

A Comprehensive Review on the Generalized Sylvester Equation $AX - YB = C$

Qing-Wen Wang ^{1,2,*}  and Jiale Gao ¹

¹ Department of Mathematics, Shanghai University, Shanghai 200444, China; jiale_gao819su@shu.edu.cn

² Collaborative Innovation Center for the Marine Artificial Intelligence, Shanghai 200444, China

* Correspondence: wqw@t.shu.edu.cn

Abstract

Since Roth's work on the generalized Sylvester equation (GSE) $AX - YB = C$ in 1952, related research has consistently attracted significant attention. Building on this, this review systematically summarizes relevant research on GSE from five perspectives: research methods, constrained solutions, various generalizations, iterative algorithms, and applications. Furthermore, we provide comments on current research, put forward several intriguing questions, and offer prospects for future research trends. We hope this work can fill the gap in the review literature on GSE and offer some inspiration for subsequent studies in the field.

Keywords: generalized Sylvester equation; Moore–Penrose inverse; matrix decompositions; semi-tensor product; constrained solutions; operator equations; tensor equations; polynomial matrix equations; iterative algorithms; hand–eye calibration; encryption and decryption schemes

MSC: 15A03; 15A09; 15A24; 15B33; 15B57; 65F10; 65F45



Academic Editor: Calogero Vetro

Received: 12 August 2025

Revised: 14 September 2025

Accepted: 18 September 2025

Published: 8 October 2025

Citation: Wang, Q.-W.; Gao, J. A Comprehensive Review on the Generalized Sylvester Equation $AX - YB = C$. *Symmetry* **2025**, *17*, 1686. <https://doi.org/10.3390/sym17101686>

Copyright: © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

Contents

1. Introduction	3
2. Preliminaries	4
3. Roth's Equivalence Theorem	6
4. Different Methods on GSE	7
4.1. Method by Linear Transformations and Subspace Dimensions	7
4.2. Method by Generalized Inverses	8
4.3. Method by Singular Value Decompositions	10
4.4. Method by Simultaneous Decompositions	12
4.5. Method by Real (Complex) Representations	13
4.6. Method by Determinable Representations	15
4.7. Method by Semi-Tensor Products	20
5. Constrained Solutions of GSE	24
5.1. Chebyshev Solutions and l_p -Solutions	25
5.2. $\star$ -Congruent Solutions	26

5.3. (Minimum-Norm Least-Squares) Symmetric Solutions	27
5.4. Self-Adjoint and Positive (Semi)Definite Solutions	28
5.5. Per(Skew)Symmetric and Bi(Skew)Symmetric Solutions	29
5.6. Maximal and Minimal Ranks of the General Solution	31
5.7. Re-(Non)negative and Re-(Non)positive Definite Solutions	32
5.8. η -Hermitian and η -Skew-Hermitian Solutions	35
5.9. ϕ -Hermitian Solutions	37
5.10. Equality-Constrained Solutions	38
6. Various Generalizations of GSE	40
6.1. Generalizing RET over Different Rings	40
6.1.1. Generalizing RET over Unit Regular Rings	40
6.1.2. Generalizing RET over Principal Ideal Domains	41
6.1.3. Generalizing RET over Division and Module-Finite Rings	42
6.1.4. Generalizing RET over Commutative Rings	43
6.1.5. Generalizing RET over Artinian and Noncommutative Rings	44
6.2. Generalizing RET to a Rank Minimization Problem	45
6.3. GSE over Dual Numbers and Dual Quaternions	46
6.4. Linear Operator Equations on Hilbert Spaces	49
6.5. Tensor Equations	52
6.6. Polynomial Matrix Equations	56
6.6.1. By the Divisibility of Polynomials	56
6.6.2. By Skew-Prime Polynomial Matrices	57
6.6.3. By the Realization of Matrix Fraction Descriptions	58
6.6.4. By the Unilateral Polynomial Matrix Equation	59
6.6.5. By the Equivalence of Block Polynomial Matrices	61
6.6.6. By Jordan Systems of Polynomial Matrices	62
6.6.7. By Linear Matrix Equations	63
6.6.8. By Root Functions of Polynomial Matrices	65
6.7. Sylvester-Polynomial-Conjugate Matrix Equations	66
6.8. Generalized Forms of GSE	70
7. Iterative Algorithms	75
8. Applications to GSE	83
8.1. Theoretical Applications	84
8.1.1. Solvability of Matrix Equations	84
8.1.2. UTV Decomposition of Dual Matrices	85
8.1.3. Microlocal Triangularization of Pseudo-Differential Systems	85
8.2. Practical Applications	86
8.2.1. Calibration Problems	86
8.2.2. Encryption and Decryption Schemes for Color Images	87
9. Conclusions	89
10. References	91

1. Introduction

This review commences with the famous Sylvester equation

$$AX - XB = C$$

with unknown X , named after the British mathematician James Joseph Sylvester (1814–1897) [1]. For a detailed discussion of this equation, refer to the review article [2] by Rajendra Bhatia, a Fellow of the Indian National Science Academy. The core of our review revolves around the generalized Sylvester equation (abbreviated as GSE):

$$AX - YB = C$$

with unknown X and Y .

In 1952, Roth established the necessary and sufficient conditions for the solvability of GSE over a field by means of block matrix equivalence. Since then, a large number of papers on GSE have been published in rapid succession. The solvability conditions and explicit expressions of the general solution for GSE have been investigated from diverse perspectives: linear transformations, generalized inverses, matrix decompositions, real (complex) representations, determinant representations, and semi-tensor products, etc. Extensive studies have been conducted on various constrained solutions (e.g., $\star$ -congruent, symmetric, self-adjoint, positive (semi)definite, per(skew)symmetric, bi(skew)symmetric, re-(non)negative definite, re-(non)positive definite, η -Hermitian, η -skew-Hermitian, ϕ -Hermitian, and equality-constrained solutions) of GSE, as well as its various best approximate solutions when the solvability conditions are not satisfied. Furthermore, the generalizations of GSE in different algebraic structures (e.g., unit regular rings, principal ideal domains, division rings, module-finite rings, commutative rings, Artinian rings, noncommutative rings, dual numbers, dual quaternions), operator equations, tensor equations, polynomial matrix equations, Sylvester-polynomial-conjugate matrix equations, and formal aspects have also shown remarkable vitality. Iterative algorithms for solving various solutions to GSE and its extended forms have also attracted considerable attention, with continuous updates and optimizations. Finally, the relevant results of GSE have demonstrated extraordinary significance in both theoretical applications (e.g., solvability of matrix equations, dual number matrix factorizations, and microlocal triangularization of pseudo-differential systems) and practical applications (e.g., hand-eye calibration problems, and encryption and decryption of color images).

Qing-Wen Wang, the first author of this paper, has been engaged in researching and teaching on the theory of matrix equations since the early 1990s, and has published over 100 related articles. Numerous scholars both domestically and internationally have suggested that we systematically present the theory of solving linear matrix equations. To date, our team has published three review articles, focusing, respectively, on solving the equations $AXB = C$ [3], $AX = C$ and $XB = D$ [4], as well as $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$ [5].

As mentioned earlier, in the development history of research on GSE, its conclusions are interrelated, mutually inspiring, mutually promotive, and progressively advanced, forming a complex, intertwined, and extensive network. By reviewing numerous literature, analyzing and comparing, summarizing, questioning, and prospecting, we strive to identify the commonalities, main threads, and approaches in these studies, aiming to provide insights and inspiration for subsequent research on GSE. Thus, this work is of significant value. In this paper, we intend to focus on GSE as the core theme, and unfold its solution methods, solution categories, generalizations, iterative algorithms, as well as theoretical and practical applications in a step-by-step manner, so as to provide a comprehensive overview.

The remainder of this paper consists of eight sections. Section 2 presents some necessary notations and notes. In Section 3, we introduce Roth's work on GSE published in [6], which serves as the starting point for this paper. In Section 4, we summarize seven methods for studying GSE, demonstrating diverse perspectives and research schemes. Various solutions to GSE are discussed in detail in Section 5. We devote Section 6 to introducing the generalizations of GSE in certain algebraic aspects. In Section 7, we enumerate several classic iterative algorithms for solving various solutions to GSE and its generalized forms. The theoretical and practical applications of GSE are presented in Section 8. Conclusions are stated in Section 9.

2. Preliminaries

This section recalls some notations and definitions used throughout the paper. Additionally, other necessary terminology for subsequent (sub)sections will be introduced within each. A few symbols, due to their bulk, will be referenced via specific indices in original sources, rather than being reproduced. This not only does not hinder readers' understanding but also makes the paper more concise and readable.

Let $\mathbb{F}$ be a ring; let $\mathbb{F}[\lambda]$ be the polynomial ring over $\mathbb{F}$ with the variable λ ; let $\mathbb{F}^{m \times n}$ be the set of all $m \times n$ matrices over $\mathbb{F}$; let $\mathbb{F}^{m \times n}[\lambda]$ be the set of all $m \times n$ polynomial matrices over $\mathbb{F}$ with the variable λ . In particular, $\mathbb{F}^m = \mathbb{F}^{m \times 1}$. Let $\deg f(\lambda)$ be the degree of $f(\lambda) \in \mathbb{F}[\lambda]$; let $\text{rank}(A(\lambda))$ be the rank of $A(\lambda) \in \mathbb{F}^{m \times n}[\lambda]$; let A^T be the transpose of $A \in \mathbb{F}^{m \times n}$. The symbol $\det(A(\lambda))$ denotes the determinant of $A(\lambda) \in \mathbb{F}^{n \times n}[\lambda]$ for a field $\mathbb{F}$. The component-wise representation of $A \in \mathbb{F}^{m \times n}$ can be denoted in the three following forms:

$$A = [a_{ij}] = [a_{i,j}] = [a_{i,j}]_{m \times n} \in \mathbb{F}^{m \times n}.$$

The Kronecker product of $A = [a_{ij}] \in \mathbb{F}^{m \times n}$ and $B \in \mathbb{F}^{s \times t}$ is defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix} \in \mathbb{F}^{ms \times nt}.$$

The matrices $A \in \mathbb{F}^{m \times n}$ and $B \in \mathbb{F}^{m \times n}$ are equivalent if there exist two nonsingular matrices $P \in \mathbb{F}^{m \times m}$ and $Q \in \mathbb{F}^{n \times n}$ such that $A = PBQ$. Moreover, $A(\lambda) \in \mathbb{F}^{m \times n}[\lambda]$ and $B(\lambda) \in \mathbb{F}^{m \times n}[\lambda]$ are equivalent if there exist two invertible polynomial matrices $P(\lambda) \in \mathbb{F}^{m \times m}[\lambda]$ and $Q(\lambda) \in \mathbb{F}^{n \times n}[\lambda]$ such that $A(\lambda) = P(\lambda)B(\lambda)Q(\lambda)$.

For a subspace $\mathcal{T} \subseteq \mathbb{F}^n$, let $P_{\mathcal{T}}$ be the orthogonal projector onto $\mathcal{T}$; and let $\mathcal{T}^{\perp}$ and $\dim(\mathcal{T})$ be the orthogonal complement and dimension of $\mathcal{T}$, respectively. In addition, denote $A\mathcal{T} = \{At \mid t \in \mathcal{T}\}$ for $A \in \mathbb{F}^{m \times n}$. For two vector spaces V and W over a field F , let τ be a linear transformation from V to W ; then $\text{Ker}(\tau)$ and $\text{Im}(\tau)$ represent the kernel space and the image of τ , respectively.

Let $\mathbb{Z}^+$, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ be the sets of all positive integers, natural numbers, real numbers, and complex numbers, respectively. The symbol $\text{sign}(a)$ stands for the sign of $a \in \mathbb{R}$, and $\emptyset$ represents the empty set. Let the set of all (real) quaternions be

$$\mathbb{H} = \{q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k} \mid \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \mathbf{ijk} = -1, q_1, q_2, q_3, q_4 \in \mathbb{R}\},$$

which is a four-dimensional noncommutative division algebra over $\mathbb{R}$ [7]. Let the conjugate of a quaternion $q = q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k} \in \mathbb{H}$ be

$$\bar{q} = q_1 - q_2\mathbf{i} - q_3\mathbf{j} - q_4\mathbf{k}.$$

Let $A = [a_{ij}] \in \mathbb{H}^{m \times n}$. The conjugate of A is $\bar{A} = [\bar{a}_{ij}] \in \mathbb{H}^{m \times n}$, and the conjugate transpose of A is $A^* = \bar{A}^T$. Let

$$A^\eta = -\eta A \eta \text{ and } A^{\eta*} = -\eta A^* \eta,$$

where $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. If $A = A^{\eta*}$ ($A = -A^{\eta*}$) for $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, then A is called η -Hermitian (η -anti-Hermitian) [8]. An η -anti-Hermitian matrix is also called an η -skew-Hermitian matrix. Moreover, $A^{\mathbf{i}*} = A^*$ and $A^{\mathbf{j}*} = A^{\mathbf{k}*} = A^T$ for $A \in \mathbb{C}^{m \times n}$.

Denote

$$P_A = A^\dagger A, Q_A = AA^\dagger, L_A = I - A^\dagger A, R_A = I - AA^\dagger,$$

where $A^\dagger$ is the Moore–Penrose inverse [9] of a matrix A . The symbols A^{-1} , $\text{rank}(A)$, $\mathcal{R}(A)$, and $\mathcal{N}(A)$ denote the inverse, rank, range, and null space of a matrix A , respectively. In addition, $\bar{A}$, A^* , and $\|A\|_F$ are the conjugate, conjugate transpose, and Frobenius norm of $A \in \mathbb{C}^{m \times n}$, respectively. Let I and 0 be the identity matrix and the null matrix, respectively, with appropriate orders. Specifically, I_n denotes the $n \times n$ identity matrix. For a symmetric matrix $A \in \mathbb{R}^{m \times m}$, $A > 0$ and $A \geq 0$ denote the positive definite matrix and the positive semidefinite matrix, respectively. In addition, denote $A \geq B$ if $A - B \geq 0$ for matrices A and B . The matrix

$$A = [A_1, A_2, \dots, A_n] = \begin{bmatrix} A_1 & A_2 & \dots & A_n \end{bmatrix}$$

denotes a new matrix formed by arranging matrix blocks $A_1, A_2, \dots, A_n$ (with the same number of rows) column-wise. The symbol $\text{diag}(\sigma_1, \dots, \sigma_r)$ denotes a (block) diagonal matrix with diagonal entries $\sigma_1, \dots, \sigma_r$.

For a ring $\mathbb{F}$ and $a \in \mathbb{F}$, a is regular (or inner-invertible) if there exists $a^- \in \mathbb{F}$ such that $aa^-a = a$; a is $(1, 2)$ -invertible (or reflexive) if there exists $a^{(1,2)} \in \mathbb{F}$ such that $aa^{(1,2)}a = a$ and $a^{(1,2)}aa^{(1,2)} = a^{(1,2)}$ (see [10]). The characteristic of a field $\mathbb{F}$ is denoted as $\text{char}(\mathbb{F})$.

An equation or a system of equations is said to be solvable (consistent) if it has at least one solution. The symbols “ $\Rightarrow$ ” and “ $\Leftrightarrow$ ” denote “imply” and “if and only if”, respectively. For the reader’s convenience, we provide the main abbreviations used in this paper along with their full names in Table 1.

Table 1. Abbreviations and their full names.

Full Name	Abbreviation
Generalized Sylvester equation	GSE
Roth’s equivalence theorem	RET
Moore–Penrose inverse	MP inverse
Singular value decomposition	SVD
Generalized singular value decomposition	GSVD
Canonical correlation decomposition	CCD
Semi-tensor product	STP
Second matrix–matrix semi-tensor product	MM-2 STP
Conjugate gradient least-squares algorithm	CGLSA
Alternating direction method	ADM
Relaxed gradient-based iterative algorithm	RGI algorithm

At the end of this section, we present three specific notes regarding this review.

- (1) The selection of references follows the core principle of focusing on the theoretical research and practical applications of GSE, with specific criteria as follows:

- (i) Time frame: It covers 19th-century foundational studies to 2025's latest achievements. It includes classic works like Hamilton's quaternion research (1844) and Sylvester's matrix equation study (1884), and emphasizes 2015–2025 recent studies (over 30% of total);
 - (ii) Publication venues: Priority is given to peer-reviewed works, including top journals (e.g., *SIAM J. Matrix Anal. Appl.*), authoritative monographs (e.g., by Gohberg), and key conference papers (e.g., from IEEE ICMA);
 - (iii) Content types: Original research is the main focus, with a small number of GSE-related review literature included. Only a few preprints/arXiv works are selected, due to their significance for subsequent research.
 - (iv) Relevance scope: Though some research does not focus directly on GSE itself, its traces of relevance to GSE are easily detectable. Thus, we regard such content as an integral part of GSE-related research.
- (2) Results closely related to the theme are rigorously presented as theorems, while less relevant conclusions are briefly summarized narratively. Furthermore, the proofs of these theorems are omitted here.
 - (3) The remarks in this paper include comments and suggestions on relevant results, encompassing both previous researchers' views and our reflections, questions, and prospects.

3. Roth's Equivalence Theorem

First, we specifically present the paper's core equation:

$$AX - YB = C, \quad (1)$$

where $A = [a_{ij}] \in \mathbb{F}^{m \times r}$, $B = [b_{ij}] \in \mathbb{F}^{s \times n}$, and $C = [c_{ij}] \in \mathbb{F}^{m \times n}$ are given over a ring $\mathbb{F}$.

Let $\mathbb{F}$ be a field. In 1952, Roth [6] first studied the necessary and sufficient conditions for the solvability to the polynomial matrix form of Equation (1), i.e.,

$$A(\lambda)X(\lambda) - Y(\lambda)B(\lambda) = C(\lambda), \quad (2)$$

where $A(\lambda), B(\lambda), C(\lambda) \in \mathbb{F}^{n \times n}[\lambda]$, by using the normal form of polynomial matrices.

Theorem 1 ([6]). Equation (2) has a solution pair $X(\lambda), Y(\lambda) \in \mathbb{F}^{n \times n}[\lambda]$ if and only if the polynomial matrices

$$\begin{bmatrix} A(\lambda) & C(\lambda) \\ 0 & B(\lambda) \end{bmatrix} \text{ and } \begin{bmatrix} A(\lambda) & 0 \\ 0 & B(\lambda) \end{bmatrix}$$

are equivalent.

Roth [6] has stated that Theorem 1 remains valid for rectangular polynomial matrices of appropriate orders. Thus, the following theorem can be derived immediately.

Theorem 2 (Theorem 1, [6]). Let $A \in \mathbb{F}^{m \times r}$, $B \in \mathbb{F}^{s \times n}$, and $C \in \mathbb{F}^{m \times n}$. Then,

$$\text{Equation (1) has a solution pair } X \in \mathbb{F}^{r \times n} \text{ and } Y \in \mathbb{F}^{m \times s} \quad (3)$$

if and only if

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \text{ and } \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \text{ are equivalent.} \quad (4)$$

We call this theorem Roth's equivalence theorem (abbreviated as RET).

Remark 1. It is easy to find that the study of Equation (1) is equivalent to that of the equation

$$AX + YB = C \quad (5)$$

with given $A = [a_{i,j}] \in \mathbb{F}^{m \times r}$, $B = [b_{i,j}] \in \mathbb{F}^{s \times n}$, and $C = [c_{i,j}] \in \mathbb{F}^{m \times n}$, by regarding $-B$ in Equation (1) as B in Equation (5). Thus, in this paper, we collectively refer to Equations (1) and (5) as GSE.

4. Different Methods on GSE

Roth [6] considered Equation (1) based on the canonical forms of the polynomial matrices. Subsequently, many scholars have provided different proofs of RET from various perspectives. These proofs demonstrate the effectiveness and uniqueness of different mathematical methods in considering Equation (1). In addition, these methods have also had a profound impact on the subsequent study of different Sylvester-type equations. This is precisely one of the enchantments of mathematics: reaching the same endpoint through different paths, or even completely unrelated ones. Furthermore, mathematicians ceaselessly pursue groundbreaking innovations and perpetually strive to discover the most elegant path.

Remark 2. It is notable that it is easy to see that (3) implies (4) by

$$\begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & AX - YB \\ 0 & B \end{bmatrix}.$$

Therefore, one only needs to consider the sufficiency of RET, i.e., (4) $\Rightarrow$ (3).

4.1. Method by Linear Transformations and Subspace Dimensions

Let $\mathbb{F}$ be a field. Flanders and Wimmer [11] proved RET for $m = r$ and $s = n$ by means of linear transformations and subspace dimension arguments. This method is more fundamental and elementary than Roth's method.

Proof of RET [11].

Step 1: Define $\psi_i : M_{r+s, 2(r+s)} \rightarrow M_{r+s}$ by

$$\psi_0(U, W) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} U - W \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \text{ and } \psi_1(U, W) = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} U - W \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Then, the condition (4) yields

$$\dim(\text{Ker}(\psi_0)) = \dim(\text{Ker}(\psi_1)).$$

Step 2: Let

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \text{ and } W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}.$$

Then,

$$\begin{aligned} \text{Ker}(\psi_0) &= \left\{ (U, W) \left| \begin{array}{l} AU_{11} = W_{11}A, \quad AU_{12} = W_{12}B, \\ BU_{21} = W_{21}A, \quad BU_{22} = W_{22}B, \end{array} \right. \right\}, \\ \text{Ker}(\psi_1) &= \left\{ (U, W) \left| \begin{array}{l} AU_{11} + CU_{21} = W_{11}A, \quad AU_{12} + CU_{22} = W_{12}B, \\ BU_{21} = W_{21}A, \quad BU_{22} = W_{22}B, \end{array} \right. \right\}. \end{aligned}$$

Let

$$Z = \left\{ \begin{bmatrix} U_{21} & U_{22} & W_{21} & W_{22} \end{bmatrix} \mid BU_{21} = W_{21}A, BU_{22} = W_{22}B \right\}.$$

For $i = 0, 1$, define

$$v_i : \text{Ker}(\psi_i) \rightarrow Z, v_i(U, W) = \begin{bmatrix} U_{21} & U_{22} & W_{21} & W_{22} \end{bmatrix}.$$

Then, $\text{Im}(v_1) \subseteq \text{Im}(v_0) = Z$ and $\text{Ker}(v_1) = \text{Ker}(v_0)$. So, $\text{Im}(v_1) = \text{Im}(v_0)$.

Step 3: Since $(U, W) \in \text{Ker}(\psi_0)$ with $U_{22} = -I$, there also exists such in $\text{Ker}(\psi_1)$. Therefore, $AU_{12} - W_{12}B = C$, i.e., (3) holds.

□

Remark 3.

- (1) In [11], Flanders and Wimmer mentioned that, by making small modifications to the above proof, one can similarly obtain the proof of RET under the condition of rectangular matrices A , B , and C .
- (2) In terms of linear transformations and subspace dimensions, Dmytryshyn et al. discussed two highly complex systems of equations (see Theorem 1.1 in [12] and Theorem 1 in [13]).

4.2. Method by Generalized Inverses

Let $\mathbb{F}$ be a field. In 1955, Penrose [9] concisely defined the Moore–Penrose inverse of any rectangular matrix through four matrix equations.

Definition 1 ([9,10]). Let $A \in \mathbb{F}^{m \times n}$. If there exists $X \in \mathbb{F}^{n \times m}$ such that

$$(1) AXA = A, (2) XAX = X, (3) (AX)^* = AX, (4) (XA)^* = XA,$$

then X is called the Moore–Penrose (abbreviated as MP) inverse of A and is denoted by $A^\dagger$. Especially if $X \in \mathbb{F}^{n \times m}$ satisfies the above Equation (1), then X is called an inner inverse of A and is denoted by A^- . Clearly, $A^\dagger$ is a special inner inverse of A .

Since then, the theory of generalized inverses has flourished (see [10,14–18]), and it has been widely used to study the solvability conditions of linear equations and represent the explicit expressions of their general solutions when solvable. Penrose's study on the matrix equation $AXB = C$ [9] is the most renowned.

In 1979, Baksalary and Kala [19] naturally utilized inner inverses to establish solvability conditions and representations of the general solution for Equation (1).

Theorem 3 ([19]). Equation (1) has a solution pair $X \in \mathbb{F}^{r \times n}$ and $Y \in \mathbb{F}^{m \times s}$ if and only if

$$(I - AA^-)C(I - B^-B) = 0, \quad (6)$$

in which case

$$\begin{aligned} X &= A^-C + A^-ZB + (I - A^-A)W, \\ Y &= -(I - AA^-)CB^- + Z - (I - AA^-)ZBB^-, \end{aligned}$$

where W and Z are arbitrary matrices with appropriate orders.

Remark 4. In views of RET and Theorem 3, Baksalary and Kala (Remark 2, [19]) noted that

$$(6) \Leftrightarrow (4) \Leftrightarrow \operatorname{rank} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \operatorname{rank}(A) + \operatorname{rank}(B).$$

They also pointed out that this result can be directly derived by using (Formula (8.7), [20]) in a particular case, i.e.,

$$\operatorname{rank} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \operatorname{rank}(A) + \operatorname{rank}(B) + \operatorname{rank}((I - AA^-)C(I - B^-B)).$$

Moreover, they said that proof using generalized inverses is simpler than Roth's [6] and Flanders et al.'s [11]. In fact, this is indeed the case when considering the simplicity of the proof and its length.

Remark 5. Under the hypotheses of Theorem 3, it is easy to obtain that

$$\begin{aligned} (6) &\Leftrightarrow \mathcal{R}(C(I - B^-B)) \subseteq \mathcal{N}((I - AA^-)) \\ &\Leftrightarrow C\mathcal{N}(B) \subseteq \mathcal{R}(A), \end{aligned} \quad (7)$$

which is also proved by Woude in (Lemma 3.2, [21]) according to an elementary method. In addition, Woude applied this result to a control problem that occurs in almost non-stationary stochastic processes via measurement feedback (see Theorems 3.3 and 4.1, [21]).

Remark 6. It is easy to show that the solvability of Equation (1) is equivalent to the existence of X and Y such that

$$\begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}. \quad (8)$$

In terms of a geometrical method, Olshevsky gave a cyclic argument over $\mathbb{C}$ as follows:

$$(3) \Rightarrow (8) \Rightarrow (4) \Rightarrow (7) \Rightarrow (3),$$

(see the proof of Theorem 1.2 in [22]).

Let $\mathbb{F} = \mathbb{C}$. Meyer [23] revealed an interesting result, that is, the solvability of Equation (1) is equivalent to the existence of the upper block inner inverse of a certain block matrix.

Theorem 4 (Theorems 1 and 2, [23]). The block triangular complex matrix

$$T = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$$

has an upper block triangular inner inverse if and only if

$$\operatorname{rank}(T) = \operatorname{rank}(A) + \operatorname{rank}(B),$$

in which case,

$$T^- = \begin{bmatrix} A^- & -A^-CB^- \\ 0 & B^- \end{bmatrix},$$

is an inner inverse of T for any A^- and B^- .

4.3. Method by Singular Value Decompositions

Let $\mathbb{F} = \mathbb{R}$. The singular value decomposition (abbreviated as SVD) [24], as an important tool for the study of matrix theory, also plays a significant role in the research on solutions of matrix equations. Chu [25] utilized the SVD to study the solvability conditions and the representation of the general solution of Equation (5).

In fact, let the SVDs of A and B given in Equation (5) be

$$A = U_A D_A V_A^T \text{ and } B = U_B D_B V_B^T, \quad (9)$$

where U_A, U_B, V_A , and V_B are orthogonal matrices with appropriate orders,

$$D_A = \begin{bmatrix} \Sigma_A & 0 \\ 0 & 0 \end{bmatrix}, \quad D_B = \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix},$$

$\Sigma_A = \text{diag}(\alpha_1, \dots, \alpha_k)$ with $\alpha_i > 0$ ($1 \leq i \leq k$), and $\Sigma_B = \text{diag}(\delta_1, \dots, \delta_l)$ with $\delta_j > 0$ ($1 \leq j \leq l$). Then,

$$(5) \Leftrightarrow U_A D_A V_A^T X + Y U_B D_B V_B^T = C \Leftrightarrow D_A \tilde{X} + \tilde{Y} D_B = \tilde{C},$$

where $\tilde{X} = V_A^T X V_B$, $\tilde{Y} = U_A^T Y U_B$, and $\tilde{C} = U_A^T C V_B$. Partitioning the above equation analogously to the partitioning of D_A and D_B , we obtain

$$\begin{aligned} & \begin{bmatrix} \Sigma_A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{21} & \tilde{X}_{22} \end{bmatrix} + \begin{bmatrix} \tilde{Y}_{11} & \tilde{Y}_{12} \\ \tilde{Y}_{21} & \tilde{Y}_{22} \end{bmatrix} \begin{bmatrix} \Sigma_B & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} \\ \tilde{C}_{21} & \tilde{C}_{22} \end{bmatrix} \\ \Leftrightarrow & \begin{bmatrix} \Sigma_A \tilde{X}_{11} + \tilde{Y}_{11} \Sigma_B & \Sigma_A \tilde{X}_{12} \\ \tilde{Y}_{21} \Sigma_B & 0 \end{bmatrix} = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} \\ \tilde{C}_{21} & \tilde{C}_{22} \end{bmatrix}. \end{aligned} \quad (10)$$

Based on the above discussion, the following theorem can be obtained.

Theorem 5 (Theorem 2, [25]). Under the notations in (9) and (10), let

$$U_A = \begin{bmatrix} U_{A_1} & U_{A_2} \end{bmatrix} \text{ and } V_B = \begin{bmatrix} V_{B_1} & V_{B_2} \end{bmatrix}.$$

(1) Then, Equation (5) is solvable if and only if

$$\tilde{C}_{22} = 0, \text{ i.e., } U_{A_2}^T C V_{B_2} = 0. \quad (11)$$

(2) Suppose that (11) holds. Denote

$$\tilde{C} = (\tilde{c}_{ij}) \text{ and } M_{ij} = \begin{bmatrix} \alpha_i & \delta_j \end{bmatrix}.$$

Then, $\tilde{X}_{21}$, $\tilde{X}_{22}$, $\tilde{Y}_{12}$, and $\tilde{Y}_{22}$ are arbitrary,

$$\tilde{X}_{12} = \Sigma_A^{-1} \tilde{C}_{12}, \quad \tilde{Y}_{21} = \tilde{C}_{21} \Sigma_B^{-1}, \quad \tilde{X}_{11} = (\tilde{x}_{ij}), \text{ and } \tilde{Y}_{11} = (\tilde{y}_{ij}),$$

where

$$\begin{bmatrix} \tilde{x}_{ij} \\ \tilde{y}_{ij} \end{bmatrix} = M_{ij}^\dagger \tilde{c}_{ij} + (I - M_{ij}^\dagger M_{ij}) Z_{ij}$$

for arbitrary Z_{ij} . Moreover, if $\tilde{Y}_{11}$ is arbitrary, then

$$\tilde{X}_{11} = \Sigma_A^{-1} (\tilde{C}_{11} - \tilde{Y}_{11} \Sigma_B).$$

(3) If (11) holds, then

$$X = V_A \begin{bmatrix} \Sigma_A^{-1}(U_{A_1}^T C V_{B_1} - Z_1 \Sigma_B) & \Sigma_A^{-1} U_{A_1}^T C V_{B_1} \\ Z_2 & Z_3 \end{bmatrix} V_B^T,$$

$$Y = U_A \begin{bmatrix} Z_1 & Z_4 \\ U_{A_2}^T C V_{B_1} \Sigma_B^{-1} & Z_5 \end{bmatrix} U_B^T,$$

where $Z_1, \dots, Z_5$ are arbitrary matrices with appropriate orders.

Remark 7. Interestingly, Chu in Theorem 1 of [25] utilized the generalized singular value decomposition (abbreviated as GSVD) [26] to study the extended form of Equation (5) over $\mathbb{R}$, i.e.,

$$AXE + FYB = C, \quad (12)$$

where A, E, F, B , and C are given real matrices with appropriate orders. Eleven years later, Xu et al. [27] once again discussed the solvability conditions, the general solution, and least-squares solutions of Equation (12) over $\mathbb{C}$ by using the canonical correlation decomposition (abbreviated as CCD) introduced in [28].

Remark 8. Inspired by solving Equation (5) via SVD, one can utilize equivalent normal forms of A and B to study Equation (5) over a field, along with an approach similar to Theorem 5. In fact, we only need to regard orthogonal matrices U_A, U_B, V_A , and V_B in (9) as invertible matrices, the transpose V_A^T and V_B^T as the inverse V_A^{-1} and V_B^{-1} , and Σ_A and Σ_B as identity matrices of appropriate orders.

It can be found that using the equivalent normal form of matrices to solve matrix equations is also a very convenient method. This viewpoint has been confirmed multiple times in subsequent research. Wang et al. used the elementary row and column transformations of matrices to give the equivalent normal form of a matrix triplet

$$\begin{bmatrix} A & B & C \end{bmatrix}$$

with the same row number over an arbitrary division ring $\mathbb{F}$ (see Theorem 2.1, [29]). Similarly, the equivalent normal form of another matrix triplet

$$\begin{bmatrix} D^T & E^T & F^T \end{bmatrix}^T$$

with the same column number can also be obtained (see Theorem 2.2, [29]). The two equivalent normal forms are applied to solve the matrix equation

$$AXD + BYE + CZF = G, \quad (13)$$

where X, Y , and Z are unknown (see Theorem 3.2, [29]).

Interestingly, He et al. utilized Theorem 2.1 of [29] to propose a simultaneous decomposition of seven matrices over $\mathbb{H}$ (see Theorem 2.3, [30]), i.e.,

$$\begin{bmatrix} G & A & B & C \\ D \\ E \\ F \end{bmatrix}$$

and discussed Equation (13) once again by this decomposition (see Theorem 3.1, [30]). Compared with the method in [29], which directly applies the equivalent normal forms of two matrix

triplets to matrix G , the simultaneous decomposition is more concise. In addition, Ref. [31] once again brilliantly demonstrates the important role of simultaneous decomposition in solving matrix equations.

4.4. Method by Simultaneous Decompositions

Let $\mathbb{F}$ be a field. Remark 8 notes using equivalent canonical forms of the matrices A and B to solve Equation (1). Can we find an equivalent canonical form that is simultaneously related to the matrix triplet

$$\begin{bmatrix} C & A \\ B & \end{bmatrix} \quad (14)$$

so as to discuss Equation (1) more simply? Gustafson [32] gave a positive answer to this problem.

Theorem 6 ([32]). Let A , B , and C be given in Equation (1). Then, there exist invertible matrices T , U , V , and W of appropriate orders such that

$$\begin{bmatrix} TCW & TAU \\ VBW & \end{bmatrix} = \begin{bmatrix} C' & A' \\ B' & \end{bmatrix}, \quad (15)$$

where

$$A' = \begin{bmatrix} I_z & 0 & 0 & 0 \\ 0 & I_{t_3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{r_2} \\ 0 & 0 & 0 & 0 \end{bmatrix}, B' = \begin{bmatrix} I_z & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{t_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{r_1} & 0 & 0 \end{bmatrix}, C' = \begin{bmatrix} I_z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{r_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{r_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{t_2} \end{bmatrix},$$

$z + t_3 + r_2 = \text{rank}(A)$, $z + t_1 + r_1 = \text{rank}(B)$, and $z + t_2 + r_1 + r_2 = \text{rank}(C)$. We call (15) the simultaneous decomposition of (14).

Remark 9. By elementary matrix row and column transformations, Gustafson designed the 12-step algorithm to obtain the simultaneous decomposition of (14) (see (Section 3, [32])).

Applying the simultaneous decomposition of (14) to Equation (1) yields

$$\begin{aligned} (1) &\Leftrightarrow T^{-1}A'U^{-1}X - YV^{-1}B'W^{-1} = T^{-1}C'W^{-1} \\ &\Leftrightarrow A'X' - Y'B' = C', \end{aligned} \quad (16)$$

where $X' = U^{-1}XW$ and $Y' = TYV^{-1}$. Thus, $X = UX'W^{-1}$ and $Y = T^{-1}Y'V$.

Theorem 7 ([32]). Equation (16) is solvable if and only if $t_2 = 0$, in which case,

$$\begin{aligned} A' &= \begin{bmatrix} I_z & 0 & 0 & 0 & 0 \\ 0 & I_{t_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{r_2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B' = \begin{bmatrix} I_z & 0 & 0 & 0 & 0 \\ 0 & I_{t_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{r_1} & 0 \end{bmatrix}, C' = \begin{bmatrix} I_z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{r_1} & 0 \\ 0 & 0 & 0 & 0 & I_{r_2} \end{bmatrix}, \\ X' &= \begin{bmatrix} X_{11} & X_{12} & 0 & X_{14} & 0 \\ X_{21} & X_{22} & 0 & X_{24} & 0 \\ X_{31} & X_{32} & X_{33} & X_{35} & X_{35} \\ X_{41} & X_{42} & 0 & X_{44} & I_{r_2} \end{bmatrix}, Y' = \begin{bmatrix} X_{11} - I_z & X_{12} & Y_{13} & X_{14} \\ X_{21} & X_{22} - I_{t_1} & Y_{23} & X_{24} \\ 0 & 0 & Y_{33} & 0 \\ 0 & 0 & Y_{43} & -I_{r_1} \\ X_{41} & X_{42} & Y_{53} & X_{44} \end{bmatrix}, \end{aligned}$$

where the X_{ij} and Y_{ij} are arbitrary matrices with appropriate orders.

Remark 10. The quiver theory introduced by Gabriel [33] is an important tool in the representation theory of algebras. Gustafson [32] gave a novel interpretation of the existence of the simultaneous decomposition of (14) from the perspective of the quiver theory. Finally, he gave a necessary and sufficient condition for Equation (1) to have a solution by using the corresponding representation of arrows (see Section 10, [32]). Note that refs. [12,34] also make use of the graph theory to discuss linear matrix equations.

Remark 11. Wang et al. in Theorem 2.1 of [35] continued with the idea of simultaneous decomposition and decomposed the following matrices over a division ring $\mathbb{F}$:

$$\begin{bmatrix} A & B & C \\ D & & \end{bmatrix}, \quad (17)$$

where A , B , and C are of the same row number, and A and D are of the same column number. So, Theorem 6 is a corollary of Theorem 2.1 of [35] (see Corollary 2.2, [35]). Also, it should be noted that Theorem 2.1 of [29] is indeed a special case of Theorem 2.1 of [35] (see Corollary 2.3, [35]). In addition, Wang et al. applied the simultaneous decomposition of (17) to solving two types of systems of matrix equations. Interestingly, He et al. [36] further refined the simultaneous decomposition presented by [35].

He et al. [37] further considered the simultaneous decompositions of two more general forms over $\mathbb{H}$:

$$\begin{bmatrix} A & & & \\ B & C & D & \\ & & E & \end{bmatrix} \text{ and } \begin{bmatrix} A & & \\ B & C & \\ & D & E \end{bmatrix},$$

where matrices in the same row (or column) have the same number of rows (or columns), and applied them to solving systems of quaternion matrix equations. Recently, Huo et al. [38] generalized the simultaneous decomposition of (17) to quaternion tensors under the Einstein product.

4.5. Method by Real (Complex) Representations

It is well known that an associative algebra $\mathcal{A}$ with finite dimensions over a field $\mathbb{F}$ is isomorphic to a subalgebra of the algebra $\mathbb{F}^{n \times n}$, where n is the dimension of $\mathcal{A}$ over $\mathbb{F}$ (see [39]). We now consider the case of complex numbers, i.e., $\mathbb{F} = \mathbb{C}$.

Let $A = A_0 + A_1\mathbf{i} \in \mathbb{C}^{m \times r}$, where $A_0, A_1 \in \mathbb{R}^{m \times r}$, and $\mathbf{i}$ is the imaginary unit such that $\mathbf{i}^2 = -1$. Define a map

$$\phi: \mathbb{C}^{m \times r} \rightarrow \mathbb{R}^{2m \times 2r} \text{ with } \phi(A) = \phi(A_0 + A_1\mathbf{i}) = \begin{bmatrix} A_0 & -A_1 \\ A_1 & A_0 \end{bmatrix}.$$

We call $\phi(A)$ a real representation of the complex matrix A [40]. Then, one can check that $\phi(\cdot)$ is an isomorphism of the real algebra $\mathbb{C}^{m \times r}$ onto the real subalgebra

$$\left\{ \begin{bmatrix} S_0 & -S_1 \\ S_1 & S_0 \end{bmatrix} \mid S_0, S_1 \in \mathbb{R}^{m \times r} \right\}.$$

Then,

$$(5) \Leftrightarrow \phi(A)\phi(X) + \phi(Y)\phi(B) = \phi(C).$$

On the other hand, suppose that a real matrix pair

$$\hat{X} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \text{ and } \hat{Y} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$$

is a solution pair of the following equation

$$\phi(A)\hat{X} + \hat{Y}\phi(B) = \phi(C). \quad (18)$$

Then, $K_{2r}^{-1}\hat{X}K_{2n}$ and $K_{2m}^{-1}\hat{Y}K_{2s}$ are also a solution pair of Equation (18), where

$$K_{2t} = \begin{bmatrix} 0 & I_t \\ -I_t & 0 \end{bmatrix} \text{ for } t = m, n, r, s.$$

Let $\bar{X} = \frac{1}{2}(X_{11} + X_{22}) + \frac{1}{2}(X_{21} - X_{12})\mathbf{i}$ and $\bar{Y} = \frac{1}{2}(Y_{11} + Y_{22}) + \frac{1}{2}(Y_{21} - Y_{12})\mathbf{i}$. Then,

$$\begin{aligned} \phi(\bar{X}) &= \frac{1}{2} \begin{bmatrix} X_{11} + X_{22} & X_{12} - X_{21} \\ X_{21} - X_{12} & X_{11} + X_{22} \end{bmatrix} = \frac{1}{2} (\hat{X} + K_{2r}^{-1}\hat{X}K_{2n}), \\ \text{and } \phi(\bar{Y}) &= \frac{1}{2} \begin{bmatrix} Y_{11} + Y_{22} & Y_{12} - Y_{21} \\ Y_{21} - Y_{12} & Y_{11} + Y_{22} \end{bmatrix} = \frac{1}{2} (\hat{Y} + K_{2m}^{-1}\hat{Y}K_{2s}) \end{aligned}$$

satisfy (18), so $\bar{X}$ and $\bar{Y}$ satisfy Equation (5).

Based on the above analysis, the problem of a complex matrix equation is transformed into a problem of a real matrix equation by using the real representation of complex matrices. Based on this consideration, Liu [40] proposed the following theorem for solving Equation (5) over $\mathbb{C}$.

Theorem 8 (Lemma 1.3, [40]). *Let*

$$A = A_0 + A_1\mathbf{i}, B = B_0 + B_1\mathbf{i}, \text{ and } C = C_0 + C_1\mathbf{i}$$

be given in Equation (5). Then, Equation (5) is consistent over $\mathbb{C}$ if and only if

$$\begin{bmatrix} A_0 & -A_1 \\ A_1 & A_0 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} + \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} B_0 & -B_1 \\ B_1 & B_0 \end{bmatrix} = \begin{bmatrix} C_0 & -C_1 \\ C_1 & C_0 \end{bmatrix} \quad (19)$$

is consistent over $\mathbb{R}$, in which case,

$$\begin{cases} X = X_0 + X_1\mathbf{i} = \frac{1}{2}(X_{11} + X_{22}) + \frac{1}{2}(X_{21} - X_{12})\mathbf{i}, \\ Y = Y_0 + Y_1\mathbf{i} = \frac{1}{2}(Y_{11} + Y_{22}) + \frac{1}{2}(Y_{21} - Y_{12})\mathbf{i}, \end{cases} \quad (20)$$

where $X_{11}, X_{12}, X_{21}, X_{22}, Y_{11}, Y_{12}, Y_{21}$, and Y_{22} constitute the general solution of Equation (19). Furthermore, the explicit forms of X and Y given in (20) are

$$\begin{aligned} X_0 &= \frac{1}{2}P_1\phi(A)^-\phi(C)Q_1 + \frac{1}{2}P_2\phi(A)^-\phi(C)Q_2 + [U_1, U_2] \begin{bmatrix} \phi(B)Q_1 \\ \phi(B)Q_2 \end{bmatrix} + [P_1F_{\phi(A)}, P_2F_{\phi(A)}] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \\ X_1 &= \frac{1}{2}P_2\phi(A)^-\phi(C)Q_1 - \frac{1}{2}P_1\phi(A)^-\phi(C)Q_2 + [U_1, U_2] \begin{bmatrix} -\phi(B)Q_2 \\ \phi(B)Q_1 \end{bmatrix} + [P_2F_{\phi(A)}, -P_1F_{\phi(A)}] \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \\ Y_0 &= \frac{1}{2}S_1E_{\phi(A)}\phi(C)\phi(B)^-T_1 + \frac{1}{2}S_2E_{\phi(A)}\phi(C)\phi(B)^-T_2 - [S_1\phi(A), S_2\phi(A)] \begin{bmatrix} \hat{U}_1 \\ \hat{U}_2 \end{bmatrix} + [W_1, W_2] \begin{bmatrix} E_{\phi(B)}T_1 \\ E_{\phi(B)}T_2 \end{bmatrix}, \\ Y_1 &= \frac{1}{2}S_2E_{\phi(A)}\phi(C)\phi(B)^-T_1 - \frac{1}{2}S_1E_{\phi(A)}\phi(C)\phi(B)^-T_2 + [-S_2\phi(A), S_1\phi(A)] \begin{bmatrix} \hat{U}_1 \\ \hat{U}_2 \end{bmatrix} + [W_1, W_2] \begin{bmatrix} E_{\phi(B)}T_1 \\ -E_{\phi(B)}T_2 \end{bmatrix}, \end{aligned}$$

where $P_1 = [I_r, 0]$, $P_2 = [0, I_r]$, $S_1 = [I_m, 0]$, $S_2 = [0, I_m]$,

$$Q_1 = \begin{bmatrix} I_n \\ 0 \end{bmatrix}, Q_2 = \begin{bmatrix} 0 \\ I_n \end{bmatrix}, T_1 = \begin{bmatrix} I_s \\ 0 \end{bmatrix}, T_2 = \begin{bmatrix} 0 \\ I_s \end{bmatrix},$$

and $U_i, V_i, \hat{U}_i, W_i$ ($i = 1, 2$) are arbitrary matrices over $\mathbb{R}$ with appropriate orders.

Remark 12. Liu [40] further used Theorem 8 to discuss the maximal and minimal ranks of Equation (5)'s solutions, as seen in Section 5.6 of this paper.

Remark 13. We know that the quaternion algebra (or the quaternion division ring) cannot be a field because it does not satisfy the commutative law. However, when studying matrix equations over $\mathbb{H}$, the method of converting the quaternion matrix equations to the real (or complex) matrix equations by using the real (or complex) representation of quaternions [7] is widely applied. For example, refs. [41,42] studied the special least-squares solutions of a class of quaternion matrix equations by using the real representation and the complex representation of quaternions, respectively. Moreover, the real (or complex) representation of some other quaternions mentioned in Remark 43 have been explored in [43–46].

On the other hand, because real (or complex) representations have the drawback of a significant increase in computational load, Wei et al. [47] introduced real structure-preserving methods over $\mathbb{H}$ for LU decomposition, QR decomposition, and SVD, thereby solving quaternion linear systems.

4.6. Method by Determinable Representations

As is well known, Cramer's rule for the linear equation $Ax = b$ for unknown vector x is an effective means of expressing its unique solution. In addition, it can be found in Section 4.2 that the theory of generalized inverses is closely related to the study of Equation (1). Naturally, we can consider whether Cramer's rule for the solutions of Equation (1) can be obtained through the determinant representations of generalized inverses.

Generalized inverses (especially the MP inverse) over fields (particularly the complex field) have been thoroughly discussed and applied to characterize various solutions of matrix equations (see [10,15,48]). Notably, Kyrchei [49] presented the determinantal representations of the MP inverse and the Drazin inverse [50] from the new perspective of limit representations of generalized inverses in 2008.

When we consider the determinantal representations of generalized inverses in the quaternion algebra, it relies on the theory of determinants of quaternion matrices. However, due to the noncommutativity of quaternions, the determinants of quaternion matrices become much more complicated (see [51,52]). It was not until several decades later, after Kyrchei [53,54] introduced the theory of column-row determinants over $\mathbb{H}$, that this problem was effectively solved.

Definition 2 (Column-row determinants over $\mathbb{H}$). (Definitions 2.4 and 2.5, [53]) Let $A = [a_{ij}] \in \mathbb{H}^{n \times n}$, and let S_n be the symmetric group on $I_n = \{1, 2, \dots, n\}$.

(1) For $i = 1, 2, \dots, n$, the i -th row determinant of A is defined by

$$\text{rdet}_i A = \sum_{\sigma \in S_n} (-1)^{n-r} (a_{i i_{k_1}} a_{i_{k_1} i_{k_1+1}} \dots a_{i_{k_1+l_1} i}) \dots (a_{i_{k_r} i_{k_r+1}} \dots a_{i_{k_r+l_r} i_{k_r}}),$$

$$\sigma = (i \ i_{k_1} \ i_{k_1+1} \ \dots \ i_{k_1+l_1}) (i_{k_2} \ i_{k_2+1} \ \dots \ i_{k_2+l_2}) \dots (i_{k_r} \ i_{k_r+1} \ \dots \ i_{k_r+l_r}),$$

where $i_{k_2} < i_{k_3} < \dots < i_{k_r}$, and $i_{k_t} < i_{k_t+s}$ for $t = 2, \dots, r$ and $s = 1, \dots, l_t$.

