebra rank condition is usually hard to check even with the help of computational devices, the following theorem turns out to be interesting on its own, since all of its hypotheses can be quite readily verified.

Theorem 5. *Let $A, B \in \mathfrak{sl}(n, \mathbb{H})$ be matrices satisfying the following conditions*

1. $B = \mathrm{diag}\{\alpha_1 + \mathrm{i}\beta_1, \ldots, \alpha_n + \mathrm{i}\beta_n\}$ *with*
   - *(i)* $\alpha_1 > \alpha_2 > \cdots > \alpha_{n-1} > \alpha_n$;
   - *(ii)* $\alpha_i - \alpha_j \neq \alpha_k - \alpha_m$ *if* $(i, j) \neq (k, m)$.
   - *(iii)* $\beta_i \neq 0$ *for* $i = 1, \ldots, n$;
   - *(iv)* $\beta_i / \beta_j \notin \mathbb{Q}$ *if* $i \neq j$.
2. *Denote by a_{ij} the entries of the matrix A. Let $\mathbb{H}_{1,\mathbf{i}}$ and $\mathbb{H}_{\mathbf{j},\mathbf{k}}$ be the (real) subspaces of $\mathbb{H}$ spanned by $\{1, \mathbf{i}\}$ and $\{\mathbf{j}, \mathbf{k}\}$ respectively. Then*
   - *(i)* $a_{i1}, a_{1i} \notin \mathbb{H}_{1,\mathbf{i}} \cup \mathbb{H}_{\mathbf{j},\mathbf{k}}$ *for* $i = 2, \ldots, n$;
   - *(ii)* $a_{in}, a_{ni} \notin \mathbb{H}_{1,\mathbf{i}} \cup \mathbb{H}_{\mathbf{j},\mathbf{k}}$ *for* $i = 1, \ldots, n-1$.

Under these conditions, the bilinear control system $\Gamma = A + \mathbb{R}B$ is controllable.

Proof. As $A + uB \in \mathfrak{c}(S)$ for all $u \in \mathbb{R}$, it immediately follows that $A \in \mathrm{LS}(\Gamma)$, and if $u \neq 0$, then

$$\frac{1}{|u|}A + \frac{u}{|u|}B = \frac{1}{|u|}(A + uB) \in \mathrm{LS}(\Gamma).$$

Taking limits as $u \to \pm\infty$, we see that $\pm B \in \mathrm{LS}(\Gamma)$. It follows that $e^{t\mathrm{ad}(B)}A \in \mathrm{LS}(\Gamma)$ and hence

$$e^{-t(\alpha_1-\alpha_n)}e^{t\mathrm{ad}(B)}A \in \mathrm{LS}(\Gamma)$$

for all $t \in \mathbb{R}$, where $\alpha_1, \ldots, \alpha_n$ are the real parts of the entries of B. Now by assumption $\alpha_1 > \alpha_2 > \cdots > \alpha_n$ so that as $t \to +\infty$, the entries $e^{-t(\alpha_1-\alpha_n)}e^{t\mathrm{ad}(B)}A$ converge to 0 except for the $(1,n)$-entry. The $(1,n)$-entry of $e^{-t(\alpha_1-\alpha_n)}e^{t\mathrm{ad}(B)}A$ is $e^{\mathbf{i}t(\beta_1-\beta_n)}p$, where p is as in the statement of the theorem and $\beta_1, \ldots, \beta_n$ are the imaginary parts of the entries of B. Choosing a sequence $t_k \to +\infty$ such that $e^{\mathbf{i}t(\beta_1-\beta_n)} \to 1$, we conclude that

$$X = \begin{bmatrix} 0 & \cdots & p \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \in \mathrm{LS}(\Gamma).$$

Using again the properties of the Lie saturate, we have that for all $t, s \in \mathbb{R}$,

$$e^{-t(\alpha_1-\alpha_n)}e^{t\mathrm{ad}(B)}X = \begin{bmatrix} 0 & \cdots & e^{\mathbf{i}t\beta_1}pe^{-\mathbf{i}t\beta_n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \in \mathrm{LS}(\Gamma).$$