(2) For $j = 1, 2, \dots, n$, the j -th column determinant of A is defined by

$$\text{cdet}_j A = \sum_{\tau \in S_n} (-1)^{n-r} (a_{j_{k_r} j_{k_r+l_r}} \dots a_{j_{k_r+1} j_{k_r}}) \dots (a_{j_{k_1+l_1} j_{k_1}} \dots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j}),$$

$$\tau = (j_{k_r+l_r} \dots j_{k_r+1} j_{k_r}) \dots (j_{k_2+l_2} \dots j_{k_2+1} j_{k_2}) (j_{k_1+l_1} \dots j_{k_1+1} j_{k_1} j),$$

where $j_{k_2} < j_{k_3} < \dots < j_{k_r}$ and $j_{k_t} < j_{k_t+s}$ for $t = 2, \dots, r$ and $s = 1, \dots, l_t$.

Remark 14. Kyrchei in Theorem 3.1 of [53] showed that if $A \in \mathbb{H}^{n \times n}$ is Hermitian, i.e., $A = A^*$, then

$$\text{rdet}_1 A = \dots = \text{rdet}_n A = \text{cdet}_1 A = \dots = \text{cdet}_n A \in \mathbb{R}.$$

Thus, Remark 3.1 [53] defines the determinant of a Hermitian matrix A by

$$\det A = \text{rdet}_i A = \text{cdet}_i A, \quad (i = 1, 2, \dots, n).$$

We now introduce some necessary notations. Let $A \in \mathbb{H}^{m \times n}$. Then, $\mathcal{R}_l(A)$, $\mathcal{R}_r(A)$, $\mathcal{N}_l(A)$, and $\mathcal{N}_r(A)$ denote the left row space, the right column space, the left null space, and the right null space of A , respectively. Let

$$\alpha = \{\alpha_1, \dots, \alpha_k\} \subseteq \{1, \dots, m\} \text{ and } \beta = \{\beta_1, \dots, \beta_k\} \subseteq \{1, \dots, n\},$$

where $1 \leq k \leq \min\{m, n\}$. By A_{β}^{α} , we denote the submatrix of A whose rows are indexed by α and whose columns are indexed by β . If A is Hermitian, then $|A_{\alpha}^{\alpha}|$ is the corresponding principal minor of A . For $1 \leq k \leq n$, let

$$L_{k,n} = \{\alpha \mid \alpha = (\alpha_1, \dots, \alpha_k), 1 \leq \alpha_1 < \dots < \alpha_k \leq n\}.$$

And, for $i \in \alpha$ and $j \in \beta$, let

$$I_{r,m}\{i\} = \{\alpha \mid \alpha \in L_{r,m}, i \in \alpha\} \text{ and } J_{r,n}\{j\} = \{\beta \mid \beta \in L_{r,n}, j \in \beta\}.$$

We denote the j -th column of A by $\mathbf{a}_j$ and its i -th row by $\mathbf{a}_i$. And, $A_{j,j}(\mathbf{b})$ is the matrix that is obtained from A by replacing its j -th column by the column vector $\mathbf{b} \in \mathbb{H}^{m \times 1}$, and $A_{i,i}(\mathbf{b})$ is the matrix obtained from A by replacing its i -th row by the row vector $\mathbf{b} \in \mathbb{H}^{1 \times n}$. Symbols $\mathbf{a}_j^*$ and $\mathbf{a}_i^*$ denoted the j -th column and the i -th row of A^* , respectively.

Kyrchei [55] obtained the Cramer's rules for some left, right and two-sided quaternion matrix equations by the theory of the column-row determinants over $\mathbb{H}$. Song et al. [56] then utilized results in [55] to further study the Cramer's rule for the quaternion matrix equation:

$$AXE + FYB = C, \quad (21)$$

where $A \in \mathbb{H}^{m \times n}$, $E \in \mathbb{H}^{s \times q}$, $F \in \mathbb{H}^{m \times q}$, $B \in \mathbb{H}^{t \times q}$, and $C \in \mathbb{H}^{m \times q}$ are given.

Theorem 9 (Theorem 3.1, [56]). Suppose that Equation (21) is consistent. Let

$$T = A^*(I + R_F)A, S = E(I + L_B)E^*, A_{11} = A^*R_F A T^{\dagger}, A_{22} = A^* A T^{\dagger},$$

$$E_{11} = S^{\dagger} E E^*, E_{22} = S^{\dagger} E L_B E^*, Y_{10} = A_{22}^{\dagger} A_{11} A^* C L_B E^* + L_{A_{22}} A^* R_F C E^* E_{22} E_{11}^{\dagger},$$

$$Y_{20} = A_{11}^{\dagger} A_{22} A^* R_F C E^* + L_{A_{11}} A^* C L_B E^* E_{11} E_{22}^{\dagger}.$$

Let K^* , L , M^* , and N be of full column rank matrices over $\mathbb{H}$ such that

$$\mathcal{N}_r(T) = \mathcal{R}_r(K^*), \mathcal{N}_r(S) = \mathcal{R}_r(L), \mathcal{N}_r(F) = \mathcal{R}_r(M^*), \mathcal{N}_r(B) = \mathcal{R}_r(N).$$

Then, the general solution of Equation (21) is

$$x_{ij} = \frac{\text{rdet}_j(S + LL^*)_j(c_i^A)}{\det(T + K^*K)\det(S + LL^*)} = \frac{\text{cdet}_i(T + K^*K)_{,i}(c_j^E)}{\det(T + K^*K)\det(S + LL^*)},$$

$$y_{hl} = \frac{\text{rdet}_l(BB^* + NN^*)_l(g_h^A)}{\det(F^*F + M^*M)\det(BB^* + NN^*)} = \frac{\text{cdet}_h(F^*F + M^*M)_{,h}(g_l^E)}{\det(F^*F + M^*M)\det(BB^* + NN^*)},$$

where

$$c_i^A = [\text{cdet}_i(T + K^*K)_{,i}(d_{1.}), \dots, \text{cdet}_i(T + K^*K)_{,i}(d_{s.})],$$

$$g_h^A = [\text{cdet}_h(F^*F + M^*M)_{,h}(k_{1.}), \dots, \text{cdet}_h(F^*F + M^*M)_{,h}(k_{t.})],$$

$$c_j^E = [\text{rdet}_j(S + LL^*)_j(d_{1.}), \dots, \text{rdet}_j(S + LL^*)_j(d_{n.})]^T,$$

$$g_l^E = [\text{rdet}_l(BB^* + NN^*)_l(k_{1.}), \dots, \text{rdet}_l(BB^* + NN^*)_l(k_{q.})]^T$$

with $d_{i.}$ and $d_{.j}$ are the i -th row and j -th column of

$$(T + K^*K)T^\dagger(A^*R_F C E^* + A^*C L_B E^* + Y_{10} + Y_{20})S^\dagger(S + LL^*) + M + W,$$

respectively, and $k_{.i}$ and $k_{j.}$ are the i -th column and j -th row of

$$(F^*F + M^*M)(F^\dagger(C - A X E)B^\dagger)(BB^* + NN^*) + Q,$$

respectively, for $i = 1, \dots, m$, $j = 1, \dots, s$, $h = 1, \dots, q$, and $l = 1, \dots, t$, where

$$M = (T + K^*K)T^\dagger(L_{A_{22}}V_1R_{E_{11}} + L_{A_{11}}V_2R_{E_{22}})S^\dagger(S + LL^*),$$

$$W = TZLL^* + K^*KZS + K^*KZLL^*, \quad Q = F^*FHNN^* + M^*MH(BB^* + NN^*)$$

for arbitrary V_1 , V_2 , Z , and H with appropriate orders.

Remark 15. When E and F in Equation (21) are identity matrices, by Theorem 9, we immediately derived the Cramer's rule to the matrix equation over $\mathbb{H}$:

$$AX + YB = C. \quad (22)$$

Note that Theorem 9 uses the auxiliary matrices, i.e., K , L , M , and N , to derive the determinant representation of the general solution of Equation (22). However, these auxiliary matrices are not always easy to obtain in practical applications. Then, can we obtain Cramer's rule for Equation (22) only through its given coefficient matrices?

In order to answer the above problem, let us first take a look at another work of Kyrchei [57]. Based on the column-row determinant theory and the limit representation of the MP inverse, he gave the determinant representation of the MP inverse over $\mathbb{H}$ in [57]. This research has greatly promoted the study of Cramer's rules to the various solutions of matrix equations over $\mathbb{H}$ (see [58–64]). Denote

$$\mathbb{H}_r^{m \times n} = \{A \in \mathbb{H}^{m \times n} \mid \text{rank}(A) = r\}.$$

Theorem 10 (Theorem 5, [57]). Let $A \in \mathbb{H}_r^{m \times n}$ and $A^\dagger = [a_{ij}^+] \in \mathbb{H}^{n \times m}$. Then,

$$a_{ij}^+ = \frac{\sum_{\beta \in I_{r,n}\{i\}} \text{cdet}_i \left((A^* A)_{\cdot i} (\mathbf{a}_{\cdot j}^*) \right)_\beta^\beta}{\sum_{\beta \in I_{r,n}} |(A^* A)_\beta^\beta|} = \frac{\sum_{\alpha \in I_{r,m}\{j\}} \text{rdet}_j \left((AA^*)_j (\mathbf{a}_{i \cdot}^*) \right)_\alpha^\alpha}{\sum_{\alpha \in I_{r,m}} |(AA^*)_\alpha^\alpha|},$$

where $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

A result similar to Theorem 3 was given by Kyrchei in [58].

Theorem 11 (Lemma 5.1, [58]). The following are equivalent:

- (1) Equation (22) is solvable;
- (2) $R_A C L_B = 0$;
- (3) $\text{rank} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \text{rank} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$;

in which case,

$$X = A^\dagger C - A^\dagger V B + L_A U \text{ and } Y = R_A C B^\dagger + A A^\dagger V + W R_B, \quad (23)$$

where U , V , and W are arbitrary matrices of appropriate orders over $\mathbb{H}$.

It is easy to see that

$$X = A^\dagger C \text{ and } Y = C B^\dagger - A A^\dagger C B^\dagger \quad (24)$$

are the partial solution pair of Equation (22) by taking U , V , and W as zero matrices in (23). Then, by applying Theorem 10 to (24), Kyrchei [58] presented a new Cramer's rule for the partial solution of Equation (22), which only makes use of the coefficient matrices, i.e., A , B , and C .

Theorem 12 (Theorem 5.2, [58]). Let $A \in \mathbb{H}_{r_1}^{m \times n}$ and $B \in \mathbb{H}_{r_2}^{t \times q}$. Then, the partial solution pair $X = [x_{ij}] \in \mathbb{H}^{n \times q}$ and $Y = [y_{gf}] \in \mathbb{H}^{m \times t}$ in (24) can be expressed as

$$x_{ij} = \frac{\sum_{\beta \in I_{r_1,n}\{i\}} \text{cdet}_i \left((A^* A)_{\cdot i} (\mathbf{c}_{\cdot j}^{(1)}) \right)_\beta^\beta}{\sum_{\beta \in I_{r_1,n}} |(A^* A)_\beta^\beta|},$$

where $\mathbf{c}_{\cdot j}^{(1)}$ is the j -th columns of $A^* C$, and

$$y_{gf} = \frac{\sum_{\alpha \in I_{r_2,t}\{f\}} \text{rdet}_f \left((BB^*)_{\cdot f} (\mathbf{c}_{g \cdot}^{(2)}) \right)_\alpha^\alpha}{\sum_{\alpha \in I_{r_2,t}} |(BB^*)_\alpha^\alpha|} - \frac{\sum_{l=1}^m \sum_{\alpha \in I_{r_1,m}\{l\}} \text{rdet}_l \left((AA^*)_{\cdot l} (\mathbf{a}_{g \cdot}^{(1)}) \right)_\alpha^\alpha \sum_{\alpha \in I_{r_2,t}\{f\}} \text{rdet}_f \left((BB^*)_{\cdot f} (\mathbf{c}_{l \cdot}^{(2)}) \right)_\alpha^\alpha}{\sum_{\alpha \in I_{r_1,m}} |(AA^*)_\alpha^\alpha| \sum_{\alpha \in I_{r_2,t}} |(BB^*)_\alpha^\alpha|},$$

where $\mathbf{c}_{g \cdot}^{(2)}$ and $\mathbf{a}_{g \cdot}^{(1)}$ are the g -th rows of CB^* and AA^* , respectively.

Remark 16. Note that although Kyrchei [58] gave the determinant representation of the particular solution of Equation (22), the problem proposed in Remark 15 is not solved completely. That is to say, Cramer's rule for representing the general solution of Equation (22) only using the coefficient matrices remains an unsolved problem.

Interestingly, Song [65] considered the determinant representation for the general solution of Equation (22) with the restricted conditions, i.e.,

$$\text{Equation (22) subject to } \begin{cases} \mathcal{R}_r(X) \subseteq T_1, \mathcal{N}_r(X) \supseteq S_1, \\ \mathcal{R}_l(Y) \subseteq T_2, \mathcal{N}_l(Y) \supseteq S_2, \end{cases} \quad (25)$$

where $T_1 \subseteq \mathbb{H}^n$, $S_1 \subseteq \mathbb{H}^p$, $T_2 \subseteq \mathbb{H}^{1 \times t}$, and $S_2 \subseteq \mathbb{H}^{1 \times m}$. Let $P_{\mathcal{T}} \in \mathbb{H}^{n \times n}$ ($Q_{\mathcal{T}} \in \mathbb{H}^{n \times n}$) denote the right (left) $\mathbb{H}$ -orthogonal projector onto a right (left) $\mathbb{H}$ -vector subspace $\mathcal{T} \in \mathbb{H}^{n \times 1}$ ($\mathcal{T} \in \mathbb{H}^{1 \times n}$) along $\mathcal{T}^\perp$.

Theorem 13 (Theorem 3.2, [65]). *Let A, B, C, T_1, T_2, S_1 , and S_2 be given in (25), and let*

$$M = R_{AP_{T_1}} Q_{S_2^\perp}, N = Q_{T_2} B P_{S_1^\perp}, T = A^* R_{Q_{S_2^\perp}} A + A^* A, S = I + L_{Q_{T_2} B},$$

and Y_1 be such that

$$A^* R_{Q_{S_2^\perp}} A T^\dagger Y_1 = A^* A T^\dagger A^* R_{Q_{S_2^\perp}} C \text{ and } Y_1 L_{Q_{T_2} B} = A^* C L_{Q_{T_2} B}.$$

(1) *The restricted Equation (25) is solvable if and only if*

$$R_M R_{AP_{T_1}} C = 0, R_{AP_{T_1}} C L_{Q_{T_2} B} = 0, C L_{P_{S_1^\perp}} L_N = 0, R_{Q_{S_2^\perp}} C L_{Q_{T_2} B} = 0,$$

in which case,

$$\begin{aligned} X &= (T P_{T_1})^\dagger \tilde{C} P_{S_1^\perp} + P_{T_1} L_{AP_{T_1}} U_1 P_{S_1^\perp} + P_{T_1} V_1 P_{S_1^\perp} \\ &= (T P_{T_1})^\dagger \tilde{C} P_{S_1^\perp} + P_{T_1} Z_1 P_{S_1^\perp} - P_{T_{11}} Z_1 P_{S_1^\perp} \\ &= (T P_{T_1})^\dagger \tilde{C} P_{S_1^\perp} + P_{T_1 \cap \mathcal{N}_r(A)} U_2 P_{S_1^\perp} + P_{T_1} V_2 P_{S_1^\perp}, \\ Y &= Q_{S_2^\perp} (C - A X) (Q_{T_2} B)^\dagger + Q_{S_2^\perp} V_3 R_{Q_{T_2} B} Q_{T_2}, \end{aligned}$$

where

$$\tilde{C} = \left(A^* R_{Q_{S_2^\perp}} C + A^* C L_{Q_{T_2} B} + A^* R_{Q_{S_2^\perp}} C L_{Q_{T_2} B} + Y_1 \right) S^{-1},$$

and Z_1, V_1, U_1, V_2, U_2 , and V_3 are arbitrary matrices over $\mathbb{H}$ with appropriate dimensions.

(2) *Let C_1^*, K_1^*, C_1 , and K_2 be full column rank matrices such that*

$$T_1 = \mathcal{N}_r(C_1), T_1 \cap \mathcal{N}_r(T) = \mathcal{R}_r(K_1^*), T_2 = \mathcal{N}_l(C_2), T_2 \cap \mathcal{N}_r(B) = \mathcal{R}_l(K_2^*).$$

Denote $X = [x_{ij}] \in \mathbb{H}^{n \times q}$ and $Y = [y_{kl}] \in \mathbb{H}^{m \times t}$. If Equation (25) is solvable, then

$$\begin{aligned} x_{ij} &= \frac{\text{cdet}_i(A^* A + C_1^* C_1 + K_1^* K_1)_{\cdot i}(d_{\cdot j})}{\det(A^* A + C_1^* C_1 + K_1^* K_1)}, \\ y_{kl} &= \frac{\text{rdet}_l(B B^* + C_2 C_2^* + K_2 K_2^*)_{\cdot l}(d_{k \cdot})}{\det(B B^* + C_2 C_2^* + K_2 K_2^*)} \end{aligned}$$

with $d_{\cdot j}$ is the j -th column of

$$T \left(A^* R_{Q_{S_2^\perp}} C + A^* C L_{Q_{T_2} B} + A^* R_{Q_{S_2^\perp}} C L_{Q_{T_2} B} + Y_1 \right) + K_1^* K_1 Z_1,$$

and $d_{k \cdot}$ is the i -th row of

$$(C - A X) B^* + Z_2 K_2 K_2^*,$$

where $i = 1, \dots, n$, $j = 1, \dots, q$, $k = 1, \dots, m$, $l = 1, \dots, t$, and Z_1 and Z_2 are arbitrary matrices over $\mathbb{H}$ with appropriate orders.

4.7. Method by Semi-Tensor Products

To effectively handle multidimensional arrays and nonlinear problems, Chinese mathematician Daizhan Cheng [66] pioneered a new matrix product in 2001: the semi-tensor product (abbreviated as STP). It exactly coincides with traditional matrix products when the two factor matrices meet the dimension requirements. Due to the excellent properties of STP [67], namely

- (1) It is applied to any two matrices;
- (2) It has certain commutative properties;
- (3) It inherits all properties of the conventional matrix product;
- (4) It enables easy expression of multilinear functions (mappings);

STP has been effectively applied to Boolean (control) networks, logical dynamic systems, system biology, graph theory, formation control, finite automata and symbolic dynamics, circuit design and failure detection, coding and cryptography, fuzzy control, engineering, game theory, and so on (see [67–71] and references therein).

Definition 3 ([66]). Let $\mathbb{F}$ be a ring, $A \in \mathbb{F}^{m \times n}$, and $B \in \mathbb{F}^{p \times q}$. The left STP of A and B is defined as

$$A \ltimes B = \left(A \otimes I_{\frac{t}{n}} \right) \left(B \otimes I_{\frac{t}{p}} \right),$$

where $t = \text{lcm}(n, p)$ is the least common multiple of n and p .

Remark 17. Similar to the left STP, Cheng [68] defined the right STP of $A \in \mathbb{F}^{m \times n}$ and $B \in \mathbb{F}^{p \times q}$, i.e.,

$$A \rtimes B = \left(I_{\frac{t}{n}} \otimes A \right) \left(I_{\frac{t}{p}} \otimes B \right),$$

where $t = \text{lcm}(n, p)$. Both (Proposition 1.3, [67]) and (Proposition 2.3.2, [70]) show that the block multiplication of matrices can be extended to the left STP. Unfortunately, the right STP does not satisfy the block multiplication rule, which is a significant difference between the left STP and right STP. Next, we use a simple example from [70] to demonstrate this limitation of the right STP: let $A = [a_1 \ a_2 \ a_3 \ a_4]$ and $B = [b_1 \ b_2]^T$. Then,

$$A \ltimes B = [a_1 b_1 + a_3 b_2 \ a_2 b_1 + a_4 b_2] \text{ and } A \rtimes B = [a_1 b_1 + a_2 b_2 \ a_3 b_1 + a_4 b_2].$$

Partition A and B equally into $A = [A_1 \ A_2]$ and $B = [B_1 \ B_2]^T$, respectively. According to the block multiplication rule, we have

$$\begin{aligned} A_1 \ltimes B_1 + A_2 \ltimes B_2 &= [a_1 \ a_2] \ltimes b_1 + [a_3 \ a_4] \ltimes b_2 = [a_1 b_1 + a_3 b_2 \ a_2 b_1 + a_4 b_2], \\ A_1 \rtimes B_1 + A_2 \rtimes B_2 &= [a_1 \ a_2] \rtimes b_1 + [a_3 \ a_4] \rtimes b_2 = [a_1 b_1 + a_3 b_2 \ a_2 b_1 + a_4 b_2]. \end{aligned}$$

Obviously,

$$A \ltimes B = A_1 \ltimes B_1 + A_2 \ltimes B_2, \text{ but } A \rtimes B \neq A_1 \rtimes B_1 + A_2 \rtimes B_2.$$

This drawback makes the right STP unable to replace the left STP in many applications, greatly limiting its use. Therefore, most scholars focus exclusively on research into the left STP. Based on this, the STP referred to in this paper specifically denotes the left STP.

Remark 18. As a generalization of the traditional matrix product, and given its extensive application significance, the study of matrix equations under STP has emerged as an inevitable and pivotal research direction. In 2016, Yao et al. [72] were the first to conduct research on the classical complex matrix equation under STP:

$$A \ltimes X = B,$$

where X is unknown. Subsequently, building on the research framework of [72], investigations into diverse matrix equations under STP have flourished (see [73–78]).

Regrettably, no studies directly addressing Equation (5) under STP have been published to date. However, when the dimension of Y is constrained to equal that of X^T , the exploration of Sylvester-transpose matrix Equation [74]

$$A \ltimes X + X^T \ltimes B = C$$

with unknown X may provide valuable guidance for this problem.

Notably, in 2019, Cheng and Liu [79], inspired by the research on cross-dimensional linear systems [80], further proposed the second matrix–matrix semi-tensor product (abbreviated as MM-2 STP), denoted by $\circ_l$. Subsequently, Wang [81] investigated the complex matrix equation under the MM-2 STP:

$$A \circ_l X = B,$$

where X is unknown, and A and B are given. So, exploring Equation (1) under MM-2 STP emerges as a potential research topic. Additionally, Cheng [67] introduced the dimension-free matrix theory, which systematically elucidates the deep mathematical insights underlying STP.

Since 2020, Liaocheng University’s team led by Professors Ying Li and Jianli Zhao has utilized STP to propose several novel matrix representations, including complex, quaternion, and octonion matrices. These representations have also been applied to solving corresponding matrix equations, and have yielded effective numerical experimental results.

Ding et al. [82] first proposed the real vector representation for quaternion matrices, whose properties were characterized by STP, as follows:

Definition 4 (Definitions 3.1–3.3, [82]). Let $x = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k} \in \mathbb{H}$. Denote

$$v^R(x) = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T.$$

Let $x = \begin{bmatrix} x^1 & \dots & x^n \end{bmatrix} \in \mathbb{H}^{1 \times n}$ and $y = \begin{bmatrix} y^1 & \dots & y^n \end{bmatrix}^T \in \mathbb{H}^n$. Denote

$$v^R(x) = \begin{bmatrix} v^R(x^1) \\ \vdots \\ v^R(x^n) \end{bmatrix} \text{ and } v^R(y) = \begin{bmatrix} v^R(y^1) \\ \vdots \\ v^R(y^n) \end{bmatrix}.$$

For $A \in \mathbb{H}^{m \times n}$, the real column stacking form $v_c^R(A)$ and the real row stacking form $v_r^R(A)$ of A are defined as

$$v_c^R(A) = \begin{bmatrix} v^R(\text{Col}_1(A)) \\ v^R(\text{Col}_2(A)) \\ \vdots \\ v^R(\text{Col}_n(A)) \end{bmatrix} \text{ and } v_r^R(A) = \begin{bmatrix} v^R(\text{Row}_1(A)) \\ v^R(\text{Row}_2(A)) \\ \vdots \\ v^R(\text{Row}_m(A)) \end{bmatrix},$$

where $\text{Col}_j(A)$ ($j = 1, \dots, n$) and $\text{Row}_i(A)$ ($j = 1, \dots, m$) are the j -th column and i -th row of A , respectively.

Remark 19. For $A \in \mathbb{H}^{m \times n}$ and $B \in \mathbb{H}^{n \times p}$, Theorem 3.3(3) of [82] shows

$$v_r^R(AB) = G(v_r^R(A) \times v_c^R(B)),$$

where

$$G = \begin{bmatrix} F \times (\delta_m^1)^T \times [I_{4mn} \otimes (\delta_p^1)^T] \\ \vdots \\ F \times (\delta_m^1)^T \times [I_{4mn} \otimes (\delta_p^p)^T] \\ \vdots \\ F \times (\delta_m^m)^T \times [I_{4mn} \otimes (\delta_p^1)^T] \\ \vdots \\ F \times (\delta_m^m)^T \times [I_{4mn} \otimes (\delta_p^p)^T] \end{bmatrix}, \quad F = M_Q \times \left(\sum_{i=1}^n (\delta_n^i)^T \times (I_{4n} \otimes (\delta_n^i)^T) \right),$$

$$M_Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

and δ_m^i ($i = 1, \dots, m$) is the i -th column of I_m .

Using the real vector representation for quaternion matrices, Ding et al. [82] further discussed the special least-squares solutions of quaternion matrix equation

$$AXB + CYD = E, \quad (26)$$

where $A \in \mathbb{H}^{m \times n}$, $B \in \mathbb{H}^{n \times s}$, $C \in \mathbb{H}^{m \times k}$, $D \in \mathbb{H}^{k \times s}$, and $E \in \mathbb{H}^{m \times s}$. Denote

$$\delta_k[i_1, \dots, i_s] = [\delta_k^{i_1}, \dots, \delta_k^{i_s}],$$

$$W_{[m,n]} = \delta_{mn}[1, \dots, (n-1)m+1, \dots, m, \dots, nm],$$

and

$$J^\eta = \begin{bmatrix} J_1^\eta \\ \vdots \\ J_m^\eta \\ \vdots \\ J_n^\eta \end{bmatrix}, \quad J_m^\eta = \begin{bmatrix} J_{1m}^\eta \\ \vdots \\ J_{rm}^\eta \\ \vdots \\ J_{nm}^\eta \end{bmatrix}, \quad R^\eta = \begin{bmatrix} R_1^\eta \\ \vdots \\ R_m^\eta \\ \vdots \\ R_n^\eta \end{bmatrix}, \quad R_m^\eta = \begin{bmatrix} R_{1m}^\eta \\ \vdots \\ R_{rm}^\eta \\ \vdots \\ R_{nm}^\eta \end{bmatrix}, \quad m = 1, 2, \dots, n,$$

where

$$\begin{aligned}
 \text{for } \eta = \mathbf{i}, J_{rm}^{\mathbf{i}} &= \begin{cases} \left(\delta_{n(n+1)/2}^{\frac{(r-1)(2n-r+2)}{2} + m - r + 1} \right)^T \otimes R_4, & r < m, \\ \left(\delta_{n(n+1)/2}^{\frac{(m-1)(2n-m+2)}{2} + r - m + 1} \right)^T \otimes I_4, & r \geq m, \end{cases} & R_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \\
 \text{for } \eta = \mathbf{j}, J_{rm}^{\mathbf{j}} &= \begin{cases} \left(\delta_{n(n+1)/2}^{\frac{(r-1)(2n-r+2)}{2} + m - r + 1} \right)^T \otimes L_4, & r < m, \\ \left(\delta_{n(n+1)/2}^{\frac{(m-1)(2n-m+2)}{2} + r - m + 1} \right)^T \otimes I_4, & r \geq m, \end{cases} & L_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \\
 \text{for } \eta = \mathbf{k}, J_{rm}^{\mathbf{k}} &= \begin{cases} \left(\delta_{n(n+1)/2}^{\frac{(r-1)(2n-r+2)}{2} + m - r + 1} \right)^T \otimes S_4, & r < m, \\ \left(\delta_{n(n+1)/2}^{\frac{(m-1)(2n-m+2)}{2} + r - m + 1} \right)^T \otimes I_4, & r \geq m, \end{cases} & S_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}; \\
 \text{for } \eta = \mathbf{i}, R_{rm}^{\mathbf{i}} &= \begin{cases} \left(\delta_{n(n+1)/2}^{\frac{(r-1)(2n-r+2)}{2} + m - r + 1} \right)^T \otimes R'_4, & r < m, \\ \left(\delta_{n(n+1)/2}^{\frac{(m-1)(2n-m+2)}{2} + r - m + 1} \right)^T \otimes I_4, & r \geq m, \end{cases} & R'_4 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}; \\
 \text{for } \eta = \mathbf{j}, R_{rm}^{\mathbf{j}} &= \begin{cases} \left(\delta_{n(n+1)/2}^{\frac{(r-1)(2n-r+2)}{2} + m - r + 1} \right)^T \otimes L'_4, & r < m, \\ \left(\delta_{n(n+1)/2}^{\frac{(m-1)(2n-m+2)}{2} + r - m + 1} \right)^T \otimes I_4, & r \geq m, \end{cases} & L'_4 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}; \\
 \text{for } \eta = \mathbf{k}, R_{rm}^{\mathbf{k}} &= \begin{cases} \left(\delta_{n(n+1)/2}^{\frac{(r-1)(2n-r+2)}{2} + m - r + 1} \right)^T \otimes S'_4, & r < m, \\ \left(\delta_{n(n+1)/2}^{\frac{(m-1)(2n-m+2)}{2} + r - m + 1} \right)^T \otimes I_4, & r \geq m, \end{cases} & S'_4 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
 \end{aligned}$$

Theorem 14 (Theorem 4.3 and Corollary 4.4, [82]). Let A , B , C , D , and E be given in Equation (26), and let

$$\hat{M} = \begin{bmatrix} M_1 & M_2 \end{bmatrix},$$

where

$$\begin{aligned}
 M_1 &= G_2 \times G_3 \times v_r^R(A) \times W_{[4ns, 4n^2]} \times v_c^R(B) \times J^\eta, \\
 M_2 &= G_4 \times G_5 \times v_r^R(C) \times W_{[4ks, 4k^2]} \times v_c^R(D) \times R^\eta,
 \end{aligned}$$

and G_i has the same structure as G given in Remark 19 but differs in orders.

(1) Then, Equation (26) is consistent if and only if

$$(\hat{M}\hat{M}^\dagger - I_{4ms})v_r^R(E) = 0.$$

(2) Let

$$S_M = \left\{ (X, Y) \mid X = X^{\eta*}, Y = -Y^{\eta*}, \|AXB + CYD - E\| = \min \right\}.$$

Then,

$$S_M = \left\{ (X, Y) \mid \begin{bmatrix} v_s^R(X) \\ v_s^R(Y) \end{bmatrix} = \hat{M}^\dagger v_r^R(E) + (I_{2(n^2+k^2)+2(n+k)} - \hat{M}^\dagger \hat{M})y, y \in \mathbb{R}^{2(n^2+k^2)+2(n+k)} \right\}.$$

(3) If $(\hat{X}, \hat{Y}) \in S_M$ satisfies

$$\|\hat{X}\|^2 + \|\hat{Y}\|^2 = \min_{(X,Y) \in S_M} (\|X\|^2 + \|Y\|^2),$$

then

$$\begin{bmatrix} v_s^R(\hat{X}) \\ v_s^R(\hat{Y}) \end{bmatrix} = \hat{M}^\dagger v_r^R(E).$$

Remark 20. Setting B and C in Equation (26) as identity matrices, Theorem 14 yields the corresponding results for Equation (5) over $\mathbb{H}$.

Remark 21. During the same period, Ding et al. [83] and Wang et al. [84] also developed the real vector representation of quaternion matrices and applied this method to address problems in quaternion matrix equations. Inspired by this idea, Liu et al. [85] proposed the left and right real element representations of octonion matrices to solve the classical octonion matrix equation

$$AXB = C,$$

where X is unknown. Recently, Chen and Song [86] used the real vector representation of quaternion matrices to study the least-squares lower (upper) triangular Toeplitz solutions and (anti)centrosymmetric solutions of quaternion matrix equations.

Remark 22. It is known that real representations of quaternion matrices are not unique. For instance, Liu et al. [87] defined three real representations of quaternion matrices. Interestingly, Fan et al. first [88,89] proposed the $\mathcal{L}$ -representation for quaternion matrices by STP to systematically study the real representations of quaternion matrices. Moreover, $\mathcal{L}$ -representation serves as an effective tool in solving quaternion matrix equations. Meanwhile, Zhang et al. [90] also defined the $\mathcal{L}$ -representation for commutative quaternion matrices and applied it to solving the corresponding matrix equations.

Following this idea, Fan et al. [91] established the $\mathcal{C}$ -representation of quaternion matrices, which generalizes the complex representation of quaternion matrices. This new representation is also applied to study η -Hermitian solutions of the quaternion matrix equation

$$AXB = C,$$

with unknown X . Similarly, Xi et al. [92] defined the $\mathcal{L}_C$ -representation of reduced biquaternion matrices to investigate the mixed solutions of the reduced biquaternion matrix equation

$$\sum_{i=1}^n A_i X_i B_i = E, \quad (27)$$

where X_i ($i = 1, \dots, n$) is unknown. Evidently, Equation (5) is the special case of Equation (27) over reduced biquaternions.

Contemporaneously with [88,89], Fan et al. [93] also derived the minimal norm least-squares (anti)-Hermitian solution of the quaternion matrix equation directly by the vectorization properties of STP (i.e., Theorems 2.9 and 2.10, [93]) and the complex representation of quaternion matrix. Recently, Liu et al. [94] directly used the vectorization properties of STP to investigate (skew) bisymmetric solutions of the generalized Lyapunov quaternion matrix equation.

We contend that the four aforementioned STP-based methods—specifically, $\mathcal{L}$ -representation, $\mathcal{C}$ -representation, $\mathcal{L}_C$ -representation, and vectorization properties of STP—provide distinct perspectives for the investigation of Equation (1).

5. Constrained Solutions of GSE

When GSE fails to satisfy solvability conditions, find its least-squares solution under a certain matrix norm; since such solutions are always non-unique, further seeking for the minimum-norm least-squares solution (also known as the best approximate solution) is a common research approach. Furthermore, in practical applications, specific constraints are often imposed on the solutions of GSE (e.g., symmetric solutions, re-(non)positive definite solutions, equality-constrained solutions). Thus, this section focuses on various solutions to GSE.

5.1. Chebyshev Solutions and l_p -Solutions

Let $\mathbb{F} = \mathbb{R}$. Ziętak studied the Chebyshev solutions [95] and the l_p -solutions [96] of Equation (5) by using the Chebyshev norm and the l_p -norm of a matrix, respectively.

Definition 5. Let $A = [a_{ij}] \in \mathbb{R}^{m \times n}$ and $1 < p < \infty$. The Chebyshev norm of A , denoted by $\|A\|_\infty$, is defined as

$$\|A\|_\infty = \max_{1 \leq i \leq m, 1 \leq j \leq n} |a_{ij}|,$$

where $|a_{ij}|$ is the absolute value of a_{ij} for $1 \leq i \leq m$ and $1 \leq j \leq n$. And, the l_p -norm of A , denoted by $\|A\|_p$, is defined as

$$\|A\|_p = \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^p \right)^{1/p}.$$

Theorem 15 (Theorem 2.2, [95]). Let $r < m$ and $s < n$. Suppose that (4) is not satisfied. Then, the matrices $X_\infty = [x_{kj}]$ and $Y_\infty = [y_{il}]$ are a Chebyshev solution pair of Equation (5), i.e.,

$$\|AX_\infty + Y_\infty B - C\|_\infty = \min_{X,Y} \|AX + YB - C\|_\infty,$$

if and only if there exists $V = [v_{ij}] (\neq 0) \in \mathbb{R}^{m \times n}$ such that $V^T A = 0$, $VB^T = 0$,

$$v_{ij} \text{sign}(r_{ij}) > 0 \text{ for } (i, j) \in \mathcal{J}_1, \text{ and } v_{ij} = 0 \text{ for } (i, j) \notin \mathcal{J}_1,$$

where

$$r_{ij} = \sum_{k=1}^r a_{ik} x_{kj} + \sum_{l=1}^s y_{il} b_{lj} - c_{ij},$$

and $\mathcal{J}_1$ is an appropriate subset of $\mathcal{J} = \{(i, j) \mid |r_{ij}| = \|AX + YB - C\|_\infty\}$.

Remark 23. Moreover, Ziętak formulated the equivalent conditions for the Chebyshev solution of Equation (5) by Theorem 3.3 of [95] and Theorem 4.1 of [95] under the assumption:

$$m = r + 1, n = s + 1, \text{rank}(A) = r, \text{rank}(B) = s, \quad (28)$$

and another assumption:

$$A \neq 0, B \neq 0, r = s = 1, \quad (29)$$

respectively.

Theorem 16 (Theorem 2.1, [96]). Let $r < m$, $s < n$, and $1 < p < \infty$. Then, the matrices $X_p = [x_{ij}]$ and $Y_p = [y_{ij}]$ are an l_p -solution of Equation (5), i.e.,

$$\|AX_p + Y_p B - C\|_p = \min_{X,Y} \|AX + YB - C\|_p,$$

if and only if

$$V^T A = 0 \text{ and } VB^T = 0,$$

where

$$V = [v_{ij}]_{m \times n} = \left[\text{sign}(r_{ij}) |r_{ij}|^{p-1} \right]_{m \times n} \text{ and } r_{ij} = \sum_{k=1}^r a_{ik} x_{kj} + \sum_{l=1}^s y_{il} b_{lj} - c_{ij}$$

for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

Ziętak [96] further presented additional characterizations of the l_p -solutions of Equation (5). Based on these characterizations, an algorithm was designed for solving such solutions (see Section 7 for details).

Theorem 17 (Theorems 2.2 and 2.3, [96]). *Let A , B , and C be given in Equation (5) and let $1 < p < \infty$. Then, the following are equivalent:*

- (1) *The matrices $X_* \in \mathbb{R}^{r \times n}$ and $Y_* \in \mathbb{R}^{m \times s}$ are the l_p -solution of Equation (5);*
- (2) *The following equalities hold:*

$$\min_X \|AX + Y_*B - C\|_p = \min_Y \|AX_* + YB - C\|_p = \|AX_* + Y_*B - C\|_p;$$

- (3) *The columns $x_j^{(*)}$ of $X_* \in \mathbb{R}^{r \times n}$ are the l_p -solutions of the linear systems*

$$Ax = c_j - Y_*b_j \quad (j = 1, 2, \dots, n)$$

and the columns $y_i^{()}$ of $Y_*^T \in \mathbb{R}^{s \times m}$ are the l_p -solutions of the linear systems*

$$B^T y = d_i - X_*^T a_i \quad (i = 1, 2, \dots, m),$$

where $x \in \mathbb{R}^r$, $y \in \mathbb{R}^s$, c_j , b_j , d_i and a_i are the appropriate columns of C , B , C^T , and A^T , respectively.

Remark 24. *Since*

$$A(X + (I - A^-A)W - A^-ZB) + (Y + Z(I - BB^-) + AA^-ZBB^-)B = AX + YB$$

for arbitrary W and Z , neither the Chebyshev solution nor the l_p -solution of Equation (5) is unique.

Remark 25. *Note that, in Theorem 2.1 of [27], Xu et al. gave the explicit expression of the l_2 -solutions of Equation (12). Moreover, in Theorems 4.2 and 4.3 of [97], Liao et al. also consider the best approximate solution of (12) to a given matrix pair (X_f, Y_f) by using GSVD and CCD. Therefore, when both E and F are identity matrices, we can immediately obtain decomposed expressions of l_2 -solutions and the best approximate solution of Equation (5).*

5.2. $\star$ -Congruent Solutions

We now discuss Equation (5) under the constraint condition $Y = X^*$, i.e.,

$$AX + X^*B = C, \tag{30}$$

where X^* denotes either X^T or X^* . We call X satisfying Equation (30) a $\star$ -congruent solution of Equation (5).

Wimmer is the first to study the necessary and sufficient conditions for the solvability of Equation (30) under $\star = *$ over $\mathbb{C}$ (see Theorem 2, [98]). After that, De Terán and Dopico [99] generalize the Wimmer's work to a field $\mathbb{F}$ with $\text{char}(\mathbb{F}) \neq 2$, in which case, X^* denotes the transpose of X , except in the particular case $\mathbb{F} = \mathbb{C}$, where it may be either the transpose or the conjugate transpose of X . Moreover, we call $A \in \mathbb{F}^{n \times n}$ and $B \in \mathbb{F}^{n \times n}$ are $\star$ -congruent if there exists a nonsingular matrix $P \in \mathbb{F}^{n \times n}$ such that $P^*AP = B$.

Theorem 18 (Theorem 2.3, [99]). *Let $\mathbb{F}$ be a field with $\text{char}(\mathbb{F}) \neq 2$. Then, Equation (30) is solvable if and only if*

$$\begin{bmatrix} C & A \\ B & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$$

are $\star$ -congruent.

Remark 26. Additionally, in Lemma 5.10 of [100], Byers and Kressner established the equivalent conditions for the existence of a unique solution to Equation (30) only when $\star = T$. Kressner et al. generalized this result in Lemma 8 of [101] to include the case where $\star = *$. In Theorems 2.1 and 2.2 of [102], Cvetković-Ilić investigated Equation (30) for the bounded linear operators under certain conditions, in which case X^* denotes the adjoint operator of X .

5.3. (Minimum-Norm Least-Squares) Symmetric Solutions

Let $\mathbb{F} = \mathbb{R}$, $s = m$, $r = n$, and $A = B$ for Equation (5), i.e.,

$$AX + YA = C. \quad (31)$$

Chang and Wang [103] studied the symmetric, minimum-2-norm symmetric, least-squares symmetric, and minimum-2-norm least-squares symmetric solutions of Equation (31) by SVD over $\mathbb{R}$. Let $\mathbb{SR}^{m \times m}$ be the set of all $m \times m$ real symmetric matrices, and let

$$A \circ B = [a_{ij}b_{ij}] \in \mathbb{F}^{m \times n}$$

denote the Hadamard product of $A = [a_{ij}] \in \mathbb{F}^{m \times n}$ and $B = [b_{ij}] \in \mathbb{F}^{m \times n}$.

Theorem 19 (Theorem 2.1, [103]). Let the SVD of A be

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T,$$

where $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_r) > 0$, $r = \text{rank}(A)$, and

$$U = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \in \mathbb{R}^{m \times m} \text{ and } V = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

are real orthogonal with

$$U_1 \in \mathbb{R}^{m \times r}, U_2 \in \mathbb{R}^{m \times (m-r)}, V_1 \in \mathbb{R}^{n \times r}, \text{ and } V_2 \in \mathbb{R}^{n \times (n-r)}.$$

Denote

$$W_1 = \Sigma U_1^T C V_1 - V_1^T C^T U_1 \Sigma, W_2 = \Sigma^{-1} (U_1^T C V_1 + V_1^T C^T U_1), \\ M_1 = \Sigma V_1^T C^T U_1 - U_1^T C V_1 \Sigma, M_2 = \Sigma^{-1} U_1^T C V_1.$$

Define

$$\Phi = [\varphi_{ij}] \in \mathbb{R}^{r \times r} \text{ and } \Psi = [\psi_{ij}] \in \mathbb{R}^{r \times r},$$

where

$$\varphi_{ij} = \begin{cases} \frac{1}{\sigma_i^2 - \sigma_j^2}, & \sigma_i \neq \sigma_j, \\ 0, & \sigma_i = \sigma_j, \end{cases} \text{ and } \psi_{ij} = 1 - |\text{sign}(\sigma_i - \sigma_j)|$$

for $i, j = 1, \dots, r$.

(1) Let

$$\mathcal{L}_1 = \{[X, Y] \mid X \in \mathbb{SR}^{n \times n}, Y \in \mathbb{SR}^{m \times m}, AX + YA = C\}.$$

Then, $\mathcal{L}_1 \neq \emptyset$ if and only if

$$U_2^T C V_2 = 0 \text{ and } \Psi \circ (U_1^T C V_1 - V_1^T C^T U_1) = 0,$$

in which case

$$\mathcal{L}_I = \left\{ \begin{bmatrix} V \begin{bmatrix} \Phi \circ W_1 + \Psi \circ (M_2 - Y_{11}) & \Sigma^{-1} U_1^T C V_2 \\ V_2^T C^T U_1 \Sigma^{-1} & X_{22} \end{bmatrix} V^T, \\ U \begin{bmatrix} \Phi \circ M_1 + \Psi \circ Y_{11} & \Sigma^{-1} V_1^T C^T U_2 \\ U_2^T C V_1 \Sigma^{-1} & Y_{22} \end{bmatrix} U^T \end{bmatrix} \right. \\ \left. \mid X_{22} \in \mathbb{S}\mathbb{R}^{(n-r) \times (n-r)}, Y_{11} \in \mathbb{S}\mathbb{R}^{r \times r}, Y_{22} \in \mathbb{S}\mathbb{R}^{(m-r) \times (m-r)} \right\}.$$

(2) If $[\hat{X}, \hat{Y}] \in \mathcal{L}_I$ satisfies

$$\|[\hat{X}, \hat{Y}]\|_F = \min_{[X, Y] \in \mathcal{L}_I} (\|\hat{X}\|_F^2 + \|\hat{Y}\|_F^2)^{\frac{1}{2}},$$

then $[\hat{X}, \hat{Y}] \in \mathcal{L}_I$ is unique and

$$\hat{X} = V \begin{bmatrix} \Phi \circ W_1 + \frac{1}{2} \Psi \circ M_2 & \Sigma^{-1} U_1^T C V_2 \\ V_2^T C^T U_1 \Sigma^{-1} & 0 \end{bmatrix} V^T, \\ \hat{Y} = U \begin{bmatrix} \Phi \circ M_1 + \frac{1}{2} \Psi \circ M_2 & \Sigma^{-1} V_1^T C^T U_2 \\ U_2^T C V_1 \Sigma^{-1} & 0 \end{bmatrix} U^T.$$

(3) Let

$$\mathcal{L}_{ILS} = \{[X, Y] \mid X \in \mathbb{S}\mathbb{R}^{n \times n}, Y \in \mathbb{S}\mathbb{R}^{m \times m}, \|AX + YA - C\|_F = \min\}.$$

Then,

$$\mathcal{L}_{ILS} = \left\{ \begin{bmatrix} V \begin{bmatrix} \Phi \circ W_1 + \Psi \circ (\frac{1}{2} W_2 - Y_{11}) & \Sigma^{-1} U_1^T C V_2 \\ V_2^T C^T U_1 \Sigma^{-1} & X_{22} \end{bmatrix} V^T, \\ U \begin{bmatrix} \Phi \circ M_1 + \Psi \circ Y_{11} & \Sigma^{-1} V_1^T C^T U_2 \\ U_2^T C V_1 \Sigma^{-1} & Y_{22} \end{bmatrix} U^T \end{bmatrix} \right. \\ \left. \mid X_{22} \in \mathbb{S}\mathbb{R}^{(n-r) \times (n-r)}, Y_{11} \in \mathbb{S}\mathbb{R}^{r \times r}, Y_{22} \in \mathbb{S}\mathbb{R}^{(m-r) \times (m-r)} \right\}.$$

(4) If $[\hat{X}, \hat{Y}] \in \mathcal{L}_{ILS}$ satisfies

$$\|[\hat{X}, \hat{Y}]\|_F = \min_{[X, Y] \in \mathcal{L}_{ILS}} (\|\hat{X}\|_F^2 + \|\hat{Y}\|_F^2)^{\frac{1}{2}},$$

then $[\hat{X}, \hat{Y}]$ is unique and

$$\hat{X} = V \begin{bmatrix} \Phi \circ W_1 + \frac{1}{4} \Psi \circ W_2 & \Sigma^{-1} U_1^T C V_2 \\ V_2^T C^T U_1 \Sigma^{-1} & 0 \end{bmatrix} V^T, \\ \hat{Y} = U \begin{bmatrix} \Phi \circ M_1 + \frac{1}{4} \Psi \circ W_2 & \Sigma^{-1} V_1^T C^T U_2 \\ U_2^T C V_1 \Sigma^{-1} & 0 \end{bmatrix} U^T.$$

5.4. Self-Adjoint and Positive (Semi)Definite Solutions

In this subsection, we consider Equation (1) with $r = n, s = m$ and $C = 0$, i.e.,

$$AX - YB = 0, \quad (32)$$

which is required in optimal control theory [104].

Jameson et al. [105] explored the symmetric, positive semidefinite, and positive definite real solutions of Equation (32) over $\mathbb{R}$. Subsequently, Dobovišek [106] further studied the self-adjoint, positive semidefinite, positive definite, minimal, and extreme solutions of Equation (32) over $\mathbb{C}$.

Definition 6 ([106]). *If the nonnegative matrix Y_m is such that (X, Y_m) is a solution pair of Equation (32), and $Y_m \leq Y$ (or $Y_m \geq Y$) for all other solution pairs (X, Y) of Equation (32) with the same X , then Y_m is called a minimal (or maximal) solution of Equation (32). The minimal and maximal solutions of Equation (32) are collectively referred to as the extreme solutions of Equation (32).*

Theorem 20 ([106]). *Let A be of full column rank, $m \geq n$,*

$$A = \begin{bmatrix} 0 \\ I \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \text{ and } Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & Y_{22} \end{bmatrix}.$$

- (1) (Theorem 6, [106]) *Assume that AB^* has at least one real eigenvalue or at least one conjugate pair of eigenvalues. Then, Equation (32) has a non-zero self-adjointed solution pair, i.e., $X^* = X$ and $Y^* = Y$.*
- (2) (Theorem 7, [106]) *Equation (32) has a nonzero positive semidefinite solution Y , i.e., $Y \geq 0$, if and only if AB^* has at least one real eigenvalue.*
- (3) (Theorem 8, [106]) *Equation (32) has a positive definite solution Y , i.e., $Y > 0$, if and only if AB^* has only real eigenvalues and is diagonalizable.*
- (4) (Theorem 9, [106]) *Equation (32) has a positive semidefinite solution, i.e., $X \geq 0$, if and only if the matrix A has at least one real eigenvalue, in which case, $X \geq 0$ is nonzero if $\text{rank}(AB^*) > \text{rank}(B_2)$.*
- (5) (Theorem 10, [106]) *Equation (32) has a positive definite solution, i.e., $X > 0$, if and only if all eigenvalues of B_2^* are real, B_2^* is diagonalizable, and $\text{rank}(AB^*) = \text{rank}(A)$. Moreover, if $Y \geq 0$, then $X > 0$ if and only if all eigenvalues of B^* are positive and B^* is diagonalizable.*
- (6) (Theorem 11, [106]) *If $\text{rank}(B) < m$, then Equation (32) with a fixed solution X does not have an extreme solution for Y . If $\text{rank}(B) = m$, the solution Y is unique.*
- (7) (Theorem 12, [106]) *If Equation (32) has a solution pair (X, Y) and $Y \geq 0$, then there exists a minimal solution $Y_m \geq 0$.*

Remark 27. *The expressions of self-adjoint, positive semidefinite, and positive definite solutions are also presented when the solvability conditions are met (see (Theorems 6–10, [106])). Additionally, Dobovišek discussed these solutions for $m < n$ (see (Theorems 13–17, [106])) and for matrices A and B without full rank (see (Theorems 18–20, [106])).*

5.5. Per(Skew)Symmetric and Bi(Skew)Symmetric Solutions

It is well known that (skew)selfconjugate, per(skew)symmetric, and centro(skew)symmetric matrices be applied to information theory, linear system theory, and numerical analysis theory (see [107–110]). Let $\mathbb{F} = \Omega$ be a finite dimensional central algebra with an involution σ and $\text{char}(\Omega) \neq 2$ (see [111]). Wang et al. in [112,113] gave necessary and sufficient conditions for the existence of per(skew)symmetric solutions and bi(skew)symmetric solutions to Equation (1) over Ω , respectively.

Definition 7. *For $A = [a_{i,j}] \in \Omega^{m \times n}$, let*

$$A^* = [\sigma(a_{i,j})] \in \Omega^{n \times m}, A^{(*)} = [\sigma(a_{n-j+1, m-i+1})] \in \Omega^{n \times m}, A^\# = [a_{m-i+1, n-j+1}] \in \Omega^{m \times n}.$$

Then, A is called to be (skew)selfconjugate if $A = A^*(-A^*)$, per(skew)symmetric if $A = A^{(*)}(-A^{(*)})$, and centro(skew)symmetric if $A = A^\#(-A^\#)$. If A is both (skew)selfconjugate and per(skew)symmetric, then A is called to be bi(skew)symmetric. Moreover, a solution pair (X, Y) of Equation (1) is called to be per(skew)symmetric (or bi(skew)symmetric) if both X and Y are per(skew)symmetric (or bi(skew)symmetric).

Theorem 21 ([112]). Let $A, B, C \in \Omega^{m \times n}[\lambda]$.

- (1) (Corollary 2.3, [112]) Equation (1) has a persymmetric solution pair (X, Y) if and only if there exist invertible matrices $P \in \Omega^{2n \times 2n}$ and $Q \in \Omega^{2m \times 2m}$ such that

$$Q \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} P^{-1} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \quad (33)$$

$$P \begin{bmatrix} -I_n & 0 \\ 0 & I_n \end{bmatrix} P^{(*)} = \begin{bmatrix} -I_n & 0 \\ 0 & I_n \end{bmatrix}, \quad (34)$$

$$Q \begin{bmatrix} -I_m & 0 \\ 0 & I_m \end{bmatrix} Q^{(*)} = \begin{bmatrix} -I_m & 0 \\ 0 & I_m \end{bmatrix}. \quad (35)$$

- (2) (Corollary 2.8, [112]) Equation (1) has a perskewsymmetric solution pair (X, Y) if and only if there exist invertible matrices $P \in \Omega^{2n \times 2n}$ and $Q \in \Omega^{2m \times 2m}$ such that (33), $P^{(*)}P = I_{2n}$, and $Q^{(*)}Q = I_{2m}$.
- (3) (Corollary 2.13, [112]) Equation (1) has a solution pair (X, Y) such that X is persymmetric and Y is perskewsymmetric if and only if there exist invertible matrices $P \in \Omega^{2n \times 2n}$ and $Q \in \Omega^{2m \times 2m}$ such that (33), (34), and $Q^{(*)}Q = I_{2m}$.
- (4) (Corollary 2.16, [112]) Equation (1) has a solution pair (X, Y) such that X is perskewsymmetric and Y is persymmetric if and only if there exist invertible matrices $P \in \Omega^{2n \times 2n}$ and $Q \in \Omega^{2m \times 2m}$ such that (33), (35), and $P^{(*)}P = I_{2n}$.

Theorem 22 ([113]). Let $A, B, C \in \Omega^{m \times n}[\lambda]$.

- (1) (Corollary 2, [113]) Equation (1) has a bisymmetric solution pair (X, Y) if and only if there exist invertible matrices $P \in \Omega^{2n \times 2n}$ and $Q \in \Omega^{2m \times 2m}$ such that

$$Q \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} P^{-1} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \quad (36)$$

$$P \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} P^* = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \quad (37)$$

$$P \begin{bmatrix} -I_n & 0 \\ 0 & I_n \end{bmatrix} P^{(*)} = \begin{bmatrix} -I_n & 0 \\ 0 & I_n \end{bmatrix}, \quad (38)$$

$$P^\# \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}, \quad (39)$$

$$Q \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} Q^* = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}, \quad (40)$$

$$Q \begin{bmatrix} -I_m & 0 \\ 0 & I_m \end{bmatrix} Q^{(*)} = \begin{bmatrix} -I_m & 0 \\ 0 & I_m \end{bmatrix}, \quad (41)$$

$$Q^\# \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} Q^{-1} = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}. \quad (42)$$

- (2) (Corollary 5, [113]) Equation (1) has a biskewsymmetric solution pair (X, Y) if and only if there exist invertible matrices $P \in \Omega^{2n \times 2n}$ and $Q \in \Omega^{2m \times 2m}$ such that (36),

$$P^{(*)}P = I_{2n}, \quad P \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} P^* = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} = P^\# \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} P^{-1}, \quad (43)$$

$$Q^{(*)}Q = I_{2m}, \quad Q \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} Q^* = \begin{bmatrix} I_m & 0 \\ 0 & I_m \end{bmatrix} = Q^\# \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} Q^{-1}. \quad (44)$$

- (3) (Corollary 8, [113]) Equation (1) has a solution pair (X, Y) such that X is bisymmetric and Y is biskewsymmetric if and only if there exist invertible matrices $P \in \Omega^{2n \times 2n}$ and $Q \in \Omega^{2m \times 2m}$ such that (36)–(39), and (44).
- (4) (Corollary 10, [113]) Equation (1) has a solution pair (X, Y) such that X is biskewsymmetric and Y is bisymmetric if and only if there exist invertible matrices $P \in \Omega^{2n \times 2n}$ and $Q \in \Omega^{2m \times 2m}$ such that (36), (40)–(42), and (43).

5.6. Maximal and Minimal Ranks of the General Solution

Let $\mathbb{F} = \mathbb{C}$, and let a solution pair (X, Y) of Equation (5) be

$$X = X_0 + X_1 i \in \mathbb{C}^{r \times n} \text{ and } Y = Y_0 + Y_1 i \in \mathbb{C}^{m \times s},$$

where $X_0, X_1 \in \mathbb{R}^{r \times n}$ and $Y_0, Y_1 \in \mathbb{R}^{m \times s}$. Liu [40] determined the maximal and minimal ranks for X, Y, X_0, X_1, Y_0 , and Y_1 .

Theorem 23 (Theorems 2.1 and 2.2, [40]). *Let Equation (5) be consistent.*

- (1) Then,

$$\begin{aligned} \max_{AX+YB=C} \text{rank}(X) &= \min \left\{ n, r, r - \text{rank}(A) + \text{rank} \begin{bmatrix} B \\ C \end{bmatrix} \right\}, \\ \max_{AX+YB=C} \text{rank}(Y) &= \min \{ m, s, s - \text{rank}(B) + \text{rank}[A, C] \}, \\ \min_{AX+YB=C} \text{rank}(X) &= \text{rank} \begin{bmatrix} B \\ C \end{bmatrix} - \text{rank}(B), \\ \min_{AX+YB=C} \text{rank}(Y) &= \text{rank}[A, C] - \text{rank}(A). \end{aligned}$$

- (2) Let

$$\begin{aligned} \mathbb{S}_1 &= \{ X_0 \in \mathbb{R}^{r \times n} \mid A(X_0 + iX_1) + (Y_0 + iY_1)B = C \}, \\ \mathbb{S}_2 &= \{ X_1 \in \mathbb{R}^{r \times n} \mid A(X_0 + iX_1) + (Y_0 + iY_1)B = C \}. \end{aligned}$$

Then,

$$\begin{aligned} \max_{X_0 \in \mathbb{S}_1} r(X_0) &= \min \left\{ r, n, \text{rank} \begin{bmatrix} B_0 & 0 \\ B_1 & 0 \\ C_1 & A_0 \\ -C_0 & A_1 \end{bmatrix} - 2\text{rank}(A) + r \right\}, \\ \min_{X_0 \in \mathbb{S}_1} r(X_0) &= \text{rank} \begin{bmatrix} B_0 & 0 \\ B_1 & 0 \\ C_1 & A_0 \\ -C_0 & A_1 \end{bmatrix} - \text{rank} \begin{bmatrix} A_0 \\ A_1 \end{bmatrix} - \text{rank} \begin{bmatrix} B_0 \\ B_1 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned}\max_{X_1 \in \mathbb{S}_2} r(X_1) &= \min \left\{ r, n, \text{rank} \begin{bmatrix} B_0 & 0 \\ B_1 & 0 \\ C_0 & A_0 \\ C_1 & A_1 \end{bmatrix} - 2\text{rank}(A) + r \right\}, \\ \min_{X_1 \in \mathbb{S}_2} r(X_1) &= \text{rank} \begin{bmatrix} B_0 & 0 \\ B_1 & 0 \\ C_0 & A_0 \\ C_1 & A_1 \end{bmatrix} - \text{rank} \begin{bmatrix} A_0 \\ A_1 \end{bmatrix} - \text{rank} \begin{bmatrix} B_0 \\ B_1 \end{bmatrix}.\end{aligned}$$

(3) Let

$$\begin{aligned}\mathbb{S}_3 &= \{Y_0 \in \mathbb{R}^{m \times s} \mid A(X_0 + iX_1) + (Y_0 + iY_1)B = C\}, \\ \mathbb{S}_4 &= \{Y_1 \in \mathbb{R}^{m \times s} \mid A(X_0 + iX_1) + (Y_0 + iY_1)B = C\}.\end{aligned}$$

Then,

$$\begin{aligned}\max_{Y_0 \in \mathbb{S}_3} r(Y_0) &= \min \left\{ m, s, \text{rank} \begin{bmatrix} A_0 & A_1 & C_1 & -C_0 \\ 0 & 0 & B_0 & B_1 \end{bmatrix} - 2\text{rank}(B) + s \right\}, \\ \min_{Y_0 \in \mathbb{S}_3} r(Y_0) &= \text{rank} \begin{bmatrix} A_0 & A_1 & C_1 & -C_0 \\ 0 & 0 & B_0 & B_1 \end{bmatrix} - \text{rank}[A_0, A_1] - \text{rank}[B_0, B_1], \\ \max_{Y_1 \in \mathbb{S}_4} r(Y_1) &= \min \left\{ m, s, \text{rank} \begin{bmatrix} A_0 & A_1 & C_0 & C_1 \\ 0 & 0 & B_0 & B_1 \end{bmatrix} - 2\text{rank}(B) + s \right\}, \\ \min_{Y_1 \in \mathbb{S}_4} r(Y_1) &= \text{rank} \begin{bmatrix} A_0 & A_1 & C_0 & C_1 \\ 0 & 0 & B_0 & B_1 \end{bmatrix} - \text{rank}[A_0, A_1] - \text{rank}[B_0, B_1].\end{aligned}$$

Remark 28. In (Corollary 2.3, [40]), Liu also presented equivalent conditions for Equation (5) to have a (all) real solution pair(s), i.e.,

$$X = X_0 \text{ and } Y = Y_0,$$

and a (all) pure imaginary solution pair(s), i.e.,

$$X = iX_1 \text{ and } Y = iY_1.$$

However, in (Section 3, [114]), Wang et al. provided two counterexamples to illustrate that the items (a) and (c) in (Corollary 2.3, [40]) are incorrect.

5.7. Re-(Non)negative and Re-(Non)positive Definite Solutions

Let $\mathbb{F} = \mathbb{C}$. For a Hermitian matrix $A \in \mathbb{C}^{n \times n}$, $i_+(A)$, $i_-(A)$, and $i_0(A)$ represent the numbers of the positive, negative, and zero eigenvalues of A , respectively. In Corollary 5.7 of [115], Wang and He established the maximal and minimal values of

$$i_{\pm}(X + X^*) \text{ and } i_{\pm}(Y + Y^*)$$

for a solution pair (X, Y) of the complex matrix Equation

$$AXB + CYD = E, \quad (45)$$

where $A \in \mathbb{C}^{m \times k_2}$, $B \in \mathbb{C}^{k_2 \times q}$, $C \in \mathbb{C}^{m \times k_3}$, $D \in \mathbb{C}^{k_3 \times q}$, and $E \in \mathbb{C}^{m \times q}$ are given. This result directly yields the equivalent conditions for re-positive definite, re-negative definite, re-nonnegative definite, and re-nonpositive definite solutions to Equation (45).

Definition 8 ([115]). Let $A \in \mathbb{C}^{n \times n}$ and $H(A) = A + A^*$. The A is called to be re-positive definite if $H(A) > 0$, re-nonnegative definite if $H(A) \geq 0$, re-negative definite if $H(A) < 0$, and re-nonpositive definite if $H(A) \leq 0$.

Theorem 24 (Corollary 5.8, [115]). Let $X \in \mathbb{C}^{k_2 \times k_2}$ and $Y \in \mathbb{C}^{k_3 \times k_3}$ be a solution pair of Equation (45). Denote

$$\begin{aligned}
 G_1 &= \begin{bmatrix} 0 & -D & C^* & 0 \\ -D^* & 0 & E^* & 0 \\ C & E & 0 & A \\ 0 & 0 & A^* & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 & -C^* & D & 0 \\ -C & 0 & E & 0 \\ D^* & E^* & 0 & B^* \\ 0 & 0 & B & 0 \end{bmatrix}, \\
 G_3 &= \begin{bmatrix} 0 & -B & A^* & 0 \\ -B^* & 0 & E^* & 0 \\ A & E & 0 & C \\ 0 & 0 & C^* & 0 \end{bmatrix}, \quad G_4 = \begin{bmatrix} 0 & -A^* & B & 0 \\ -A & 0 & E & 0 \\ B^* & E^* & 0 & D^* \\ 0 & 0 & D & 0 \end{bmatrix}, \\
 s_1 &= \text{rank} \begin{bmatrix} 0 & -A^* & B & 0 & 0 \\ -A & 0 & E & C & 0 \\ B^* & E^* & 0 & 0 & D^* \end{bmatrix}, \quad s_2 = \text{rank} \begin{bmatrix} 0 & -B & A^* & 0 & 0 \\ -A & E & 0 & C & 0 \\ B^* & 0 & E^* & 0 & D^* \\ 0 & 0 & C^* & 0 & 0 \end{bmatrix}, \\
 s_3 &= \text{rank} \begin{bmatrix} 0 & -B & A^* & 0 & 0 \\ -A & E & 0 & C & 0 \\ B^* & 0 & E^* & 0 & D^* \\ 0 & D & 0 & 0 & 0 \end{bmatrix}, \quad s_4 = \text{rank} \begin{bmatrix} A \\ B^* \end{bmatrix}, \\
 w_1 &= \text{rank} \begin{bmatrix} 0 & -C^* & D & 0 & 0 \\ -C & 0 & E & A & 0 \\ D^* & E^* & 0 & 0 & B^* \end{bmatrix}, \quad w_2 = \text{rank} \begin{bmatrix} 0 & -D & C^* & 0 & 0 \\ -C & E & 0 & A & 0 \\ D^* & 0 & E^* & 0 & B^* \\ 0 & 0 & A^* & 0 & 0 \end{bmatrix}, \\
 w_3 &= \text{rank} \begin{bmatrix} 0 & -D & C^* & 0 & 0 \\ -C & E & 0 & A & 0 \\ D^* & 0 & E^* & 0 & B^* \\ 0 & B & 0 & 0 & 0 \end{bmatrix}, \quad w_4 = \text{rank} \begin{bmatrix} C \\ D^* \end{bmatrix}.
 \end{aligned}$$

Then,

(1) X is re-positive definite if and only if

$$\begin{aligned}
 i_+(G_3) &= \text{rank} \begin{bmatrix} A & C \end{bmatrix} + \text{rank}(B) \text{ and } i_+(G_3) \geq \text{rank} \begin{bmatrix} D \\ B \end{bmatrix} + \text{rank}(A), \\
 \text{or } i_+(G_3) &\geq \text{rank} \begin{bmatrix} A & C \end{bmatrix} + \text{rank}(B) \text{ and } i_+(G_4) = \text{rank} \begin{bmatrix} D \\ B \end{bmatrix} + \text{rank}(A).
 \end{aligned}$$

(2) X is re-negative definite if and only if

$$i_-(G_3) = \text{rank} \begin{bmatrix} A & C \end{bmatrix} + \text{rank}(B) \text{ and } i_-(G_4) \geq \text{rank} \begin{bmatrix} D \\ B \end{bmatrix} + \text{rank}(A),$$

$$\text{or } i_-(G_3) \geq \text{rank} \begin{bmatrix} A & C \end{bmatrix} + \text{rank}(B) \text{ and } i_-(G_4) = \text{rank} \begin{bmatrix} D \\ B \end{bmatrix} + \text{rank}(A).$$

(3) X is re-nonnegative definite if and only if

$$s_1 - s_4 + i_-(G_3) - s_2 = 0 \text{ and } s_1 - s_4 + i_-(G_4) - s_3 \leq 0,$$

$$\text{or } s_1 - s_4 + i_-(G_3) - s_2 \leq 0 \text{ and } s_1 - s_4 + i_-(G_4) - s_3 = 0.$$

(4) X is re-nonpositive definite if and only if

$$s_1 - s_4 + i_+(G_3) - s_2 = 0 \text{ and } s_1 - s_4 + i_+(G_3) - s_3 \leq 0,$$

$$\text{or } s_1 - s_4 + i_+(G_3) - s_2 \leq 0 \text{ and } s_1 - s_4 + i_+(G_4) - s_3 = 0.$$

(5) Y is re-positive definite if and only if

$$i_+(G_1) = \text{rank} \begin{bmatrix} A & C \end{bmatrix} + \text{rank}(D) \text{ and } i_+(G_2) \geq \text{rank} \begin{bmatrix} D \\ B \end{bmatrix} + \text{rank}(C),$$

$$\text{or } i_+(G_1) \geq \text{rank} \begin{bmatrix} A & C \end{bmatrix} + \text{rank}(D) \text{ and } i_+(G_2) = \text{rank} \begin{bmatrix} D \\ B \end{bmatrix} + \text{rank}(C).$$

(6) Y is re-negative definite if and only if

$$i_-(G_1) = \text{rank} \begin{bmatrix} A & C \end{bmatrix} + \text{rank}(D) \text{ and } i_-(G_2) \geq \text{rank} \begin{bmatrix} D \\ B \end{bmatrix} + \text{rank}(C),$$

$$\text{or } i_-(G_1) \geq \text{rank} \begin{bmatrix} A & C \end{bmatrix} + \text{rank}(D) \text{ and } i_-(G_2) = \text{rank} \begin{bmatrix} D \\ B \end{bmatrix} + \text{rank}(C).$$

(7) Y is re-nonnegative definite if and only if

$$w_1 - w_4 + i_-(G_1) - w_2 = 0 \text{ and } w_1 - w_4 + i_-(G_2) - w_3 \leq 0,$$

$$\text{or } w_1 - w_4 + i_-(G_1) - w_2 \leq 0 \text{ and } w_1 - w_4 + i_-(G_2) - w_3 = 0.$$

(8) Y is re-nonpositive definite if and only if

$$w_1 - w_4 + i_+(G_1) - w_2 = 0 \text{ and } w_1 - w_4 + i_+(G_2) - w_3 \leq 0,$$

$$\text{or } w_1 - w_4 + i_+(G_1) - w_2 \leq 0 \text{ and } w_1 - w_4 + i_+(G_2) - w_3 = 0.$$

Remark 29. When both B and C in Theorem 24 are taken as identity matrices with $k_2 = q$ and $m = k_3$, we can immediately obtain the equivalent conditions for the existence of re-positive definite, re-negative definite, re-nonnegative definite, and re-nonpositive definite solutions of Equation (5).

5.8. η -Hermitian and η -Skew-Hermitian Solutions

Let $\mathbb{F} = \mathbb{H}$. The η -Hermitian matrices have been employed in statistical multichannel processing and widely linear modeling (see [116,117]). Yuan and Wang [118] considered the least-squares η -Hermitian solution with the least norm to the quaternion matrix equation

$$AXE + FXB = C$$

with unknown X . After that, He and Wang [119] investigated the η -Hermitian solutions of the matrix equation

$$AXA^{\eta*} + BYB^{\eta*} = C, \quad (46)$$

where A , B , and C are given quaternion matrices with appropriate orders. Moreover, a solution pair (X, Y) of Equation (46) is called to be η -Hermitian (η -skew-Hermitian) if both X and Y are η -Hermitian (η -skew-Hermitian).

Theorem 25 (Corollaries 3.5 and 4.3, [119]). *Let A , B , and C be given over $\mathbb{H}$ such that $C = C^{\eta*}$. Set*

$$M = R_A B \text{ and } S = BL_M.$$

Then, the following are equivalent:

- (1) Equation (46) has an η -Hermitian solution pair (X, Y) ;
- (2) $R_M R_A C = 0$ and $R_A C (R_B)^{\eta*} = 0$;
- (3) $\text{rank} \begin{bmatrix} A & C \\ 0 & B^{\eta*} \end{bmatrix} = \text{rank}(A) + \text{rank}(B)$ and $\text{rank} \begin{bmatrix} A & B & C \end{bmatrix} = \text{rank} \begin{bmatrix} A & B \end{bmatrix}$.

In this case,

$$\begin{aligned} X &= A^\dagger C (A^\dagger)^{\eta*} - \frac{1}{2} A^\dagger B M^\dagger C [I + (B^\dagger)^{\eta*} S^{\eta*}] (A^\dagger)^{\eta*} \\ &\quad - \frac{1}{2} A^\dagger (I + S B^\dagger) C (M^\dagger)^{\eta*} B^{\eta*} (A^\dagger)^{\eta*} - A^\dagger S W_2 S^{\eta*} (A^\dagger)^{\eta*} + L_A U + U^{\eta*} (L_A)^\eta, \\ Y &= \frac{1}{2} M^\dagger C (B^\dagger)^{\eta*} [I + (S^\dagger S)^\eta] + \frac{1}{2} (I + S^\dagger S) B^\dagger C (M^\dagger)^{\eta*} \\ &\quad + L_M W_2 (L_M)^\eta + V L_B^\eta + L_B V^{\eta*} + L_M L_S W_1 + W_1^{\eta*} (L_S)^\eta (L_M)^\eta, \end{aligned}$$

where W_1 , U , V , and $W_2 = W_2^{\eta*}$ are arbitrary quaternion matrices with appropriate sizes, and

$$\begin{aligned} \min_{AXA^{\eta*} + BYB^{\eta*} = C} \text{rank}(X) &= 2\text{rank} \begin{bmatrix} C & B \end{bmatrix} - \text{rank} \begin{bmatrix} 0 & B^{\eta*} \\ B & C \end{bmatrix}, \\ \min_{AXA^{\eta*} + BYB^{\eta*} = C} \text{rank}(Y) &= 2\text{rank} \begin{bmatrix} A & C \end{bmatrix} - \text{rank} \begin{bmatrix} 0 & A^{\eta*} \\ A & C \end{bmatrix}. \end{aligned}$$

Remark 30. Inspired by Theorem 25, we now consider the η -Hermitian solutions of Equation (5) over $\mathbb{H}$, namely finding X and Y over $\mathbb{H}$ such that

$$AX + YB = C, \quad X = X^{\eta*}, \text{ and } Y = Y^{\eta*}, \quad (47)$$

where A , B , and C are given over $\mathbb{H}$. By means of

$$X = \frac{1}{2}(\hat{X} + \hat{X}^{\eta*}) \text{ and } Y = \frac{1}{2}(\hat{Y} + \hat{Y}^{\eta*}),$$

it is easy to check that the following are equivalent:

- (1) The statement (47) holds.
- (2) There exist the matrices $\hat{X}$ and $\hat{Y}$ over $\mathbb{H}$ such that

$$A\hat{X} + \hat{Y}B = C \text{ and } \hat{X}A^{\eta*} + B^{\eta*}\hat{Y} = C^{\eta*}.$$

- (3) There exist the matrices $\hat{X}$ and $\hat{Y}$ over $\mathbb{H}$ such that

$$\begin{bmatrix} A & I \end{bmatrix} \hat{X} \begin{bmatrix} I & 0 \\ 0 & A^{\eta*} \end{bmatrix} + \begin{bmatrix} I & B^{\eta*} \end{bmatrix} \hat{Y} \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} C & C^{\eta*} \end{bmatrix}.$$

Therefore, solving the η -Hermitian solutions of Equation (5) reduces to solving an equation of the form

$$AXB + CYD = E,$$

which has been solved by Baksalary and Kala [120]. By the same method, we have

- (1) There exists a matrix pair (X, Y) such that

$$AX + YB = C \text{ and } X = X^{\eta*},$$

if and only if there exist the matrices $\hat{X}$, $\hat{Y}$, and $\hat{Z}$ such that

$$A\hat{X} + \hat{Y}B = C \text{ and } \hat{X}A^{\eta*} + B^{\eta*}\hat{Z} = C^{\eta*}. \quad (48)$$

- (2) There exists a matrix pair (X, Y) such that

$$AX + YB = C \text{ and } Y = Y^{\eta*},$$

if and only if there exist the matrices $\hat{X}$, $\hat{Y}$, and $\hat{Z}$ such that

$$A\hat{X} + \hat{Y}B = C \text{ and } \hat{Z}A^{\eta*} + B^{\eta*}\hat{Y} = C^{\eta*}. \quad (49)$$

Moreover, solvability conditions and expressions of the general solution to Equations (48) and (49) can be obtained by Theorem 2.1 of [121].

Kyrchei [122] further discussed the η -skew-Hermitian solutions of Equation (46) under $C = -C^{\eta*}$ using a method similar to Theorem 25.

Theorem 26 (Corollary 4.4, [122]). Under the hypotheses of Theorem 25, the following are equivalent:

- (1) Equation (46) has an η -skew-Hermitian solution pair (X, Y) ;
- (2) $R_M R_A C = 0$ and $R_A C (R_B)^{\eta*} = 0$;
- (3) $\text{rank} \begin{bmatrix} A & C \\ 0 & B^{\eta*} \end{bmatrix} = \text{rank}(A) + \text{rank}(B)$ and $\text{rank} \begin{bmatrix} A & B & C \end{bmatrix} = \text{rank} \begin{bmatrix} A & B \end{bmatrix}$;

in which case, the η -skew-Hermitian solutions of Equation (46) are

$$\begin{aligned} X &= A^{\dagger} C (A^{\dagger})^{\eta*} - \frac{1}{2} \left[A^{\dagger} B M^{\dagger} C (A^{\dagger})^{\eta*} + A^{\dagger} C (A^{\dagger} B M^{\dagger})^{\eta*} \right] \\ &\quad - \frac{1}{2} \left[A^{\dagger} B M^{\dagger} C (A^{\dagger} S B^{\dagger})^{\eta*} + A^{\dagger} S B^{\dagger} C (A^{\dagger} B M^{\dagger})^{\eta*} \right] - A^{\dagger} S W_2 (A^{\dagger} S)^{\eta*} - L_A U + (L_A U)^{\eta*}, \\ Y &= \frac{1}{2} \left[M^{\dagger} C (B^{\dagger})^{\eta*} + B^{\dagger} C (M^{\dagger})^{\eta*} \right] + \frac{1}{2} \left[M^{\dagger} C (B^{\dagger})^{\eta*} (Q_S)^{\eta*} + Q_S B^{\dagger} C (M^{\dagger})^{\eta*} \right] \\ &\quad + L_M W_2 L_M^{\dagger} + V L_B^{\dagger} - L_B V^{\eta*} + L_M L_S W_1 - W_1^{\eta*} L_S^{\dagger} L_M^{\eta*}, \end{aligned}$$

where W_1 , U , V , and $W_2 = W_2^{\eta*}$ are arbitrary over $\mathbb{H}$ with appropriate sizes.

Remark 31. Using a method similar to that in Remark 30, one can also consider the η -skew-Hermitian solutions of Equation (5) over $\mathbb{H}$, i.e., finding X and Y such that

$$AX + YB = C, X = -X^{\eta*}, \text{ and } Y = -Y^{\eta*}.$$

Remark 32. In terms of the determinant representations of the MP inverse over $\mathbb{H}$ (i.e., Theorem 10), Kyrchei [122] also showed the Cramer's rules for the partial η -Hermitian and η -skew-Hermitian solution to Equation (46) under $C = C^{\eta*}$ and $C = -C^{\eta*}$, respectively.

5.9. ϕ -Hermitian Solutions

Let $\mathbb{F} = \mathbb{H}$. Rodman [7] introduced the quaternion matrix A_ϕ over $\mathbb{H}$, a generation of A^* and $A^{\eta*}$, and defined the ϕ -Hermitian quaternion matrix as follows:

Definition 9 ([7]). Let $\phi : \mathbb{H} \rightarrow \mathbb{H}$ be a map.

(1) We call ϕ an anti-endomorphism if for any $\alpha, \beta \in \mathbb{H}$, ϕ satisfies

$$\phi(\alpha\beta) = \phi(\beta)\phi(\alpha) \text{ and } \phi(\alpha + \beta) = \phi(\alpha) + \phi(\beta).$$

An anti-endomorphism ϕ is called an involution if ϕ^2 is the identity map.

(2) Let ϕ be a nonzero involution. Then, ϕ can be represented as a matrix in $\mathbb{R}^{4 \times 4}$ with respect to the basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, i.e.,

$$\phi = \begin{bmatrix} 1 & 0 \\ 0 & T \end{bmatrix},$$

where either $T = -I_3$ (in which case ϕ is called a standard involution), or $T \in \mathbb{R}^{3 \times 3}$ is an orthogonal symmetric matrix with the eigenvalues $\{1, 1, -1\}$ (in which case ϕ is called a nonstandard involution).

(3) Let ϕ be a nonstandard involution and $A = [a_{i,j}] \in \mathbb{H}^{m \times n}$. Define

$$\phi(A) = [\phi(a_{i,j})] \in \mathbb{H}^{m \times n} \text{ and } A_\phi = \phi(A^T) \in \mathbb{H}^{n \times m}.$$

If $A = A_\phi$ with $m = n$, then A is called a ϕ -Hermitian matrix.

He et al. [123] considered the ϕ -Hermitian solution $Z = Z_\phi$ of the following system

$$\begin{cases} A_1 X - Y B_1 = C_1, \\ A_2 Z - Y B_2 = C_2, \end{cases} \quad (50)$$

where A_i , B_i , and C_i ($i = 1, 2$) are given matrices over $\mathbb{H}$ with appropriate orders.

Theorem 27 (Theorem 4.5, [123]). Let

$$A_{11} = R_{B_2} B_1, B_{11} = R_{A_2} A_2, C_{11} = B_1 L_{A_{11}}, D_{11} = R_{A_1} (R_{A_2} C_2 B_2^\dagger B_1 - C_1) L_{A_{11}}, A_{22} = [L_{A_2}, -(R_{C_{11}} B_2)_\phi],$$

$$B_{22} = \begin{bmatrix} R_{C_{11}} B_2 \\ -(L_{A_2})_\phi \end{bmatrix}, C_{22} = (A_2^\dagger C_2 L_{B_2})_\phi + (B_2)_\phi (C_{11}^\dagger)_\phi D_{22} (B_{11}^\dagger)_\phi - A_2^\dagger C_2 - B_{11}^\dagger D_{11} C_{11}^\dagger B_2,$$

$$A = R_{A_{22}} L_{B_{11}}, B = B_2 L_{B_{22}}, C = -R_{A_{22}} (B_2)_\phi, D = (L_{B_{11}})_\phi L_{B_{22}}, E = R_{A_{22}} C_{22} L_{B_{22}}, M = R_A C,$$

Then, the following are equivalent:

(1) The system (50) has a solution (X, Y, Z) such that $Z = Z_\phi$.

(2) The following rank equalities hold:

$$\begin{aligned} \text{rank} \begin{bmatrix} C_i & A_i \\ B_i & 0 \end{bmatrix} &= \text{rank}(A_i) + \text{rank}(B_i), i = 1, 2, \\ \text{rank} \begin{bmatrix} C_1 & C_2 & A_1 & A_2 \\ B_1 & B_2 & 0 & 0 \end{bmatrix} &= \text{rank} \begin{bmatrix} A_1 & A_2 \end{bmatrix} + \text{rank} \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \\ \text{rank} \begin{bmatrix} C_1 & C_2(A_2)_\phi - A_2(C_2)_\phi & A_1 & A_2(B_2)_\phi \\ B_1 & B_2(A_2)_\phi & 0 & 0 \end{bmatrix} &= \text{rank} \begin{bmatrix} A_1 & A_2(B_2)_\phi \end{bmatrix} + \text{rank} \begin{bmatrix} B_1 & B_2(A_2)_\phi \end{bmatrix}, \\ \text{rank} \begin{bmatrix} C_2(A_2)_\phi - A_2(C_2)_\phi & A_2(B_2)_\phi \\ B_2(A_2)_\phi & 0 \end{bmatrix} &= 2\text{rank}(A_2(B_2)_\phi), \\ \text{rank} \begin{bmatrix} C_1 & C_2(A_2)_\phi - A_2(C_2)_\phi & A_1 & A_2(B_2)_\phi \\ 0 & -(C_1)_\phi & 0 & (B_1)_\phi \\ B_1 & B_2(A_2)_\phi & 0 & 0 \\ 0 & (A_1)_\phi & 0 & 0 \end{bmatrix} &= 2\text{rank} \begin{bmatrix} A_1 & A_2(B_2)_\phi \\ 0 & (B_1)_\phi \end{bmatrix}. \end{aligned}$$

(3) The following equations hold:

$$R_{A_2}C_2L_{B_2} = 0, D_{11}L_{C_{11}} = 0, R_{B_{11}}D_{11} = 0, R_MR_AE = 0, R_CEL_B = 0, R_AEL_D = 0.$$

In this case,

$$X = \frac{1}{2}(X_1 + (X_5)_\phi), Y = \frac{1}{2}(X_2 + (X_4)_\phi), \text{ and } Z = \frac{1}{2}(X_3 + (X_3)_\phi),$$

where $X_1, X_2, \dots, X_5$ are given in (Formulas (4.24)–(4.39), [123]).

Remark 33.

(1) When $A_1 = B_1 = C_1 = 0$, Theorem 27 yields the result for

$$A_2Z - YB_2 = C_2 \text{ subject to } Z = Z_\phi,$$

which can be regarded as Equation (1) under the constrain that X is ϕ -Hermitian, i.e.,

$$AX - YB = C \text{ subject to } X = X_\phi. \quad (51)$$

(2) Note that ϕ -Hermitian matrices are a generalization of Hermitian matrices. In Theorems 5.1 and 5.2 of [121], He and Wang have investigated the following problem over $\mathbb{C}$:

$$AX - YB = C \text{ subject to } X = X^* \text{ (or } Y = Y^*),$$

which is clearly similar to the problem (51).

(3) By the same method as in Remark 30, we can also discuss the following problem:

$$AX - YB = C \text{ subject to } X = X_\phi \text{ and } Y = Y_\phi.$$

5.10. Equality-Constrained Solutions

Let $\mathbb{F} = \mathbb{C}$. Wang et al. [124] considered the solvability conditions and the general solution for Equation (5) over $\mathbb{C}$ under the following equality constraints:

$$A_1X = C_1, YB_2 = C_2, A_3XB_3 = C_3, \text{ and } A_4YB_4 = C_4, \quad (52)$$

where $A_1, A_3, A_4, B_2, B_3, B_4, C_1, C_2, C_3$, and C_4 are given.

Theorem 28 (Theorem 3.2, [124]). *Let $A_1, B_1, C_1, B_2, C_2, A_3, B_3, C_3, A_4, B_4, C_4, A, B$, and C be given matrices over $\mathbb{C}$ with appropriate sizes. Set*

$$\begin{aligned} T &= A_3 L_{A_1}, K = R_{B_2} B_4, \phi_1 = A_1^\dagger C_1 + L_{A_1} T^\dagger (C_3 - A_3 A_1^\dagger C_1 B_3) B_3^\dagger, \\ \phi_2 &= C_2 B_2^\dagger + A_4^\dagger (C_4 - A_4 C_2 B_2^\dagger B_4) K^\dagger R_{B_2}, A_{11} = A L_{A_1} L_T, B_{11} = R_K R_{B_2} B, \\ C_{33} &= A L_{A_1}, D_{33} = R_{B_3}, C_{44} = L_{A_4}, D_{44} = R_{B_2} B, E_{11} = C - A \phi_1 - \phi_2 B, \\ A_a &= R_{A_{11}} C_{33}, B_b = D_{33} L_{B_{11}}, C_c = R_{A_a} C_{44}, D_d = D_{44} L_{B_{11}}, \\ E &= R_{A_{11}} E_{11} L_{B_{11}}, M = R_{A_a} C_c, N = D_d L_{B_b}, S = C_c L_M. \end{aligned}$$

Then, the following are equivalent:

- (1) Equation (5) under the constraints (52) is consistent.
- (2) The following rank equations hold:

$$\begin{aligned} \text{rank}[A_1, C_1] &= \text{rank}(A_1), \text{rank} \begin{bmatrix} C_3 \\ B_3 \end{bmatrix} = \text{rank}(B_3), \text{rank} \begin{bmatrix} A_1 & C_1 B_3 \\ A_3 & C_3 \end{bmatrix} = \text{rank} \begin{bmatrix} A_1 \\ A_3 \end{bmatrix}, \\ \text{rank} \begin{bmatrix} C_2 \\ B_2 \end{bmatrix} &= \text{rank}(B_2), \text{rank}[A_4, C_4] = \text{rank}(A_4), \text{rank} \begin{bmatrix} C_4 & A_4 C_2 \\ B_4 & B_2 \end{bmatrix} = \text{rank}[B_4, B_2], \\ \text{rank} \begin{bmatrix} 0 & B_2 & B B_3 \\ A & C_2 & C B_3 \\ A_3 & 0 & C_3 \\ A_1 & 0 & C_1 B_3 \end{bmatrix} &= \text{rank} \begin{bmatrix} 0 & B_2 & B B_3 \\ A & 0 & 0 \\ A_3 & 0 & 0 \\ A_1 & 0 & 0 \end{bmatrix}, \text{rank} \begin{bmatrix} 0 & B & B_2 \\ A & C & C_2 \\ A_1 & C_1 & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & B & B_2 \\ A & 0 & 0 \\ A_1 & 0 & 0 \end{bmatrix}, \\ \text{rank} \begin{bmatrix} 0 & B & B_4 & B_2 \\ A_4 A & A_4 C & C_4 & A_4 C_2 \\ A_1 & C_1 & 0 & 0 \end{bmatrix} &= \text{rank} \begin{bmatrix} 0 & B & B_4 & B_2 \\ A_4 A & 0 & 0 & 0 \\ A_1 & 0 & 0 & 0 \end{bmatrix}, \\ \text{rank} \begin{bmatrix} 0 & B_4 & B_2 & B B_3 \\ A_3 & 0 & 0 & C_3 \\ A_1 & 0 & 0 & C_1 B_3 \\ A_4 A & C_4 & A_4 C_2 & A_4 C B_3 \end{bmatrix} &= \text{rank} \begin{bmatrix} 0 & B_4 & B_2 & B B_3 \\ A_3 & 0 & 0 & 0 \\ A_1 & 0 & 0 & 0 \\ A_4 A & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

- (3) The following equations hold:

$$\begin{aligned} R_{A_1} C_1 &= 0, R_T (C_3 - A_3 A_1^\dagger C_1 B_3) = 0, C_3 L_{B_3} = 0, \\ C_2 L_{B_2} &= 0, R_{A_4} C_4 = 0, (C_4 - A_4 C_2 B_2^\dagger B_4) L_K = 0, \\ R_M R_{A_a} E &= 0, E L_{B_b} L_N = 0, R_{A_a} E L_{D_d} = 0, R_{C_c} E L_{B_b} = 0. \end{aligned}$$

In this case,

$$\begin{aligned} X &= A_1^\dagger C_1 + L_{A_1} T^\dagger (C_3 - A_3 A_1^\dagger C_1 B_3) B_3^\dagger + L_{A_1} L_T Z_1 + L_{A_1} W_1 R_{B_3}, \\ Y &= C_2 B_2^\dagger + A_4^\dagger (C_4 - A_4 C_2 B_2^\dagger B_4) K^\dagger R_{B_2} + L_{A_4} W_2 R_{B_2} + Z_2 R_K R_{B_2}, \\ Z_1 &= A_{11}^\dagger (E_{11} - C_{33} W_1 D_{33} - C_{44} W_2 D_{44}) - A_{11}^\dagger V_7 B_1 + L_{A_{11}} V_6, \\ Z_2 &= R_{A_{11}} (E_{11} - C_{32} W_1 D_{33} - C_{44} W_2 D_{44}) B_{11}^\dagger + A_{11} A_{11}^\dagger V_7 + V_8 R_{B_{11}}, \\ W_1 &= A_a^\dagger E B_b^\dagger - A_a^\dagger C_c M^\dagger E B_b^\dagger - A_a^\dagger S C_c^\dagger E N^\dagger D_d B_b^\dagger - A_a^\dagger S V_4 R_N D_d B_b^\dagger + L_{A_a} V_1 + V_2 R_{B_b}, \\ W_2 &= M^\dagger E D_d^\dagger + S^\dagger S C_c^\dagger E N^\dagger + L_M L_S V_3 + L_M V_4 R_N + V_5 R_{D_d}, \end{aligned}$$

where $V_1, \dots, V_8$ are arbitrary matrices over $\mathbb{C}$ with appropriate orders.

Remark 34. Inspired by Theorem 28, using the Kronecker product and the vectorization operation, Wang et al. [125] investigated the minimum-norm least-squares solution for the quaternion tensor system under the Einstein product:

$$\begin{aligned} & \mathcal{A} *_N \mathcal{X} + \mathcal{Y} *_N \mathcal{B} = \mathcal{C} \\ \text{subject to } & \begin{cases} \mathcal{A}_1 *_N \mathcal{X} = \mathcal{C}_1, \mathcal{A}_3 *_N \mathcal{X} *_N \mathcal{B}_3 = \mathcal{C}_3, \\ \mathcal{Y} *_N \mathcal{B}_2 = \mathcal{C}_2, \mathcal{A}_4 *_N \mathcal{Y} *_N \mathcal{B}_4 = \mathcal{C}_4, \end{cases} \end{aligned} \quad (53)$$

where $\mathcal{X}$ and $\mathcal{Y}$ are unknown tensors and the others are given tensors over $\mathbb{H}$. Thus, the minimum-norm least-squares solution of the tensor Equation (53) is directly given in (Corollary 3.4, [125]), which is also expounded in Section 6.5.

6. Various Generalizations of GSE

This section shows the generalizations of GSE to diverse domains such as various rings, dual numbers, dual quaternions, linear operators, tensors, and matrix polynomials, as well as its more general forms. This embodies another enchantment of mathematics: constantly exploring more general problems to ultimately reveal the most essential conclusions.

6.1. Generalizing RET over Different Rings

RET given in Section 3 characterizes the equivalent condition for the solvability of GSE over a field. This subsection mainly generalizes RET over the following algebraic structures: unit regular rings, principal ideal rings, commutative rings, division rings, Artinian rings, etc. To simplify expressions, we first introduce Guralnick's definition in [126].

Definition 10 ([126]). If (3) is equivalent to (4) over a ring $\mathbb{F}$, then we say that $\mathbb{F}$ has the equivalence property.

6.1.1. Generalizing RET over Unit Regular Rings

A ring $\mathbb{F}$ is called to be unit regular if for any $a \in \mathbb{F}$, there exists a unit $u \in \mathbb{F}$ such that

$$a = aua.$$

Hartwig [127] generalized RET to a unit regular ring, also extending Theorems 3 and 4.

Theorem 29 ([127]). Let $\mathbb{F}$ be a unit regular ring, and

$$M = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} \in \mathbb{F}^{2 \times 2}.$$

Then, the following statements are equivalent:

- (1) M has an inner inverse with the form of $\begin{bmatrix} r & s \\ 0 & t \end{bmatrix} \in \mathbb{F}^{2 \times 2}$;
- (2) $ax - yb = c$ has a solution pair $x, y \in \mathbb{F}$;
- (3) $(1 - aa^-)c(1 - b^-b) = 0$ for all a^- and b^- ;
- (4) $a_1cb_1 = 0$, where $a_1 \in \{x \in \mathbb{F} \mid xa = 0\}$ and $b_1 \in \{x \in \mathbb{F} \mid bx = 0\}$;
- (5) $M = p \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} q$, where $p, q \in \mathbb{F}$ are invertible;
- (6) $(1 - aa^{(1,2)})c(1 - b^{(1,2)}b) = 0$ for all $a^{(1,2)}$ and $b^{(1,2)}$;
- (7) $M^{(1,2)} = \begin{bmatrix} a^{(1,2)} & -a^{(1,2)}cb^{(1,2)} \\ 0 & b^{(1,2)} \end{bmatrix}$ is a reflexive inverse of M .

If $\mathbb{F}$ is a skewfield (or a commutative ring without zero divisors) and $a, c, b \in \mathbb{F}^{n \times n}$, then items (1)–(7) are also equivalent to

$$(5a) \quad \text{rank}(M) = \text{rank}(a) + \text{rank}(b);$$

$$(5b) \quad \text{rank}\left((1 - aa^{(1,2)})c(1 - b^{(1,2)}b)\right) = 0.$$

Remark 35. In the conclusions of [127], Hartwig mentioned that RET also holds for matrices over Euclidean domains and unit regular rings. These rings are finite and elementary divisor rings satisfying the cancellation law. However, whether these properties are sufficient to ensure that the validity of RET remains an open problem. In [126], Guralnick considered parts of this problem.

6.1.2. Generalizing RET over Principal Ideal Domains

Building on Theorem 2 of [128], Feinberg [129] considered RET over principal ideal domains and further extended it to a more general form.

Theorem 30 (Theorems 1 and 2, [129]). *Let $\mathbb{F}$ be a principal ideal domain.*

(1) *Let $A \in \mathbb{F}^{r \times r}$, $B \in \mathbb{F}^{s \times s}$, and $C \in \mathbb{F}^{r \times s}$. Then, the matrix equation*

$$AX + YB = C$$

is consistent if and only if

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \text{ and } \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

are equivalent.