Applying Corollary 1 to this curve, it follows that the subspace $\mathrm{span}_{\mathbb{H}}\{X\}$ is contained in $\mathrm{LS}(\Gamma)$. By taking limits as $t \to -\infty$ of $e^{-t(\alpha_1-\alpha_n)}e^{t\mathrm{ad}(B)}A$, it follows that

$$Y = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ q & \cdots & 0 \end{bmatrix} \in \mathrm{LS}(\Gamma),$$

and applying again Corollary 1, we conclude that $\mathrm{span}_{\mathbb{H}}\{Y\} \subset \mathrm{LS}(\Gamma)$.

Claim 1. $\pm({}^nA^1 - {}_1A_n), \pm({}^1A^n - {}_nA_1) \in \mathrm{LS}(\Gamma)$.

Proof of Claim 1. In fact, since $\mathrm{span}_{\mathbb{H}}\{X\} \subset \mathrm{LS}(\Gamma)$, we have that $\pm E_{1n} \in \mathrm{LS}(\Gamma)$; this means that

$$\alpha(t) = \exp(-tE_{1n})A\exp(tE_{1n}) \in \mathrm{LS}(\Gamma), \quad \forall t \in \mathbb{R}.$$

But

$$\begin{aligned} \alpha(t) &= (Id - tE_{1n})A(Id + tE_{1n}) \\ &= A + tAE_{1n} - tE_{1n}A - t^2E_{1n}AE_{1n} \\ &= A + t({}^nA^1) - t({}_1A_n) - t^2a_{nn}E_{1n}. \end{aligned}$$

Again, as $\mathrm{span}_{\mathbb{H}}\{X\} \subset \mathrm{LS}(\Gamma)$, we have that $\beta(t) := \alpha(t) + t^2a_{nn}E_{1n} \in \mathrm{LS}(\Gamma)$ for all $t \in \mathbb{R}$. Thus

$$\frac{1}{|t|}\beta(t) = \frac{1}{|t|}A + \frac{t}{|t|}({}^nA^1 - {}_1A_n) \in \mathrm{LS}(\Gamma), \quad \forall t \in \mathbb{R}.$$

By taking limits as $t \to \pm\infty$, we get that $\pm({}^nA^1 - {}_1A_n) \in \mathrm{LS}(\Gamma)$. The same proof as above with E_{n1} rather than E_{1n} shows us that $\pm({}^1A^n - {}_nA_1) \in \mathrm{LS}(\Gamma)$. □

Claim 2. $\mathrm{span}_{\mathbb{H}}\{E_{1i}, E_{ni}, E_{i1}, E_{in}\} \subset \mathrm{LS}(\Gamma)$ *for* $i = 2, \ldots, n-1$.

Proof of Claim 2. In fact, we already know that $\text{span}_{\mathbb{H}}\{E_{1n}\} \subset \text{LS}(\Gamma)$. So, let us consider the matrix $C := ({}^nA^1 - {}_1A_n) - (a_{11} - a_{nn})E_{1n}$ which has null $(1,n)$-entry. Then the curves

$$\gamma_1(t) = \pm \exp(t\text{ad}B)C \quad (t \in \mathbb{R}),$$

are contained in $\text{LS}(\Gamma)$. Further, the (k,m)-entries of these curves are given by

$$\gamma_1(t)_{km} = \pm e^{t(\alpha_k - \alpha_m)} e^{\mathbf{i}t(\beta_k - \beta_m)} c_{km}.$$

Let (k_0, m_0) be a nondiagonal entry such that $\alpha_{k_0} - \alpha_{m_0}$ is the maximum (or minimum) of the differences $\alpha_k - \alpha_m$ such that $c_{km} \neq 0$. Thus, as $t \to \infty$ $(-\infty)$, all the entries of $e^{-t(\alpha_{k_0} - \alpha_{m_0})}\gamma(t)$ tend to 0 except the (k_0, m_0)-entry, which is given by $e^{\mathbf{i}t(\beta_{k_0} - \beta_{m_0})} c_{k_0 m_0}$. Taking a sequence $t_l \to \infty$ such that $e^{\mathbf{i}t_l(\beta_{k_0} - \beta_{m_0})} \to 1$, we obtain that $c_{k_0 m_0} E_{k_0 m_0} \in \text{LS}(\Gamma)$. Recalling that $c_{k_0 m_0}$ is one of that terms listed in Condition 2 of the theorem, we can apply Corollary 1 and conclude that $\text{span}_{\mathbb{H}}\{c_{k_0 m_0} E_{k_0 m_0}\} \subset \text{LS}(\Gamma)$. That is,

$$\text{span}_{\mathbb{H}}\{E_{k_0 m_0}\} \subset \text{LS}(\Gamma).$$

Now, consider the matrix $C_1 := C - c_{k_0 m_0} E_{k_0 m_0}$. We can iterate the reasoning just made until we get

$$\text{span}_{\mathbb{H}}\{E_{12}, E_{13}, \ldots, E_{1(n-1)}, E_{2n}, E_{3n}, \ldots, E_{(n-1)n}\} \subset \text{LS}(\Gamma).$$

In the same fashion, we define the matrix $D = ({}^1A^n - {}_nA_1) - (a_{nn} - a_{11})$ and the curves

$$\gamma_2(t) = \pm \exp(t\text{ad}B)D \quad (t \in \mathbb{R}),$$

and proceed in a similar way as before to conclude that

$$\text{span}_{\mathbb{H}}\{E_{21}, E_{31}, \ldots, E_{(n-1)1}, E_{n2}, E_{n3}, \ldots, E_{n(n-1)}\} \subset \text{LS}(\Gamma).$$

□

We turn now our attention to the diagonal matrices. Direct computations show us that

$$\pm[E_{1i}, E_{i1}] = \pm(E_{11} - E_{ii}) \in \text{LS}(\Gamma), \quad i = 2, 3, \ldots, n.$$

We also have that

$$\pm[E_{1i}^{\mathbf{j}}, E_{i1}^{\mathbf{k}}] = \pm(E_{11}^{\mathbf{i}} + E_{ii}^{\mathbf{i}}) \in \text{LS}(\Gamma) \quad i = 1, 2, \ldots, n,$$

and

$$\pm[E_{1i}^{\mathbf{i}}, E_{i1}] = \pm(E_{11}^{\mathbf{i}} - E_{ii}^{\mathbf{i}}) \in \text{LS}(\Gamma) \quad i = 1, 2, \ldots, n,$$

which implies that $\pm E_{ii}^{\mathbf{i}} \in \text{LS}(\Gamma)$ for $i = 1, 2, \ldots, n$. Interchanging the roles of $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ in the above calculations, we get

$$\pm E_{ii}^{\mathbf{i}}, \pm E_{ii}^{\mathbf{j}}, \pm E_{ii}^{\mathbf{k}} \in \text{LS}(\Gamma), \quad i = 1, 2, \ldots, n.$$

Finally, we can prove that the remaining matrices are in $\text{LS}(\Gamma)$ by the following relations. For $i, j \neq 1$, $i, j \neq n$, and $i \neq j$,

$$\pm[E_{i1}, E_{1j}] = \pm E_{ij},$$

$$\pm[E_{i1}, E_{1j}^{\mathbf{i}}] = \pm E_{ij}^{\mathbf{i}}, \quad \pm[E_{i1}, E_{1j}^{\mathbf{j}}] = \pm E_{ij}^{\mathbf{j}}, \quad \pm[E_{i1}, E_{1j}^{\mathbf{k}}] = \pm E_{ij}^{\mathbf{k}}.$$

Consequently, $\mathrm{LS}(\Gamma) = \mathfrak{sl}(n, \mathbb{H})$, proving that the bilinear control system Γ is controllable on $\mathrm{Sl}(n, \mathbb{H})$. This concludes the proof of Theorem 5.