(2) *Let $M_{ij} \in \mathbb{F}^{r_j \times r_j}$ for $1 \leq i \leq j \leq t$. Then,*

$$\begin{bmatrix} M_{11} & M_{12} & \dots & M_{1t} \\ 0 & M_{22} & \dots & M_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & M_{tt} \end{bmatrix} \text{ and } \begin{bmatrix} M_{11} & 0 & \dots & 0 \\ 0 & M_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & M_{tt} \end{bmatrix}$$

are equivalent if and only if there exist $X_{ij}, Y_{ij} \in \mathbb{F}^{r_i \times r_j}$ such that

$$M_{ij} = M_{ii}X_{ij} + \sum_{k=i+1}^j Y_{ik}M_{kj}$$

for $1 \leq i < j \leq t$.

Remark 36. Feinberg [129] also noted that it is easy to generalize Theorem 1 to a principal ideal domain by a method similar to that of Theorem 1 of [129].

Let $\mathbb{F} = \mathbb{C}$. In view of Remark 6, we can see that (4) implies that there exist the equivalence matrices to be of the upper block triangular, i.e.,

$$\begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix} \text{ and } \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}.$$

Olshevsky [22] generalized the above conclusion by showing that, if any block upper triangular matrix is equivalent to its block diagonal part, then the equivalent matrices can also be chosen in the form of a block upper triangular matrix.

Theorem 31 (Theorem 3.2, [22]). Let $A_{ij} \in \mathbb{C}^{n_i \times n_j}$ for $1 \leq i \leq j \leq k$. If

$$G = \begin{bmatrix} A_{11} & 0 & \dots & \dots & 0 \\ 0 & A_{22} & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & \dots & 0 & A_{kk} \end{bmatrix} \text{ and } H = \begin{bmatrix} A_{11} & A_{12} & \dots & \dots & A_{1k} \\ 0 & A_{22} & A_{23} & \dots & A_{2k} \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & A_{kk} \end{bmatrix}$$

are equivalent, then there exist $X_{ij}, Y_{ij} \in \mathbb{C}^{n_i \times n_j}$ ($1 \leq i < j \leq k$) such that

$$\begin{bmatrix} I_{n_1} & Y_{12} & \dots & \dots & Y_{1k} \\ 0 & I_{n_2} & Y_{23} & \dots & Y_{2k} \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & I_{n_k} \end{bmatrix} G \begin{bmatrix} I_{n_1} & X_{12} & \dots & \dots & X_{1k} \\ 0 & I_{n_2} & X_{23} & \dots & X_{2k} \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & I_{n_k} \end{bmatrix} = H.$$

Remark 37. Theorem 31 can also be derived from Theorem 30 (30).

6.1.3. Generalizing RET over Division and Module-Finite Rings

Let $\mathbb{F}$ be a ring with identity. For $G \in \mathbb{F}^{a \times b}$ and $H \in \mathbb{F}^{c \times d}$, let

$$E_R(G, H) = \left\{ (T, S) \mid T \in \mathbb{F}^{c \times a}, S \in \mathbb{F}^{d \times b} \text{ and } TG = HS \right\}.$$

Denote

$$M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix},$$

where A, B , and C are given in Equation (1). For $(T, S) \in E_R(M, A)$, denote

$$T = \begin{bmatrix} T_1 & T_2 \end{bmatrix} \text{ and } S = \begin{bmatrix} S_1 & S_2 \end{bmatrix},$$

where $T_1 \in \mathbb{F}^{m \times m}$, $T_2 \in \mathbb{F}^{m \times s}$, $S_1 \in \mathbb{F}^{r \times r}$, and $S_2 \in \mathbb{F}^{r \times n}$. Then,

$$TM = AS \Leftrightarrow \begin{cases} T_1A = AS_1, \\ T_1C + T_2B = AS_2. \end{cases}$$

Define a map

$$g_{M,R} : E_R(M, A) \rightarrow E_R(A, A) \text{ by } g((T_1, T_2), (S_1, S_2)) = (T_1, S_1). \quad (54)$$

Gustafson et al. [130] gave a general characterization of the solvability of Equation (1) over $\mathbb{F}$ based on the map $g_{M,R}$, which is essentially similar to the method used by Hartwig in [127].

Theorem 32 (Lemma 1, [130]). Equation (1) is consistent if and only if the map $g_{M,R}$ defined in (54) is surjective.

Using Theorem 32, Gustafson et al. [130] further proved that RET also holds for a division ring and for a ring which is finitely generated as modules over its center.

Theorem 33 ([130]). *A division ring (or a ring that is module-finite over its center) has the equivalence property.*

6.1.4. Generalizing RET over Commutative Rings

Let $\mathbb{F}$ be a commutative ring with identity. Gustafson [131] showed that RET remain valid over the commutative ring $\mathbb{F}$. By reducing this problem to Artinian rings, Gustafson employed a simple argument with the composition length. This approach parallels that of Flanders and Wimmer [11], who used linear transformations and subspace dimensions to discuss RET over fields.

Theorem 34 (Theorem 1, [131]). *A commutative ring with identity has the equivalence property.*

Guralnick [132] further generalized Theorem 34 to finite sets of matrices over the commutative ring $\mathbb{F}$. Let $\mathbb{F}[x_1, x_2, \dots, x_t]$ be the polynomial ring over $\mathbb{F}$, where x_i ($i = 1, \dots, t$) is commute with each other and any element in $\mathbb{F}$. For

$$\tilde{C} = \{C_i \in \mathbb{F}^{m \times n} \mid 1 \leq i \leq r\} \text{ and } \tilde{D} = \{D_i \in \mathbb{F}^{m \times n} \mid 1 \leq i \leq r\},$$

we call that $\tilde{C}$ and $\tilde{D}$ are simultaneously equivalent if there exist invertible matrices $U \in \mathbb{F}^{m \times m}$ and $V \in \mathbb{F}^{n \times n}$ such that $UC_iV = D_i$ for any $1 \leq i \leq r$ (see [132,133]).

Theorem 35 (Theorem B(i), [132]). *For $A_i \in \mathbb{F}^{m \times r}$, $B_i \in \mathbb{F}^{s \times n}$, and $C_i \in \mathbb{F}^{m \times n}$, let*

$$M_i = \begin{bmatrix} A_i & C_i \\ 0 & B_i \end{bmatrix} \text{ and } N_i = \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix},$$

where $1 \leq i \leq t$. Suppose that the polynomial ring $\mathbb{F}[x_1, x_2, \dots, x_t]$ has the equivalence property. Then, the system of matrix equations

$$\begin{cases} A_1X - YB_1 = C_1, \\ A_2X - YB_2 = C_2, \\ \dots \\ A_rX - YB_r = C_r, \end{cases} \quad (55)$$

has a common solution pair $X \in \mathbb{F}^{r \times n}$ and $Y \in \mathbb{F}^{m \times s}$ if and only if

$$\tilde{M} = \{M_i \mid 1 \leq i \leq r\} \text{ and } \tilde{N} = \{N_i \mid 1 \leq i \leq r\}$$

are simultaneously equivalent.

Remark 38. Dmytryshyn and their collaborators [12,13] have done significant contributions on the further generalizations of the system (55). Moreover, Dmytryshyn was awarded the SIAM Student Paper Prize 2015 for their work [12].

Let $\mathbb{F}$ be a field with $\text{char}(\mathbb{F}) \neq 2$. We stipulate that, in the following systems from [12,13], except for the unknown matrices, the remaining matrices are given with appropriate orders over $\mathbb{F}$. In Theorem 4.1 of [12], Dmytryshyn et al. first study the system

$$A_iX_k \pm X_jB_i = C_i, \quad (56)$$

where $i = 1, \dots, n$, $k, j \in \{1, \dots, m\}$, $1 \leq m \leq 2n$, and $X_1, \dots, X_m$ are unknown. Their method for solving the system (56) extends that used in [11]. Based on the system (56), they then considered the main research subject of [12], i.e.,

$$\begin{cases} A_i X_k \pm X_j B_i = C_i, & i = 1, \dots, n_1, \\ F_{i'} X_{k'} \pm X_{j'}^* G_{i'} = H_{i'}, & i' = 1, \dots, n_2, \end{cases} \quad (57)$$

where

- (i) $k, j, k', j' \in \{1, \dots, m\}$, $1 \leq m \leq (2n_1 + 2n_2)$, and $X_1, \dots, X_m$ are unknown;
- (ii) For $1 \leq l \leq m$, the symbol X_l^* denotes the matrix transpose X_l^T and for the complex number field, also the matrix conjugate transpose X_l^* ,

(see Theorem 1.1, [12]). In Theorem 6.1 of [12], they further generalized the system (57) to the following form:

$$\begin{cases} A_i X_k K_i - L_i X_j B_i = C_i, & i = 1, \dots, n_1, \\ F_{i'} X_{k'} M_{i'} + N_{i'} X_{j'}^* G_{i'} = H_{i'}, & i' = 1, \dots, n_2, \end{cases} \quad (58)$$

where $j, k, j', k' \in \{1, \dots, m\}$, $1 \leq m \leq (2n_1 + 2n_2)$, and $X_1, X_2, \dots, X_m$ are unknown.

Let $\mathbb{F}$ be a skew field of $\text{char}(\mathbb{F}) \neq 2$ that is finite dimensional over its center. Interestingly, two years later, Dmytryshyn et al. [13] generalized the system (58) over $\mathbb{F}$ including the complex conjugation of unknown matrices, i.e.,

$$A_i X_{i'}^{\varepsilon_i} M_i - N_i X_{i''}^{\delta_i} B_i = C_i,$$

- (i) Of complex matrix equations, in which $\varepsilon_i, \delta_i \in \{1, C, T, *\}$ and $X^C = \bar{X}$ is the complex conjugate of X ;
- (ii) Of quaternion matrix equations, in which $\varepsilon_i, \delta_i \in \{1, *\}$ and X^* is the quaternion conjugate transpose of X ,

where $i', i'' \in \{1, \dots, t\}$, $i = 1, \dots, s$, $1 \leq t \leq 2s$, and $X_1, \dots, X_t$ are unknown (see Theorem 2, [13]). The system (57) is also extended over $\mathbb{F}$ (see Theorem 1, [13]).

6.1.5. Generalizing RET over Artinian and Noncommutative Rings

Let N be a subgroup of a finitely generated Abelian group M . If M and $N \oplus M/N$ are isomorphic, then N is a direct summand of M ? This problem is raised by H. Matsumura, and it is subsequently answered affirmatively by H. Toda. Miyata [134] showed that this result can also be generalized to the more general case of module, i.e., if $\mathbb{F}$ is a commutative Noetherian ring and M is a finitely presented $\mathbb{F}$ -module with a submodule N , then

$$M \cong N \oplus M/N \Leftrightarrow N \text{ is a summand of } M. \quad (59)$$

We say that $\mathbb{F}$ has the extension property if (59) holds.

Guralnick [126] proved the equivalence between the equivalence property and the extension property for an Artinian ring $\mathbb{F}$.

Theorem 36 (Corollary 2.7, [126]). *Let $\mathbb{F}$ be a right Artinian ring. Then, $\mathbb{F}$ has the extension property if and only if $\mathbb{F}$ has the equivalence property.*

Remark 39. In the proof of Theorem 3.4 of [126], Guralnick proposed a new perspective to prove Theorem 34. Moreover, Guralnick in Theorem 3.5 of [126] showed that a commutative ring has the extension property. Differing from Miyata [134] and Gustafson [131], this proof avoids the completion of a local Noetherian ring.

Guralnick [126] found that two special classes of Artinian rings (i.e., semisimple Artinian rings and Artinian principal ideal rings) possess the equivalence property.

Theorem 37 (Theorem 2.4 and Corollary 4.6, [126]).

- (1) *A semisimple Artinian ring has the equivalence property.*
- (2) *An Artinian principal ideal ring has the equivalence property.*

Guralnick [126] finally discussed the more general case where $\mathbb{F}$ is a regular ring, which generalizes Theorem 37 as well items (2) and (5) in Theorem 29.

Theorem 38 (Theorem 4.3, [126]). *Let $\mathbb{F}$ be a regular ring. Then, $\mathbb{F}$ has the equivalence property if and only if $\mathbb{F}^{n \times n}$ is directly finite for all n .*

Interestingly, Guralnick [135] gave a generalized definition of the equivalence property, i.e., the generalized equivalence property.

Definition 11 ([135]). *Let $\mathbb{F}$ be a ring with identity. For $A_{ij} \in \mathbb{F}^{n_i \times m_j}$ ($1 \leq i \leq j \leq k$), denote*

$$\tilde{A} = \begin{bmatrix} A_{11} & & 0 \\ & \ddots & \\ 0 & & A_{kk} \end{bmatrix} \text{ and } \tilde{B} = \begin{bmatrix} A_{11} & & A_{1j} \\ & \ddots & \\ 0 & & A_{kk} \end{bmatrix}.$$

We define that $\mathbb{F}$ has the generalized equivalence property if $\tilde{A}$ and $\tilde{B}$ are and equivalent imply that there exist $X_{ij} \in \mathbb{F}^{n_i \times n_j}$ and $Y_{ij} \in \mathbb{F}^{m_i \times m_j}$ such that

$$\begin{bmatrix} I & & X_{ij} \\ & \ddots & \\ 0 & & I \end{bmatrix} \tilde{A} = \tilde{B} \begin{bmatrix} I & & Y_{ij} \\ & \ddots & \\ 0 & & I \end{bmatrix}.$$

Guralnick [135] then proved that not only semisimple Artinian rings and Artinian principal ideal rings but also module finite R -algebras for a commutative ring R possess the generalized equivalence property. This evidently generalizes Theorem 37.

Theorem 39 (Theorems 3.3, 3.6 and 3.7, [135]). *A semisimple Artinian ring, an Artinian principal ideal ring, or a module finite R -algebra for a commutative ring R has the generalized equivalence property.*

Remark 40. *It is worth noting that the generalized form of Equation (5), i.e.,*

$$AXB + CYD = E$$

with unknown X and Y , has also been discussed over fields, principal ideal domains, simple Artinian rings, regular rings with identity, and the associative rings with unit by [120,136–139], respectively.

6.2. Generalizing RET to a Rank Minimization Problem

In a brief three-page article [140], Lin and Wimmer revealed that RET is essentially a special case of a rank minimization problem over a field $\mathbb{F}$. Let $\text{GI}(k)$ be the set of all invertible matrices of order k .

Theorem 40 (Theorem 2, [140]). Let $\mathbb{F}$ be a field, and let $A \in \mathbb{F}^{m \times m}$, $B \in \mathbb{F}^{n \times n}$, and $C \in \mathbb{F}^{m \times n}$. Then,

$$\begin{aligned} & \min\{\text{rank}(AX - YB - C) \mid X, Y \in \mathbb{F}^{m \times n}\} \\ &= \min\left\{\text{rank}\left(P \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} - \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} Q\right) \mid P, Q \in \text{GI}(m+n)\right\}. \end{aligned}$$

Subsequently, Ito and Wimmer [141] generalized Theorem 40 to Bezout domains under the condition that A and B are regular. An integral domain $\mathbb{F}$ with identity is called a Bezout domain if it satisfies that all finitely generated ideals are principal.

Theorem 41 (Theorem 3.1, [141]). Let $\mathbb{F}$ be a Bezout domain, and let $A \in \mathbb{F}^{m \times m}$, $B \in \mathbb{F}^{n \times n}$, and $C \in \mathbb{F}^{m \times n}$. If A and B are regular over $\mathbb{F}$, then

$$\begin{aligned} & \min\{\text{rank}(AX - YB - C) \mid X, Y \in \mathbb{F}^{m \times n}\} \\ &= \min\left\{\text{rank}\left(P \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} - \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} Q\right) \mid P, Q \in \text{GI}(m+n)\right\} \\ &= \text{rank} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} - \text{rank} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \\ &= \dim(\mathcal{R}(A) + C\mathcal{N}(B)) - \text{rank}(A). \end{aligned}$$

Remark 41. Theorem 41 also yields the equivalence of (3), (4), and (7) over a Bezout domain directly (see (Corollary 3.2, [141])).

6.3. GSE over Dual Numbers and Dual Quaternions

In 1843, the Irish mathematician William Rowan Hamilton [142] invented quaternions $\mathbb{H}$ (also called Hamilton quaternions or real quaternions). The set $\mathbb{H}$ is a noncommutative associative division algebra, and it also generalizes the real field $\mathbb{R}$ and the complex field $\mathbb{C}$. The quaternion algebra has been effectively applied to mechanics, optics, color image processing, signal processing, computer graphics, flight mechanics, quantum physics, and so on (see [7,47,143–147]).

On the other hand, the British mathematician William Kingdon Clifford [148] invented dual numbers and dual quaternions in 1873. Up to now, dual numbers have been widely used in fields such as kinematics, statics, dynamics, robotics, and brain dynamics (see [149–154]).

Definition 12 ([148]). A dual number is defined as

$$\hat{a} = a_0 + a_1 \varepsilon,$$

where $a_0, a_1 \in \mathbb{R}$, and ε is the dual unit such that

$$\varepsilon \neq 0, 0\varepsilon = \varepsilon 0 = 0, 1\varepsilon = \varepsilon 1 = \varepsilon \text{ and } \varepsilon^2 = 0. \quad (60)$$

In this case, a_0 is called primal/real/standard part of $\hat{a}$, and a_1 is called dual/infinitesimal part of $\hat{a}$. The set of all dual numbers is denoted by $\mathbb{D}$, which is a commutative ring.

Fan et al. [155] established solvability conditions and expressions of the general solution to Equation (5) over $\mathbb{D}$ by using the MP inverse and SVD.

Theorem 42 (Theorem 3, [155]). *Let*

$$A = A_0 + A_1\epsilon \in \mathbb{D}^{m \times r}, B = B_0 + B_1\epsilon \in \mathbb{D}^{s \times n}, C = C_0 + C_1\epsilon \in \mathbb{D}^{m \times n},$$

where $A_i \in \mathbb{R}^{m \times r}$, $B_i \in \mathbb{R}^{s \times n}$, and $C_i \in \mathbb{R}^{m \times n}$ ($i = 0, 1$). The SVDs of A_0 and B_0 are given by

$$A_0 = P \begin{bmatrix} \Omega & 0 \\ 0 & 0 \end{bmatrix} Q^T \text{ and } B_0 = M \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} N^T,$$

where $\Omega = \text{diag}(\omega_1, \dots, \omega_l)$, $l = \text{rank}(A_0)$, $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_t)$, $t = \text{rank}(B_0)$, and $P \in \mathbb{R}^{m \times m}$, $Q \in \mathbb{R}^{r \times r}$, $M \in \mathbb{R}^{s \times s}$, and $N \in \mathbb{R}^{n \times n}$ are orthogonal. Let

$$P = \begin{bmatrix} P_1 & P_2 \end{bmatrix}, Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}, M = \begin{bmatrix} M_1 & M_2 \end{bmatrix}, N = \begin{bmatrix} N_1 & N_2 \end{bmatrix},$$

where $P_2 \in \mathbb{R}^{m \times (m-l)}$, $Q_2 \in \mathbb{R}^{r \times (r-l)}$, $M_2 \in \mathbb{R}^{s \times (s-t)}$, and $N_2 \in \mathbb{R}^{n \times (n-t)}$. Denote

$$J = C_1 - A_1 A_0^\dagger C_0 - R_{A_0} C_0 B_0^\dagger B_1, K_1 = P_2^T A_1 Q_2, K_2 = M_2^T B_1 N_2.$$

Then, Equation (5) has a solution pair $X \in \mathbb{D}^{r \times n}$ and $Y \in \mathbb{D}^{m \times s}$ if and only if

$$R_{A_0} C_0 L_{B_0} = 0 \text{ and } R_{K_1} P_2^T J N_2 L_{K_2} = 0,$$

in which case,

$$X = X_0 + X_1\epsilon \text{ and } Y = Y_0 + Y_1\epsilon$$

with

$$\begin{aligned} X_0 &= A_0^\dagger C_0 + Q_2 K_1^\dagger P_2^T J L_{B_0} - A_0^\dagger P_1 V_{11} M_1^T B_0 + Q_2 V_{23} N_1^T + Q_2 (L_{K_1} W_4 - K_1^\dagger W_3 K_2) N_2^T, \\ Y_0 &= R_{A_0} C_0 B_0^\dagger + P_2 R_{K_1} P_2^T J N_2 K_2^\dagger M_2^T + P_1 (V_{11} M_1^T + V_{12} M_2^T) + P_2 (W_3 - R_{K_1} W_3 K_2 K_2^\dagger) M_2^T, \\ X_1 &= A_0^\dagger J - A_0^\dagger A_1 Q_2 (K_1^\dagger P_2^T J N_2 - K_1^\dagger W_3 K_2 + L_{K_1} W_4) N_2^T - A_0^\dagger P_1 (V_{11} M_1^T + V_{12} M_2^T) B_1 \\ &\quad - A_0^\dagger R_1 B_0 + L_{A_0} R_2 + A_0^\dagger A_1 (A_0^\dagger P_1 V_{11} M_1^T B_0 - Q_2 V_{23} N_1^T), \\ Y_1 &= R_{A_0} J B_0^\dagger + R_{A_0} A_1 (A_0^\dagger P_1 V_{11} M_1^T B_0 - Q_2 V_{23} N_1^T) B_0^\dagger - P_2 R_{K_1} P_2^T J N_2 K_2^\dagger M_2^T B_1 B_0^\dagger \\ &\quad - P_2 (W_3 - R_{K_1} W_3 K_2 K_2^\dagger) M_2^T B_1 B_0^\dagger + R_1 - R_{A_0} R_1 B_0 B_0^\dagger, \end{aligned}$$

where V_{11} , V_{12} , V_{23} , W_3 , W_4 , R_1 , and R_2 are arbitrary matrices over $\mathbb{D}$ with appropriate sizes.

At the same time, due to the excellent property that dual quaternions can represent both rotation and translation, the theory of dual quaternions is not only one of the most powerful tools for handling rigid-body motion but also finds applications in computer graphics, medical procedures, neural networks, proximity operations in spacecraft, modern robotics, and so on (see [156–158]).

Definition 13 ([148]). *A dual quaternion is defined as*

$$\hat{q} = q_0 + q_1\epsilon,$$

where $q_0, q_1 \in \mathbb{H}$. The set of all dual quaternions is denoted by $\mathbb{DH}$, which is a noncommutative ring with zero divisors.

Recently, Xie et al. [159], inspired by the hand–eye calibration problem in robotics research, studied Equation (1) over $\mathbb{DH}$.

Theorem 43 (Theorem 3.1, [159]). *Let*

$$A = A_0 + A_1\epsilon \in \mathbb{DH}^{m \times r}, B = B_0 + B_1\epsilon \in \mathbb{DH}^{s \times n}, C = C_0 + C_1\epsilon \in \mathbb{DH}^{m \times n},$$

where $A_i \in \mathbb{R}^{m \times r}$, $B_i \in \mathbb{R}^{s \times n}$, and $C_i \in \mathbb{R}^{m \times n}$ ($i = 0, 1$). Set

$$\begin{aligned} A_{11} &= A_1 L_{A_0}, A_2 = R_{A_0} A_{11}, A_3 = R_{A_0}, C_3 = -R_{A_0} C_0, B_2 = R_{B_0} B_1, \\ A_4 &= R_{A_2} R_{A_0}, A_5 = -A_4 A_1 A_0^\dagger, C_4 = A_4 (A_1 A_0^\dagger C_0 + C_0 B_0^\dagger B_1 - C_1), \\ A_6 &= R_{A_4} A_5, C_5 = R_{A_4} C_4, B_3 = B_2 L_{B_0}, B_4 = C_4 L_{B_0}. \end{aligned}$$

Then, the following are equivalent:

- (1) Equation (1) has a solution pair $X \in \mathbb{DH}^{r \times n}$ and $Y \in \mathbb{DH}^{m \times s}$;
- (2) $C_3 L_{B_0} = 0$ and $B_4 L_{B_3} = 0$;
- (3) The following rank equations hold:

$$\begin{aligned} \text{rank} \begin{bmatrix} B_0 & 0 \\ -C_0 & A_0 \end{bmatrix} &= \text{rank}(B_0) + \text{rank}(A_0), \\ \text{rank} \begin{bmatrix} B_0 & 0 & 0 & 0 \\ B_1 & B_0 & 0 & 0 \\ -C_1 & -C_0 & A_0 & A_1 \\ 0 & 0 & 0 & A_0 \end{bmatrix} &= \text{rank} \begin{bmatrix} B_0 & 0 \\ B_1 & B_0 \end{bmatrix} + \text{rank} \begin{bmatrix} A_0 & A_1 \\ 0 & A_0 \end{bmatrix}; \end{aligned}$$

in which case,

$$X = X_0 + X_1 \varepsilon \text{ and } Y = Y_0 + Y_1 \varepsilon$$

with

$$X_0 = A_0^\dagger (C_0 + Y_0 B_0) + L_{A_0} W, \quad (61)$$

$$X_1 = A_0^\dagger [C_1 + Y_0 B_1 + Y_1 B_0 - A_1 A_0^\dagger (C_0 + Y_0 B_0) - A_{11} W] + L_{A_0} W_1, \quad (62)$$

$$Y_0 = A_3^\dagger C_3 B_0^\dagger + L_{A_3} U_1 + U_2 R_{B_0},$$

$$Y_1 = A_4^\dagger (C_4 - A_5 U_1 B_0 - A_4 U_2 B_2) B_0^\dagger + L_{A_4} W_3 + W_4 R_{B_0},$$

$$W = A_2^\dagger A_3 [C_1 + Y_0 B_1 + Y_1 B_0 - A_1 A_0^\dagger (C_0 + Y_0 B_0)] + L_{A_2} W_2,$$

$$U_1 = A_6^\dagger C_5 B_0^\dagger + L_{A_6} W_5 + W_6 R_{B_0}, \quad U_2 = A_4^\dagger B_4 B_3^\dagger + L_{A_4} W_7 + W_8 R_{B_3},$$

where $W_1, W_2, \dots, W_8$ are arbitrary matrices over $\mathbb{DH}$ with appropriate sizes.

Remark 42. Subsequently, Xie and Wang [160] further investigated a more general form of Equation (1) over $\mathbb{DH}$, namely

$$AX + EXF = CY + D, \quad (63)$$

where X and Y are unknown. Additionally, the systematic introduction to Equation (63) and its more general forms can be found in the book [161].

Remark 43. After the Hamilton quaternions, different concepts of quaternions were proposed, greatly enriching the quaternion theory, such as biquaternion (also called complexified quaternions), split quaternions, commutative quaternions (also called Segre biquaternions or reduced biquaternions), generalized commutative quaternions, degenerate quaternions, degenerate pseudo-quaternions, doubly degenerate quaternions, and quaternion algebras over a field (see [43,162–167]).

Interestingly, in Remark 8.2.7 of [168], Pottmann and Wallner introduced the concept of the generalized quaternions, which, in specific cases, coincide with Hamilton quaternions, split quaternions, degenerate quaternions, pseudo-degenerate quaternions, and doubly degenerate quaternions.

The theory of dual numbers has also developed rapidly, giving rise to three-dimensional dual numbers (also known as hyper-dual numbers), n -dimensional dual numbers (also known as higher

dimensional dual numbers), interval dual numbers, fuzzy dual numbers, neutrosophic dual numbers, and finite complex modulo integer neutrosophic dual numbers (see [169] and references therein).

One can see that dual quaternions are essentially a combination of dual numbers and quaternions. It is then natural to ask: given such a rich variety of quaternions and dual numbers, what sparks will fly when they interact, and what applications will emerge?

6.4. Linear Operator Equations on Hilbert Spaces

Generalizing matrix equations to operator equations on Hilbert spaces or Hilbert C^* -modules has been a mainstream research direction. For a C^* -algebra $\mathcal{A}$, a Hilbert C^* -module $\mathcal{E}$ [170,171] is a right $\mathcal{A}$ -module equipped with an $\mathcal{A}$ -valued inner product

$$\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{A}$$

such that its induced norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ is complete.

The theory of generalized inverses also serves as an effective tool for studying operator equations. Notably, most research requires the condition that the ranges of related operators are closed to ensure the existence of their MP inverses (see [172,173]). Interestingly, Douglas [174] pioneered an alternative approach in Hilbert spaces without the strong condition of closed ranges, which is known as the Douglas theorem. This work has provided valuable inspiration for subsequent research on operator equations on Hilbert C^* -modules (see [171,175]).

Let $\mathcal{A}$ be a C^* -algebra, and let $\mathcal{E}$ and $\mathcal{F}$ be Hilbert C^* -modules. The set of all bounded $\mathcal{A}$ -linear maps

$$A : \mathcal{E} \rightarrow \mathcal{F}$$

is denoted by $\mathcal{L}(\mathcal{E}, \mathcal{F})$. Particularly, $\mathcal{L}(\mathcal{E}) = \mathcal{L}(\mathcal{E}, \mathcal{E})$. The adjoint of $A \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ is a map $A^* \in \mathcal{L}(\mathcal{F}, \mathcal{E})$ such that

$$\langle Ax, y \rangle = \langle x, A^*y \rangle \text{ for all } x \in \mathcal{E}, y \in \mathcal{F}.$$

The range and the null space of an operator A are denoted by $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively. A closed submodule $\mathcal{L}$ of $\mathcal{E}$ is called to be orthogonally complemented [170] if

$$\mathcal{E} = \mathcal{L} \oplus \mathcal{L}^\perp,$$

where $\mathcal{L}^\perp = \{x \in \mathcal{E} : \langle x, y \rangle = 0 \text{ for all } y \in \mathcal{L}\}$. And, $\overline{\mathcal{L}}$ is the closure of $\mathcal{L}$. Let P_{A^*} be the projection of $\mathcal{E}$ onto $\overline{\mathcal{R}(A^*)}$. And, $R_A = I - P_{A^*}$, where I is the identity operator on $\mathcal{E}$.

Let $A, B, C \in \mathcal{L}(\mathcal{E})$ be such that A and B are adjointable. Mousavi et al. [175] investigated Equation (5) in Hilbert C^* -modules $\mathcal{E}$, where only the range closures of adjointable operators need to be orthogonally complemented.

Theorem 44 (Theorem 3.3, [175]). *Let $\overline{\mathcal{R}(A)}$, $\overline{\mathcal{R}(B)}$, $\overline{\mathcal{R}(A^*)}$, and $\overline{\mathcal{R}(B^*)}$ be orthogonally complemented. If*

$$\mathcal{R}(CR_B) \subseteq \mathcal{R}(A) \text{ and } \mathcal{R}(P_{B^*}C^*) \subseteq \mathcal{R}(B^*),$$

then Equation (5) is consistent, in which case,

$$X = X_h + X_p \text{ and } Y = Y_h + Y_p,$$

*where X_p and Y_p satisfying $AX_p = P_A CR_B$ and $B^*Y_p^* = P_{B^*}C^*$,*

$$X_h = R_A W_1 + W_2 P_{B^*} \text{ and } Y_h = P_A W_3 + W_4 R_{B^*},$$

where $W_1, W_2, W_3, W_4 \in \mathcal{L}(\mathcal{E})$ are arbitrary satisfying $AW_2P_{B^*} + P_AW_3B = 0$.

Remark 44. In Example 2.1 of [175], Mousavi et al. gave the example to show that, in a Hilbert C^* -module, an operator's range closure being orthogonally complemented is weaker than its range being closed.

Let $A \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ satisfy that $\mathcal{R}(A)$ is closed. In view of the orthogonal decompositions of closed submodules, i.e.,

$$\mathcal{E} = \mathcal{R}(A^*) \oplus \mathcal{N}(A) \text{ and } \mathcal{F} = \mathcal{R}(A) \oplus \mathcal{N}(A^*),$$

Karizaki et al. in Corollary 1.2 of [176] showed that the operator A can be decomposed into the following matrix form

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where A_1 is invertible, and thus the MP inverse $A^+ \in \mathcal{L}(\mathcal{F}, \mathcal{E})$ of A is

$$A^+ = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

Interestingly, four years after [175], Moghani et al. [177] supplemented the conclusions on solving operator Equation (5) on Hilbert C^* -modules using the matrix forms of adjointable operators and generalized inverses.

Theorem 45 (Theorem 3.2, [177]). Let $A \in \mathcal{L}(\mathcal{E}, \mathcal{F})$, $B \in \mathcal{L}(\mathcal{F}, \mathcal{E})$, and $C \in \mathcal{L}(\mathcal{F})$ be such that $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed, $\mathcal{R}(A) = \mathcal{R}(B^*)$, and $\mathcal{R}(A^*) = \mathcal{R}(B)$. Then, the operator Equation (5) has a solution pair $X \in \mathcal{L}(\mathcal{F}, \mathcal{E})$ and $Y \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ if and only if

$$(I - AA^+)C(I - B^+B) = 0,$$

in which case,

$$\begin{aligned} X &= \frac{1}{2}A^+C + \frac{1}{2}A^+C(I - B^+B) + \frac{1}{2}WB + (I - A^+A)Z, \\ Y &= \frac{1}{2}AA^+CB^+ + (I - AA^+)CB^+ - \frac{1}{2}AWBB^+ + V(I - BB^+), \end{aligned}$$

where $Z \in \mathcal{L}(\mathcal{F}, \mathcal{E})$ and $V \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ are arbitrary, and $W \in \mathcal{L}(\mathcal{E})$ satisfies

$$(I - AA^+)WBB^+ = 0.$$

Remark 45. In Section 4 of [177], Moghani et al. further studied the following operator equation

$$AXE + FYB = C,$$

where X and Y are unknown operators between Hilbert C^* -modules.

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space. Recently, An et al. [178] revisited the operator Equation (1) in $\mathcal{H}$. For $A \in \mathcal{L}(\mathcal{H})$, a $(1, 2)$ -inverse $A^{(1,2)}$ of A is an operator in $\mathcal{L}(\mathcal{H})$ satisfying $AA^{(1,2)}A = A$ and $A^{(1,2)}AA^{(1,2)} = A^{(1,2)}$.

Theorem 46 (Theorem 2.1, [178]). Let A, B , and $C \in \mathcal{L}(\mathcal{H})$. Then, Equation (1) has a solution pair

$$X = A^{(1,2)}C \text{ and } Y = -(I - AA^{(1,2)})CB^{(1,2)}$$

if and only if

$$(I - AA^{(1,2)})C(I - B^{(1,2)}B) = 0.$$

For $A, B \in \mathcal{L}(\mathcal{H})$, we call that the operator pair (A, B) has the generalized Fuglede–Putnam property [178,179] if

$$AX = YB \text{ for } X, Y \in \mathcal{L}(\mathcal{H}) \Rightarrow A^*X = YB^*.$$

An et al. [178] then presented an interesting connection between the generalized orthogonality and the solvability of the operator Equation (1) in $\mathcal{H}$.

Theorem 47 (Theorem 2.9, [178]). Let A, B , and $C \in \mathcal{L}(\mathcal{H})$.

(1) If the operator Equation (1) is consistent, then invertible operators $U, V \in \mathcal{H} \oplus \mathcal{H}$ exist such that

$$U \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} V.$$

(2) Suppose that (B, A) and (B, B) satisfy the generalized Fuglede–Putnam property. If there exist invertible operators $T, S \in \mathcal{H} \oplus \mathcal{H}$ such that

$$T \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} S,$$

and the $(2, 2)$ -entry of ST^* is invertible, then the operator Equation (1) is consistent.

Example 1. Note that Olshevsky in Section 2 of [22] designed an example to show that RET does not hold in infinite dimensional spaces. Indeed, let $\mathcal{H}$ be an infinite dimensional separable Hilbert space with the orthonormal basis $\{e_i\}_{i=1}^\infty$. Define the operators $A, C \in \mathcal{L}(\mathcal{H})$ as

$$Ae_{3k+1} = 0, Ae_{3k+2} = 0, Ae_{3k+3} = e_{3k+2} \ (k = 0, 1, 2, \dots), \\ Ce_1 = e_1, \text{ and } Ce_i = 0 \ (i \neq 1).$$

Put $B = A$. Let

$$E = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \text{ and } F = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}.$$

Then, one can observe that the operator E has the only the eigenvalue $\lambda_0 = 0$. Corresponding to this eigenvalue, there are a countable number of Jordan chains of lengths 1 and 2. Additionally, the vectors of these chains form the orthonormal basis of $\mathcal{H} \oplus \mathcal{H}$. So, F has the same properties. Thus, E and F are equivalent. On the other hand, the operator Equation (1) is not solvable. In fact, assume that X and Y satisfy Equation (1). Then, $AXe_1 = e_1$, which is a contradiction with $e_1 \notin \mathcal{R}(A)$.

Inspired by Bhatia's characterizations of the unique solution of Sylvester equation (i.e., Theorem VII.2.3, [180]), An et al. [178] further proposed an integral expression for the solution of the operator Equation (1) under the specific conditions. For $A \in \mathcal{L}(\mathcal{H})$, $\sigma(A)$ denotes the spectrum of A .

Theorem 48 (Theorem 2.17, [178]). Let A, B , and $C \in \mathcal{L}(\mathcal{H})$.

- (1) If the spectra of A and B are contained in the open right half-plane and the open left half-plane, respectively, then the operator Equation (1) has the solution pair

$$X = \int_0^\infty e^{-2tA} C dt \text{ and } Y = - \int_0^\infty C e^{-2tB} dt.$$

- (2) Suppose that A and B are Hermitian operators such that

$$\sigma(A) \cap \sigma(B) = \emptyset \text{ and } \alpha + \frac{1}{2} = \beta,$$

where α and β are eigenvalues of A and B , respectively. Assume that, for an absolutely integrable function f defined on $\mathbb{R}$, its Fourier transform $\hat{f}(s)$ satisfies

$$\hat{f}(s) = \frac{1}{s},$$

where $s \in \sigma(A) - \sigma(B)$. Then, the operator Equation (1) has the solution pair

$$X = \int_{-\infty}^\infty e^{-itA} C f(t) dt \text{ and } Y = \int_{-\infty}^\infty C e^{itB} f(t) dt.$$

6.5. Tensor Equations

From the end of the 19th century to the present, the understanding of tensors in multilinear algebra and physics has essentially gone through three ways: as multi-indexed objects satisfying certain transformation rules, as multilinear maps, and as elements in the tensor product of vector spaces (see [181] and references therein). On the other hand, the direct interpretation of “tensors” as multidimensional arrays (or hypermatrices) has also been widely accepted by many scholars (see [182–184] and references therein). Following this habit, the tensors in this paper are referred to as multidimensional arrays.

Definition 14. Let $\mathbb{F}$ be a ring. An N -order $I_1 \times \dots \times I_N$ -dimension tensor $\mathcal{A}$ over $\mathbb{F}$ is defined as a multidimensional array with $I_1 I_2 \dots I_N$ entries, i.e.,

$$\mathcal{A} = [a_{i_1 \dots i_N}]_{1 \leq i_j \leq I_j} \quad (j = 1, \dots, N),$$

where $a_{i_1 \dots i_N} \in \mathbb{F}$ for $1 \leq i_j \leq I_j$ and $j = 1, \dots, N$. Moreover, denote

$$(\mathcal{A})_{i_1 \dots i_N} = a_{i_1 \dots i_N}.$$

The set of all N -order $I_1 \times \dots \times I_N$ -dimensional tensors over $\mathbb{F}$ is denoted by $\mathbb{F}^{I_1 \times \dots \times I_N}$.

Tensor theory has been effectively applied in diverse fields: image processing [185], handwritten digit classification [186], hypergraphs [182], extreme learning machines [187], signal processing and machine learning [188], quantum physics and mechanics [189], etc. In addition, the review article [190] by Kolda and Bader introduced the theory of tensor decomposition and its applications in psychometrics, chemometrics, numerical linear algebra, computer vision, numerical analysis, neuroscience, and so on.

In the 2017 preprint [191], He et al. first discussed SVD and the MP inverse of quaternion tensors under the Einstein product, establishing a fundamental framework for solving quaternion tensor equations under the Einstein product.

Definition 15 ([192]). Let

$$\mathcal{A} = [a_{i_1 \dots i_N j_1 \dots j_N}] \in \mathbb{H}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N} \text{ and } \mathcal{B} = [b_{j_1 \dots j_N k_1 \dots k_M}] \in \mathbb{H}^{J_1 \times \dots \times J_N \times K_1 \times \dots \times K_M}.$$

The Einstein product of $\mathcal{A}$ and $\mathcal{B}$ is defined as

$$\mathcal{A} *_N \mathcal{B} = [c_{i_1 \dots i_N k_1 \dots k_M}] \in \mathbb{H}^{I_1 \times \dots \times I_N \times K_1 \times \dots \times K_M},$$

where

$$c_{i_1 \dots i_N k_1 \dots k_M} = \sum_{j_1 \dots j_N} a_{i_1 \dots i_N j_1 \dots j_N} b_{j_1 \dots j_N k_1 \dots k_M},$$

for $1 \leq i_j \leq I_j, j = 1, \dots, N, 1 \leq k_t \leq K_t$, and $t = 1, \dots, M$.

The conjugate transpose $\mathcal{A}^*$ of $\mathcal{A} = [a_{i_1 \dots i_N j_1 \dots j_M}] \in \mathbb{H}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_M}$ is

$$\mathcal{A}^* = [b_{j_1 \dots j_M i_1 \dots i_N}] \in \mathbb{H}^{J_1 \times \dots \times J_M \times I_1 \times \dots \times I_N} \text{ with } b_{j_1 \dots j_M i_1 \dots i_N} = \bar{a}_{i_1 \dots i_N j_1 \dots j_M}.$$

The MP inverse [193] of $\mathcal{A} \in \mathbb{H}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$ is $\mathcal{A}^\dagger \in \mathbb{H}^{J_1 \times \dots \times J_N \times I_1 \times \dots \times I_N}$ satisfying

$$\mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{A} = \mathcal{A}, \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^\dagger = \mathcal{A}^\dagger, (\mathcal{A} *_N \mathcal{A}^\dagger)^* = \mathcal{A} *_N \mathcal{A}^\dagger, (\mathcal{A}^\dagger *_N \mathcal{A})^* = \mathcal{A}^\dagger *_N \mathcal{A}.$$

Moreover, let the unit (or identity) tensor be

$$\mathcal{I}_{I_N} = [e_{i_1 \dots i_N j_1 \dots j_N}] \in \mathbb{H}^{I_1 \times \dots \times I_N \times I_1 \times \dots \times I_N},$$

where all diagonal entries $e_{i_1 \dots i_N i_1 \dots i_N}$ are 1 and all off-diagonal entries are 0. Define

$$\mathcal{L}_{\mathcal{A}} = \mathcal{I} - \mathcal{A}^\dagger *_N \mathcal{A} \text{ and } \mathcal{R}_{\mathcal{A}} = \mathcal{I} - \mathcal{A} *_N \mathcal{A}^\dagger,$$

where $\mathcal{I}$ denotes the unit tensor with appropriate dimensions.

Theorem 49 (Corollary 5.3, [191]). Let

$$\mathcal{A} \in \mathbb{H}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}, \mathcal{B} \in \mathbb{H}^{H_1 \times \dots \times H_M \times L_1 \times \dots \times L_M}, \mathcal{C} \in \mathbb{H}^{I_1 \times \dots \times I_N \times L_1 \times \dots \times L_M}.$$

Then, Equation (5) over quaternion tensors under the Einstein product, i.e.,

$$\mathcal{A} *_N \mathcal{X} + \mathcal{Y} *_M \mathcal{B} = \mathcal{C} \quad (64)$$

has a solution pair

$$\mathcal{X} \in \mathbb{H}^{J_1 \times \dots \times J_N \times L_1 \times \dots \times L_M} \text{ and } \mathcal{Y} \in \mathbb{H}^{I_1 \times \dots \times I_N \times H_1 \times \dots \times H_M}$$

if and only if

$$\mathcal{R}_{\mathcal{A}} *_N \mathcal{C} *_M \mathcal{L}_{\mathcal{B}} = 0,$$

in which case,

$$\begin{aligned} \mathcal{X} &= \mathcal{A}^\dagger *_N \mathcal{C} - \mathcal{U}_1 *_M \mathcal{B} + \mathcal{L}_{\mathcal{A}} *_N \mathcal{U}_2, \\ \mathcal{Y} &= \mathcal{R}_{\mathcal{A}} *_N \mathcal{C} *_M \mathcal{B}^\dagger + \mathcal{A} *_N \mathcal{U}_1 + \mathcal{U}_3 *_M \mathcal{R}_{\mathcal{B}}, \end{aligned}$$

where $\mathcal{U}_1, \mathcal{U}_2$, and $\mathcal{U}_3$ are arbitrary tensors over $\mathbb{H}$ with appropriate dimensions.

Remark 46.

- (1) Theorem 49 is a direct corollary of Theorem 5.1 of [191], which establishes the solvability conditions and the general solution for the following quaternion tensor equation:

$$\mathcal{A} *_N \mathcal{X} *_M \mathcal{D} + \mathcal{E} *_N \mathcal{Y} *_M \mathcal{B} = \mathcal{C},$$

where $\mathcal{X}$ and $\mathcal{Y}$ are unknown and other tensors are given over $\mathbb{H}$.

- (2) Inspired by the transformation between tensors and matrices over $\mathbb{R}$ (see (Definition 2.8, [194])), He et al. [191,195] defined an analogous transformation over $\mathbb{H}$, i.e., the transformation f is a map defined as

$$f : \mathbb{H}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N} \rightarrow \mathbb{H}^{(I_1 I_2 \dots I_N) \times (J_1 J_2 \dots J_N)} \\ \mathcal{A} \in \mathbb{H}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N} \mapsto A = f(\mathcal{A}) \in \mathbb{H}^{(I_1 I_2 \dots I_N) \times (J_1 J_2 \dots J_N)},$$

where the components of A are given by

$$(\mathcal{A})_{i_1 \dots i_N j_1 \dots j_N} \xrightarrow{f} (A)_{[i_1 + \sum_{k=2}^N (i_k - 1) \prod_{s=1}^{k-1} I_s] [j_1 + \sum_{k=2}^N (j_k - 1) \prod_{s=1}^{k-1} J_s]}.$$

Lemma 2.2 of [191] shows that the transformation f is a bijection satisfying

$$f(\mathcal{A} + \mathcal{B}) = f(\mathcal{A}) + f(\mathcal{B}) \text{ and } f(\mathcal{A} *_N \mathcal{C}) = f(\mathcal{A})f(\mathcal{C})$$

for $\mathcal{A}, \mathcal{B} \in \mathbb{H}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$ and $\mathcal{C} \in \mathbb{H}^{J_1 \times \dots \times J_N \times L_1 \times \dots \times L_N}$. The transformation f ingeniously bridges quaternion tensors under the Einstein product and quaternion matrices under the ordinary product. By virtue of its isomorphism property, f serves as a powerful tool for studying problems related to quaternion tensors under the Einstein product.

- (3) The work [191] on quaternion tensor equations has profoundly influenced subsequent research on tensor equations over $\mathbb{H}$ (see [196–202]).

Subsequently, Wang et al. [125] discussed the minimum-norm least-squares solution of the quaternion tensor equation of the form (64), i.e.,

$$\mathcal{A} *_N \mathcal{X} + \mathcal{Y} *_N \mathcal{B} = \mathcal{D}$$

with the unknown $\mathcal{X}$ and $\mathcal{Y}$. We now introduce some notations.

Let $\mathcal{A} = [a_{i_1 \dots i_N j_1 \dots j_M}] \in \mathbb{H}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_M}$. The transpose $\mathcal{A}^T$ of $\mathcal{A}$ is

$$\mathcal{A}^T = [b_{j_1 \dots j_M i_1 \dots i_N}] \in \mathbb{H}^{J_1 \times \dots \times J_M \times I_1 \times \dots \times I_N},$$

where $b_{j_1 \dots j_M i_1 \dots i_N} = a_{i_1 \dots i_N j_1 \dots j_M}$, and the conjugate $\overline{\mathcal{A}}$ of $\mathcal{A}$ is

$$\overline{\mathcal{A}} = [\overline{a}_{i_1 \dots i_N j_1 \dots j_M}] \in \mathbb{H}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_M}.$$

The symbol

$$\mathcal{A}_{(i_1 \dots i_N | :)} = [a_{i_1 \dots i_N : \dots :}] \in \mathbb{H}^{J_1 \times \dots \times J_M}$$

stands for a subblock of $\mathcal{A}$, and $\text{Vec}(\mathcal{A})$ is a new tensor obtained by lining up all subtensor in a column, where the t -th subblock of $\text{Vec}(\mathcal{A})$ is $\mathcal{A}_{(i_1 \dots i_N | :)}$ for

$$t = i_N + \sum_{K=1}^{N-1} \left[(i_K - 1) \prod_{L=K+1}^N I_L \right].$$

Since $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 \mathbf{j}$ for $\mathcal{A}_1, \mathcal{A}_2 \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_M}$, the complex representation tensor of $\mathcal{A}$ is defined as

$$f(\mathcal{A}) = \begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ -\overline{\mathcal{A}_2} & \overline{\mathcal{A}_1} \end{bmatrix} \in \mathbb{C}^{2I_1 \times \dots \times 2I_N \times 2J_1 \times \dots \times 2J_M}.$$

Let $\Theta_{\mathcal{A}} = [\mathcal{A}_1, \mathcal{A}_2]$, $\vec{\mathcal{A}} = (\text{Re}\mathcal{A}_1, \text{Im}\mathcal{A}_1, \text{Re}\mathcal{A}_2, \text{Im}\mathcal{A}_2)$,

$$\text{Vec}(\vec{\mathcal{A}}) = \begin{bmatrix} \text{Vec}(\text{Re}\mathcal{A}_1) \\ \text{Vec}(\text{Im}\mathcal{A}_1) \\ \text{Vec}(\text{Re}\mathcal{A}_2) \\ \text{Vec}(\text{Im}\mathcal{A}_2) \end{bmatrix}, \text{ and } \mathcal{K}_{J_M} = \begin{bmatrix} \mathcal{I} & \mathbf{i}\mathcal{I}_{J_M} & 0 & 0 \\ 0 & 0 & \mathcal{I}_{J_M} & \mathbf{i}\mathcal{I}_{J_M} \\ \mathcal{I}_{J_M} & -\mathbf{i}\mathcal{I}_{J_M} & 0 & 0 \\ 0 & 0 & \mathcal{I}_{J_M} & -\mathbf{i}\mathcal{I}_{J_M} \end{bmatrix},$$

where $\mathbf{i}$ is imaginary unit such that $\mathbf{i}^2 = -1$.

Let $\mathcal{A} \in \mathbb{R}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$. The Frobenius norm $\|\cdot\|_F$ of $\mathcal{A}$ is

$$\|\mathcal{A}\|_F = \left(\sum_{i_1 \dots i_N j_1 \dots j_N} |a_{i_1 \dots i_N j_1 \dots j_N}|^2 \right)^{1/2}.$$

For $\mathcal{B} = B_1 + B_2 \mathbf{j} \in \mathbb{H}^{J_1 \times \dots \times J_N \times K_1 \times \dots \times K_M}$, define

$$\mathcal{A} \bullet f(\mathcal{B}) = \begin{bmatrix} \mathcal{A} \otimes B_1 & \mathcal{A} \otimes B_2 \\ \mathcal{A} \otimes (-\overline{B_2}) & \mathcal{A} \otimes \overline{B_1} \end{bmatrix},$$

where $\otimes$ denotes the Kronecker product of two tensors. The the inverse of $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$ is the tensor $\mathcal{A}^{-1} \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$ satisfying

$$\mathcal{A} *_N \mathcal{A}^{-1} = \mathcal{A}^{-1} *_N \mathcal{A} = \mathcal{I}.$$

Theorem 50 (Corollary 3.4, [125]). Let $\mathcal{A}, \mathcal{B}, \mathcal{D} \in \mathbb{H}^{J_1 \times \dots \times J_N \times J_1 \times \dots \times J_N}$, and let

$$\mathbb{H}_{L1} = \{[\mathcal{X}, \mathcal{Y}]\}$$

be the set of all $\mathcal{X}, \mathcal{Y} \in \mathbb{H}^{J_1 \times \dots \times J_N \times J_1 \times \dots \times J_N}$ such that

$$\|\mathcal{A} *_N \mathcal{X} + \mathcal{Y} *_N \mathcal{B} - \mathcal{D}\|_F^2 = \min_{\mathcal{X}_1, \mathcal{Y}_1 \in \mathbb{H}^{J_1 \times \dots \times J_N \times J_1 \times \dots \times J_N}} \|\mathcal{A} *_N \mathcal{X}_1 + \mathcal{Y}_1 *_N \mathcal{B} - \mathcal{D}\|_F^2.$$

Denote $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 \mathbf{j}$. Put

$$\begin{aligned} \mathcal{P}_{01} &= [\mathcal{A}_1 \bullet f(\mathcal{I}_{J_N})^T \quad \mathcal{A}_2 \bullet f(\mathcal{I}_{J_N} \mathbf{j})^*] *_N \mathcal{K}_{J_N}, \quad \mathcal{Q}_{01} = [\mathcal{I}_{J_N} \bullet f(\mathcal{B})^T \quad 0] *_N \mathcal{K}_{J_N}, \\ \mathcal{T}_{11} &= [\text{Re}\mathcal{P}_{01} \quad \text{Re}\mathcal{Q}_{01}], \quad \mathcal{T}_{12} = [\text{Im}\mathcal{P}_{01} \quad \text{Im}\mathcal{Q}_{01}], \quad \mathcal{E}_1 = \begin{bmatrix} \text{Vec}(\text{Re}\Theta_{\mathcal{D}}) \\ \text{Vec}(\text{Im}\Theta_{\mathcal{D}}) \end{bmatrix}, \\ \mathcal{R}_1 &= (\mathcal{I} - \mathcal{T}_{11}^\dagger *_N \mathcal{T}_{11}) *_N \mathcal{T}_{12}^T, \\ \mathcal{H}_1 &= \mathcal{R}_1^\dagger + (\mathcal{I} - \mathcal{R}_1^\dagger *_N \mathcal{R}_1) *_N \mathcal{Z}_1 *_N \mathcal{T}_{12} *_N \mathcal{T}_{11}^\dagger *_N (\mathcal{T}_{11}^\dagger)^T *_N (\mathcal{I} - \mathcal{T}_{12}^T *_N \mathcal{R}_1^\dagger), \\ \mathcal{Z}_1 &= (\mathcal{I} + (\mathcal{I} - \mathcal{R}_1^\dagger *_N \mathcal{R}_1) *_N \mathcal{T}_{12} *_N \mathcal{T}_{11}^\dagger *_N (\mathcal{T}_{11}^\dagger)^T *_N \mathcal{T}_{12}^T *_N (\mathcal{I} - \mathcal{R}_1^\dagger *_N \mathcal{R}_1))^{-1}. \end{aligned}$$

(1) Then,

$$\mathbb{H}_{L1} = \left\{ [\mathcal{X}, \mathcal{Y}] \mid \begin{bmatrix} \text{Vec}(\vec{\mathcal{X}}) \\ \text{Vec}(\vec{\mathcal{Y}}) \end{bmatrix} = \begin{bmatrix} \mathcal{T}_{11}^\dagger - \mathcal{H}_1^\top *_N \mathcal{T}_{12} *_N \mathcal{T}_{11}^\dagger & \mathcal{H}_1^\top \\ \mathcal{I} - \mathcal{T}_{11}^\dagger *_N \mathcal{T}_{11} - \mathcal{R}_1 *_N \mathcal{R}_1^\dagger & *_N \mathcal{W}_1 \end{bmatrix} *_N \mathcal{E}_1 \right\},$$

where $\mathcal{W}_1$ is arbitrary with appropriate dimensions.

(2) If $[\mathcal{X}_{l1}, \mathcal{Y}_{l1}] \in \mathbb{H}_{L1}$ satisfies

$$\|[\mathcal{X}_{l1}, \mathcal{Y}_{l1}]\|_F^2 = \min_{[\mathcal{X}, \mathcal{Y}] \in \mathbb{H}_{L1}} (\|\mathcal{X}\|_F^2 + \|\mathcal{Y}\|_F^2),$$

then $[\mathcal{X}_{l1}, \mathcal{Y}_{l1}] \in \mathbb{H}_{L1}$ is unique and

$$\begin{bmatrix} \text{Vec}(\vec{\mathcal{X}}_{l1}) \\ \text{Vec}(\vec{\mathcal{Y}}_{l1}) \end{bmatrix} = \begin{bmatrix} \mathcal{T}_{11}^\dagger - \mathcal{H}_1^\top *_N \mathcal{T}_{12} *_N \mathcal{T}_{11}^\dagger & \mathcal{H}_1^\top \\ \mathcal{I} - \mathcal{T}_{11}^\dagger *_N \mathcal{T}_{11} - \mathcal{R}_1 *_N \mathcal{R}_1^\dagger & *_N \mathcal{W}_1 \end{bmatrix} *_N \mathcal{E}_1.$$

Remark 47. Recently, some scholars have extended quaternion tensor equations under the Einstein product to different categories. For instance, Jia and Wang [203] investigated split quaternion tensor equations, while Yang et al. [204] explored dual split quaternion tensor equations. On the other hand, tensor theory encompasses a variety of product operations, including the Einstein product [192], k-mode Product [182], contracted product [205], T-product [206], Qt-product [207], general product [208], cosine transform product (c-product) [209], and M-product [210,211]. Combined with Remark 43, the investigation of tensor equations over diverse quaternion algebras under various tensor products reveals significant untapped research potential.

6.6. Polynomial Matrix Equations

As described in Section 3, Roth was the first to study the solvability conditions of polynomial matrix Equation (2) via the equivalence of two block polynomial matrices. Since then, the theoretical research and practical applications regarding this polynomial matrix equation have gradually become more extensive and enriched. For instance, it is successfully applied to multivariable linear discrete systems in stochastic control [212], the algebraic regulator problem [213], etc. Next, we mainly discuss different approaches to studying this polynomial matrix.

6.6.1. By the Divisibility of Polynomials

Let F be a field. Cheng and Pearson [214], in their research on the regulator problem with internal stability, provided an equivalent characterization of the solvability of the polynomial matrix equation

$$BX + YD = P \quad (65)$$

with given polynomial matrix $B \in \mathbb{F}^{p \times n}[\lambda]$, $D \in \mathbb{F}^{p \times p}[\lambda]$, and $P \in \mathbb{F}^{p \times p}[\lambda]$, by the divisibility of a series of polynomials. Assume that $\text{rank}(B) = r$ and $\text{rank}(D) = p$. Using Lemma 2 of [214] (i.e., Smith normal form theorem for polynomial matrices), there exist unimodular matrices M_b, N_b, M_d , and N_d such that

$$M_b B N_b = B_d = \begin{bmatrix} B_0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } M_d D N_d = \text{diag}(d_1, d_2, \dots, d_p), \quad (66)$$

where $B_0 = \text{diag}(b_1, b_2, \dots, b_r)$, $b_1 b_2 \dots b_p \neq 0$, and $d_1 d_2 \dots d_p \neq 0$. Left-multiplying (65) by M_b and right-multiplying (65) by N_d yield that

$$B_d X' + Y' D_d = P', \quad (67)$$

where $X' = N_b^{-1} X N_d$, $Y' = M_b Y M_d^{-1}$, and $P' = M_b P N_d$. For $a, b \in \mathbb{F}[\lambda]$, $a|b$ denotes a divides b .

Theorem 51 (Lemma 6, [214]). Let B , D , and P be given in (65) with $\text{rank}(B) = r$ and $\text{rank}(D) = p$. Let $b_1, b_2, \dots, b_r$ and $d_1, d_2, \dots, d_p$ be given in (66), and $P' = [p'_{ij}]$ be given in (67). Then, Equation (65) has a solution pair (X, Y) if and only if

- (1) $g_{ij} | p'_{ij}$ for $i = 1, \dots, r$ and $j = 1, \dots, p$;
- (2) $d_j | p'_{ij}$ if $p > r$, for $i = r + 1, \dots, p$ and $j = 1, \dots, p$.

where g_{ij} is the monic greatest common divisor of b_i and d_j .

Remark 48. Notably, in Theorem 2 of [214], Cheng and Pearson equivalently transform the solvability of a restricted regulator problem with internal stability into the solvability of Equation (65) in a special form.

6.6.2. By Skew-Prime Polynomial Matrices

Let F be a field. In 1978, Wolovich [215] proposed a new approach to studying the polynomial matrix equation

$$A(\lambda)X(\lambda) + Y(\lambda)B(\lambda) = C(\lambda) \quad (68)$$

with given $A(\lambda) \in \mathbb{F}^{p \times m}[\lambda]$, $B(\lambda) \in \mathbb{F}^{q \times t}[\lambda]$, and $C(\lambda) \in \mathbb{F}^{p \times t}[\lambda]$, based on skew-prime polynomial matrices.

Definition 16 ([215,216]). Let $A(\lambda) \in \mathbb{F}^{p \times m}[\lambda]$ and $B(\lambda) \in \mathbb{F}^{q \times p}[\lambda]$ with $q + m > p$. Assume that there exist $M(\lambda) \in \mathbb{F}^{m \times p}[\lambda]$ and $N(\lambda) \in \mathbb{F}^{p \times q}[\lambda]$ such that

$$A(\lambda)M(\lambda) + N(\lambda)B(\lambda) = I_p.$$

Then, $A(\lambda)$ and $B(\lambda)$ are called externally skew prime (or $B(\lambda)$ and $A(\lambda)$ are called internally skew prime). Moreover, $A(\lambda)$ and $N(\lambda)$ are called relatively left prime while $M(\lambda)$ and $B(\lambda)$ are called relatively right prime.

Suppose that $A(\lambda)$ is nonsingular with $p = m$. Then, $A^{-1}(s)C(\lambda)$ can be factored in dual prime form:

$$A^{-1}(s)C(\lambda) = \bar{C}(s)\mathcal{A}^{-1}(s), \quad (69)$$

where $\bar{C}(s) \in \mathbb{F}^{p \times t}[\lambda]$ and $\mathcal{A}(s) \in \mathbb{F}^{t \times t}[\lambda]$ are relatively right prime.

Theorem 52 (Theorem 3 and Corollary 3, [215]). Let $A(\lambda) \in \mathbb{F}^{p \times p}[\lambda]$ be nonsingular and let $\mathcal{A}(s)$ be given in (69).

- (1) If $\mathcal{A}(s)$ and $B(\lambda)$ are externally skew, then Equation (68) is consistent.
- (2) Suppose that $A(\lambda)$ and $C(\lambda)$ are relatively left prime. Then, Equation (68) is consistent if and only if $\mathcal{A}(s)$ and $B(\lambda)$ are externally skew prime.

Remark 49. Note that Wolovich proved the sufficiency of the item (2) in Theorem 52 via a constructive method, implying a new procedure to find a solution pair of Equation (68). Moreover,

when $C(\lambda) = I$, all solutions of Equation (68) are characterized in (Section 5, [215]), and this characterization is further used to obtain the unique solution of Equation (68).

6.6.3. By the Realization of Matrix Fraction Descriptions

Let $\mathcal{F}$ be a field. For given $Q \in \mathbb{F}^{p \times q}[\lambda]$, $R \in \mathbb{F}^{m \times t}[\lambda]$, and $\Phi \in \mathbb{F}^{p \times t}[\lambda]$, consider the polynomial matrix equation

$$XR + QY = \Phi, \quad (70)$$

where $X \in \mathbb{F}^{p \times m}[\lambda]$ and $Y \in \mathbb{F}^{q \times t}[\lambda]$ are unknown. According to the realization of matrix fraction descriptions presented in [217], Emre and Silverman [218] transformed Equation (70) into a set of linear matrix equations when Q is nonsingular.

Let $\mathbb{F}^q[\lambda]$ and $\mathbb{F}^q(\lambda)$ be the sets of all q -tuples of polynomials in λ with coefficients in $\mathbb{F}$ and all q -tuples of rational functions in λ over $\mathbb{F}$, respectively. Assume that Q is nonsingular with $p = q$. Let

$$\mathbb{F}_Q = \{x \in \mathbb{F}^p[\lambda] : Q^{-1}x \text{ is strictly proper}\}.$$

For $X_1 \in \mathbb{F}^{p \times m}[\lambda]$ satisfying that $Q^{-1}X_1$ is strictly proper, define the following $\mathbb{F}$ -linear maps:

$$\begin{aligned} G : \mathbb{F}^m &\rightarrow \mathbb{F}_Q, u \mapsto X_1 u \text{ for } u \in \mathbb{F}^m, \\ \pi : \mathbb{F}^p(\lambda) &\rightarrow \mathbb{F}^p(\lambda), q \mapsto \text{strictly proper part of } q, \\ \pi_Q : \mathbb{F}^p[\lambda] &\rightarrow \mathbb{F}^p[\lambda], x \mapsto Q\pi(Q^{-1}x), \\ F : \mathbb{F}_Q &\rightarrow \mathbb{F}_Q, x \mapsto \pi_Q(x), \\ H : \mathbb{F}_Q &\rightarrow \mathbb{F}^p, x \mapsto (Q^{-1}x)_{-1}, \end{aligned}$$

where $(Q^{-1}x)_{-1}$ is the coefficient of λ^{-1} in the formal power series of $Q^{-1}x$ in λ^{-1} . For $Z = Q^{-1}X_1$, we call $\Sigma = (F, G, H)$ the Q -realization of Z [217].

Let $S \in \mathbb{F}^{p \times n}[\lambda]$ be such that its columns are a basis of $\mathbb{F}_Q$. Let (F, G_1, H) be the Q -realization of $Q^{-1}S$. Let $\hat{F}$, $\hat{G}_1$, and $\hat{H}$ denote the matrix representations of F , G_1 , and H , respectively, with respect to the canonical bases of $\mathbb{F}^m$ and $\mathbb{F}^p$, and the columns of S serving as a basis of $\mathbb{F}_Q$. Put

$$R = \sum_{j=0}^r u_{-j} \lambda^j,$$

where $u_{-j} \in \mathbb{F}^{m \times p}$. Define $\hat{\Phi} \in \mathbb{F}^{n \times p}$ uniquely by $\pi_Q(\Phi) = S\hat{\Phi}$, and for the unique polynomial matrix Φ_1 , express Φ as

$$\Phi = Q\Phi_1 + S\hat{\Phi}. \quad (71)$$

Moreover, let the linear equations be as follows:

$$\sum_{j=0}^r \hat{F}^j \hat{G}_1 u_{-j} = \hat{\Phi}, \quad (72)$$

where $\hat{G} \in \mathbb{F}^{n \times m}$ are unknown.

Theorem 53 (Theorem 2.5, [218]). *Let R , Q , and Φ be given in (70). Suppose that Q is nonsingular with $p = q$. Denote*

$$\bar{E}(Q, R) = \{(X_1, Y_1) \mid X_1 R + QY_1 = \Phi \text{ and } Q^{-1}X_1 \text{ is strictly proper}\}.$$

The following are equivalent:

- (1) $(X_1, Y_1) \in \bar{E}(Q, R)$;
- (2) Equation (72) has a solution $\hat{G}$ such that

$$X_1 = S\hat{G} \text{ and } Y_1 = \Phi_1 - Q_p,$$

where Φ_1 satisfies (71), and Q_p is the polynomial part of $Q^{-1}X_1R$.

Remark 50.

- (1) Under the hypotheses of Theorem 53, let

$$E(Q, R) = \{(X, Y) \mid XR + QY = \Phi\}.$$

In terms of Lemma 2.2 of [218], Emre and Silverman have shown that

$$E(Q, R) = \{(X_1, Y_1) + (\bar{X}, \bar{Y}) \mid (X_1, Y_1) \in \bar{E}(Q, R) \text{ and } (\bar{X}, \bar{Y}) \in \bar{H}(Q, R)\},$$

where $\bar{H}(Q, R) = \{(QQ_1, -Q_1R) \mid Q_1 \text{ is an arbitrary polynomial matrix}\}$. This implies that, to characterize $E(Q, R)$, it is sufficient to characterize $\bar{E}(Q, R)$.

- (2) In Section 3, [218], Equation (70) is further generalized to the case where Q is a general polynomial matrix. In fact, for $Q \in \mathbb{F}^{p \times q}[\lambda]$, there exist unimodular polynomial matrices M_1 and M_2 such that

$$M_1QM_2 = \begin{bmatrix} \hat{Q} & 0 \\ 0 & 0 \end{bmatrix},$$

where $\hat{Q}$ is the nonsingular polynomial matrix. Let

$$\hat{X} = M_1X = \begin{bmatrix} \hat{X}_1 \\ \hat{X}_2 \end{bmatrix}, \hat{Y} = M_2^{-1}Y = \begin{bmatrix} \hat{Y}_1 \\ \hat{Y}_2 \end{bmatrix}, M_1\Phi = \begin{bmatrix} \tilde{\Phi}_1 \\ \tilde{\Phi}_2 \end{bmatrix}.$$

Then,

$$\begin{aligned} \text{Equation (70)} &\Leftrightarrow \hat{X}R + \begin{bmatrix} \hat{Q} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{Y}_1 \\ \hat{Y}_2 \end{bmatrix} = \begin{bmatrix} \tilde{\Phi}_1 \\ \tilde{\Phi}_2 \end{bmatrix} \\ &\Leftrightarrow \begin{cases} \hat{X}_2R = \tilde{\Phi}_2, \\ \hat{X}_1R + \hat{Q}\hat{Y}_1 = \tilde{\Phi}_1. \end{cases} \end{aligned} \quad (73)$$

So, Theorem 53 can be applied to the second equation of (73); see [219] for the first Equation of (73).

6.6.4. By the Unilateral Polynomial Matrix Equation

Let $\mathbb{F}$ be a field. For given $A, B, C \in \mathbb{F}^{n \times n}[\lambda]$, consider the following polynomial matrix Equation (also called the bilateral polynomial matrix equation):

$$AX + YB = C, \quad (74)$$

where $X, Y \in \mathbb{F}^{n \times n}[\lambda]$ are unknown. Žak [220] proposed an algorithm for finding the unique solution pair (X, Y) of Equation (74). In fact, let

$$A = \sum_{i=0}^N \lambda^i A_i \text{ and } B = \sum_{j=0}^M \lambda^j B_j.$$

For $C = [c_{ij}]_{n \times n} \in \mathbb{F}^{n \times n}[\lambda]$, denote

$$\text{vec}_r(C) = \begin{bmatrix} c_{11} & \dots & c_{1n} & c_{21} & \dots & c_{2n} & \dots & c_{n1} & \dots & c_{nn} \end{bmatrix}^T.$$

Using the Kronecker product, Equation (74) can be transformed into the following unilateral polynomial matrix equation:

$$\mathfrak{A}\mathfrak{X} + \mathfrak{B}\mathfrak{Y} = \mathfrak{C}, \quad (75)$$

where $\mathfrak{X} = \text{vec}_r(X)$, $\mathfrak{Y} = \text{vec}_r(Y)$, $\mathfrak{C} = \text{vec}_r(C)$,

$$\mathfrak{A} = \sum_{i=0}^N \lambda^i (A_i \otimes I_n), \text{ and } \mathfrak{B} = \sum_{j=0}^M \lambda^j (I_n \otimes B_j^T).$$

Let

$$X = \sum_{i=0}^{M_1-1} \lambda^i X_i \text{ and } Y = \sum_{i=0}^{N_1-1} \lambda^i Y_i.$$

Denote

$$\mathfrak{A}_i = A_i \otimes I_n \text{ and } \mathfrak{B}_j = I_n \otimes B_j^T,$$

where $i = 0, 1, \dots, N$ and $j = 0, 1, \dots, M$. Then, by comparing of like powers, (75) can be rewritten as

$$\begin{bmatrix} \mathfrak{B}_0 & & \mathfrak{A}_0 & & \\ \mathfrak{B}_1 & \mathfrak{B}_0 & \mathfrak{A}_1 & \mathfrak{A}_0 & \\ \vdots & \mathfrak{B}_1 & \mathfrak{B}_0 & \vdots & \mathfrak{A}_1 & \mathfrak{A}_0 \\ \mathfrak{B}_M & \vdots & \mathfrak{B}_1 & \mathfrak{A}_N & \vdots & \mathfrak{A}_1 \\ & \mathfrak{B}_M & \vdots & \mathfrak{A}_N & \vdots & \\ & & \mathfrak{B}_M & & \mathfrak{A}_N & \end{bmatrix} \begin{bmatrix} Y_0 \\ \vdots \\ Y_{N_1-1} \\ X_0 \\ \vdots \\ X_{M_1-1} \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}. \quad (76)$$

$N_1 + M_1$ blocks

Let us assume without loss of generality that B_0 is nonsingular, which implies that $\mathfrak{B}_0$ is also nonsingular. As shown by Feinstein and Bar-Ness in [221], performing a series of elementary row operations on Equation (76) yields the following form:

$$\left[\begin{array}{ccc|ccc} I_n & & & L_0 & & \\ & \ddots & & L_1 & & \\ & & I_n & \vdots & & \\ \hline & & 0 & L_0 & \dots & L_0 \end{array} \right] \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}. \quad (77)$$

Therefore, a necessary and sufficient condition for the solvability of Equation (74) is

$$\text{rank} \begin{bmatrix} L & p_2 \end{bmatrix} = \text{rank}(L),$$

in which case, we can obtain x by solving $Lx = p_2$, and then compute y recursively. Furthermore, the upper bound on the degree of $\mathfrak{Y}$ in (75) are also given as follows:

Theorem 54 ([220]). For $\mathfrak{A}$ and $\mathfrak{B}$ given in (75), assume that

- (1) $\mathfrak{A}$ and $\mathfrak{B}$ are relatively left prime;
- (2) $\mathfrak{B}$ is nonsingular and satisfies that $\mathfrak{B}^{-1}$ is strictly proper;
- (3) $\mathfrak{A}_1(\lambda)\mathfrak{B}_1^{-1}(\lambda)$ is the right coprime factorization of $\mathfrak{B}^{-1}\mathfrak{A}$, where $\mathfrak{B}_1(\lambda)$ is row reduced.

If Equation (75) is consistent, then it has a solution pair $(\mathfrak{X}(\lambda), \mathfrak{Y}(\lambda))$ such that

$$\deg_{ri} \mathfrak{X}(\lambda) < \deg_{ri} \mathfrak{B}_1(\lambda) \text{ and } \deg \mathfrak{Y}(\lambda) < \deg \mathfrak{A}_1(\lambda),$$

where $\deg_{ri}$ denotes the degree of the i -th row.

Remark 51. Zak [220] also noted that the number of equations in (77) can be further reduced by using additional information about A and B given in (74). For instance, if a row of B has a degree of zero, then the corresponding row of X is identically zero, so discard the relevant column and row of L given in (77).

6.6.5. By the Equivalence of Block Polynomial Matrices

Let $\mathbb{F}$ be a field. Building upon Theorem 1, Wimmer [222] investigated the constant solutions of the following polynomial matrix equation:

$$A(\lambda)X - YB(\lambda) = C(\lambda), \quad (78)$$

where $A(\lambda) \in \mathbb{F}^{m \times n}[\lambda]$, $B(\lambda) \in \mathbb{F}^{p \times k}[\lambda]$, and $C(\lambda) \in \mathbb{F}^{m \times k}[\lambda]$ are given.

Theorem 55 (Lemma 2.1, [222]). Equation (78) has a constant solution pair

$$X \in \mathbb{F}^{n \times k} \text{ and } Y \in \mathbb{F}^{m \times p}$$

if and only if there exist two nonsingular constant matrices

$$R \in \mathbb{F}^{(n+k) \times (n+k)} \text{ and } S \in \mathbb{F}^{(m+p) \times (m+p)}$$

such that

$$\begin{bmatrix} A(\lambda) & 0 \\ 0 & B(\lambda) \end{bmatrix} R = S \begin{bmatrix} A(\lambda) & C(\lambda) \\ 0 & B(\lambda) \end{bmatrix}.$$

Remark 52. Theorem 55 also appeared as a lemma in the earlier article (Lemma 3, [223]), though no complete proof was provided.

Remark 53. Let $A_i \in \mathbb{F}^{m \times n}$, $B_i \in \mathbb{F}^{p \times k}$, and $C_i \in \mathbb{F}^{m \times k}$ for $i = 1, 2$. Using Theorem 55, Wimmer (Theorem 1.1, [222]) showed that the system of matrix equations

$$\begin{cases} A_1 X - Y B_1 = C_1, \\ A_2 X - Y B_2 = C_2, \end{cases}$$

has a solution pair $X \in \mathbb{F}^{n \times k}$ and $Y \in \mathbb{F}^{m \times p}$ if and only if there exist two nonsingular matrices $R \in \mathbb{F}^{(n+k) \times (n+k)}$ and $S \in \mathbb{F}^{(m+p) \times (m+p)}$ such that

$$S \left(\begin{bmatrix} A_1 & C_1 \\ 0 & B_1 \end{bmatrix} - \lambda \begin{bmatrix} A_2 & C_2 \\ 0 & B_2 \end{bmatrix} \right) = \left(\begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix} - \lambda \begin{bmatrix} A_2 & 0 \\ 0 & B_2 \end{bmatrix} \right) R.$$

Obviously, this result is also a generalization of RET. A similar research idea is also introduced in Section 8.1.

6.6.6. By Jordan Systems of Polynomial Matrices

Let $A \in \mathbb{C}^{m \times m}[\lambda]$, $B \in \mathbb{C}^{n \times n}[\lambda]$, and $C \in \mathbb{C}^{m \times n}[\lambda]$ be such that $\det(A) \neq 0$ and $\det(B) \neq 0$. According to Jordan systems of polynomial matrices, Wimmer [224] discussed the solvability conditions for the polynomial matrix equations

$$AX - YB = C, \quad (79)$$

where $X, Y \in \mathbb{C}^{m \times n}[\lambda]$ are unknown. Let

$$\sigma(B) = \{\lambda \in \mathbb{C} \mid \det(B(\lambda)) = 0\},$$

and let the elementary divisors corresponding to $\lambda_1 \in \sigma(B)$ be

$$(\lambda - \lambda_1)^{l_1}, \dots, (\lambda - \lambda_1)^{l_q},$$

where $l_1 \geq \dots \geq l_q \geq 1$ and $l = l_1 + \dots + l_q$ satisfying

$$\det(B) = (\lambda - \lambda_1)^l c(\lambda) \text{ and } c(\lambda_1) \neq 0.$$

For $r \in \mathbb{Z}^+$, denote

$$N_r = \begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix}_{r \times r}.$$

Let the Jordan matrix of B associated to λ_1 be

$$J = \text{diag}(\lambda_1 I - N_{l_1}, \dots, \lambda_1 I - N_{l_q}). \quad (80)$$

Then, there exist $H \in \mathbb{C}^{n \times l}$ and $\hat{H} \in \mathbb{C}^{n \times l}[\lambda]$ such that

$$BH = \hat{H}(\lambda I - J),$$

where the columns of $\hat{H}$ are $\mathbb{C}$ -linearly independent (see [225]). Thus, H is called a right Jordan system of B corresponding to λ_1 . Then, G is called a left Jordan system of A corresponding to $\lambda_1 \in \sigma(A)$ if G^T is a right Jordan system of A^T .

For $\lambda_1 \in \sigma(A) \cap \sigma(B)$, let G and H be left and right Jordan systems of A and B corresponding to λ_1 , respectively. Then, (G, H) is called a pair of Jordan systems of (A, B) corresponding to λ_1 . According to (80), partition

$$H = \begin{bmatrix} H_1 & H_2 & \dots & H_q \end{bmatrix}$$

with $H_j = [h_{j0}, \dots, h_{j,l_j-1}]$ for $j = 1, \dots, q$. Similarly, assume that the elementary divisors belonging to $\lambda_1 \in \sigma(A)$ are

$$(\lambda - \lambda_1)^{k_1}, \dots, (\lambda - \lambda_1)^{k_p},$$

where $k_1 \geq \dots \geq k_p \geq 1$. Partition

$$G = \begin{bmatrix} G_1 \\ \vdots \\ G_p \end{bmatrix} \text{ with } G_i = \begin{bmatrix} g_{i0} \\ \vdots \\ g_{i,k_i-1} \end{bmatrix} \text{ for } i = 1, \dots, p.$$

The k -th derivative of $C(\lambda)$ is denoted by $C^{(k)}(\lambda)$. We call that (G, H) has the property (Σ) if

$$\sum_{\nu+\sigma+\tau=r_{ij}} g_{iv} \frac{C^{(\sigma)}(\lambda_1)}{\sigma!} h_{j\tau} = 0,$$

where $r_{ij} = 0, 1, \dots, \min(k_i, l_j) - 1$, $i = 1, \dots, p$, and $j = 1, \dots, q$.

Theorem 56 (Theorem 1.1, [224]). *Let A , B , and C be given in (79). Then, the following are equivalent:*

- (1) Equation (79) is consistent;
- (2) There exists a pair of Jordan systems (G, H) of (A, B) with property (Σ) for each $\lambda \in \sigma(A) \cap \sigma(B)$;
- (3) All pairs of Jordan systems of (A, B) have property (Σ) for each $\lambda \in \sigma(A) \cap \sigma(B)$.

Remark 54. Wimmer [224] pointed out that Theorem 56 can also be extended to matrices over the ring $\mathcal{O}(\mathbb{G})$ of complex holomorphic functions in a domain $\mathbb{G}$.

6.6.7. By Linear Matrix Equations

Let $\mathbb{F}$ be a field. Assume that

$$A(\lambda) = \sum_{i=0}^m A_i \lambda^i, \quad B(\lambda) = \sum_{i=0}^n B_i \lambda^i, \quad \text{and} \quad C(\lambda) = \sum_{i=0}^k C_i \lambda^i, \quad (81)$$

where $A_i, B_i, C_i \in \mathbb{F}^{r \times r}$ satisfying $A_m \neq 0$, $B_n \neq 0$, and $C_k \neq 0$. Consider the polynomial matrix equation:

$$A(\lambda)X(\lambda) + Y(\lambda)B(\lambda) = C(\lambda), \quad (82)$$

where $X(\lambda), Y(\lambda) \in \mathbb{F}^{r \times r}[\lambda]$ are unknown.

Barnett [226] provided an equivalent condition for Equation (82) to have a unique solution pair $(X(\lambda), Y(\lambda))$ with

$$\deg X(\lambda) < n \quad \text{and} \quad \deg Y(\lambda) < m. \quad (83)$$

For $H(\lambda) = \sum_{i=0}^m H_i \lambda^i$ with $H_m \neq 0$, we say that $H(\lambda)$ is regular if $\det(H_m) \neq 0$.

Theorem 57 ([226]). *Let $A(\lambda)$, $B(\lambda)$, and $C(\lambda)$ given in (81) be such that $A(\lambda)$ and $B(\lambda)$ are regular and that $\deg C(\lambda) \leq n + m - 1$. Then, Equation (82) has a unique solution pair $(X(\lambda), Y(\lambda))$ such that (83) if and only if $\det(A(\lambda))$ and $\det(B(\lambda))$ are relatively prime, (i.e., the greatest common divisor is a constant independent λ).*

Remark 55. Note that the condition in Theorem 57, i.e., that $\det(A(\lambda))$ and $\det(B(\lambda))$ are relatively prime, implies that $A(\lambda)$ and $B(\lambda)$ are nonsingular (see [227]).

Subsequently, Feinstein and Bar-Ness [227] conducted further research on the solutions to Equation (82) with (83); such special solutions are also called the minimal solutions.

Theorem 58 (Theorem II, [227]). *Let $A(\lambda)$, $B(\lambda)$, and $C(\lambda)$ given in (81) be such that $A(\lambda)$ and $B(\lambda)$ are nonsingular and $\deg C(\lambda) \leq n + m - 1$. Assume that $A(\lambda)$ (or $B(\lambda)$) is regular. Then, Equation (82) has a unique minimal solution if and only if $\det(A(\lambda))$ and $\det(B(\lambda))$ are relatively prime.*

Remark 56. In Theorem III of [227], Feinstein and Bar-Ness showed that, if Equation (82) has a minimal solution $(X(\lambda), Y(\lambda))$, then $(X(\lambda), Y(\lambda))$ is not unique if and only if both $A(\lambda)$ and $B(\lambda)$ are not regular.

Motivated by Theorem 57, Chen and Tian [228] proved that Equation (81) can be reduced to a linear matrix equation. For $A(\lambda)$ and $B(\lambda)$ given in (81), let

$$A_R = \begin{bmatrix} 0 & 0 & \dots & 0 & -A_0 \\ I & 0 & \dots & 0 & -A_1 \\ 0 & I & \dots & 0 & -A_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I & -A_{m-1} \end{bmatrix} \text{ and } B_L = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I \\ -B_0 & -B_1 & -B_2 & \dots & -B_{n-1} \end{bmatrix}.$$

Theorem 59 (Lemma 3.1 and Theorems 1.1 and 1.3, [228]). Let $A(\lambda)$, $B(\lambda)$, and $C(\lambda)$ given in (81) be such that $A_m = B_n = I_r$, and let $\mathbb{H}_{C(\lambda)}$ be the set of all solutions $(X(\lambda), Y(\lambda))$ of Equation (82) with (83).

- (1) Let $k < m + n$. If Equation (82) is solvable, then $\mathbb{H}_{C(\lambda)} \neq \emptyset$.
- (2) Let $k < m$. There exists $X(\lambda)$ satisfying $(X(\lambda), Y(\lambda)) \in \mathbb{H}_{C(\lambda)}$ if and only if

$$A_R^n Y + \sum_{i=0}^{n-1} A_R^i Y B_i = C, \quad (84)$$

where

$$Y(\lambda) = \sum_{i=0}^{m-1} Y_i \lambda^i, \quad Y = \begin{bmatrix} Y_0 \\ \vdots \\ Y_{m-1} \end{bmatrix}, \quad C = \begin{bmatrix} C_0 \\ \vdots \\ C_{m-1} \end{bmatrix}.$$

- (3) Let $k < n$. There exists $Y(\lambda)$ satisfying $(X(\lambda), Y(\lambda)) \in \mathbb{H}_{C(\lambda)}$ if and only if

$$X B_L^m + \sum_{i=0}^{n-1} A_i X B_L^i = \tilde{C}, \quad (85)$$

where $X(\lambda) = \sum_{i=0}^{n-1} X_i \lambda^i$, $X = \begin{bmatrix} X_0 & \dots & X_{n-1} \end{bmatrix}$, and $\tilde{C} = \begin{bmatrix} C_0 & \dots & C_{n-1} \end{bmatrix}$.

Remark 57.

- (1) For $A = [a_{ij}] \in \mathbb{F}^{p \times q}$, let

$$\text{row}(A) = \begin{bmatrix} a_{11} & \dots & a_{p1} & a_{12} & \dots & a_{p2} & \dots & a_{1q} & \dots & a_{pq} \end{bmatrix}.$$

and $\text{vec}(A) = \text{row}(A)^T$. Then,

$$(84) \Leftrightarrow \left(I \otimes A_R^n + \sum_{i=0}^{n-1} B_i^T \otimes A_R^i \right) \text{vec}(Y) = \text{vec}(C),$$

$$(85) \Leftrightarrow \text{row}(X) \left(I \otimes B_L^m + \sum_{i=0}^{m-1} A_i \otimes B_L^i \right) = \text{row}(\tilde{C}).$$

- (2) The explicit solutions to Equations (84) and (85) have been studied in [229,230], which also serve as a starting point of Section 6.7 in this paper.
- (3) Moreover, Sheng and Tian [228] mentioned that Theorem 59 still holds when the field $\mathbb{F}$ is extended to a commutative ring with identity.

6.6.8. By Root Functions of Polynomial Matrices

Let $L_1^{n \times n}[a, b]$ denote the space of $n \times n$ matrix-valued functions that are Lebesgue integrable over the interval $[a, b]$. Let

$$\mathcal{B}(\lambda) = I_n + \int_{-\omega}^0 e^{i\lambda t} b(t) dt, \mathcal{D}(\lambda) = I_n + \int_0^{\omega} e^{i\lambda t} d(t) dt, G(\lambda) = \int_{-\omega}^{\omega} e^{i\lambda t} g(t) dt,$$

where $b \in L_1^{n \times n}[-\omega, 0]$, $d \in L_1^{n \times n}[0, \omega]$, and $g \in L_1^{n \times n}[-\omega, \omega]$. Consider the following linear entire matrix function equation

$$X(\lambda)\mathcal{B}(\lambda) + \mathcal{D}(\lambda)Y(\lambda) = G(\lambda), \lambda \in \mathbb{C}, \quad (86)$$

where X and Y are unknown $n \times n$ matrix functions

$$X(\lambda) = \int_0^{\omega} e^{i\lambda t} x(t) dt \text{ and } Y(\lambda) = \int_{-\omega}^0 e^{i\lambda t} y(t) dt$$

with $x \in L_1^{n \times n}[0, \omega]$ and $y \in L_1^{n \times n}[-\omega, 0]$. Gohberg [231] proposed necessary and sufficient conditions for the solvability of Equation (86) using the root functions of the coefficients.

Definition 17 ([231,232]). Let an $n \times n$ matrix function $H(\lambda)$ be analytic at $\lambda_0 \in \mathbb{C}$. A $\mathbb{C}^n$ -valued function φ is called a root function of $H(\lambda)$ at λ_0 of order (at least) $k \in \mathbb{Z}^+$ if φ is analytic at λ_0 , $\varphi(\lambda_0) \neq 0$, and $H(\lambda)\varphi(\lambda)$ has a zero at λ_0 of order (at least) k .

Theorem 60 (Theorem 1.1, [231]). Equation (86) is consistent if and only if for each common zero λ_0 of $\det(\mathcal{B}(\lambda))$ and $\det(\mathcal{D}(\lambda))$, if φ is a root function of $\mathcal{B}(\lambda)$ at λ_0 of order p and ψ is a root function of $\mathcal{D}(\lambda)^T$ at λ_0 of order q , then $\psi(\lambda)^T G(\lambda) \varphi(\lambda)$ has a zero at λ_0 of order at least $\min\{q, p\}$.

Let $H(\lambda)$ be an analytic $r \times r$ matrix function, and let λ_0 be a point in the domain of analyticity of $H(\lambda)$. If $\det(H(\lambda_0)) = 0$, then λ_0 is called an eigenvalue of $H(\lambda)$. Let a $\mathbb{C}^r$ -vector valued function $\phi(\lambda)$ be analytic in a neighborhood of an eigenvalue λ_0 of $F(\lambda)$. If $\phi(\lambda_0) \neq 0$ and $F(\lambda_0)\phi(\lambda_0) = 0$, then $\phi(\lambda)$ is called a right root function of $F(\lambda)$ at λ_0 . The order (at least) k of the right root function $\phi(\lambda)$ at λ_0 is the order (at least) k of λ_0 as a zero of the analytic function $F(\lambda)\phi(\lambda)$. Similarly, left root functions can be defined (see [233]).

Utilizing the right and left root functions of polynomial matrices, Kaashoek and Lerer [233] then presented a discrete version of Theorem 60. Let

$$L(\lambda) = \sum_{j=0}^l \lambda^j L_j, M(\lambda) = \sum_{j=0}^m \lambda^j M_j, G(\lambda) = \sum_{j=0}^{\ell+m-1} \lambda^j G_j, \quad (87)$$

where $L_j, M_j, G_j \in \mathbb{C}^{r \times r}$ with $L_l \neq 0$ and $M_m \neq 0$. Consider the following polynomial matrix equation:

$$X(\lambda)L(\lambda) + M(\lambda)Y(\lambda) = G(\lambda), \quad (88)$$

where $X(\lambda)$ and $Y(\lambda)$ are unknown. Define

$$\begin{aligned} \widehat{L}(\lambda) &= \lambda^l L(\lambda^{-1}) = \sum_{j=0}^l \lambda^j L_{l-j}, \widehat{M}(\lambda) = \lambda^m M(\lambda^{-1}) = \sum_{j=0}^m \lambda^j M_{m-j}, \\ \overline{G}(\lambda) &= \lambda^{l+m-1} G(\lambda^{-1}) = \sum_{j=0}^{l+m-1} \lambda^j G_{l+m-1-j}. \end{aligned}$$

Theorem 61 (Theorem 1.1, [233]). For $L(\lambda)$ and $M(\lambda)$ given in (87), assume that both $\det(L(\lambda))$ and $\det(M(\lambda))$ do not vanish identically. Then, Equation (88) has a solution pair $(X(\lambda), Y(\lambda))$ such that

$$\deg X(\lambda) \leq m - 1 \text{ and } \deg Y(\lambda) \leq l - 1$$

if and only if both of the following two conditions hold:

- (1) For each $\lambda_0 \in \mathbb{C}$ satisfying $\det(L(\lambda_0)) = \det(M(\lambda_0)) = 0$, if $f(\lambda)$ is a right root function of $L(\lambda)$ at λ_0 of order s and $h(\lambda)$ is a left root function of $M(\lambda)$ at λ_0 of order t , then $h(\lambda)G(\lambda)f(\lambda)$ has a zero at λ_0 of order at least $\min\{s, t\}$;
- (2) If $f_{\circ}(\lambda)$ is a right root function of $\hat{L}(\lambda)$ at zero of order $s_{\circ}$ and $h_{\circ}(\lambda)$ is a left root function of $\hat{M}(\lambda)$ at zero of order $t_{\circ}$, then $h_{\circ}(\lambda)\bar{G}(\lambda)f_{\circ}(\lambda)$ has a zero of order at least $\min\{s_{\circ}, t_{\circ}\}$.

Remark 58. Kaashoek and Lerer proved Theorem 61 by means of Theorem 1.2 of [233], which is a direct corollary of Theorems 3.2 and 4.1 of [234]. Although this proof strategy can be regarded as a discrete version of the proof of Theorem 60, in the discrete case, the common spectrum at infinity plays a crucial role, whereas in [231] there are no common root functions at infinity.

6.7. Sylvester-Polynomial-Conjugate Matrix Equations

For given matrices $A, B_i (i = 1, \dots, k)$, and C , the matrix equation

$$\sum_{i=0}^k A^i X B_i = C$$

has been thoroughly investigated for its important role in control theory (see [229,230,235–237]). Based on results in [230], Wu et al. [238] defined Sylvester sums and Kronecker maps, and used them to discuss the following matrix equation over $\mathbb{R}$:

$$\sum_{i=0}^{n_1} A_i X F^i = \sum_{i=0}^{n_2} B_i R F^i,$$

where X is unknown and other matrices are given over $\mathbb{R}$. Building on the work in [238], Wu et al. [239–241] established the framework of conjugate products and Sylvester-conjugate sums over $\mathbb{C}$. Specifically, for $A \in \mathbb{C}^{m \times n}$ and $k \in \mathbb{N}$, define

$$A^{*k} = \begin{cases} A, & \text{for even } k, \\ \bar{A}, & \text{for odd } k. \end{cases}$$

The conjugate product [239,241] of

$$A(\lambda) = \sum_{i=0}^m A_i \lambda^i \in \mathbb{C}^{p \times q}[\lambda] \text{ and } B(\lambda) = \sum_{j=0}^n B_j \lambda^j \in \mathbb{C}^{q \times r}[\lambda]$$

is defined as

$$A(\lambda) \otimes B(\lambda) = \sum_{i=0}^m \sum_{j=0}^n A_i B_j^* \lambda^{i+j}.$$

For $A \in \mathbb{C}^{n \times n}$ and $k \in \mathbb{N}$, define

$$A^{\overleftarrow{k}} = A^{k-2\lfloor \frac{k}{2} \rfloor} (\bar{A} A)^{\lfloor \frac{k}{2} \rfloor},$$

where $\lfloor \cdot \rfloor$ is the floor function (downward rounding). For $Z \in \mathbb{C}^{r \times p}$, $F \in \mathbb{C}^{p \times p}$, and

$$T(\lambda) = \sum_{i=0}^t T_i \lambda^i \in \mathbb{C}^{n \times r}[\lambda],$$

we define the Sylvester-conjugate sum [241] of $T(\lambda)$ and Z with respect to F by

$$T(\lambda) \boxplus^F Z = \sum_{i=0}^t T_i Z^{*i} F^{\leftarrow i}. \quad (89)$$

Remark 59. Note that Lemma 7 in ref. [241] reveals an intriguing relationship between the conjugate product and the Sylvester-conjugate sum. Specifically, if $A(\lambda) \in \mathbb{C}^{l \times q}[\lambda]$, $B(\lambda) \in \mathbb{C}^{q \times r}[\lambda]$, $F \in \mathbb{C}^{p \times p}$, and $Z \in \mathbb{C}^{r \times p}$, then

$$(A(\lambda) \otimes B(\lambda)) \boxplus^F Z = A(\lambda) \boxplus^F (B(\lambda) \boxplus^F Z).$$

On this basis, Wu et al. [241] investigated the following Sylvester-polynomial-conjugate matrix equation:

$$\sum_{i=0}^{\phi_1} A_i X^{*i} F^{\leftarrow i} + \sum_{j=0}^{\phi_2} B_j Y^{*j} F^{\leftarrow j} = \sum_{k=0}^{\phi_3} C_k R^{*k} F^{\leftarrow k}, \quad (90)$$

where $A_i \in \mathbb{C}^{n \times n}$ ($i = 0, 1, \dots, \phi_1$), $B_j \in \mathbb{C}^{n \times r}$ ($j = 0, 1, \dots, \phi_2$), $C_k \in \mathbb{C}^{n \times m}$ ($k = 0, 1, \dots, \phi_3$), $R \in \mathbb{C}^{m \times p}$, and $F \in \mathbb{C}^{p \times p}$ are given, and $X \in \mathbb{C}^{n \times p}$ and $Y \in \mathbb{C}^{r \times p}$ are unknown. Denote

$$A(\lambda) = \sum_{i=0}^{\phi_1} A_i \lambda^i, \quad B(\lambda) = \sum_{j=0}^{\phi_2} B_j \lambda^j, \quad C(\lambda) = \sum_{k=0}^{\phi_3} C_k \lambda^k, \quad (91)$$

where A_i , B_i , and C_i are given in (90). Thus, by conjugate products and Sylvester-conjugate sums, Equation (90) can be directly rewritten as

$$A(\lambda) \boxplus^F X + B(\lambda) \boxplus^F Y = C(\lambda) \boxplus^F R. \quad (92)$$

Additionally, we say that $A(\lambda) \in \mathbb{C}^{n \times n}[\lambda]$ and $B(\lambda) \in \mathbb{C}^{n \times r}[\lambda]$ are left coprime [239] if all their greatest common left divisors are unimodular, which is also equivalent to the existence of $C(\lambda) \in \mathbb{C}^{n \times r}[\lambda]$ and $D(\lambda) \in \mathbb{C}^{r \times r}[\lambda]$ such that

$$A(\lambda) \otimes C(\lambda) + B(\lambda) \otimes D(\lambda) = I.$$

Theorem 62 (Theorem 2, [241]). Assume that $A(\lambda)$ and $B(\lambda)$ given in (91) are left coprime. Suppose that the unimodular polynomial matrix $U(\lambda) \in \mathbb{C}^{(n+r) \times (n+r)}[\lambda]$ satisfies

$$\begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix} \otimes U(\lambda) = \begin{bmatrix} I & 0 \end{bmatrix}.$$

Then, all solutions of Equation (92) (or (90)) are

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \left(U(\lambda) \otimes \begin{bmatrix} C(\lambda) & 0 \\ 0 & I \end{bmatrix} \right) \boxplus^F \begin{bmatrix} R \\ Z \end{bmatrix}, \quad (93)$$

where $Z \in \mathbb{C}^{r \times p}$ is arbitrary. Furthermore, if partition

$$U(\lambda) = \begin{bmatrix} H(\lambda) & N(\lambda) \\ L(\lambda) & D(\lambda) \end{bmatrix}$$

with $N(\lambda) \in \mathbb{C}^{n \times r}[\lambda]$ and $D(\lambda) \in \mathbb{C}^{r \times r}[\lambda]$, then (93) can be rewritten as

$$\begin{cases} X = (H(\lambda) \otimes C(\lambda)) \overset{F}{\boxplus} R + N(\lambda) \overset{F}{\boxplus} Z, \\ Y = (L(\lambda) \otimes C(\lambda)) \overset{F}{\boxplus} R + D(\lambda) \overset{F}{\boxplus} Z, \end{cases}$$

for arbitrary $Z \in \mathbb{C}^{r \times p}$.