□

5. Conclusions

We investigated controllability of right-invariant bilinear systems on the groups $\mathrm{Sl}(n, \mathbb{C})$ and $\mathrm{Sl}(n, \mathbb{H})$ using the *Lie saturate* $\mathrm{LS}(\Gamma)$. This approach avoids the full Lie algebra computation by exploiting convexity, adjoint invariance, and closure properties.

For $\mathrm{Sl}(n, \mathbb{C})$ with a strongly regular diagonal control matrix, we proved that controllability is equivalent to the irreducibility of the drift A, verifiable through strong connectivity of its associated graph, in line with recent graph-based controllability results for bilinear systems over Lie groups [10]. For $\mathrm{Sl}(n, \mathbb{H})$, we derived explicit conditions based on quaternionic entries and $\mathbb{T}^2$-orbit convexity; in the 2×2 case, these conditions are necessary and sufficient, while for general n, they provide practical sufficient tests.

The results illustrate two complementary aspects of the *Lie saturate* framework: the *algebraic* criteria are reflected in matrix-irreducibility and graph-connectivity conditions, ensuring the generation of the full Lie algebra, while the *geometric* criteria arise from convexity arguments and torus-orbit structures that characterize reachable sets and invariance properties on homogeneous spaces.

Compared with recent studies on controllability of networked and structured systems [10,11], the present work extends these ideas to complex and quaternionic Lie groups, providing explicit constructive tests within a unified theoretical framework. Further progress in the analysis of quaternionic control systems and applications of the Lie saturate to other Lie groups continues to be made, and the methods introduced here may serve as a basis for such developments.

Overall, the Lie saturate framework unifies algebraic and geometric perspectives, yielding constructive controllability certificates that scale with dimension. Future developments may further relate these results to graph-theoretic controllability characterizations on structured networks and extend them to other real forms and nonlinear invariant systems.

6. Future Work

Several directions arise from this study:

- **Spectral relaxation.** Extend the results to drifts with resonant or clustered eigenvalues, and analyze robustness margins near resonance.
- **Other Lie groups.** Apply the Lie saturate method to real forms such as $\mathrm{Sp}(2n, \mathbb{R})$ and $\mathrm{SO}(p, q)$, using Cartan decompositions and flag-manifold techniques.
- **Semi-direct product Lie groups.** Consider an affine control systems on $\mathbb{R}^d$ of the form

$$\dot{x}(t) = A_0 x(t) + a_0 + \sum_{i=1}^{m} u_i(t)[A_i x(t) + a_i],\ u \in \mathcal{U}, \tag{3}$$

where $A_i \in \mathbb{R}^{d \times d}, a_i \in \mathbb{R}^d, i \in \{0, 1, \ldots, m\}$ and the controls u are in

$$\mathcal{U} := \{u \in L^\infty(\mathbb{R}, \mathbb{R}^m) | u(t) \in \mathbb{R}^m\}. \tag{4}$$

Note the intuition that the bounded inhomogeneous term should become less relevant for states far from the origin. In this direction, ref. [21] showed that the controllability of an affine control system depends only on the controllability of its homogeneous (bilinear) part. Within this context, the first question is under which conditions the Lie saturate of the affine control system depends solely on its homogeneous part. A

second question is how to demonstrate, using the Lie saturate approach, that the controllability of the affine control system indeed depends only on the controllability of its homogeneous part.

- **Quaternionic case.** Strengthen the sufficiency conditions for $\mathrm{Sl}(n, \mathbb{H})$ to obtain necessary and sufficient criteria for general n.
- **Applications.** Develop algorithmic tests for controllability in quantum systems, robotics, and rigid-body dynamics, particularly in settings where sparse patterns and symmetries constrain the system dynamics. In this direction, exploring connections between controllability and graph-theoretic structures, such as the zero-forcing approach considered in [22], may yield new insights into how network topology influences the ability to steer complex or structured systems. Such investigations could also provide a bridge between algebraic methods, like the Lie saturate framework, and combinatorial techniques for analyzing controllability in networked dynamical systems.

These problems highlight the potential of the Lie saturate framework as both a theoretical tool and a basis for practical controllability algorithms.

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