Remark 60.

- (1) Theorem 9 in ref. [239] guarantees the existence of the polynomial matrix $U(\lambda)$ in Theorem 62.
- (2) Taking

$$\phi_1 = 0, \phi_2 = 1, B_0 = 0, B_1 = I, \text{ and } \phi_3 = 0,$$

Equation (90) over $\mathbb{R}$ reduces to

$$AX + YB = C,$$

where $A = A_0$, $B = F$, and $C = C_0R$. Clearly, Theorem 62 is also a generalization of RET over $\mathbb{R}$.

- (3) In Theorem 1 of [241], Wu et al. characterized the homogeneous case of Equation (90) more specifically via a pair of right coprime polynomial matrices. Moreover, in Remark 4 of [241], they utilized the same method to discuss a more general form of Equation (90), i.e.,

$$\sum_{k=1}^{\theta} \sum_{i=0}^{\omega_k} A_{ki} X_k F^{\overleftarrow{i}} = \sum_{j=0}^c C_j R F^{\overleftarrow{j}},$$

where $X_k (k = 1, \dots, \theta)$ are unknown and others are given.

- (4) It can be observed that Lemmas 11 and 12 of [241] are crucial for proving Theorem 62 and Theorem 1 of [239]. Meanwhile, it should be noted that Lemmas 11 and 12 of [241] provide only necessary conditions for left and right coprimeness, respectively. Thus, we contend that exploring the converse problems of these two lemmas is interesting.
- (5) Equation (90) generalizes a class of complex conjugate matrix equations (see [242–248]). For systematic research on complex conjugate matrix equations and their applications in discrete-time antilinear systems, refer to the monograph [249] by Wu and Zhang.

Inspired by [241], Mazurek [250] recently generalized Theorem 62 based on groupoids and vector spaces.

Theorem 63 (Theorem 2, [250]). Let M_{11}, M_{12}, M_{21} , and M_{22} be groupoids with binary operations commonly denoted by $\oplus$, and let V_1 and V_2 be finite-dimensional vector spaces over a field $\mathbb{F}$. Assume that for any $i, j, k \in \{1, 2\}$, two operations

$$\boxtimes : M_{ij} \times V_j \rightarrow V_i \text{ and } \odot : M_{ij} \times M_{jk} \rightarrow M_{ik}$$

are given such that

- (1) $(s \oplus t) \boxtimes vs. = s \boxtimes vs. + t \boxtimes vs.$ for $i, j \in \{1, 2\}$, $s, t \in M_{ij}$, and $vs. \in V_j$;
- (2) $s \boxtimes (ku + lv) = k(s \boxtimes u) + l(s \boxtimes v)$ for $i, j \in \{1, 2\}$, $s \in M_{ij}$, $k, l \in \mathbb{F}$, and $u, vs. \in V_j$;

(3) $s \boxtimes (t \boxtimes v) = (s \odot t) \boxtimes vs.$ for $i, j, k \in \{1, 2\}$, $s \in M_{ij}$, $t \in M_{jk}$, and $vs. \in V_k$.

Suppose that for $a \in M_{11}$ and $b \in M_{12}$, there exist $p \in M_{11}$, $g \in M_{12}$, $d, q \in M_{21}$, and $h, w \in M_{22}$ such that

(i) $((a \odot p) \oplus (b \odot q)) \boxtimes vs. = vs.$ for any $vs. \in V_1$;

(ii) $((d \odot g) \oplus (w \odot h)) \boxtimes u = u$ for any $u \in V_2$;

(iii) $((a \odot g) \oplus (b \odot h)) \boxtimes u = 0$ for any $u \in V_2$.

Then, for $c \in V_1$, the all solutions of the equation

$$a \boxtimes x + b \boxtimes y = c$$

are

$$(x, y) = (p \boxtimes c + g \boxtimes z, q \boxtimes c + h \boxtimes z),$$

where $z \in V_2$ is arbitrary.

For a ring $\mathbb{F}$ with unity and a ring endomorphism σ of $\mathbb{F}$, the skew polynomial ring $\mathbb{F}[\lambda; \sigma]$ is the set of polynomials over $\mathbb{F}$ in the indeterminate λ with the usual addition and multiplication subject to $\lambda a = \sigma(a)\lambda$ for any $a \in \mathbb{F}$ (see [251]). Specifically, the multiplication in $\mathbb{F}[\lambda; \sigma]$ is given by

$$\left(\sum_{i=0}^n a_i \lambda^i \right) \left(\sum_{j=0}^m b_j \lambda^j \right) = \sum_{i=0}^n \sum_{j=0}^m a_i \sigma^i(b_j) \lambda^{i+j} \quad (94)$$

for $\sum_{i=0}^n a_i \lambda^i, \sum_{j=0}^m b_j \lambda^j \in \mathbb{F}[\lambda; \sigma]$. The set of $m \times n$ matrices over the skew polynomial ring $\mathbb{F}[\lambda; \sigma]$ is denoted by $\mathbb{F}^{m \times n}[\lambda; \sigma]$. Assume that $\mathbb{F}$ is also a finite-dimensional vector space over a field $\mathbb{K}$, and put

$$M_{11} = \mathbb{F}^{n \times n}[\lambda; \sigma], \quad M_{12} = \mathbb{F}^{n \times m}[\lambda; \sigma], \quad M_{21} = \mathbb{F}^{m \times n}[\lambda; \sigma], \\ M_{22} = \mathbb{F}^{m \times m}[\lambda; \sigma], \quad V_1 = \mathbb{F}^{n \times p}, \quad V_2 = \mathbb{F}^{m \times p},$$

with the usual addition (denoted by $\oplus$) and the skew multiplication (94) (denoted by $\odot$) of polynomial matrices. Then, applying Theorem 63, Mazurek obtained the relevant result (i.e., Theorem 3, [250]) for Equation

$$A(\lambda) \boxtimes X + B(\lambda) \boxtimes Y = C,$$

where $A(\lambda) \in \mathbb{F}^{n \times n}[\lambda; \sigma]$, $B(\lambda) \in \mathbb{F}^{n \times m}[\lambda; \sigma]$, and $C \in \mathbb{F}^{n \times p}$ are given, and $X \in \mathbb{F}^{n \times p}$ and $Y \in \mathbb{F}^{m \times p}$ are unknown.

Remark 61. In the proof of Theorem 1 in [250], Mazurek showed that Theorem 62 is an immediate corollary of Theorem 3 of [250].

Moreover, analogous to the conjugate product of complex matrices, Wu et al. [252] further defined the $\mathbf{j}$ -conjugate product of quaternion matrices. The $\mathbf{j}$ -conjugate of $A \in \mathbb{H}^{m \times n}$ is

$$\overleftarrow{A} = -\mathbf{j}A\mathbf{j}.$$

For $k \in \mathbb{Z}^+$, inductively define $A^{\circ k} = \overleftarrow{A^{\circ(k-1)}}$ with $A^{\circ 0} = A$. Then, the $\mathbf{j}$ -conjugate product [252] of

$$A(\lambda) = \sum_{i=0}^m A_i \lambda^i \in \mathbb{H}^{p \times q}[\lambda] \text{ and } B(\lambda) = \sum_{j=0}^n B_j \lambda^j \in \mathbb{H}^{q \times r}[\lambda]$$

is defined as

$$A(\lambda) \circledast B(\lambda) = \sum_{i=0}^m \sum_{j=0}^n A_i B_j^{\circ i} \lambda^{i+j}.$$

We say that $A(\lambda) \in \mathbb{H}^{n \times n}[\lambda]$ and $B(\lambda) \in \mathbb{H}^{n \times m}[\lambda]$ are $\circledast$ -left coprime [250] if there exists a unimodular polynomial matrix $U(\lambda) \in \mathbb{H}^{(n+m) \times (n+m)}[\lambda]$ such that

$$\begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix} \circledast U(\lambda) = \begin{bmatrix} I & 0 \end{bmatrix}.$$

Let the map σ be $\sigma: \mathbb{H} \rightarrow \mathbb{H}$ with $\sigma(h) = -\mathbf{j}h\mathbf{j}$ for $h \in \mathbb{H}$. Then, σ is an automorphism on $\mathbb{H}$. So, the $\mathbf{j}$ -conjugate product is the product of matrices over $\mathbb{H}[\lambda; \sigma]$. Similarly to (89), Mazurek [250] defined the Sylvester- $\mathbf{j}$ -conjugate sum over $\mathbb{H}$,

$$T(\lambda) \overset{F}{\boxplus} V = T_0 V + \sum_{m=1}^t T_m \sigma^m(V) \sigma^{m-1}(F) \dots \sigma^1(F) \sigma^0(F)$$

for $T(\lambda) = \sum_{i=0}^t T_i \lambda^i \in \mathbb{H}^{n \times r}[\lambda]$, $V \in \mathbb{H}^{r \times p}$, and $F \in \mathbb{H}^{p \times p}$. Applying Theorem 3 of [250] to matrices over $\mathbb{H}[\lambda; \sigma]$ yields the following result immediately.

Theorem 64 (Theorem 4, [250]). *Let $A(\lambda) \in \mathbb{H}^{n \times n}[\lambda]$ and $B(\lambda) \in \mathbb{H}^{n \times m}[\lambda]$ be $\circledast$ -left coprime. Then, there exist*

$$P(\lambda) \in \mathbb{H}^{n \times n}[\lambda], G(\lambda) \in \mathbb{H}^{n \times m}[\lambda], D(\lambda), Q(\lambda) \in \mathbb{H}^{m \times n}[\lambda], H(\lambda), W(\lambda) \in \mathbb{H}^{m \times m}[\lambda]$$

such that

$$\begin{aligned} A(\lambda) \circledast P(\lambda) + B(\lambda) \circledast Q(\lambda) &= I_n, D(\lambda) \circledast G(\lambda) + W(\lambda) \circledast H(\lambda) = I_m, \\ A(\lambda) \circledast G(\lambda) + B(\lambda) \circledast H(\lambda) &= 0. \end{aligned}$$

Moreover, given $F \in \mathbb{H}^{p \times p}$ and $C \in \mathbb{H}^{n \times p}$, the general solution of the matrix equation

$$A(\lambda) \overset{F}{\boxplus} X + B(\lambda) \overset{F}{\boxplus} Y = C$$

is

$$\begin{cases} X = P(\lambda) \overset{F}{\boxplus} C + G(\lambda) \overset{F}{\boxplus} Z, \\ Y = Q(\lambda) \overset{F}{\boxplus} C + H(\lambda) \overset{F}{\boxplus} Z, \end{cases}$$

where $Z \in \mathbb{H}^{m \times p}$ is arbitrary.

Remark 62. Theorem 64 not only generalizes Theorem 62 to $\mathbb{H}$, but also presents a more general result than the relevant results in [87,253–261].

6.8. Generalized Forms of GSE

In Section 4.3, we can see that the SVD plays an important role in solving Equation (1). Then, will the development of the SVD promote research on a more general form of Equation (1)? The work in [31,262,263] shows that it is affirmative.

De Moor and Zha [262] established GSVD of a finite number $k \in \mathbb{Z}^+$ of matrices over $\mathbb{C}$.

Theorem 65 (GSVD for any k matrices (Theorem 1, [262])). *Let*

$$A_1 \in \mathbb{C}^{n_0 \times n_1}, A_2 \in \mathbb{C}^{n_1 \times n_2}, \dots, A_{k-1} \in \mathbb{C}^{n_{k-2} \times n_{k-1}}, \text{ and } A_k \in \mathbb{C}^{n_{k-1} \times n_k}.$$

Then, there exist unitary matrices $U_1 \in \mathbb{C}^{n_0 \times n_0}$ and $V_k \in \mathbb{C}^{n_k \times n_k}$, and nonsingular matrices $X_j \in \mathbb{C}^{n_j \times n_j}$ ($j = 1, \dots, k-1$) such that

$$U_1^* A_1 X_1 = \Lambda_1, Z_1 A_2 X_2 = \Lambda_2, \dots, Z_{i-1} A_i X_i = \Lambda_i, \dots, Z_{k-1} A_k V_k = \Lambda_k,$$

where $Z_j = X_j^*$ (or $= X_j^{-1}$) for $j = 1, 2, \dots, k-1$ (i.e., both choices are always possible),

$$\Lambda_j = \begin{bmatrix} I & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & I & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & I & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & I & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \begin{matrix} r_j^1 \\ r_{j-1}^1 - r_j^1 \\ r_j^2 \\ r_{j-1}^2 - r_j^2 \\ r_j^3 \\ r_{j-1}^3 - r_j^3 \\ \dots \\ r_j^j \\ n_{j-1} - r_{j-1} - r_j^j \end{matrix}$$

$$\begin{matrix} r_j^1 & r_j^2 & r_j^3 & r_j^4 & \dots & r_j^j & n_j - r_j \end{matrix}$$

($j = 1, 2, \dots, k-1$) with $r_0 = 0$ and $r_j = \sum_{i=1}^j r_i^i = \text{rank}(A_j)$,

$$\Lambda_k = \begin{bmatrix} \Lambda_k^1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \Lambda_k^2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \Lambda_k^3 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \Lambda_k^k & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix} \begin{matrix} r_k^1 \\ r_{k-1}^1 - r_k^1 \\ r_k^2 \\ r_{k-1}^2 - r_k^2 \\ r_k^3 \\ r_{k-1}^3 - r_k^3 \\ \dots \\ r_k^k \\ n_{k-1} - r_{k-1} - r_k^k \end{matrix} \quad (95)$$

$$\begin{matrix} r_k^1 & r_k^2 & r_k^3 & r_k^4 & \dots & r_k^k & n_k - r_k \end{matrix}$$

with $r_k = \sum_{i=1}^k r_i^i = \text{rank}(A_k)$, and $\Lambda_k^i \in \mathbb{C}^{r_i^i \times r_i^i}$ ($i = 1, 2, \dots, k$) are diagonal with positive diagonal elements.

It is easy to see that there exists an elementary column transformation that turns Λ_k given in (95) into the matrix Λ'_k consisting of only zero and identity matrix blocks. So, under the hypotheses of Theorem 65, there exist nonsingular matrices $U_1 \in \mathbb{C}^{n_0 \times n_0}$, $V'_k \in \mathbb{C}^{n_k \times n_k}$, and $X_j \in \mathbb{C}^{n_j \times n_j}$ ($j = 1, \dots, k-1$) such that

$$U_1^* A_1 X_1 = \Lambda_1, Z_1 A_2 X_2 = \Lambda_2, \dots, Z_{i-1} A_i X_i = \Lambda_i, \dots, Z_{k-1} A_k V'_k = \Lambda'_k. \quad (96)$$

In this case, $\Lambda_1, \dots, \Lambda_{k-1}$, and Λ'_k only have zero and identity blocks.

Following the idea analogous to that in (96), He (Lemma 2.1, [263]) considered the pure product singular value decomposition (PSVD) for four quaternion matrices, i.e.,

$$\begin{bmatrix} A_1 & & & \\ I & A_2 & & \\ & I & A_3 & \\ & & I & A_4 \end{bmatrix},$$

where $A_1 \in \mathbb{H}^{q_1 \times q_2}$, $A_2 \in \mathbb{H}^{q_2 \times q_3}$, $A_3 \in \mathbb{H}^{q_3 \times q_4}$, and $A_4 \in \mathbb{H}^{q_4 \times q_5}$. Using the PSVD, He further investigated the system of generalized Sylvester matrix equations over $\mathbb{H}$:

$$\begin{cases} X_1 A_1 - B_1 X_2 = C_1, \\ X_2 A_2 - B_2 X_3 = C_2, \\ X_3 A_3 - B_3 X_4 = C_3, \\ X_4 A_4 - B_4 X_5 = C_4, \end{cases}$$

where $X_1, X_2, \dots, X_5$ are unknown (see Theorems 4.1 and 4.2, [263]).

Inspired by He's aforementioned work, since the GSVD of an arbitrary number k of matrices has been established, can we then consider a system of k generalized Sylvester equations? That is to say, consider

$$\begin{cases} X_1 A_1 - B_1 X_2 = C_1, \\ X_2 A_2 - B_2 X_3 = C_2, \\ \vdots \\ X_k A_k - B_k X_{k+1} = C_k, \end{cases} \quad (97)$$

where $k \in \mathbb{Z}^+$, and $X_1, X_2, \dots, X_{k+1}$ are unknown.

To answer this question, let us first take a look at the work of He et al. in [31]. In terms of Theorem 65, He et al. in Theorem 2.2 of [31] gave the simultaneous decomposition of fifteen matrices over $\mathbb{C}$, i.e.,

$$\begin{bmatrix} & B_1 & & & & & \\ A_1 & E_1 & C_1 & & & & \\ & D_1 & & B_2 & & & \\ & & A_2 & E_2 & C_2 & & \\ & & & D_2 & & B_3 & \\ & & & & A_3 & E_3 & C_3 \\ & & & & & D_3 & \end{bmatrix},$$

where A_i, B_i, C_i, D_i, E_i ($i = 1, 2, 3$) are given matrices over $\mathbb{C}$ with appropriate orders. By this simultaneous decomposition of fifteen matrices, they further studied the following system of complex matrix equations:

$$\begin{cases} A_1 X_1 B_1 + C_1 X_2 D_1 = E_1, \\ A_2 X_2 B_2 + C_2 X_3 D_2 = E_2, \\ A_3 X_3 B_3 + C_3 X_4 D_3 = E_3, \end{cases}$$

where $X_1, \dots, X_4$ are unknown (see Theorems 3.1 and 3.6, [31]). Interestingly, they also demonstrated the simultaneous decomposition of $5k$ matrices by the same means, i.e.,

$$\begin{bmatrix} & B_1 & & & & & \\ A_1 & E_1 & C_1 & & & & \\ & D_1 & & B_2 & & & \\ & & A_2 & E_2 & C_2 & & \\ & & & D_2 & & \ddots & \\ & & & & \ddots & \ddots & \\ & & & & & & B_k \\ & & & & & & A_k & E_k & C_k \\ & & & & & & & D_k & \end{bmatrix},$$

where $k \in \mathbb{Z}^+$, and A_i, B_i, C_i, D_i, E_i ($i = 1, 2, \dots, k$) are given matrices over $\mathbb{C}$ with appropriate orders (see Theorem 4.1, [31]). Next, it is natural to consider the system

$$\begin{cases} A_1 X_1 B_1 + C_1 X_2 D_1 = E_1, \\ A_2 X_2 B_2 + C_2 X_3 D_2 = E_2, \\ \vdots \\ A_k X_k B_k + C_k X_{k+1} D_k = E_k, \end{cases} \quad (98)$$

where $X_1, X_2, \dots, X_{k+1}$ are unknown. Obviously, Problem (97) is a special case of the system (98). However, only one conjecture on the solvability conditions of the system (98) was presented, i.e., (Conjecture 4.2, [31]). The conjecture was not officially solved until 2025 (see Corollary 4.4, [34]). Moreover, in Theorem 2.1 of [34], He et al. further investigated

$$(98) \text{ subject to } \begin{cases} G_1 X_1 = P_1, G_2 X_2 = P_2, \dots, G_{k+1} X_{k+1} = P_{k+1}, \\ X_1 H_1 = Q_1, X_2 H_2 = Q_2, \dots, X_{k+1} H_{k+1} = Q_{k+1}, \end{cases} \quad (99)$$

where $X_1, X_2, \dots, X_{k+1}$ are unknown. Furthermore, Xie et al. [264] studied the following system over $\mathbb{H}$:

$$\begin{cases} A_1 X_1 + Y_1 B_1 + C_1 Z_1 D_1 + F_1 Z_2 G_1 = E_1, \\ A_2 X_2 + Y_2 B_2 + C_2 Z_2 D_2 + F_2 Z_3 G_2 = E_2, \\ A_3 X_3 + Y_3 B_3 + C_3 Z_3 D_3 + F_3 Z_4 G_3 = E_3, \end{cases} \quad (100)$$

where X_i, Y_i, Z_i ($i = 1, 2, 3$), and Z_4 are unknown.

Up to this point, we can observe that the systems (99) and (100) encompass most of the current formal generalizations of Equation (1) without considering differences in number sets, such as [121,265–271].

Remark 63. However, there are currently no published studies on the more generalized system as follows:

$$\left\{ \begin{array}{l} \sum_{j=1}^l A_{1,j} X_{1j} B_{1,j} + A_{1,l+1} X_{1,l+1} B_{1,l+1} + A_{1,l+2} X_{2,l+1} B_{1,l+2} = C_1, \\ \sum_{j=1}^l A_{2,j} X_{2,j} B_{2,j} + A_{2,l+1} X_{2,l+1} B_{2,l+1} + A_{2,l+2} X_{3,l+1} B_{2,l+2} = C_2, \\ \vdots \\ \sum_{j=1}^l A_{i,j} X_{i,j} B_{i,j} + A_{i,l+1} X_{i,l+1} B_{i,l+1} + A_{i,l+2} X_{i+1,l+1} B_{i,l+2} = C_i, \\ \vdots \\ \sum_{j=1}^l A_{k,j} X_{k,j} B_{k,j} + A_{k,l+1} X_{k,l+1} B_{k,l+1} + A_{k,l+2} X_{k+1,l+1} B_{k,l+2} = C_k, \end{array} \right. \quad (101)$$

subject to $\begin{cases} G_{i,j} X_{i,j} = P_{i,j}, X_{i,j} H_{i,j} = Q_{i,j} \text{ for } 1 \leq i \leq k, 1 \leq j \leq l+1, \\ G_{k+1,l+1} X_{k+1,l+1} = P_{k+1,l+1}, X_{k+1,l+1} H_{k+1,l+1} = Q_{k+1,l+1}, \end{cases}$

where $l, k \in \mathbb{Z}^+$, $A_{i,j}, B_{i,j}, C_i$ ($1 \leq i \leq k, 1 \leq j \leq l+2$), $G_{i,j}, P_{i,j}, H_{i,j}, Q_{i,j}$ ($1 \leq i \leq k, 1 \leq j \leq l+1$), $G_{k+1,l+1}, P_{k+1,l+1}, H_{k+1,l+1}$, and $Q_{k+1,l+1}$ are given, and $X_{i,j}$ ($1 \leq i \leq k, 1 \leq j \leq l+1$) and $X_{k+1,l+1}$ are unknown. So, this is also an interesting problem.

Remark 64. Through the continuous research on the generalizations of Equation (1), it can be found that the GSVD gradually becomes ineffective, while the tool of the generalized inverse has always been an effective method. However, we also find that the generalized inverse theory has little success in studying the systems of equations that simultaneously couple multiple (≥ 2) unknown matrices.

For instance, for given matrices A , B , and C , the system

$$\begin{cases} AX - YB = C, \\ X - Y = 0, \end{cases} \quad (102)$$

is consistent if and only if the Sylvester equation

$$AX - XB = C \quad (103)$$

is consistent. At present, there are almost no articles that directly represent the general solution of the Sylvester equation using only the generalized inverses of the coefficient matrices. However, it can be intuitively observed that, using the Kronecker product, SVD, or STP to discuss the Sylvester equation is a feasible option.

It is noted that Liu put forward an open problem in [40], that is, to study the equivalent conditions for the solvability of the system of matrix equations over $\mathbb{C}$:

$$\begin{cases} A_1 X + Y B_1 = C_1, \\ A_2 X + Y B_2 = C_2, \end{cases} \quad (104)$$

where X and Y are unknown. Clearly, the system (104) is a generalization of both Equation (5) and the system (102). Later, using the rank equalities, Wang et al. [114] solved this problem over $\mathbb{H}$ under the certain condition, i.e.,

$$\text{rank} \begin{bmatrix} A_1 & A_2 \end{bmatrix} = \text{rank}(A_1) + \text{rank}(A_2) \text{ and } \text{rank} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \text{rank}(B_1) + \text{rank}(B_2). \quad (105)$$

This wonderful work is illuminating for completely solving Equation (103) and the system (104) by using rank equalities or the generalized inverses.

Theorem 66 (Theorem 2.8, [114]). Let $A_1, A_2 \in \mathbb{H}^{m \times p}$, $B_1, B_2 \in \mathbb{H}^{q \times n}$, and $C_1, C_2 \in \mathbb{H}^{m \times n}$ be such that every matrix equation in system (104) is consistent. If (105) holds, then system (104) is consistent if and only if

$$\begin{aligned} \text{rank} \begin{bmatrix} B_1 & 0 \\ B_2 & 0 \\ -C_1 & A_1 \\ C_2 & A_2 \end{bmatrix} &= \text{rank} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + \text{rank} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \\ \text{rank} \begin{bmatrix} A_1 & A_2 & -C_1 & C_2 \\ 0 & 0 & B_1 & B_2 \end{bmatrix} &= \text{rank} \begin{bmatrix} A_1 & A_2 \end{bmatrix} + \text{rank} \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \\ \text{rank} \begin{bmatrix} 0 & B_1 & B_2 \\ A_1 & 0 & 0 \\ A_2 & 0 & F \end{bmatrix} &= \text{rank} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + \text{rank} \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \\ \text{rank} \begin{bmatrix} 0 & B_1 & B_2 \\ A_1 & 0 & 0 \\ A_2 & 0 & \hat{F} \end{bmatrix} &= \text{rank} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + \text{rank} \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} F &= A_1(A_2^{(1,2)}C_2 - A_1^{(1,2)}C_1) \begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}^{(1,2)} \begin{bmatrix} B_1 \\ -B_2 \end{bmatrix} + \Omega B_1, \\ \hat{F} &= A_2(A_2^{(1,2)}C_2 - A_1^{(1,2)}C_1) \begin{bmatrix} B_1 \\ -B_2 \end{bmatrix}^{(1,2)} \begin{bmatrix} B_1 \\ -B_2 \end{bmatrix} + \Omega B_2, \\ \Omega &= \begin{bmatrix} -A_1 & A_2 \end{bmatrix} \begin{bmatrix} -A_1 & A_2 \end{bmatrix}^{(1,2)} (R_{A_2}C_2B_2^{(1,2)} - R_{A_1}C_1B_1^{(1,2)}) \end{aligned}$$

with $R_{A_1} = I - A_1A_1^{(1,2)}$.

Remark 65. For more research on Equation (104), please refer to [222,272].

7. Iterative Algorithms

Analytical solutions of GSE over various algebras have been introduced in Sections 5 and 6. However, in practical applications, challenges such as high computational complexity, stability, and robustness issues often arise. Therefore, investigating numerical solutions to GSE is imperative. In this section, we primarily present the relevant results on the numerical solutions of GSE. First, several iterative algorithms that directly address the GSE are proposed.

In 1984, Ziętak [96] proposed an algorithm (Algorithm 1) for computing the l_p -solution of Equation (5) over $\mathbb{R}$, using the equivalence between items (1) and (3) of Theorem 17.

Theorem 67. Let $\{X_t\}$ and $\{Y_t\}$ be generalized by Algorithm 1, and let $1 < p < \infty$. Then, the sequence $\{R_t = AX_t + Y_tB\}$ is convergent. Moreover, if $\hat{X} \in \mathbb{R}^{r \times n}$ and $\hat{Y} \in \mathbb{R}^{m \times s}$ satisfy $\hat{R} = A\hat{X} + \hat{Y}B$, where

$$\hat{R} = \lim_{t \rightarrow \infty} R_t,$$

then $\hat{X}$ and $\hat{Y}$ are the l_p -solution of Equation (5).

Remark 66. When $p = 2$, Ziętak (Theorem 3.3, [96]) showed that the iterative process in Algorithm 1 ends after two iterations for any arbitrary initial matrix Y_0 , i.e., matrices X_1 and Y_1 are the l_2 -solution of Equation (5).

Algorithm 1 Algorithm [96] for the l_p -solution of Equation (5) over $\mathbb{R}$

Require: Given matrices $A \in \mathbb{R}^{m \times r}$, $B \in \mathbb{R}^{s \times n}$, and $C \in \mathbb{R}^{m \times n}$; Initial matrices: $Y_0 \in \mathbb{R}^{m \times s}$;
Ensure: Sequences: $\{X_t \in \mathbb{R}^{r \times n}\}$ and $\{Y_t \in \mathbb{R}^{m \times s}\}$;

```

1: for  $t = 1, 2, \dots$  do
2:   for  $j = 1, 2, \dots, n$  do
3:     Let  $c_j$  and  $b_j$  be  $j$ -th columns of  $C$  and  $B$ , respectively;
4:     Solve the  $l_p$ -solution  $x_j^{(t)}$  of the linear system  $Ax = c_j - Y_{t-1}b_j$ ;
5:     Set the  $j$ -th column of  $X_t$  as  $x_j^{(t)}$ ;
6:   end for
7:   for  $i = 1, 2, \dots, m$  do
8:     Let  $d_i$  and  $a_i$  be the  $i$ -th columns of  $C^T$  and  $A^T$ , respectively;
9:     Solve the  $l_p$ -solution  $y_i^{(t)}$  of the linear system  $B^T y = d_i - X_t^T a_i$ ;
10:    Set the  $i$ -th column of  $Y_t^T$  as  $y_i^{(t)}$ ;
11:  end for
12:  Compute  $\gamma_{2t-1} = \|AX_t + Y_{t-1}B - C\|_p$ ;
13:  Compute  $\gamma_{2t} = \|AX_t + Y_tB - C\|_p$ ;
14:  if the sequence  $\{\gamma_t\}$  is not decreasing then
15:    Break
16:  end if
17: end for

```

In 1985, analogous to Algorithm R [273] for a nonlinear matrix equation, Ziętak [95] devised Algorithm T (Algorithm 2). Based on this algorithm, Ziętak [95] further discussed the Chebyshev solution of Equation (5) under the conditions (29) and (28).

Algorithm 2 Algorithm T [95] for the Chebyshev Solution of Equation (5) over $\mathbb{R}$

Require: Matrices $A \in \mathbb{R}^{m \times r}$, $B \in \mathbb{R}^{s \times n}$, and $C \in \mathbb{R}^{m \times n}$; Initial matrices $X_0 \in \mathbb{R}^{r \times n}$ and $Y_0 \in \mathbb{R}^{m \times s}$; Tolerance $\epsilon > 0$ (for termination criterion);

Ensure: Convergent sequences: $\{X_k \in \mathbb{R}^{r \times n}\}$ and $\{Y_k \in \mathbb{R}^{m \times s}\}$;

```

1: Initial residual matrix:  $R_0 = AX_0 + Y_0B - C$ ;
2: Initial residual norm:  $\gamma_0 = \|R_0\|_\infty$ ;
3: Set iteration index:  $k \leftarrow 0$ ;
4: Termination criterion:  $|\gamma_{2k+1} - \gamma_{2k}| < \epsilon$ ;
5: while Termination criterion not satisfied do
6:    $R_{2k} = AX_k + Y_kB - C$ ;
7:    $\gamma_{2k} = \|R_{2k}\|_\infty$ ;
8:   for  $j = 1$  to  $n$  do
9:     Extract the  $j$ -th column of  $R_{2k}$ :  $r_j^{(2k)}$ ;
10:    Solve the Chebyshev solution  $h_j^{(k)}$  of  $Ah = -r_j^{(2k)}$ ;
11:    Set the  $j$ -th column of  $X_{k+1}$  as  $x_j^{(k+1)} = x_j^{(k)} + h_j^{(k)}$ ;
12:  end for
13:   $R_{2k+1} = AX_{k+1} + Y_kB - C$ ;
14:   $\gamma_{2k+1} = \|R_{2k+1}\|_\infty$ ;
15:  for  $i = 1$  to  $m$  do
16:    Extract the  $i$ -th column of  $R_{2k+1}^T$ :  $s_i^{(2k+1)}$ ;
17:    Solve the Chebyshev solution  $g_i^{(k)}$  of  $B^T g = -s_i^{(2k+1)}$ ;
18:    Set the  $i$ -th column of  $Y_{k+1}^T$  as  $y_i^{(k+1)} = y_i^{(k)} + g_i^{(k)}$ ;
19:  end for
20:   $k \leftarrow k + 1$ 
21: end while

```

Remark 67. From Algorithm 2, it follows that $x_j^{(k+1)}$ are the Chebyshev solutions of

$$Ax = c_j - Y_k b_j \quad (j = 1, \dots, n)$$

and $y_i^{(k+1)}$ are the Chebyshev solutions of

$$B^T y = d_i - X_{k+1}^T a_i \quad (i = 1, \dots, m),$$

where c_j, b_j, d_i , and a_i are the columns of C, B, C^T and A^T , respectively. Moreover, one can derive that the sequence $\{\gamma_l\}$ given by Algorithm 2 is convergent.

Theorem 68 (Theorems 5.2 and 5.3, [95]). Let A, B , and C be given in Equation (5).

(1) Suppose that (29) is satisfied. If

$$Z_* = AX_* + Y_* B$$

is a cluster point of the sequence $\{Z_{2k}\}$ generated by Algorithm 2, then X_* and Y_* are a Chebyshev solution of Equation (5)

(2) Suppose that (28) is satisfied and matrix A satisfies Haar's condition, i.e., $\det(A_i) \neq 0$ for $i = 1, \dots, m$, where A_i is the matrix obtained from A by deletion of the i -th row. Then, the matrices X_1 and Y_1 generated by Algorithm 2 are the Chebyshev solution of Equation (5) for arbitrary initial matrices X_0 and Y_0 .

In 2008, using Kronecker products, Yang and Huang [274] derived the normwise backward errors of the approximate solutions of Equation (5) over $\mathbb{R}$, as well as their upper and lower bounds.

Definition 18 ([275]). Let $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{n \times n}$, and $C \in \mathbb{R}^{m \times n}$, and let $\tilde{X}, \tilde{Y} \in \mathbb{R}^{m \times n}$ be a numerical solution of Equation (5). Put

$$\eta = \{(\Delta A, \Delta B, \Delta C) \mid (A + \Delta A)\tilde{X} + \tilde{Y}(B + \Delta B) = C + \Delta C\}$$

$$\text{and } \eta(\tilde{X}, \tilde{Y}) = \min_{(\Delta A, \Delta B, \Delta C) \in \eta} \left\| \begin{bmatrix} \frac{1}{\theta_1} \Delta A, \frac{1}{\theta_2} \Delta B, \frac{1}{\theta_3} \Delta C \end{bmatrix} \right\|_F,$$

where $\theta_1, \theta_2, \theta_3 > 0$ are parameters. Then, $\eta(\tilde{X}, \tilde{Y})$ is referred to as the relative backward error if $\theta_1 = \|A\|_F, \theta_2 = \|B\|_F, \theta_3 = \|C\|_F$; it is termed the absolute backward error if $\theta_1 = \theta_2 = \theta_3 = 1$.

Theorem 69 (Theorem 1, [274]). Under the hypotheses Definition 18, let

$$R = C - A\tilde{X} - \tilde{Y}B \text{ and } T = \left[\theta_1(\tilde{X}^T \otimes I_m), \theta_2(I_n \otimes \tilde{Y}), -\theta_3(I_n \otimes I_m) \right].$$

Then

$$\frac{\|R\|_F}{\sqrt{n\theta_3^2 + \theta_1^2 \|\tilde{X}\|_F^2 + \theta_2^2 \|\tilde{Y}\|_F^2}} \leq \eta(\tilde{X}, \tilde{Y}) = \|T^\dagger \text{Vec}(R)\|_2 \leq \frac{\|R\|_F}{\sqrt{\theta_3^2 + \theta_1^2 \lambda_n(\tilde{X}^T \tilde{X}) + \theta_2^2 \lambda_m(\tilde{Y} \tilde{Y}^T)}},$$

where $\text{Vec}(\cdot)$ denotes the column-wise vectorization of a matrix, and $\lambda_n(\cdot)$ denotes the largest eigenvalue of a $n \times n$ positive semidefinite matrix.

In 2011, Li et al. [276] extended the classical conjugate gradient least-squares algorithm (abbreviated as CGLSA) to compute the optimal solution of Equation (5) over $\mathbb{R}$ with symmetric pattern constraints, i.e., consider

$$\min_{X \in \mathbb{S}^{m \times m}, Y \in \mathbb{S}^{n \times n}} \|AX + YB - C\|_F, \quad (106)$$

where $A, B, C \in \mathbb{R}^{n \times m}$, and $\mathbb{S}^{m \times m}$ denotes the set of all $m \times m$ real symmetric matrices.

Theorem 70 (Theorems 1 and 2, [276]). Let $A, B, C \in \mathbb{R}^{n \times m}$.

(1) For any arbitrary initial matrices $X^{(0)} \in \mathbb{S}^{m \times m}$ and $Y^{(0)} \in \mathbb{S}^{n \times n}$, the matrix sequence $\{X^{(k)}\}$ and $\{Y^{(k)}\}$ generated by Algorithm 3 will converge to a solution of Problem (106) with finitely many steps in exact arithmetic.

(2) Let

$$\mathbb{S} = \left\{ [X, Y] \mid X = \frac{1}{2}(A^T H + H^T A) \text{ and } Y = \frac{1}{2}(H B^T + B H^T) \right\}$$

for arbitrary $H \in \mathbb{R}^{n \times m}$. If $[X^{(0)}, Y^{(0)}] \in \mathbb{S}$, then by Algorithm 3 we can obtain the unique least norm solution of Problem (106).

Remark 68. Under the conditions of Theorem 70, if matrix pair $[X, Y] \in \mathbb{S}$ is a solution of Problem (106), then it is the unique least norm solution. Additionally, Li et al. (Theorem 3, [276]) characterized the minimization property of Algorithm 3, which ensures that the algorithm converges smoothly.

The classic alternating direction method (abbreviated as ADM) is an extension of the augmented Lagrangian multiplier method [277]. In 2017, inspired by [278], Ke and Ma [279] applied ADM to solve the nonnegative solutions of Equation (5) over $\mathbb{R}$, i.e.,

$$\text{find } X(\geq 0) \in \mathbb{R}^{s \times n} \text{ and } Y(\geq 0) \in \mathbb{R}^{m \times l} \text{ s.t. } AX + YB = C,$$

where $A \in \mathbb{R}^{m \times s}$, $B \in \mathbb{R}^{l \times n}$, and $C \in \mathbb{R}^{m \times n}$. Note that Problem (107) is equivalent to the following quadratic programming problem:

$$\min_{X \in \mathbb{R}^{s \times n}, Y \in \mathbb{R}^{m \times l}} \frac{1}{2} \|AX + YB - C\|_F^2 \quad \text{s.t.} \quad X \geq 0 \text{ and } Y \geq 0,$$

which is further equivalent to

$$\min_{X, U \in \mathbb{R}^{s \times n}, Y, V \in \mathbb{R}^{m \times l}} \frac{1}{2} \|AX + YB - C\|_F^2 \quad \text{s.t.} \quad X = U, Y = V, U \geq 0, \text{ and } V \geq 0. \quad (107)$$

For $A = (a_{ij}) \in \mathbb{R}^{m \times n}$, let $\mathcal{P}_+(A) = [b_{ij}] \in \mathbb{R}^{m \times n}$, where $b_{ij} = \max\{0, a_{ij}\}$.

Definition 19 ([279]). A point (X, Y, U, V) satisfies the KKT conditions for Problem (107), i.e., there exist the matrices Λ and Π such that

$$\begin{aligned} A^T(AX + YB - C) + \Lambda &= 0, (AX + YB - C)B^T + \Pi = 0, X - U = 0, \\ Y - V &= 0, \Lambda \leq 0 \leq U, \Lambda \odot U = 0, \Pi \leq 0 \leq V, \Pi \odot V = 0, \end{aligned}$$

where $\odot$ denotes the component-wise multiplication.

Theorem 71 (Theorem 4.1 and Corollary 4.1, [279]). Let the sequence $\{W_k = (X_k, Y_k, U_k, V_k, \Lambda_k, \Pi_k)\}$ be generated by Algorithm 4.

(1) Suppose that $\{W_k\}$ satisfies

$$\lim_{k \rightarrow \infty} (W_{k+1} - W_k) = 0.$$

Then, any accumulation point $(X, Y, U, V, \Lambda, \Pi)$ of $\{W_k\}$ satisfies the KKT conditions for Problem (107).

(2) If $\{W_k\}$ converges, then it converges to a KKT point of Problem (107).

Algorithm 3 Extended CGLSA [276] for the real symmetric solution of Equation (5)

Require: Given matrices: $A, B, C \in \mathbb{R}^{n \times m}$; Initial matrices: $X^{(0)} \in \mathbb{S}\mathbb{R}^{m \times m}$ and $Y^{(0)} \in \mathbb{S}\mathbb{R}^{n \times n}$; Tolerance $\epsilon > 0$ (for termination criterion);

Ensure: The symmetric solution: $X^{(k)}$ and $Y^{(k)}$;

- 1: The initial residual matrix: $R_0 = C - AX^{(0)} - Y^{(0)}B$;
- 2: $P_{0,1} = \frac{1}{2}(A^T R_0 + R_0^T A)$ and $P_{0,2} = \frac{1}{2}(R_0 B^T + B R_0^T)$;
- 3: $Q_{0,1} = P_{0,1}$ and $Q_{0,2} = P_{0,2}$;
- 4: $M_0 = A Q_{0,1} + Q_{0,2} B$;
- 5: Set iteration index: $k = 0$;
- 6: **while** $\|P_{k,1}\|_F^2 + \|P_{k,2}\|_F^2 > \epsilon$ **do**
- 7: Update iteration index: $k \leftarrow k + 1$
- 8: $M_k = A Q_{k,1} + Q_{k,2} B$
- 9: $\alpha_k = \frac{\|P_{k,1}\|_F^2 + \|P_{k,2}\|_F^2}{\|M_k\|_F^2}$
- 10: Update solution matrices: $X^{(k+1)} = X^{(k)} + \alpha_k Q_{k,1}$ and $Y^{(k+1)} = Y^{(k)} + \alpha_k Q_{k,2}$;
- 11: $R_{k+1} = R_k - \alpha_k M_k$;
- 12: $P_{k+1,1} = P_{k,1} - \frac{\alpha_k}{2}(A^T M_k + M_k^T A)$ and $P_{k+1,2} = P_{k,2} - \frac{\alpha_k}{2}(M_k B^T + B M_k^T)$;
- 13: $\beta_k = \frac{\|P_{k+1,1}\|_F^2 + \|P_{k+1,2}\|_F^2}{\|P_{k,1}\|_F^2 + \|P_{k,2}\|_F^2}$;
- 14: $Q_{k+1,1} = P_{k+1,1} + \beta_k Q_{k,1}$ and $Q_{k+1,2} = P_{k+1,2} + \beta_k Q_{k,2}$;
- 15: **end while**
- 16: Output the final solution: $X^{(k+1)}$ and $Y^{(k+1)}$.

Algorithm 4 ADM [279] for the nonnegative solution of Equation (5) over $\mathbb{R}$

Require: Given matrices: $A \in \mathbb{R}^{m \times r}$, $B \in \mathbb{R}^{s \times n}$, and $C \in \mathbb{R}^{m \times n}$; Initial matrices: $Y_0, U_0, V_0, \Lambda_0, \Pi_0 = 0$; Penalty parameters: $\alpha, \beta > 0$; $\gamma \in (0, 1.618)$; Tolerance $\epsilon > 0$;

Ensure: The nonnegative solution: $X^{(k+1)}$ and $Y^{(k+1)}$;

- 1: Set iteration index: $k = 0$;
- 2: **while** $k < \text{maxiter}$ **do**
- 3: $X_{k+1} = (A^T A + \alpha I)^{-1}(A^T C + \alpha U_k - A^T Y_k B - \Lambda_k)$;
- 4: $Y_{k+1} = (C B^T + \beta V_k - A X_{k+1} B^T - \Pi_k)(B B^T + \beta I)^{-1}$;
- 5: $U_{k+1} = \mathcal{P}_+(X_{k+1} + \frac{1}{\alpha} \Lambda_k)$;
- 6: $V_{k+1} = \mathcal{P}_+(Y_{k+1} + \frac{1}{\beta} \Pi_k)$;
- 7: $\Lambda_{k+1} = \Lambda_k + \gamma \alpha (X_{k+1} - U_{k+1})$;
- 8: $\Pi_{k+1} = \Pi_k + \gamma \beta (Y_{k+1} - V_{k+1})$;
- 9: **if** $\|A X_{k+1} + Y_{k+1} B - C\|_F \leq \epsilon$ and $X_{k+1}, Y_{k+1} \geq 0$ **then**
- 10: Output X_{k+1}, Y_{k+1} and **return**
- 11: **end if**
- 12: $k \leftarrow k + 1$;
- 13: **end while**

We observe that GSE has numerous generalizations, as discussed in Section 6. Consequently, the iterative algorithms for these generalized forms specialize to the corresponding results of GSE as special cases. Next, for convenience, we present the key algorithms related to these generalizations in an enumerated form.

- (I) In 2006, Peng et al. [280] showed an efficient iterative algorithm for solving the following matrix equation over $\mathbb{R}$:

$$AXB + CYD = E, \quad (108)$$

where X and Y are unknown. They noted that the algorithm can also be used to construct symmetric, antisymmetric, and bisymmetric solutions of (108) with only minor changes.

- (II) The condition number is an important topic in numerical analysis, characterizing the worst-case sensitivity of problems to input data perturbations. A large condition number indicates an ill-posed problem. Consider the following matrix equation:

$$\begin{cases} AX - YB = C, \\ DX - YE = F, \end{cases} \quad (109)$$

where X and Y are unknown.

- (i) In 1996, Kågström and Poromaa [281] presented LAPACK-style algorithms and software for solving Equation (109) over $\mathbb{C}$.
 - (ii) In 2007, Lin and Wei [282] studied the perturbation analysis for Equation (109) over $\mathbb{R}$, explicitly deriving the expressions and upper bounds for normwise, mixed, and componentwise condition numbers.
 - (iii) In 2013, Diao et al. [283] developed the small sample statistical condition estimation algorithm to evaluate the normwise, mixed, and componentwise condition numbers of Equation (109) over $\mathbb{R}$. In [283], they also investigated the effective condition number for Equation (109) and derived sharp perturbation bounds using this condition number.
- (III) In 2008, Dehghan and Hajarian [284] proposed an iterative algorithm to solve the reflexive solutions of Equation (109) over $\mathbb{R}$.
- (IV) In 2010, Dehghan and Hajarian [285] presented an iterative algorithm for solving the generalized bisymmetric solutions of the generalized coupled Sylvester matrix equation over $\mathbb{R}$:

$$\begin{cases} AXB + CYD = M, \\ EXF + GYH = N, \end{cases} \quad (110)$$

where X and Y are unknown generalized bisymmetric matrices.

- (V) In 2012, inspired by the least-squares QR-factorization algorithm in [286], Li and Huang [287] proposed an iterative method to find the best approximate solution of Equation (110) over $\mathbb{R}$, where unknown matrices X and Y are constrained to be symmetric, generalized bisymmetric, or (R, S) -symmetric.
- (VI) In 2018, inspired by [288,289], Lv and Ma (Section 3, [290]) proposed a parametric iterative algorithm for Equation (109) over $\mathbb{R}$. Moreover, in (Section 4, [290]), they developed an accelerated iterative algorithm based on this parametric approach. Note that Ref. [289] is a monograph on iterative algorithms for the constrained solutions of matrix equations.
- (VII) Interestingly, in 2024, Ma et al. [291] proposed a Newton-type splitting iterative method for the coupled Sylvester-like absolute value equation $\mathbb{R}$:

$$\begin{cases} A_1XB_1 + C_1|Y|D_1 = E_1, \\ A_2YB_2 + C_2|X|D_2 = E_2, \end{cases}$$

where X and Y are unknown. Here, $|A|$ means that each component of a matrix A is absolute-valued.

- (VIII) The algorithms in [292–294] suffer from parameter tuning challenges, particularly for large-scale problems. To address this limitation, in 2025, Shirilord and Dehghan [295] recently proposed an advanced gradient descent-based parameter-free method to solve Equation (109) over $\mathbb{R}$.

In the final part of this section, we further enumerate some iterative algorithms for solving the generalizations of GSE, which involve an arbitrary number of unknown (or coefficient) matrices.

- (A) In 2005–2006, using the hierarchical identification principle, Ding and Chen [293,294] presented a large family of iterative methods for the more general form of Equation (5) over $\mathbb{R}$, i.e.,

$$A_{i,1}X_1B_{i,1} + A_{i,2}X_2B_{i,2} + \dots + A_{i,p}X_pB_{i,p} = C_i, \quad i = 1, 2, \dots, p, \quad (111)$$

where $X_1, X_2, \dots, X_p$ are unknown. These iterative methods subsume the well-known Jacobi and Gauss–Seidel iterations. Subsequent scholars have conducted more extensive research on numerical algorithms for Equation (111).

- (a) In 2010, based on the conjugate gradient method, Dehghan and Hajarian [296] constructed an iterative method for Equation (111) over $\mathbb{R}$ with the generalized bisymmetric solutions $(X_1, X_2, \dots, X_p)$.
- (b) In 2012, Dehghan and Hajarian [297] introduced two iterative methods for solving (111) over $\mathbb{R}$ with the generalized centro-symmetric and central anti-symmetric solutions $(X_1, X_2, \dots, X_p)$.
- (c) In 2014, Hajarian [298] solved Equation (111) over $\mathbb{C}$ by the matrix form of the conjugate gradients squared method.
- (d) In 2016, Hajarian [299] presented the generalized conjugate direction algorithm for computing Equation (111) with the symmetric solutions $(X_1, X_2, \dots, X_p)$.
- (e) In 2017, based on the Hestenes–Stiefel version of the biconjugate residual (BCR) algorithm, Hajarian [300] solved the generalized Sylvester matrix equation

$$\sum_{i=1}^f (A_i X B_i) + \sum_{j=1}^g (C_j Y D_j) = E$$

over $\mathbb{R}$ with the generalized reflexive solutions (X, Y) . In 2018, Lv and Ma [301] introduced another Hestenes–Stiefel version of BCR method for computing the centrosymmetric or anti-centrosymmetric solutions of Equation (111) over $\mathbb{R}$.

- (f) In 2018, inspired by [302], Sheng [292] proposed a relaxed gradient-based iterative (abbreviated as RGI) algorithm to solve Equation (109), and further generalized this algorithm to Equation (111). Moreover, numerical examples in [292] demonstrate that the RGI algorithm outperforms the iterative algorithm in [294] in terms of speed, elapsed time, and iterative steps.
- (g) In 2018, Hajarian [303] extended the Lanczos version of BCR algorithm to find the symmetric solutions (X, Y, Z) of the matrix equation over $\mathbb{R}$:

$$A_i X B_i + C_i Y D_i + E_i Z F_i = G_i, \quad i = 1, 2, \dots, t.$$

In 2020, Yan and Ma [304] also used the Lanczos version of BCR algorithm to study Equation (111) over $\mathbb{R}$ with the (anti-)reflexive solutions $(X_1, X_2, \dots, X_p)$.

- (B) In 2009, from an optimization perspective, Zhou et al. [305] developed a novel iterative method for solving Equation (111) over $\mathbb{R}$ and its more general form, i.e.,

$$\sum_{j=1}^{s_{i,1}} A_{i,1,j} X_1 B_{i,1,j} + \sum_{j=1}^{s_{i,2}} A_{i,2,j} X_2 B_{i,2,j} + \dots + \sum_{j=1}^{s_{i,p}} A_{i,p,j} X_p B_{i,p,j} = C_i, \quad i = 1, 2, \dots, p$$

with unknown $X_1, X_2, \dots, X_p$, which contains iterative methods in [293,294] as special cases. In 2015, by extending the generalized product biconjugate gradient algorithms,

Hajarian gave [306] four effective matrix algorithms for the coupled matrix equation over $\mathbb{R}$:

$$\sum_{j=1}^l (A_{i,1,j}X_1B_{i,1,j} + A_{i,2,j}X_2B_{i,2,j} + \dots + A_{i,l,j}X_lB_{i,l,j}) = D_i, \quad i = 1, 2, \dots, l,$$

where $X_1, X_2, \dots, X_l$ are unknown.

- (C) In 2011, Wu et al. [307] constructed an iterative algorithm to solve the coupled Sylvester-conjugate matrix equation over $\mathbb{C}$:

$$\sum_{j=1}^p (A_{ij}X_jB_{ij} + C_{ij}\bar{X}_jD_{ij}) = F_i, \quad i = 1, 2, \dots, n,$$

where $X_1, X_2, \dots, X_p$ are unknown. In 2021, inspired by [307], Yan and Ma [308] proposed an iterative algorithm for the generalized Hamiltonian solutions of the generalized coupled Sylvester-conjugate matrix equations over $\mathbb{C}$:

$$\begin{cases} \sum_{j=1}^q (A_{ij}X_jB_{ij} + C_{ij}\bar{Y}_jD_{ij}) = M_i, \\ \sum_{j=1}^q (E_{ij}Y_jF_{ij} + G_{ij}\bar{X}_jH_{ij}) = N_i, \end{cases}$$

where $i = 1, \dots, p$, and X_j and Y_j ($j = 1, \dots, q$) are unknown generalized Hamiltonian matrices.

- (D) In 2015, inspired by [309,310], Hajarian [311] obtained an iterative method for the coupled Sylvester-transpose matrix equations over $\mathbb{R}$:

$$\begin{cases} \sum_{k=1}^l (A_{1,k}XB_{1,k} + C_{1,k}X^TD_{1,k} + E_{1,k}YF_{1,k}) = M_1, \\ \sum_{k=1}^l (A_{2,k}XB_{2,k} + C_{2,k}X^TD_{2,k} + E_{2,k}YF_{2,k}) = M_2 \end{cases}$$

with unknown X and Y , by developing the biconjugate A -orthogonal residual and the conjugate A -orthogonal residual squared methods. Based on this developed method, Hajarian [311] also considered the coupled periodic Sylvester matrix equations over $\mathbb{R}$:

$$\begin{cases} A_{1,j}X_jB_{1,j} + C_{1,j}X_{j+1}D_{1,j} + E_{1,j}Y_jF_{1,j} = M_{1,j}, \\ A_{2,j}X_jB_{2,j} + C_{2,j}X_{j+1}D_{2,j} + E_{2,j}Y_jF_{2,j} = M_{2,j}, \end{cases} \quad \text{for } j = 1, 2, \dots,$$

where X_j and Y_j are unknown periodic matrices with a period.

- (E) Discrete-time periodic matrix equations are an important tool for analyzing and designing periodic systems [312]. More related studies are as follows:

- (a) In 2017, Hajarian [313] introduced a generalized conjugate direction method for solving the general coupled Sylvester discrete-time periodic matrix equations over $\mathbb{R}$:

$$\begin{cases} \sum_{j=1}^m (A_{ij}X_iB_{ij} + C_{ij}X_{i+1}D_{ij} + E_{ij}Y_iF_{ij} + G_{ij}Y_{i+1}H_{ij}) = M_i, \quad i = 1, 2, \dots, \\ \sum_{j=1}^m (\hat{A}_{ij}X_i\hat{B}_{ij} + \hat{C}_{ij}X_{i+1}\hat{D}_{ij} + \hat{E}_{ij}Y_i\hat{F}_{ij} + \hat{G}_{ij}Y_{i+1}\hat{H}_{ij}) = \hat{M}_i, \quad i = 1, 2, \dots, \end{cases}$$

8.1. Theoretical Applications

8.1.1. Solvability of Matrix Equations

In 1988, Wimmer [223] utilized the solvability condition of the polynomial matrix form of Equation (1) (i.e., Theorem 55) to give a necessary and sufficient condition for the consistency of the matrix equation over a field $\mathbb{F}$:

$$X - AXB = C, \quad (112)$$

where $A \in \mathbb{F}^{p \times p}$, $B \in \mathbb{F}^{q \times q}$, and $C \in \mathbb{F}^{p \times q}$.

Theorem 72 (Theorem 2, [223]). Equation (112) has a solution $X \in \mathbb{F}^{p \times q}$ if and only if there exist nonsingular matrices $S, R \in \mathbb{F}^{(p+q) \times (p+q)}$ such that

$$S \left(\lambda \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix} + \begin{bmatrix} A & C \\ 0 & I \end{bmatrix} \right) R = \left(\lambda \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix} + \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \right).$$

Remark 70. The core of the proof of Theorem 72 is that

$$(112) \Leftrightarrow \begin{cases} X - AY = C, \\ Y = XB, \end{cases} \Leftrightarrow (A + \lambda I)Y - X(I + \lambda B) = -C.$$

Furthermore, using a special polynomial matrix form of Equation (5), Huang and Liu [237] considered the solvability condition for the matrix equation

$$\sum_{i=0}^k A^i X B_i = C \quad (113)$$

with unknown X over a ring with identity.

Theorem 73 (Theorems 1 and 2, [237]). Let $\mathbb{F}$ be a ring with identity, and let $A \in \mathbb{F}^{n \times n}$, $B_i \in \mathbb{F}^{m \times q}$ ($i = 0, 1, \dots, k$), and $C \in \mathbb{F}^{n \times q}$. Denote

$$B(\lambda) = \sum_{i=0}^k B_i \lambda^i \in \mathbb{F}^{m \times q}[\lambda].$$

(1) Then, Equation (113) is solvable if and only if the polynomial matrix equation

$$(\lambda I - A)X(\lambda) + Y(\lambda)B(\lambda) = C$$

has a solution pair $X(\lambda) \in \mathbb{F}^{n \times q}[\lambda]$ and $Y(\lambda) \in \mathbb{F}^{n \times m}[\lambda]$.

(2) Suppose that $\mathbb{F}$ is a division ring and A is algebraic (or $\mathbb{F}$ is a finitely generated as module over its center). Then, Equation (113) is solvable if and only if

$$\begin{bmatrix} \lambda I - A & -C \\ 0 & B(\lambda) \end{bmatrix} \text{ and } \begin{bmatrix} \lambda I - A & 0 \\ 0 & B(\lambda) \end{bmatrix}$$

are equivalent.

Remark 71.

(1) Theorem 73 remains valid when $\mathbb{F}$ is a finite dimensional central simple algebra over a field (see Theorems 3 and 4, [235]).

- (2) In Corollaries 1–3 of [237], Huang and Liu indicated that relevant results regarding the solvability of the equations $AX - XB = C$, $X - AXB = C$, and $AXB = C$ can be directly derived by Theorem 73.

8.1.2. UTV Decomposition of Dual Matrices

In 2024, Xu et al. [324] presented the UTV decomposition of dual complex matrices based on the solvability conditions and general solution representations of Equation (1) over $\mathbb{C}$ (i.e., Theorem 3).

For $a_s, a_i \in \mathbb{C}$, $a = a_s + a_i\epsilon$ represents a dual complex number, where ϵ is the dual unit given in (60). The set of all dual complex numbers is denoted by $\mathbb{DC}$. For $A = A_s + A_i\epsilon \in \mathbb{DC}^{n \times p}$, A has unitary columns if $n \geq p$ and $A^*A = I_n$, where $A^* = A_s^* + A_i^*\epsilon$ is the conjugate transpose of A . For $A = A_s + A_i\epsilon \in \mathbb{DC}^{n \times n}$, A is unitary if $A^*A = AA^* = I_n$; A is diagonal if both A_s and A_i are diagonal; and A is nonsingular if $AA^{-1} = A^{-1}A = I_n$ for some $A^{-1} \in \mathbb{DC}^{n \times n}$.

We say that $A = A_s + A_i\epsilon \in \mathbb{DC}^{m \times n}$ has the UTV decomposition [324] if

$$A = UTV^*, \quad (114)$$

where $U = U_s + U_i\epsilon \in \mathbb{DC}^{m \times k}$ has unitary columns, $T = T_s \in \mathbb{C}^{k \times k}$ is triangular and nonsingular, and $V = V_s + V_i\epsilon \in \mathbb{DC}^{n \times k}$ has a unitary standard part V_s .

Theorem 74 (Theorem 3.1, [324]). Let $A = A_s + A_i\epsilon \in \mathbb{DC}^{m \times n}$. Assume that the UTV decomposition of A_s is given by

$$A_s = U_s T_s V_s^*,$$

where both $U_s \in \mathbb{C}^{m \times k}$ and $V_s \in \mathbb{C}^{n \times k}$ have unitary columns, and $T_s \in \mathbb{C}^{k \times k}$ is triangular and nonsingular. Then, the UTV decomposition of A exists if and only if

$$(I_m - U_s U_s^*) A_i (I_n - V_s V_s^*) = 0,$$

in which case,

$$U_i = (I_m - U_s U_s^*) A_i V_s T_s^{-1} + U_s P \text{ and } V_i = A_i^* U_s (T_s^{-1})^* - V_s T_s^* P^* (T_s^{-1})^*,$$

where $P \in \mathbb{C}^{k \times k}$ is arbitrary skew-Hermitian matrix.

Remark 72. The proof of Theorem 3.1 in [324] shows that the pivotal step in proving Theorem 74 is that A has UTV decomposition if and only if the matrix equation

$$U_i T_s V_s^* + U_s T_s V_i^* = A_i$$

is consistent for unknown $U_i \in \mathbb{DC}^{m \times k}$ and $V_i \in \mathbb{DC}^{n \times k}$.

8.1.3. Microlocal Triangularization of Pseudo-Differential Systems

In 2013, using the solvability of Equation (1) over $\mathbb{C}$, Kiran [321] constructed a recursive scheme to factorize a pseudo-differential system into lower and upper triangular systems (LU factorization) independent of lower order terms.

Let $OPS_N^0(\Omega)$ be the set of all $N \times N$ pseudo-differential systems of order 0 defined on an open subset Ω of $\mathbb{R}^n$. In addition, the notation and terminology in this subsection follow that in [325,326].

Definition 20 (Definition 2.2, [321]). A matrix valued operator $A \in OPS_N^0(\Omega)$ admits LU factorization if

$$A = LU,$$

where $L \in OPS_N^0(\Omega)$ is an elliptic lower triangular matrix whose principal symbol has the identity on the diagonal entries, and $U \in OPS_N^0(\Omega)$ is an upper triangular matrix.

Theorem 75 (Theorem 2.3, [321]). Let $\lambda_1(x, \xi), \dots, \lambda_N(x, \xi)$ be N sections of eigenvalues of the principal symbol of $A \in OPS_N^0(\Omega)$, including multiplicities, in a conic neighborhood Γ of $(x_0, \xi_0) \in T^*\Omega \setminus \{0\}$. If

$$\lambda_i^{-1}(0) \cap \lambda_j^{-1}(0) \cap \Gamma = \emptyset, \quad i \neq j,$$

then A is microlocally triangularizable in Γ independent of lower order terms. Moreover, the system admits LU factorization independent of lower order terms in Γ if and only if the principal symbol of A admits an LU factorization where the first $N - 1$ eigenvalues of the upper triangular matrix do not vanish in Γ .

Remark 73.

- (1) In Sections 3.3 and 3.4 of [321], Kiran showed that the triangularization scheme in Theorem 75 can also be applied to symbolic hierarchies.
- (2) Lemma 2.5 of [321] shows that Equation (1) over $\mathbb{C}$ has a unique solution X if and only if A or B is nonsingular. However, there is a simple counterexample to its sufficiency. Indeed, if both A and B are identity matrices (and thus nonsingular), the solution X of Equation (1) is obviously not unique for a given C . For instance, take $X = C$ and $Y = 0$, or $X = 2C$ and $Y = C$. This minor error, however, does not affect the existence of solutions to Equation (1).

8.2. Practical Applications

8.2.1. Calibration Problems

The late 1970s to early 1980s witnessed a surge of interest in deploying robotic manipulators for automated manufacturing. However, integrating robots as core components in flexible manufacturing systems remained challenging across many industrial applications, prompting extensive research into the manipulator calibration [327].

Inspired by [328,329], Zhuang et al. [330] first solved a specialized form of Equation (1) to address a robot calibration problem: the calibration of the robot/world (i.e., the BASE transformation) and tool/flange (i.e., the TOOL transformation). As a matter of fact, robot manipulator calibration refers to the procedure of enhancing a robot manipulator's accuracy by adjusting its control software.

Figure 1 provides a schematic illustration of the geometry of a robotic cell. The world coordinate frame serves as an external reference frame. The base coordinate frame is defined within the robot structure. The flange coordinate frame is defined on the mounting surface of the robot end effector. The tool frame is positioned at a point inside the end effector.

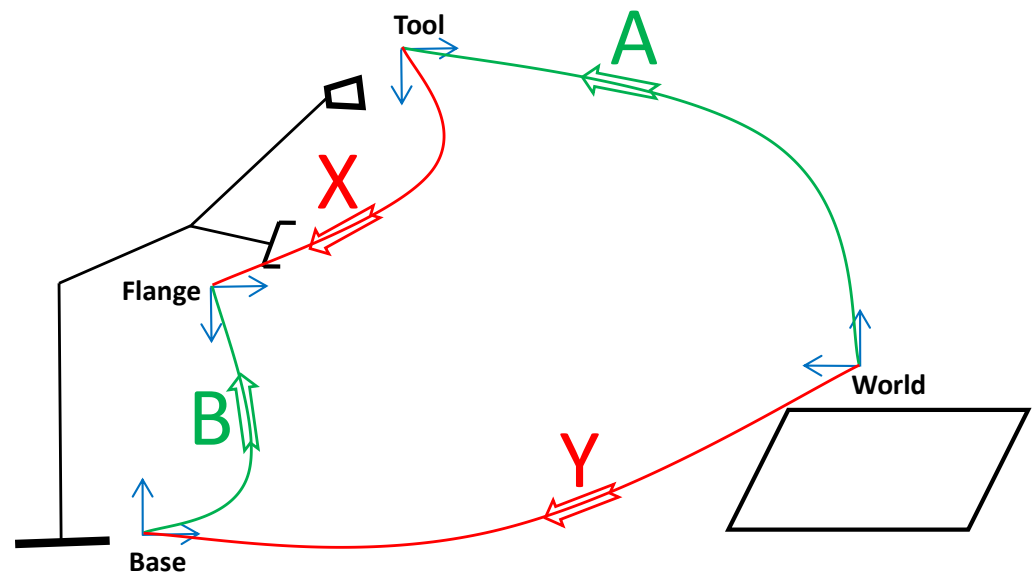


Figure 1. Geometry of a robotic system.

Then, the robot kinematic model can be transformed into the following form:

$$\mathbf{A}\mathbf{X} = \mathbf{Y}\mathbf{B}, \quad (115)$$

where

- (i) $\mathbf{A}$ is the known homogeneous transformation from end effector pose measurements,
- (ii) $\mathbf{B}$ is derived from the calibrated manipulator internal-link forward kinematics;
- (iii) $\mathbf{X}$ is the unknown transformation from the tool frame to the flange frame;
- (iv) $\mathbf{Y}$ is the unknown transformation from the world frame to the base frame.

Assume that there are n pose measurements for $i = 1, 2, \dots, n$. Thus, the calibration problem is reduced to solving the system of equations

$$\mathbf{A}_i\mathbf{X} = \mathbf{Y}_i\mathbf{B}, \quad i = 1, 2, \dots, n.$$

Subsequently, Zhuang et al. elaborated on the solution of Equation (115) using the rotational properties of quaternions in Section 3 of [330].

Subsequently, the robot manipulator calibration problem is further optimized by considering Equation (115) through different approaches: dual quaternion method [331–335], new hybrid calibration method [336], least-squares approach [337], Kronecker product method [338], 3D position measurements [339], nonlinear optimization and evolutionary computation [340], 2D positional features [341], dual Lie algebra [342], symbolic method [343], linear matrix inequality and semi-definite programming optimization [344], probabilistic framework [345], transference principle [346], etc.

8.2.2. Encryption and Decryption Schemes for Color Images

The RGB (red, green, blue) color channels can be directly mapped to the imaginary parts (i, j, k) of a pure imaginary quaternion matrix. Naturally, a dual quaternion matrix can represent two color images since both its standard and infinitesimal parts are quaternion matrices. By solving Equation (1) over $\mathbb{DH}$ (i.e., Theorem 43), Xie et al. (Section 4, [159]) proposed the encryption and decryption schemes for color images, as shown in Figure 2.

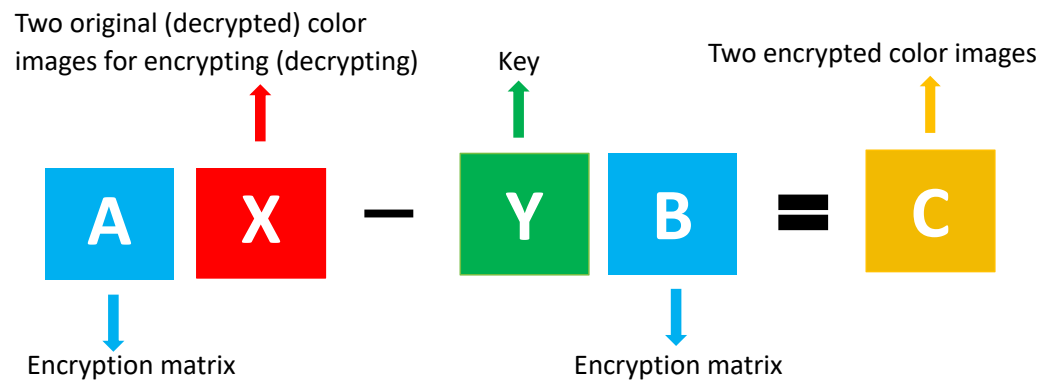


Figure 2. Encrypting and decrypting color images.

The specific encryption and decryption processes for color images are presented in Algorithms 5 and 6, respectively. To ensure the uniqueness of the decrypted color images ($\hat{X}_0, \hat{X}_1$) output by Algorithmic 6, the encryption dual quaternion matrices A and B must be restricted to satisfy Condition P , i.e., both the standard parts and the infinitesimal parts of A and B are of either full row rank or full column rank.

Algorithm 5 Color image encryption scheme

- 1: Input: two original color images X_0 and X_1 , two color images Y_0 and Y_1 as keys, and two encryption dual quaternion matrices A and B satisfying Condition P ;
- 2: Output: two encrypted color images C_0 and C_1 by

$$A(X_0 + X_1\epsilon) - (Y_0 + Y_1\epsilon)B = C_0 + C_1\epsilon.$$

Algorithm 6 Color image decryption scheme

- 1: Input: the encryption matrices A and B , the keys Y_0 and Y_1 , and the encrypted color images C_0 and C_1 from Algorithm 5.
 - 2: Output: two decrypted color images $\hat{X}_0$ and $\hat{X}_1$ by (61) and (62) in Theorem 43.
-

As illustrated in Figure 3, Xie et al. [159] selected two color images (i.e., Bike 1 and Bike 2) as the objects to be encrypted, and another two color images (i.e., Sunflower and Big-Windmill) as the keys. They first encrypted Bike 1 and Bike 2 using Algorithm 5, obtaining the two encrypted color images (see Figure 4). Subsequently, Algorithm 6 was applied to decrypt the encrypted images, resulting in the two decrypted color images (see Figure 5). Moreover, the structural similarity index measures (SSIM) of both decrypted images are shown in Table 2, which fully demonstrates the high effectiveness of their decryption scheme.

Table 2. SSIM values for the decrypted color images.

Color Image	SSIM
Bike 1	0.99
Bike 2	0.99

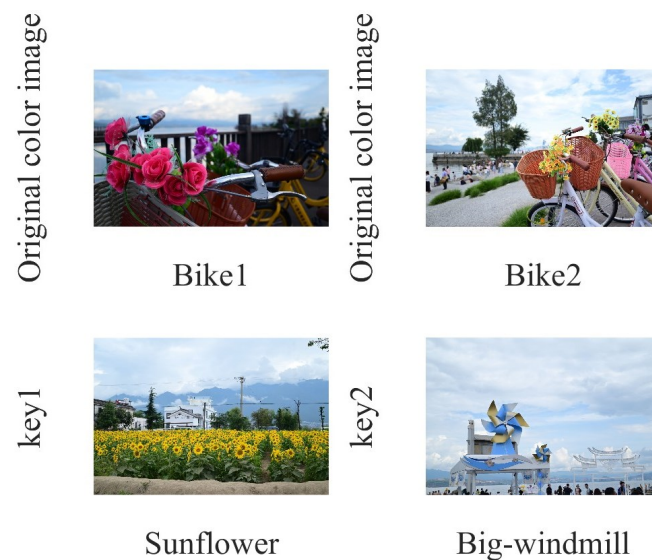


Figure 3. Two original color images and keys.

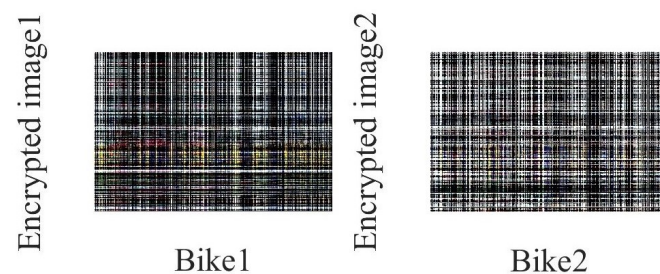


Figure 4. Two encrypted color images (generated by Algorithm 5).



Figure 5. Two decrypted color images (generated by Algorithm 6).

Remark 74. Recently, research on encrypting and decrypting color images and color videos by solving matrix or tensor equations has attracted attention in [203,204,347].

9. Conclusions

Research on GSE and its generalizations has long been a vibrant field, boasting both profound theoretical value and extensive practical application prospects. This comprehensive review encompasses 75 theorems, 74 remarks, and 347 references spanning from 1844 to 2025, covering pure mathematics (linear algebra, abstract algebra, operators, tensors, semi-tensor products, polynomial matrices, etc.), computational mathematics (iterative algorithms, condition numbers, etc.), and applied mathematics (robotics, image processing, encryption/decryption schemes, etc.). Centered on solving GSE, this paper elaborates on five dimensions—methods, constraints, generalizations, algorithms, and applications—distilling the field’s essence through point-by-point analysis and synthesis. A network diagram (i.e., Figure 6) intuitively illustrates the GSE research’s core framework.

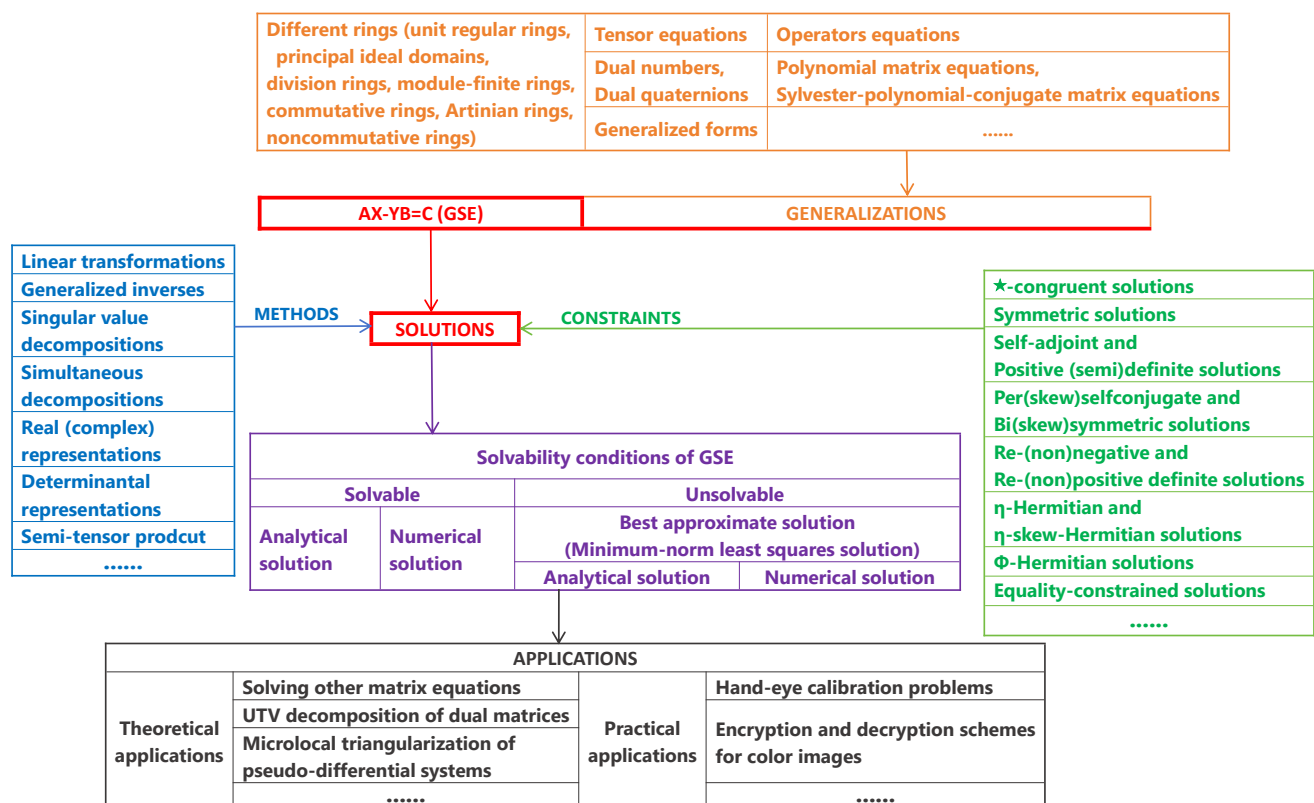


Figure 6. Core framework of GSE research.

As shown in Figure 6, let us start with “ $AX - YB = C$ (GSE)” in the red box, which is the core research object of the entire paper. The orange boxes show the various generalizations of GSE in different mathematical fields, which greatly enriches the relevant research. Then, looking down, the red box “SOLUTIONS” represents the main problem discussed in GSE and its generalizations, that is, the problem of solving equations, which is the core of the entire research. The green box on the right imposes the relevant constraint on the solution of the equation due to the demand for special matrices or specific conditions in practical applications. Further down, the purple box is a further elaboration on solving the equation: first, discuss the solvability conditions of the relevant equation; in the solvable case, study the representation of its analytical solution (explicit solution); in the unsolvable case, consider the analytical solution of its best approximation solution (minimum-norm least squares solution); finally, when the relevant analytical solution cannot be obtained, or when constrained by factors such as computational complexity, it is necessary to further study the numerical solution of related problems by designing iterative algorithms and continuously enhance computational efficiency. The blue box on the left contains various methods for tackling related solving problems, and also different approaches that can be chosen to solve the same problem. The gray box at the bottom is the purpose or end-point of all previous theoretical research, that is, to serve applications, which includes two aspects: theoretical applications and practical applications.

In the narrative of this paper, we have provided detailed introductions, in the form of remarks, to several interesting problems worthy of further exploration. For the convenience of readers, we have hereby compiled Table 3 to briefly summarize these problems.

Table 3. Open problems for further research on GSE.

Number	Remark Number	Open Problem
1	Remarks 15 and 16	Cramer’s rule for GSE only through coefficient matrices (partially solved)
2	Remark 18	Solving GSE under STP (or MM-2 STP)
3	Remark 22	Solving the GSE via $\mathcal{L}$ -representation, $\mathcal{C}$ -representation, $\mathcal{L}_C$ -representation, and vectorization properties of STP, respectively,
4	Remark 35	Discussing RET from a single common property of Euclidean domains and unit regular rings (partially solved)
5	Remark 43	Research on the fundamental properties and applications of combinations of different types of quaternions and dual numbers
6	Remark 47	Investigating GSE tensor equations under different tensor products and quaternion algebras.
7	Remark 60 (4)	Exploring converse problems for Lemmas 11 and 12 of [241]
8	Remark 63	Solving the restricted system (101)
9	Remark 64	Solving the system (104) (partially solved)

Numerous researchers worldwide have made significant contributions to GSE-related studies. Owing to the authors’ limitations, we have not been able to cover all GSE research findings, and there may inevitably be inadequacies. We offer our apologies here. Moreover, as GSE research advances rapidly with continuous innovations, this paper’s framework and content will grow increasingly rich and substantial as developments unfold. Finally, we kindly request experts and readers to offer their valuable insights, assisting us in further refining the summarization work in this field.

Author Contributions: Conceptualization, Q.-W.W. and J.G.; Methodology, Q.-W.W. and J.G.; Investigation, Q.-W.W. and J.G.; Writing—Original Draft Preparation, Q.-W.W. and J.G.; Writing—Review and Editing, Q.-W.W. and J.G.; Supervision, Q.-W.W.; Funding Acquisition, Q.-W.W. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the National Natural Science Foundation of China grant number [12371023].

Data Availability Statement: No new data were created or analyzed in this study.

Acknowledgments: The authors sincerely appreciate the editor and the anonymous reviewers for their insightful comments and valuable suggestions.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Sylvester, J.J. Sur l’équation en matrices $px = xq$. *C. R. Acad. Sci. Paris* **1884**, *99*, 67–71, 115–116. (In French)
2. Bhatia, R.; Rosenthal, P. How and why to solve the operator equation $AX - XB = Y$. *Bull. Lond. Math. Soc.* **1997**, *29*, 1–21.
3. Wang, Q.W.; Xie, L.M.; Gao, Z.H. A survey on solving the matrix equation $AXB = C$ with applications. *Mathematics* **2025**, *13*, 450. [CrossRef]
4. Wang, Q.W.; Gao, Z.H.; Gao, J. A comprehensive review on solving the system of equations $AX = C$ and $XB = D$. *Symmetry* **2025**, *17*, 625.
5. Wang, Q.W.; Gao, Z.H.; Li, Y.F. An overview of methods for solving the system of matrix equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$. *Symmetry* **2025**, *17*, 1307. [CrossRef]
6. Roth, W.E. The equations $AX - YB = C$ and $AX - XB = C$ in matrices. *Proc. Am. Math. Soc.* **1952**, *3*, 392–396.

7. Rodman, L. *Topics in Quaternion Linear Algebra*; Princeton University Press: Princeton, NJ, USA, 2014.
8. Horn, R.A.; Zhang, F. A generalization of the complex Autonne-Takagi factorization to quaternion matrices. *Linear Multilinear Algebra* **2012**, *60*, 1239–1244. [\[CrossRef\]](#)
9. Penrose, R. A generalized inverse for matrices. *Math. Proc. Camb. Philos. Soc.* **1955**, *51*, 406–413. [\[CrossRef\]](#)
10. Ben-Israel, A.; Greville, T.N.E. *Generalized Inverses: Theory and Applications*, 2nd ed.; Springer: New York, NY, USA, 2003.
11. Flanders, H.; Wimmer, H.K. On the matrix equations $AX - XB = C$ and $AX - YB = C$. *SIAM J. Appl. Math.* **1977**, *32*, 707–710.
12. Dmytryshyn, A.; Kågström, B. Coupled Sylvester-type matrix equations and block diagonalization. *SIAM J. Matrix Anal. Appl.* **2015**, *36*, 580–593. [\[CrossRef\]](#)
13. Dmytryshyn, A.; Futorny, V.; Klymchuk, T.; Sergeichuk, V.V. Generalization of Roth's solvability criteria to systems of matrix equations. *Linear Algebra Appl.* **2017**, *527*, 294–302. [\[CrossRef\]](#)
14. Rao, C.R.; Mitra, S.K. *Generalized Inverse of Matrices and Its Applications*; Wiley: New York, NY, USA, 1971.
15. Wang, G.; Wei, Y.; Qiao, S. *Generalized Inverses: Theory and Computations*; Springer: Singapore, 2018.
16. Cvetković-Ilić, D.S.; Wei, Y. *Algebraic Properties of Generalized Inverses*; Springer: Singapore, 2017.
17. Zhang, D.; Zhao, Y.; Mosić, D. The generalized Drazin inverse of the sum of two elements in a Banach algebra. *J. Comput. Appl. Math.* **2025**, *470*, 116701. [\[CrossRef\]](#)
18. Gao, Z.H.; Wang, Q.W.; Xie, L.M. A novel Moore–Penrose inverse to dual quaternion matrices with applications. *Appl. Math. Lett.* **2026**, *172*, 109727. [\[CrossRef\]](#)
19. Baksalary, J.K.; Kala, R. The matrix equation $AX - YB = C$. *Linear Algebra Appl.* **1979**, *25*, 41–43. [\[CrossRef\]](#)
20. Marsaglia, G.; Styan, G.P.H. Equalities and inequalities for ranks of matrices. *Linear Multilinear Algebra* **1974**, *2*, 269–292. [\[CrossRef\]](#)
21. van der Woude, J.W. Almost non-interacting control by measurement feedback. *Syst. Control Lett.* **1987**, *9*, 7–16. [\[CrossRef\]](#)
22. Olshevsky, V. Similarity of block diagonal and block triangular matrices. *Integral Equ. Oper. Theory* **1992**, *15*, 853–863. [\[CrossRef\]](#)
23. Meyer, C.D., Jr. Generalized inverses of block triangular matrices. *SIAM J. Appl. Math.* **1970**, *19*, 741–750. [\[CrossRef\]](#)
24. Golub, G.H.; Loan, C.F.V. *Matrix Computations*; The Johns Hopkins University Press: Baltimore, MD, USA, 1983.
25. Chu, K.E. Singular value and generalized singular value decompositions and the solution of linear matrix equations. *Linear Algebra Appl.* **1987**, *88–89*, 83–98. [\[CrossRef\]](#)
26. Paige, C.C.; Saunders, M.A. Towards a generalized singular value decomposition. *SIAM J. Numer. Anal.* **1981**, *18*, 398–405. [\[CrossRef\]](#)
27. Xu, G.; Wei, M.; Zheng, D. On solutions of matrix equation $AXB + CYD = F$. *Linear Algebra Appl.* **1998**, *279*, 93–109. [\[CrossRef\]](#)
28. Golub, G.H.; Zha, H. Perturbation analysis of the canonical correlations of matrix pairs. *Linear Algebra Appl.* **1994**, *210*, 3–28. [\[CrossRef\]](#)
29. Wang, Q.W.; van Woude, J.W.; Yu, S.W. An equivalence canonical form of a matrix triplet over an arbitrary division ring with applications. *Sci. China-Math.* **2011**, *54*, 907–924. [\[CrossRef\]](#)
30. He, Z.H.; Wang, Q.W.; Zhang, Y. A simultaneous decomposition for seven matrices with applications. *J. Comput. Appl. Math.* **2019**, *349*, 93–113. [\[CrossRef\]](#)
31. He, Z.H.; Agudelo, O.M.; Wang, Q.W.; Moor, B.D. Two-sided coupled generalized Sylvester matrix equations solving using a simultaneous decomposition for fifteen matrices. *Linear Algebra Appl.* **2016**, *496*, 549–593. [\[CrossRef\]](#)
32. Gustafson, W.H. Quivers and matrix equations. *Linear Algebra Appl.* **1995**, *231*, 159–174. [\[CrossRef\]](#)
33. Gabriel, P. Unzerlegbare Darstellungen I. *Manuscripta Math.* **1972**, *6*, 71–103. [\[CrossRef\]](#)
34. He, Z.H.; Dmytryshyn, A.; Wang, Q.W. A new system of Sylvester-like matrix equations with arbitrary number of equations and unknowns over the quaternion algebra. *Linear Multilinear Algebra* **2025**, *73*, 1269–1309. [\[CrossRef\]](#)
35. Wang, Q.W.; Zhang, X.; Woude, J.W.v. A new simultaneous decomposition of a matrix quaternary over an arbitrary division ring with applications. *Commun. Algebra* **2012**, *40*, 2309–2342. [\[CrossRef\]](#)
36. He, Z.H.; Wang, Q.W.; Zhang, Y. The complete equivalence canonical form of four matrices over an arbitrary division ring. *Linear Multilinear Algebra* **2018**, *66*, 74–95. [\[CrossRef\]](#)
37. He, Z.H.; Xu, Y.Z.; Wang, Q.W.; Zhang, C.Q. The equivalence canonical forms of two sets of five quaternion matrices with applications. *Math. Meth. Appl. Sci.* **2025**, *48*, 5483–5505.
38. Huo, J.W.; Xu, Y.Z.; He, Z.H. A simultaneous decomposition for a quaternion tensor quaternary with applications. *Mathematics* **2025**, *13*, 1679. [\[CrossRef\]](#)
39. Flaut, C.; Shpakivskyi, V. Real matrix representations for the complex quaternions. *Adv. Appl. Clifford Algebras* **2013**, *23*, 657–671. [\[CrossRef\]](#)
40. Liu, Y.H. Ranks of solutions of the linear matrix equation $AX + YB = C$. *Comput. Math. Appl.* **2006**, *52*, 861–872. [\[CrossRef\]](#)
41. Zhang, F.; Mu, W.; Li, Y.; Zhao, J. Special least squares solutions of the quaternion matrix equation $AXB + CXD = E$. *Comput. Math. Appl.* **2016**, *72*, 1426–1435. [\[CrossRef\]](#)

42. Yuan, S. Least squares pure imaginary solution and real solution of the quaternion matrix equation $AXB + CXD = E$ with the least norm. *J. Appl. Math.* **2014**, *2014*, 857081. [\[CrossRef\]](#)
43. Liu, X.; Zhang, Y. Matrices over quaternion algebras. In *Matrix and Operator Equations and Applications*; Moslehian, M.S., Ed.; Springer: Cham, Switzerland, 2023; pp. 139–183.
44. Wang, G.; Guo, Z.; Zhang, D.; Jiang, T. Algebraic techniques for least-squares problem over generalized quaternion algebras: A unified approach in quaternionic and split quaternionic theory. *Math. Meth. Appl. Sci.* **2020**, *43*, 1124–1137. [\[CrossRef\]](#)
45. Yu, C.; Liu, X.; Zhang, Y. The generalized quaternion matrix equation $AXB + CX^*D = E$. *Math. Meth. Appl. Sci.* **2020**, *43*, 8506–8517. [\[CrossRef\]](#)
46. Ren, B.Y.; Wang, Q.W.; Chen, X.Y. The η -anti-Hermitian solution to a constrained matrix equation over the generalized segre quaternion algebra. *Symmetry* **2023**, *15*, 592. [\[CrossRef\]](#)
47. Wei, M.S.; Li, Y.; Zhang, F.; Zhao, J. *Quaternion Matrix Computations*; Nova Science Publishers: New York, NY, USA, 2018.
48. Stanimirovic, P.S. General determinantal representation of pseudoinverses of matrices. *Mat. Vesn.* **1996**, *48*, 1–9.
49. Kyrchei, I.I. Analogs of the adjoint matrix for generalized inverses and corresponding Cramer rules. *Linear Multilinear Algebra* **2008**, *56*, 453–469. [\[CrossRef\]](#)
50. Drazin, M.P. Pseudo-inverses in associative rings and semigroups. *Am. Math. Mon.* **1958**, *65*, 506–514. [\[CrossRef\]](#)
51. Aslaksen, H. Quaternionic determinants. *Math. Intell.* **1996**, *18*, 57–65. [\[CrossRef\]](#)
52. Cohen, N.; Leo, S.D. The quaternionic determinant. *Electron. J. Linear Algebra* **2000**, *7*, 100–111. [\[CrossRef\]](#)
53. Kyrchei, I.I. Cramer's rule for quaternionic systems of linear equations. *J. Math. Sci.* **2008**, *155*, 839–858. [\[CrossRef\]](#)
54. Kyrchei, I.I. The theory of the column and row determinants in a quaternion linear algebra. In *Advances in Mathematics Research*; Baswell, A.R., Ed.; Nova Science Publisher: New York, NY, USA, 2012; Volume 15, pp. 301–358.
55. Kyrchei, I.I. Cramer's rule for some quaternion matrix equations. *Appl. Math. Comput.* **2010**, *217*, 2024–2030. [\[CrossRef\]](#)
56. Song, G.J.; Wang, Q.W.; Yu, S.W. Cramer's rule for a system of quaternion matrix equations with applications. *Appl. Math. Comput.* **2018**, *336*, 490–499. [\[CrossRef\]](#)
57. Kyrchei, I.I. Determinantal representations of the Moore-Penrose inverse matrix over the quaternion skew field. *J. Math. Sci.* **2012**, *180*, 23–33. [\[CrossRef\]](#)
58. Kyrchei, I. Cramer's rules for Sylvester quaternion matrix equation and its special cases. *Adv. Appl. Clifford Algebr.* **2018**, *28*, 90. [\[CrossRef\]](#)
59. Kyrchei, I. Explicit representation formulas for the minimum norm least squares solutions of some quaternion matrix equations. *Linear Algebra Appl.* **2013**, *438*, 136–152. [\[CrossRef\]](#)
60. Kyrchei, I.I. Explicit determinantal representation formulas for the solution of the two-sided restricted quaternionic matrix equation. *J. Appl. Math. Comput.* **2018**, *58*, 335–365. [\[CrossRef\]](#)
61. Kyrchei, I. Determinantal representations of solutions to systems of quaternion matrix equations. *Adv. Appl. Clifford Algebr.* **2018**, *28*, 23. [\[CrossRef\]](#)
62. Song, G.J.; Dong, C.Z. New results on condensed Cramer's rule for the general solution to some restricted quaternion matrix equations. *J. Appl. Math. Comput.* **2017**, *53*, 321–341. [\[CrossRef\]](#)
63. Song, G.J.; Wang, Q.W. Condensed Cramer rule for some restricted quaternion linear equations. *Appl. Math. Comput.* **2011**, *218*, 3110–3121. [\[CrossRef\]](#)
64. Song, G.J.; Wang, Q.W.; Chang, H.X. Cramer rule for the unique solution of restricted matrix equations over the quaternion skew field. *Comput. Math. Appl.* **2011**, *61*, 1576–1589. [\[CrossRef\]](#)
65. Song, G.J. Determinantal expression of the general solution to a restricted system of quaternion matrix equations with applications. *Bull. Korean Math. Soc.* **2018**, *55*, 1285–1301.
66. Cheng, D. Semi-tensor product of matrices and its application to Morgan's problem. *Sci. China* **2001**, *44*, 195–212.
67. Cheng, D. *From Dimension-Free Matrix Theory to Cross-Dimensional Dynamic Systems*; Elsevier: London, UK, 2019.
68. Cheng, D. *Matrix and Polynomial Approach to Dynamic Control Systems*; Science Press: Beijing, China, 2002.
69. Cheng, D.; Qi, H.; Xue, A. A survey on semi-tensor product of matrices. *J. Syst. Sci. Complex.* **2007**, *20*, 304–322. [\[CrossRef\]](#)
70. Cheng, D.; Qi, H. *Semi-Tensor Product of Matrices—Theory and Applications*, 2nd ed.; Science Press: Beijing, China, 2011. (In Chinese)
71. Cheng, D.; Qi, H.; Li, Z. *Analysis and Control of Boolean Networks: A Semi-Tensor Product Approach*; Springer: London, UK, 2011.
72. Yao, J.; Feng, J.; Meng, M. On solutions of the matrix equation $AX = B$ with respect to semi-tensor product. *J. Franklin Inst.* **2016**, *353*, 1109–1131. [\[CrossRef\]](#)
73. Wang, J. Least squares solutions of matrix equation $AXB = C$ under semi-tensor product. *Electron. Res. Arch.* **2024**, *32*, 2976–2993. [\[CrossRef\]](#)
74. Jaiprasert, J.; Chansangiam, P. Solving the Sylvester-transpose matrix equation under the semi-tensor product. *Symmetry* **2022**, *14*, 1094. [\[CrossRef\]](#)
75. Wang, N. Solvability of the Sylvester equation $AX - XB = C$ under left semi-tensor product. *Math. Model. Control* **2022**, *2*, 81–89. [\[CrossRef\]](#)

76. Ji, Z.; Li, J.; Zhou, X.; Duan, F.; Li, T. On solutions of matrix equation $AXB = C$ under semi-tensor product. *Linear Multilinear Algebra* **2021**, *69*, 1935–1963. [\[CrossRef\]](#)
77. Li, J.; Tao, L.; Li, W.; Chen, Y.; Huang, R. Solvability of matrix equations $AX = B, XC = D$ under semi-tensor product. *Linear Multilinear Algebra* **2017**, *65*, 1705–1733. [\[CrossRef\]](#)
78. Wang, J.; Feng, J.; Huang, H. Solvability of the matrix equation $AX^2 = B$ with semi-tensor product. *Electron. Res. Arch.* **2020**, *29*, 2249–2267. [\[CrossRef\]](#)
79. Cheng, D.; Liu, Z. A new semi-tensor product of matrices. *Control Theory Technol.* **2019**, *17*, 4–12. [\[CrossRef\]](#)
80. Cheng, D.; Xu, Z.; Shen, T. Equivalence-based model of dimension-varying linear systems. *IEEE Trans. Autom. Control* **2020**, *65*, 5444–5449. [\[CrossRef\]](#)
81. Wang, J. On solutions of the matrix equation $A \circ_f X = B$ with respect to MM-2 semitensor product. *J. Math.* **2021**, *2021*, 6651434.
82. Ding, W.; Li, Y.; Wang, D.; Wei, A. Constrained least squares solution of Sylvester equation. *Math. Model. Control* **2021**, *1*, 112–120. [\[CrossRef\]](#)
83. Ding, W.; Li, Y.; Wang, D. A real method for solving quaternion matrix equation $X - \hat{A}\hat{X}B = C$ based on semi-tensor product of matrices. *Adv. Appl. Clifford Algebr.* **2021**, *31*, 78. [\[CrossRef\]](#)
84. Wang, D.; Li, Y.; Ding, W.X. Several kinds of special least squares solutions to quaternion matrix equation $AXB = C$. *J. Appl. Math. Comput.* **2022**, *68*, 1881–1899. [\[CrossRef\]](#)
85. Liu, X.; Li, Y.; Ding, W.; Tao, R. A real method for solving octonion matrix equation $AXB = C$ based on semi-tensor product of matrices. *Adv. Appl. Clifford Algebr.* **2024**, *34*, 12. [\[CrossRef\]](#)
86. Chen, W.; Song, C. STP method for solving the least squares special solutions of quaternion matrix equations. *Adv. Appl. Clifford Algebr.* **2025**, *35*, 6. [\[CrossRef\]](#)
87. Liu, X.; Wang, Q.W.; Zhang, Y. Consistency of quaternion matrix equations $AX^* - XB = C$ and $X - AX^*B = C^*$. *Electron. J. Linear Algebra* **2019**, *35*, 394–407. [\[CrossRef\]](#)
88. Fan, X.; Li, Y.; Liu, Z.; Zhao, J. Solving quaternion linear system based on semi-tensor product of quaternion matrices. *Symmetry* **2022**, *14*, 1359. [\[CrossRef\]](#)
89. Fan, X.; Li, Y.; Liu, Z.; Zhao, J. The (anti)- η -Hermitian solution of quaternion linear system. *Filomat* **2024**, *38*, 4679–4695. [\[CrossRef\]](#)
90. Zhang, M.; Li, Y.; Sun, J.; Fan, X.; Wei, A. A new method based on the semi-tensor product of matrices for solving communicative quaternion matrix equation $\sum_{i=1}^k A_i X B_i = C$ and its application. *Bull. Sci. Math.* **2025**, *199*, 103576. [\[CrossRef\]](#)
91. Fan, X.; Li, Y.; Sun, J.; Zhao, J. Solving quaternion linear system $AXB = E$ based on semi-tensor product of quaternion matrices. *Banach J. Math. Anal.* **2023**, *17*, 25. [\[CrossRef\]](#)
92. Xi, Y.; Liu, Z.; Li, Y.; Tao, R.; Wang, T. On the mixed solution of reduced biquaternion matrix equation $\sum_{i=1}^n A_i X_i B_i = E$ with sub-matrix constraints and its application. *AIMS Math.* **2023**, *8*, 27901–27923. [\[CrossRef\]](#)
93. Fan, X.; Li, Y.; Zhang, M.; Zhao, J. Solving the least squares (anti)-Hermitian solution for quaternion linear systems. *Comput. Appl. Math.* **2022**, *41*, 371. [\[CrossRef\]](#)
94. Liu, Z.; Li, Y.; Fan, X.; Ding, W. A new method of solving special solutions of quaternion generalized Lyapunov matrix equation. *Symmetry* **2022**, *14*, 1120. [\[CrossRef\]](#)
95. Ziętak, K. The Chebyshev solution of the linear matrix equation $AX + YB = C$. *Numer. Math.* **1985**, *46*, 455–478. [\[CrossRef\]](#)
96. Ziętak, K. The l_p -solution of the linear matrix equation $AX + YB = C$. *Computing* **1984**, *32*, 153–162. [\[CrossRef\]](#)
97. Liao, A.P.; Bai, Z.Z.; Lei, Y. Best approximate solution of matrix equation $AXB + CYD = E$. *SIAM J. Matrix Anal. Appl.* **2005**, *27*, 675–688. [\[CrossRef\]](#)
98. Wimmer, H.K. Roth's theorems for matrix equations with symmetry constraints. *Linear Algebra Appl.* **1994**, *199*, 357–362. [\[CrossRef\]](#)
99. Terán, F.D.; Dopico, F.M. Consistency and efficient solution of the Sylvester equation for $\star$ -congruence. *Electron. J. Linear Algebra* **2011**, *22*, 849–863. [\[CrossRef\]](#)
100. Byers, R.; Kressner, D. Structured condition numbers for invariant subspaces. *SIAM J. Matrix Anal. Appl.* **2006**, *28*, 326–347. [\[CrossRef\]](#)
101. Kressner, D.; Schröder, C.; Watkins, D.S. Implicit QR algorithms for palindromic and even eigenvalue problems. *Numer. Algor.* **2009**, *51*, 209–238. [\[CrossRef\]](#)
102. Cvetković-Ilić, D.S. The solutions of some operator equations. *J. Korean Math. Soc.* **2008**, *45*, 1417–1425. [\[CrossRef\]](#)
103. Chang, X.W.; Wang, J.S. The symmetric solution of the matrix equations $AX + YA = C$, $AXA^T + BYB^T = C$, and $(A^T X A, B^T X B) = (C, D)$. *Linear Algebra Appl.* **1993**, *179*, 171–189. [\[CrossRef\]](#)
104. Jameson, A.; Kreindler, E. Inverse problem of linear optimal control. *SIAM J. Control* **1973**, *11*, 1–19. [\[CrossRef\]](#)
105. Jameson, A.; Kreindler, E.; Lancaster, P. Symmetric, positive semidefinite, positive definite real solutions of $AX = XA^T$ and $AX = YB$. *Linear Algebra Appl.* **1992**, *160*, 189–215. [\[CrossRef\]](#)
106. Dobovišek, M. On minimal solutions of the matrix equation $AX - YB = 0$. *Linear Algebra Appl.* **2001**, *325*, 81–99. [\[CrossRef\]](#)

107. Cantoni, A.; Butler, P. Eigenvalues and eigenvectors of symmetric centrosymmetric matrices. *Linear Algebra Appl.* **1976**, *13*, 275–288. [\[CrossRef\]](#)
108. Reid, R.M. Some eigenvalues properties of persymmetric matrices. *SIAM Rev.* **1997**, *39*, 313–316. [\[CrossRef\]](#)
109. Andrew, A.L. Centrosymmetric matrices. *SIAM Rev.* **1998**, *40*, 697–698. [\[CrossRef\]](#)
110. Weaver, J.R. Centrosymmetric (cross-symmetric) matrices, their basic properties, eigenvalues, eigenvectors. *Am. Math. Mon.* **1985**, *92*, 711–717. [\[CrossRef\]](#)
111. Draxl, P.K. *Skew Field*; Cambridge University Press: London, UK, 1983.
112. Wang, Q.W.; Li, S.Z. Persymmetric and perskewsymmetric solutions to sets of matrix equations over a finite central algebra. *Acta Math. Sin.* **2004**, *47*, 27–34. (In Chinese)
113. Wang, Q.W.; Sun, J.H.; Li, S.Z. Consistency for bi(skew)symmetric solutions to systems of generalized Sylvester equations over a finite central algebra. *Linear Algebra Appl.* **2002**, *353*, 169–182. [\[CrossRef\]](#)
114. Wang, Q.W.; Zhang, H.S.; Song, G.J. A new solvable condition for a pair of generalized Sylvester equations. *Electron. J. Linear Algebra* **2009**, *18*, 289–301. [\[CrossRef\]](#)
115. Wang, Q.W.; He, Z.H. Some matrix equations with applications. *Linear Multilinear Algebra* **2012**, *60*, 1327–1353. [\[CrossRef\]](#)
116. Took, C.C.; Mandic, D.P. Augmented second-order statistics of quaternion random signals. *Signal Process.* **2011**, *91*, 214–224. [\[CrossRef\]](#)
117. Took, C.C.; Mandic, D.P.; Zhang, F. On the unitary diagonalisation of a special class of quaternion matrices. *Appl. Math. Lett.* **2011**, *24*, 1806–1809. [\[CrossRef\]](#)
118. Yuan, S.F.; Wang, Q.W. Two special kinds of least squares solutions for the quaternion matrix equation $AXB + CXD = E$. *Electron. J. Linear Algebra* **2012**, *23*, 257–274. [\[CrossRef\]](#)
119. He, Z.H.; Wang, Q.W. A real quaternion matrix equation with applications. *Linear Multilinear Algebra* **2013**, *61*, 725–740. [\[CrossRef\]](#)
120. Baksalary, J.K.; Kala, R. The matrix equation $AXB + CYD = E$. *Linear Algebra Appl.* **1980**, *30*, 141–147. [\[CrossRef\]](#)
121. He, Z.H.; Wang, Q.W. A pair of mixed generalized Sylvester matrix equations. *J. Shanghai Univ. Nat. Sci.* **2014**, *20*, 138–156.
122. Kyrchei, I. Cramer’s rules of η -(skew-)Hermitian solutions to the quaternion Sylvester-type matrix equations. *Adv. Appl. Clifford Algebras* **2019**, *29*, 56. [\[CrossRef\]](#)
123. He, Z.H.; Liu, J.; Tam, T.Y. The general ϕ -Hermitian solution to mixed pairs of quaternion matrix Sylvester equations. *Electron. J. Linear Algebra* **2017**, *32*, 475–499. [\[CrossRef\]](#)
124. Wang, L.; Wang, Q.; He, Z. The common solution of some matrix equations. *Algebra Colloq.* **2016**, *23*, 71–81. [\[CrossRef\]](#)
125. Wang, Q.W.; Lv, R.Y.; Zhang, Y. The least-squares solution with the least norm to a system of tensor equations over the quaternion algebra. *Linear Multilinear Algebra* **2022**, *70*, 1942–1962. [\[CrossRef\]](#)
126. Guralnick, R.M. Roth’s theorems and decomposition of modules. *Linear Algebra Appl.* **1980**, *39*, 155–165. [\[CrossRef\]](#)
127. Hartwig, R.E. Roth’s equivalence problem in unit regular rings. *Proc. Am. Math. Soc.* **1976**, *59*, 39–44. [\[CrossRef\]](#)
128. Newman, M. The Smith normal form of a partitioned matrix. *J. Res. Nat. Bur. Stand.-B. Math. Sci.* **1974**, *78B*, 3–6. [\[CrossRef\]](#)
129. Feinberg, R.B. Equivalence of partitioned matrices. *J. Res. Nat. Bur. Stand.-B. Math. Sci.* **1976**, *80B*, 89–97. [\[CrossRef\]](#)
130. Gustafson, W.H.; Zelmanowitz, J.M. On matrix equivalence and matrix equations. *Linear Algebra Appl.* **1979**, *27*, 219–224. [\[CrossRef\]](#)
131. Gustafson, W.H. Roth’s theorems over commutative rings. *Linear Algebra Appl.* **1979**, *23*, 245–251. [\[CrossRef\]](#)
132. Guralnick, R.M. Roth’s theorems for sets of matrices. *Linear Algebra Appl.* **1985**, *71*, 113–117. [\[CrossRef\]](#)
133. Lee, S.G.; Vu, Q.P. Simultaneous solutions of matrix equations and simultaneous equivalence of matrices. *Linear Algebra Appl.* **2012**, *437*, 2325–2339. [\[CrossRef\]](#)
134. Miyata, T. Note on direct summands of modules. *J. Math. Kyoto Univ.* **1967**, *7*, 65–69. [\[CrossRef\]](#)
135. Guralnick, R.M. Matrix equivalence and isomorphism of modules. *Linear Algebra Appl.* **1982**, *43*, 125–136. [\[CrossRef\]](#)
136. Özgüler, A.B. The matrix equation $AXB + CYD = E$ over a principal ideal domain. *SIAM J. Matrix Anal. Appl.* **1991**, *12*, 581–591. [\[CrossRef\]](#)
137. Huang, L.; Zeng, Q. The matrix equation $AXB + CYD = E$ over a simple artinian ring. *Linear Multilinear Algebra* **1995**, *38*, 225–232.
138. Wang, Q.W. A system of matrix equations and a linear matrix equation over arbitrary regular rings with identity. *Linear Algebra Appl.* **2004**, *384*, 43–54. [\[CrossRef\]](#)
139. Dajić, A. Common solutions of linear equations in a ring, with applications. *Electron. J. Linear Algebra* **2015**, *30*, 66–79. [\[CrossRef\]](#)
140. Lin, M.; Wimmer, H.K. The generalized Sylvester matrix equation, rank minimization and Roth’s equivalence theorem. *Bull. Aust. Math. Soc.* **2011**, *84*, 441–443. [\[CrossRef\]](#)
141. Ito, N.; Wimmer, H.K. Rank minimization of generalized Sylvester equations over Bezout domains. *Linear Algebra Appl.* **2013**, *439*, 592–599. [\[CrossRef\]](#)
142. Hamilton, W.R. *Lectures on Quaternions*; Hodges and Smith: Dublin, Ireland, 1853.
143. Voight, J. *Quaternion Algebras*; Springer: Cham, Switzerland, 2021.

144. Adler, S.L. *Quaternionic Quantum Mechanics and Quantum Fields*; Oxford University Press: New York, NY, USA, 1995.
145. Kuipers, J.B. *Quaternions and Rotation Sequences: A Primer with Applications to Orbits, Aerospace and Virtual Reality*; Princeton University Press: Princeton, NJ, USA, 1999.
146. Girard, P.R. *Quaternions, Clifford Algebras and Relativistic Physics*; Birkhäuser: Basel, Switzerland, 2007.
147. Li, W. *Quaternion Matrices*; National University of Defense Technology Press: Changsha, China, 2002. (In Chinese)
148. Clifford, M.A. Preliminary sketch of biquaternions. *Proc. Lond. Math. Soc.* **1873**, *4*, 381–395. [\[CrossRef\]](#)
149. Wei, T.; Ding, W.; Wei, Y. Singular value decomposition of dual matrices and its application to traveling wave identification in the brain. *SIAM J. Matrix Anal. Appl.* **2024**, *45*, 634–660. [\[CrossRef\]](#)
150. Fischer, I. *Dual-Number Methods in Kinematics, Statics and Dynamics*; CRC Press: Boca Raton, FL, USA, 1999.
151. Condurache, D.; Burlacu, A. Dual tensors based solutions for rigid body motion parameterization. *Mech. Mach. Theory* **2014**, *74*, 390–412. [\[CrossRef\]](#)
152. Gu, Y.L.; Luh, J.Y.S. Dual-number transformations and its applications to robotics. *IEEE J. Robot. Autom.* **1987**, *3*, 615–623. [\[CrossRef\]](#)
153. Udawadia, F.E.; Pennestri, E.; de Falco, D. Do all dual matrices have dual Moore-Penrose inverses? *Mech. Mach. Theory* **2020**, *151*, 103878. [\[CrossRef\]](#)
154. Udawadia, F.E. Dual generalized inverses and their use in solving systems of linear dual equations. *Mech. Mach. Theory* **2021**, *156*, 104158. [\[CrossRef\]](#)
155. Fan, R.; Zeng, M.; Yuan, Y. The solutions to some dual matrix equations. *Miskolc Math. Notes* **2024**, *25*, 679–691. [\[CrossRef\]](#)
156. Farias, J.G.; Pieri, E.D.; Martins, D. A review on the applications of dual quaternions. *Machines* **2024**, *12*, 402. [\[CrossRef\]](#)
157. Kenwright, B. A beginner's guide to dual-quaternions: What they are, how they work, how to use them for 3D character hierarchies. In Proceedings of the 20th International Conference in Central Europe on Computer Graphics, Visualization and Computer Vision, Pilsen, Czech Republic, 25–28 June 2012.
158. Mukundan, R. Quaternions: From classical mechanics to computer graphics, and beyond. In Proceedings of the 7th Asian Technology Conference in Mathematics, Melaka, Malaysia, 17–21 December 2002; pp. 97–106.
159. Xie, L.M.; Wang, Q.W.; He, Z.H. The generalized hand-eye calibration matrix equation $AX - YB = C$ over dual quaternions. *Comput. Appl. Math.* **2025**, *44*, 137. [\[CrossRef\]](#)
160. Xie, L.M.; Wang, Q.W. A generalized Sylvester dual quaternion matrix equation with applications. *Authorea* **2025**. [\[CrossRef\]](#)
161. Duan, G.R. *Generalized Sylvester Equations: Unified Parametric Solutions*; Taylor and Francis Group: Abingdon, UK; CRC Press: Boca Raton, FL, USA, 2015.
162. Hamilton, W.R. II. On quaternions; or on a new system of imaginaries in algebra. *Lond. Edinb. Dublin Philos. Mag. J. Sci.* **1844**, *25*, 10–13. [\[CrossRef\]](#)
163. Cockle, J. On systems of algebra involving more than one imaginary and on equations of the fifth degree. *Lond. Edinb. Dublin Philos. Mag. J. Sci.* **1849**, *35*, 434–437. [\[CrossRef\]](#)
164. Segre, C. The real representations of complex elements and extension to bicomplex systems. *Math. Ann.* **1892**, *40*, 413–467. (In Italian) [\[CrossRef\]](#)
165. Tian, Y.; Liu, X.; Zhang, Y. Least-squares solutions of the generalized reduced biquaternion matrix equations. *Filomat* **2023**, *37*, 863–870. [\[CrossRef\]](#)
166. Yaglom, I.M. *Complex Numbers in Geometry*; Academic Press: New York, NY, USA, 1968.
167. Lam, T.Y. *Introduction to Quadratic Forms over Fields*; American Mathematical Society: Providence, RI, USA, 2005.
168. Pottmann, H.; Wallner, J. *Computational Line Geometry*; Springer: Berlin/Heidelberg, Germany, 2001.
169. Kandasamy, W.B.V.; Smarandache, F. *Dual Numbers*; Zip Publishing: Columbus, OH, USA, 2012.
170. Lance, E.C. *Hilbert C^* -Modules: A Toolkit for Operator Algebras*; Cambridge University Press: Cambridge, UK, 1995.
171. Fang, X.; Yu, J.; Yao, H. Solutions to operator equations on Hilbert C^* -modules. *Linear Algebra Appl.* **2009**, *431*, 2142–2153. [\[CrossRef\]](#)
172. Xu, Q. Common Hermitian and positive solutions to the adjointable operator equations $AX = C$, $XB = D$. *Linear Algebra Appl.* **2008**, *429*, 1–11. [\[CrossRef\]](#)
173. Dajić, A.; Koliha, J.J. Positive solutions to the equations $AX = C$ and $XB = D$ for Hilbert space operators. *J. Math. Anal. Appl.* **2007**, *333*, 567–576. [\[CrossRef\]](#)
174. Douglas, R.G. On majorization, factorization and range inclusion of operators on Hilbert space. *Proc. Am. Math. Soc.* **1966**, *17*, 413–416. [\[CrossRef\]](#)
175. Mousavi, Z.; Eskandari, R.; Moslehian, M.S.; Mirzapour, F. Operator equations $AX + YB = C$ and $AXA^* + BYB^* = C$ in Hilbert C^* -modules. *Linear Algebra Appl.* **2017**, *517*, 85–98. [\[CrossRef\]](#)
176. Karizaki, M.M.; Hassani, M.; Amyari, M.; Khosravi, M. Operator matrix of Moore-Penrose inverse operators on Hilbert C^* -modules. *Colloq. Math.* **2015**, *140*, 171–182. [\[CrossRef\]](#)

177. Moghani, Z.N.; Karizaki, M.M.; Khanehgir, M. Solutions of the Sylvester equation in C^* -Modular operators. *Ukr. Math. J.* **2021**, *73*, 354–369. [\[CrossRef\]](#)
178. An, I.J.; Ko, E.; Lee, J.E. On the generalized Sylvester operator equation $AX - YB = C$. *Linear Multilinear Algebra* **2022**, *72*, 585–596. [\[CrossRef\]](#)
179. Jo, S.; Kim, Y.; Ko, E. On Fuglede-Putnam properties. *Positivity* **2015**, *19*, 911–925. [\[CrossRef\]](#)
180. Bhatia, R. *Matrix Analysis*; Springer: New York, NY, USA, 1997.
181. Lim, L.H. Tensors in computations. *Acta Numer.* **2021**, *30*, 555–764. [\[CrossRef\]](#)
182. Qi, L.; Luo, Z. *Tensor Analysis: Spectral Theory and Special Tensors*; SIAM: Philadelphia, PA, USA, 2017.
183. Ding, W.; Wei, Y. *Theory and Computation of Tensors: Multi-Dimensional Arrays*; Elsevier: Amsterdam, The Netherlands; Academic Press: London, UK, 2016.
184. Che, M.; Wei, Y. *Theory and Computation of Complex Tensors and its Applications*; Springer: Singapore, 2020.
185. Wu, F.; Li, C.; Li, Y. Manifold regularization nonnegative triple decomposition of tensor sets for image compression and representation. *J. Optim. Theory Appl.* **2022**, *192*, 979–1000. [\[CrossRef\]](#)
186. Savas, B.; Eldén, L. Handwritten digit classification using higher order singular value decomposition. *Pattern Recognit.* **2007**, *40*, 993–1003. [\[CrossRef\]](#)
187. Huang, S.; Zhao, G.; Chen, M. Tensor extreme learning design via generalized Moore-Penrose inverse and triangular type-2 fuzzy sets. *Neural Comput. Appl.* **2019**, *31*, 5641–5651. [\[CrossRef\]](#)
188. Sidiropoulos, N.D.; Lathauwer, L.D.; Fu, X.; Huang, K.; Papalexakis, E.E.; Faloutsos, C. Tensor decomposition for signal processing and machine learning. *IEEE Trans. Signal Process.* **2017**, *65*, 3551–3582. [\[CrossRef\]](#)
189. Qi, L.; Chen, H.; Chen, Y. *Tensor Eigenvalues and Their Applications*; Springer: Singapore, 2018.
190. Kolda, T.G.; Bader, B.W. Tensor decompositions and applications. *SIAM Rev.* **2009**, *51*, 455–500. [\[CrossRef\]](#)
191. He, Z.H.; Navasca, C.; Wang, Q.W. Tensor decompositions and tensor equations over quaternion algebra. *arXiv* **2017**, arXiv:1710.07552v1. [\[CrossRef\]](#)
192. Einstein, A. The foundation of the general theory of relativity. In *The Collected Papers of Albert Einstein*; Kox, A.J., Klein, M.J., Schulmann, R., Eds.; Princeton University Press: Princeton, NJ, USA, 1997; Volume 6, pp. 146–200.
193. Sun, L.; Zheng, B.; Bu, C.; Wei, Y. Moore-Penrose inverse of tensors via Einstein product. *Linear Multilinear Algebra* **2016**, *64*, 686–698. [\[CrossRef\]](#)
194. Brazell, M.; Li, N.; Navasca, C.; Tamon, C. Solving multilinear systems via tensor inversion. *SIAM J. Matrix Anal. Appl.* **2013**, *34*, 542–570. [\[CrossRef\]](#)
195. He, Z.H.; Navasca, C.; Wang, X.X. Decomposition for a quaternion tensor triplet with applications. *Adv. Appl. Clifford Algebras* **2022**, *32*, 9. [\[CrossRef\]](#)
196. Wang, Q.W.; Wang, X. A system of coupled two-sided Sylvester-type tensor equations over the quaternion algebra. *Taiwan J. Math.* **2020**, *24*, 1399–1416. [\[CrossRef\]](#)
197. Mehany, M.S.; Wang, Q.; Liu, L. A system of Sylvester-like quaternion tensor equations with an application. *Front. Math.* **2024**, *19*, 749–768. [\[CrossRef\]](#)
198. Wang, Q.W.; Wang, X.; Zhang, Y. A constraint system of coupled two-sided Sylvester-like quaternion tensor equations. *Comput. Appl. Math.* **2020**, *39*, 317. [\[CrossRef\]](#)
199. Qin, J.; Wang, Q.W. Solving a system of two-sided Sylvester-like quaternion tensor equations. *Comput. Appl. Math.* **2023**, *42*, 232. [\[CrossRef\]](#)
200. He, Z.H. The general solution to a system of coupled Sylvester-type quaternion tensor equations involving η -Hermiticity. *Bull. Iran. Math. Soc.* **2019**, *45*, 1407–1430. [\[CrossRef\]](#)
201. Xie, M.; Wang, Q.W. Reducible solution to a quaternion tensor equation. *Front. Math. China* **2020**, *15*, 1047–1070. [\[CrossRef\]](#)
202. Xie, M.; Wang, Q.W.; Zhang, Y. The minimum-norm least squares solutions to quaternion tensor systems. *Symmetry* **2022**, *14*, 1460. [\[CrossRef\]](#)
203. Jia, Z.R.; Wang, Q.W. The general solution to a system of tensor equations over the split quaternion algebra with applications. *Mathematics* **2025**, *13*, 644. [\[CrossRef\]](#)
204. Yang, L.; Wang, Q.W.; Kou, Z. A system of tensor equations over the dual split quaternion algebra with an application. *Mathematics* **2024**, *12*, 3571. [\[CrossRef\]](#)
205. Bader, B.W.; Kolda, T.G. Algorithm 862: Matlab tensor classes for fast algorithm prototyping. *ACM Trans. Math. Softw.* **2006**, *32*, 635–653. [\[CrossRef\]](#)
206. Kilmer, M.; Martin, C.D. Factorization strategies for third-order tensors. *Linear Algebra Appl.* **2011**, *435*, 641–658. [\[CrossRef\]](#)
207. Qin, Z.; Ming, Z.; Zhang, L. Singular value decomposition of third order quaternion tensors. *Appl. Math. Lett.* **2022**, *123*, 107597. [\[CrossRef\]](#)
208. Shao, J.Y. A general product of tensors with applications. *Linear Algebra Appl.* **2013**, *439*, 2350–2366. [\[CrossRef\]](#)

209. Kernfeld, E.; Kilmer, M.; Aeron, S. Tensor-tensor products with invertible linear transforms. *Linear Algebra Appl.* **2015**, *485*, 545–570. [\[CrossRef\]](#)
210. Kilmer, M.; Horesh, L.; Avron, H.; Newman, E. Tensor-tensor products for optimal representation and compression. *arXiv* **2019**, arXiv:2001.00046v1. [\[CrossRef\]](#)
211. Jin, H.; Xu, S.; Wang, Y.; Liu, X. The Moore-Penrose inverse of tensors via the M -product. *Comput. Appl. Math.* **2023**, *42*, 294. [\[CrossRef\]](#)
212. Kučera, V. Algebraic approach to discrete stochastic control. *Kybernetika* **1975**, *11*, 114–147.
213. Bengtsson, G. Output regulation and internal models—a frequency domain approach. *Automatica* **1977**, *13*, 333–345. [\[CrossRef\]](#)
214. Cheng, L.; Pearson, J. Frequency domain synthesis of multivariable linear regulators. *IEEE Trans. Autom. Control* **1978**, *23*, 3–15. [\[CrossRef\]](#)
215. Wolovich, W.A. Skew prime polynomial matrices. *IEEE Trans. Autom. Control* **1978**, *23*, 880–887. [\[CrossRef\]](#)
216. Wolovich, W.A. *Linear Multivariable Systems*; Springer: New York, NY, USA, 1974.
217. Fuhrmann, P.A. Algebraic system theory: An analyst's point of view. *J. Franklin Inst.* **1976**, *301*, 521–540. [\[CrossRef\]](#)
218. Emre, E.; Silverman, L.M. The equation $XR + QY = \Phi$: A characterization of solutions. *SIAM J. Control Optim.* **1981**, *19*, 33–38. [\[CrossRef\]](#)
219. Emre, E. The polynomial equation $QQ_c + RP_c = \Phi$ with application to dynamic feedback. *SIAM J. Control Optim.* **1980**, *18*, 611–620. [\[CrossRef\]](#)
220. Žak, S.H. On the polynomial matrix equation $AX + YB = C$. *IEEE Trans. Autom. Control* **1985**, *30*, 1240–1242. [\[CrossRef\]](#)
221. Feinstein, J.; Bar-Ness, Y. The solution of the matrix polynomial $A(s)X(s) + B(s)Y(s) = C(s)$. *IEEE Trans. Autom. Control* **1984**, *29*, 75–77. [\[CrossRef\]](#)
222. Wimmer, H.K. Consistency of a pair of generalized Sylvester equations. *IEEE Trans. Autom. Control* **1994**, *39*, 1014–1016. [\[CrossRef\]](#)
223. Wimmer, H.K. The matrix equation $X - AXB = C$ and an analogue of Roth's theorem. *Linear Algebra Appl.* **1988**, *109*, 145–147. [\[CrossRef\]](#)
224. Wimmer, H.K. The generalized Sylvester equation in polynomial matrices. *IEEE Trans. Autom. Control* **1996**, *41*, 1372–1376. [\[CrossRef\]](#)
225. Wimmer, H.K. The structure of nonsingular polynomial matrices. *Math. Syst. Theory* **1981**, *14*, 367–379. [\[CrossRef\]](#)
226. Barnett, S. Regular polynomial matrices having relatively prime determinants. *Proc. Camb. Philos. Soc.* **1969**, *65*, 585–590. [\[CrossRef\]](#)
227. Feinstein, J.; Bar-Ness, Y. On the uniqueness minimal solution of the matrix polynomial equation $A(\lambda)X(\lambda) + Y(\lambda)B(\lambda) = C(\lambda)$. *J. Franklin Inst.* **1980**, *310*, 131–134. [\[CrossRef\]](#)
228. Chen, S.; Tian, Y. On solutions of generalized Sylvester equation in polynomial matrices. *J. Franklin Inst.* **2014**, *351*, 5376–5385. [\[CrossRef\]](#)
229. Wimmer, H.K. Explicit solutions of the matrix equation $\sum A^i X D_i = C$. *SIAM J. Matrix Anal. Appl.* **1992**, *13*, 1123–1130. [\[CrossRef\]](#)
230. Huang, L. The explicit solutions and solvability of linear matrix equations. *Linear Algebra Appl.* **2000**, *311*, 195–199. [\[CrossRef\]](#)
231. Gohberg, I.; Kaashoek, M.A.; Lerer, L. On a class of entire matrix function equations. *Linear Algebra Appl.* **2007**, *425*, 434–442. [\[CrossRef\]](#)
232. Gohberg, I.; Kaashoek, M.A.; Schagen, F. *Partially Specified Matrices and Operators: Classification, Completion, Applications*; Birkhäuser Verlag: Basel, Switzerland, 1995.
233. Kaashoek, M.A.; Lerer, L. On a class of matrix polynomial equations. *Linear Algebra Appl.* **2013**, *439*, 613–620. [\[CrossRef\]](#)
234. Gohberg, I.; Kaashoek, M.A.; Lerer, L. The resultant for regular matrix polynomials and quasi commutativity. *Indiana Univ. Math. J.* **2008**, *57*, 2793–2813. [\[CrossRef\]](#)
235. Huang, L. The solvability of linear matrix equation over a central simple algebra. *Linear Multilinear Algebra* **1996**, *40*, 353–363. [\[CrossRef\]](#)
236. Huang, L. The quaternion matrix equation $\sum A^i X B_i = E$. *Acta Math. Sin. New Ser.* **1998**, *14*, 91–98.
237. Huang, L.; Liu, J. The extension of Roth's theorem for matrix equations over a ring. *Linear Algebra Appl.* **1997**, *259*, 229–235. [\[CrossRef\]](#)
238. Wu, A.G.; Duan, G.R.; Xue, Y. Kronecker maps and Sylvester-polynomial matrix equations. *IEEE Trans. Autom. Control* **2007**, *52*, 905–910. [\[CrossRef\]](#)
239. Wu, A.G.; Liu, W.; Duan, G.R. On the conjugate product of complex polynomial matrices. *Math. Comput. Model.* **2011**, *53*, 2031–2043. [\[CrossRef\]](#)
240. Wu, A.G.; Duan, G.R.; Feng, G.; Liu, W. On conjugate product of complex polynomials. *Appl. Math. Lett.* **2011**, *24*, 735–741. [\[CrossRef\]](#)
241. Wu, A.G.; Feng, G.; Liu, W.; Duan, G.R. The complete solution to the Sylvester-polynomial-conjugate matrix equations. *Math. Comput. Model.* **2011**, *53*, 2044–2056. [\[CrossRef\]](#)

242. Bevis, J.H.; Hall, F.J.; Hartwig, R.E. Consimilarity and the matrix equation $AX - XB = C$. In *Current Trends in Matrix Theory*; Uhlig, F., Grone, R., Eds.; North-Holland: New York, NY, USA, 1987; pp. 51–64.
243. Bevis, J.H.; Hall, F.J.; Hartwig, R.E. The matrix equation $A\bar{X} - XB = C$ and its special cases. *SIAM J. Matrix Anal. Appl.* **1988**, *9*, 348–359. [\[CrossRef\]](#)
244. Wu, A.G.; Duan, G.R.; Yu, H.H. On solutions of the matrix equations $XF - AX = C$ and $XF - A\bar{X} = C$. *Appl. Math. Comput.* **2006**, *183*, 932–941.
245. Jiang, T.; Wei, M. On solutions of the matrix equations $X - AXB = C$ and $X - A\bar{X}B = C$. *Linear Algebra Appl.* **2003**, *367*, 225–233. [\[CrossRef\]](#)
246. Wu, A.G.; Wang, H.Q.; Duan, G.R. On matrix equations $X - AXF = C$ and $X - A\bar{X}F = C$. *J. Comput. Appl. Math.* **2009**, *230*, 690–698. [\[CrossRef\]](#)
247. Wu, A.G.; Fu, Y.M.; Duan, G.R. On solutions of matrix equations $V - AVF = BW$ and $V - A\bar{V}F = BW$. *Math. Comput. Model.* **2008**, *47*, 1181–1197. [\[CrossRef\]](#)
248. Wu, A.G.; Feng, G.; Hu, J.; Duan, G.R. Closed-form solutions to the nonhomogeneous Yakubovich-conjugate matrix equation. *Appl. Math. Comput.* **2009**, *214*, 442–450. [\[CrossRef\]](#)
249. Wu, A.G.; Zhang, Y. *Complex Conjugate Matrix Equations for Systems and Control*; Springer: Singapore, 2017.
250. Mazurek, R. A general approach to Sylvester-polynomial-conjugate matrix equations. *Symmetry* **2024**, *16*, 246. [\[CrossRef\]](#)
251. Lam, T.Y. *A First Course in Noncommutative Rings*; Springer: New York, NY, USA, 1991.
252. Wu, A.G.; Liu, W.; Li, C.; Duan, G.R. On j-conjugate product of quaternion polynomial matrices. *Appl. Math. Comput.* **2013**, *219*, 11223–11232. [\[CrossRef\]](#)
253. Futorny, V.; Klymchuk, T.; Sergeichuk, V.V. Roth's solvability criteria for the matrix equations $AX - \hat{X}B = C$ and $X - A\hat{X}B = C$ over the skew field of quaternions with an involutive automorphism $q \mapsto \hat{q}$. *Linear Algebra Appl.* **2016**, *510*, 246–258. [\[CrossRef\]](#)
254. Jiang, T.; Ling, S. On a solution of the quaternion matrix equation $A\tilde{X} - XB = C$ and its applications. *Adv. Appl. Clifford Algebras* **2013**, *23*, 689–699. [\[CrossRef\]](#)
255. Song, C.; Chen, G. On solutions of matrix equation $XF - AX = C$ and $XF - A\tilde{X} = C$ over quaternion field. *J. Appl. Math. Comput.* **2011**, *37*, 57–68. [\[CrossRef\]](#)
256. Jiang, T.S.; Wei, M.S. On a solution of the quaternion matrix equation $X - A\tilde{X}B = C$ and its application. *Acta Math. Sin. Engl. Ser.* **2005**, *21*, 483–490. [\[CrossRef\]](#)
257. Song, C.; Chen, G.; Liu, Q. Explicit solutions to the quaternion matrix equations $X - AXF = C$ and $X - A\tilde{X}F = C$. *Int. J. Comput. Math.* **2012**, *89*, 890–900. [\[CrossRef\]](#)
258. Song, C.; Feng, J.; Wang, X.; Zhao, J. A real representation method for solving Yakubovich-j-conjugate quaternion matrix equation. *Abstr. Appl. Anal.* **2014**, *2014*, 285086. [\[CrossRef\]](#)
259. Yuan, S.; Liao, A. Least squares solution of the quaternion matrix equation $X - A\hat{X}B = C$ with the least norm. *Linear Multilinear Algebra* **2011**, *59*, 985–998. [\[CrossRef\]](#)
260. Song, C.; Feng, J. On solutions to the matrix equations $XB - AX = CY$ and $XB - A\hat{X} = CY$. *J. Franklin Inst.* **2016**, *353*, 1075–1088. [\[CrossRef\]](#)
261. Song, C.; Chen, G. Solutions to matrix equations $X - AXB = CY + R$ and $X - A\hat{X}B = CY + R$. *J. Comput. Appl. Math.* **2018**, *343*, 488–500. [\[CrossRef\]](#)
262. Moor, B.D.; Zha, H. A tree of generalization of the ordinary singular value decomposition. *Linear Algebra Appl.* **1991**, *147*, 469–500. [\[CrossRef\]](#)
263. He, Z.H. Pure PSVD approach to Sylvester-type quaternion matrix equations. *Electron. J. Linear Algebra* **2019**, *35*, 266–284. [\[CrossRef\]](#)
264. Xie, M.Y.; Wang, Q.W.; He, Z.H.; Saad, M.M. A system of Sylvester-type quaternion matrix equations with ten variables. *Acta Math. Sin. Engl. Ser.* **2022**, *38*, 1399–1420. [\[CrossRef\]](#)
265. Mehany, M.S.; Wang, Q.W. Three symmetrical systems of coupled Sylvester-like quaternion matrix equations. *Symmetry* **2022**, *14*, 550. [\[CrossRef\]](#)
266. He, Z.H.; Wang, Q.W.; Zhang, Y. A system of quaternary coupled Sylvester-type real quaternion matrix equations. *Automatica* **2018**, *87*, 25–31. [\[CrossRef\]](#)
267. Rehman, A.; Wang, Q.W.; Ali, I.; Akram, M.; Ahmad, M.O. A constraint system of generalized Sylvester quaternion matrix equations. *Adv. Appl. Clifford Algebr.* **2017**, *27*, 3183–3196. [\[CrossRef\]](#)
268. Wang, Q.W.; He, Z.H. Solvability conditions and general solution for mixed Sylvester equations. *Automatica* **2013**, *49*, 2713–2719. [\[CrossRef\]](#)
269. Wang, Q.W.; He, Z.H. Systems of coupled generalized Sylvester matrix equations. *Automatica* **2014**, *50*, 2840–2844. [\[CrossRef\]](#)
270. Wang, Q.W.; Rehman, A.; He, Z.H.; Zhang, Y. Constraint generalized Sylvester matrix equations. *Automatica* **2016**, *69*, 60–64. [\[CrossRef\]](#)

271. He, Z.H.; Wang, Q.W. A system of periodic discrete-time coupled Sylvester quaternion matrix equations. *Algebra Colloq.* **2017**, *24*, 169–180. [\[CrossRef\]](#)
272. Kågström, B. A perturbation analysis of the generalized Sylvester equation $(AR - LB, DR - LE) = (C, F)$. *SIAM J. Matrix Anal. Appl.* **1994**, *15*, 1045–1060. [\[CrossRef\]](#)
273. Ziętak, K. The properties of the minimax solution of a non-linear matrix equation $XY = A$. *IMA J. Numer. Anal.* **1983**, *3*, 229–244. [\[CrossRef\]](#)
274. Yang, X.; Huang, W. Backward error analysis of the matrix equations for Sylvester and Lyapunov. *J. Sys. Sci. Math. Scis.* **2008**, *28*, 524–534. (In Chinese)
275. Sun, J.G.; Xu, S.F. Perturbation analysis of the maximal solution of the matrix equation $X + A^*X^{-1} = P$. II. *Linear Algebra Appl.* **2003**, *362*, 211–228. [\[CrossRef\]](#)
276. Li, J.F.; Hu, X.Y.; Duan, X.F. A symmetric preserving iterative method for generalized Sylvester equation. *Asian J. Control* **2011**, *13*, 408–417. [\[CrossRef\]](#)
277. Rockafellar, R.T. The multiplier method of Hestenes and Powell applied to convex programming. *J. Optim. Theory Appl.* **1973**, *12*, 555–562. [\[CrossRef\]](#)
278. Hartwig, R.E. A note on light matrices. *Linear Algebra Appl.* **1987**, *97*, 153–169. [\[CrossRef\]](#)
279. Ke, Y.; Ma, C. An alternating direction method for nonnegative solutions of the matrix equation $AX + YB = C$. *Comp. Appl. Math.* **2017**, *36*, 359–365. [\[CrossRef\]](#)
280. Peng, Z.; Peng, Y. An efficient iterative method for solving the matrix equation $AXB + CYD = E$. *Numer. Linear Algebra Appl.* **2006**, *13*, 473–485. [\[CrossRef\]](#)
281. Kågström, B.; Poromaa, P. Lapack-style algorithms and software for solving the generalized Sylvester equation and estimating the separation between regular matrix pairs. *ACM Trans. Math. Softw.* **1996**, *22*, 78–103. [\[CrossRef\]](#)
282. Lin, Y.; Wei, Y. Condition numbers of the generalized Sylvester equation. *IEEE Trans. Autom. Control* **2007**, *52*, 2380–2385. [\[CrossRef\]](#)
283. Diao, H.; Shi, X.; Wei, Y. Effective condition numbers and small sample statistical condition estimation for the generalized Sylvester equation. *Sci. China-Math.* **2013**, *56*, 967–982. [\[CrossRef\]](#)
284. Dehghan, M.; Hajarian, M. An iterative algorithm for the reflexive solutions of the generalized coupled Sylvester matrix equations and its optimal approximation. *Appl. Math. Comput.* **2008**, *202*, 571–588. [\[CrossRef\]](#)
285. Dehghan, M.; Hajarian, M. An iterative method for solving the generalized coupled Sylvester matrix equations over generalized bisymmetric matrices. *Appl. Math. Modell.* **2010**, *34*, 639–654. [\[CrossRef\]](#)
286. Paige, C.C.; Saunders, M.A. LSQR: An algorithm for sparse linear equations and sparse least squares. *ACM Trans. Math. Softw.* **1982**, *8*, 43–71. [\[CrossRef\]](#)
287. Li, S.K.; Huang, T.Z. LSQR iterative method for generalized coupled Sylvester matrix equations. *Appl. Math. Modell.* **2012**, *36*, 3545–3554. [\[CrossRef\]](#)
288. Huang, J. On parameter iteration method for solving the mixed-type Lyapunov matrix equation. *Math. Numer. Sinica* **2007**, *29*, 285–292. (In Chinese)
289. Zhang, K. *Iterative Algorithms for Constrained Solutions of Matrix Equations*; National Defense Industry Press: Beijing, China, 2015. (In Chinese)
290. Lv, C.; Ma, C. Two parameter iteration methods for coupled Sylvester matrix equations. *East Asian J. Appl. Math.* **2018**, *8*, 336–351. [\[CrossRef\]](#)
291. Ma, C.; Wu, Y.; Xie, Y. The Newton-type splitting iterative method for a class of coupled Sylvester-like absolute value equation. *J. Appl. Anal. Comput.* **2024**, *14*, 3306–3331. [\[CrossRef\]](#)
292. Sheng, X. A relaxed gradient based algorithm for solving generalized coupled Sylvester matrix equations. *J. Franklin Inst.* **2018**, *355*, 4282–4297. [\[CrossRef\]](#)
293. Ding, F.; Chen, T. Iterative least-squares solutions of coupled Sylvester matrix equations. *Syst. Control Lett.* **2005**, *54*, 95–107. [\[CrossRef\]](#)
294. Ding, F.; Chen, T. On iterative solutions of general coupled matrix equations. *SIAM J. Control Optim.* **2006**, *44*, 2269–2284. [\[CrossRef\]](#)
295. Shirilord, A.; Dehghan, M. Gradient descent-based parameter-free methods for solving coupled matrix equations and studying an application in dynamical systems. *Appl. Numer. Math.* **2025**, *212*, 29–59. [\[CrossRef\]](#)
296. Dehghan, M.; Hajarian, M. The general coupled matrix equations over generalized bisymmetric matrices. *Linear Algebra Appl.* **2010**, *432*, 1531–1552. [\[CrossRef\]](#)
297. Dehghan, M.; Hajarian, M. Iterative algorithms for the generalized centro-symmetric and central anti-symmetric solutions of general coupled matrix equations. *Eng. Comput.* **2012**, *29*, 528–560. [\[CrossRef\]](#)
298. Hajarian, M. Matrix form of the CGS method for solving general coupled matrix equations. *Appl. Math. Lett.* **2014**, *34*, 37–42. [\[CrossRef\]](#)

299. Hajarian, M. Generalized conjugate direction algorithm for solving the general coupled matrix equations over symmetric matrices. *Numer. Algor.* **2016**, *73*, 591–609. [\[CrossRef\]](#)
300. Hajarian, M. Convergence of HS version of BCR algorithm to solve the generalized Sylvester matrix equation over generalized reflexive matrices. *J. Franklin Inst.* **2017**, *354*, 2340–2357. [\[CrossRef\]](#)
301. Lv, C.Q.; Ma, C.F. BCR method for solving generalized coupled Sylvester equations over centrosymmetric or anti-centrosymmetric matrix. *Comput. Math. Appl.* **2018**, *75*, 70–88. [\[CrossRef\]](#)
302. Niu, Q.; Wang, X.; Lu, L.Z. A relaxed gradient based algorithm for solving Sylvester equations. *Asian J. Control* **2011**, *13*, 461–464. [\[CrossRef\]](#)
303. Hajarian, M. Computing symmetric solutions of general Sylvester matrix equations via Lanczos version of biconjugate residual algorithm. *Comput. Math. Appl.* **2018**, *76*, 686–700. [\[CrossRef\]](#)
304. Yan, T.; Ma, C. The BCR algorithms for solving the reflexive or anti-reflexive solutions of generalized coupled Sylvester matrix equations. *J. Franklin Inst.* **2020**, *357*, 12787–12807. [\[CrossRef\]](#)
305. Zhou, B.; Duan, G.R.; Li, Z.Y. Gradient based iterative algorithm for solving coupled matrix equations. *Syst. Control Lett.* **2009**, *58*, 327–333. [\[CrossRef\]](#)
306. Hajarian, M. Matrix GPBiCG algorithms for solving the general coupled matrix equations. *IET Control Theory Appl.* **2015**, *9*, 74–81. [\[CrossRef\]](#)
307. Wu, A.G.; Li, B.; Zhang, Y.; Duan, G.R. Finite iterative solutions to coupled Sylvester-conjugate matrix equations. *Appl. Math. Modell.* **2011**, *35*, 1065–1080. [\[CrossRef\]](#)
308. Yan, T.; Ma, C. An iterative algorithm for generalized Hamiltonian solution of a class of generalized coupled Sylvester-conjugate matrix equations. *Appl. Math. Comput.* **2021**, *411*, 126491. [\[CrossRef\]](#)
309. Carpentieri, B.; Jing, Y.F.; Huang, T.Z. The BiCOR and CORS iterative algorithms for solving nonsymmetric linear systems. *SIAM J. Sci. Comput.* **2011**, *33*, 3020–3036. [\[CrossRef\]](#)
310. Jing, Y.F.; Huang, T.Z.; Zhang, Y.; Li, L.; Cheng, G.H.; Ren, Z.G.; Duan, Y.; Sogabe, T.; Carpentieri, B. Lanczos-type variants of the COCR method for complex nonsymmetric linear systems. *J. Comput. Phys.* **2009**, *228*, 6376–6394. [\[CrossRef\]](#)
311. Hajarian, M. Developing BiCOR and CORS methods for coupled Sylvester-transpose and periodic Sylvester matrix equations. *Appl. Math. Modell.* **2015**, *39*, 6073–6084. [\[CrossRef\]](#)
312. Bittanti, S.; Colaneri, P. *Periodic Systems: Filtering and Control*; Springer: London, UK, 2009.
313. Hajarian, M. Convergence analysis of generalized conjugate direction method to solve general coupled Sylvester discrete-time periodic matrix equations. *Int. J. Adapt. Control Signal Process.* **2017**, *31*, 985–1002. [\[CrossRef\]](#)
314. Ma, C.; Yan, T. A finite iterative algorithm for the general discrete-time periodic Sylvester matrix equations. *J. Franklin Inst.* **2022**, *359*, 4410–4432.
315. Dehghani-Madiseh, M.; Dehghan, M. Generalized solution sets of the interval generalized Sylvester matrix equation $\sum_{i=1}^p A_i X_i + \sum_{j=1}^q Y_j B_j = C$ and some approaches for inner and outer estimations. *Comput. Math. Appl.* **2014**, *68*, 1758–1774. [\[CrossRef\]](#)
316. Hajarian, M. Convergence properties of BCR method for generalized Sylvester matrix equation over generalized reflexive and anti-reflexive matrices. *Linear Multilinear Algebra* **2018**, *66*, 1975–1990.
317. Li, T.; Wang, Q.W.; Zhang, X.F. A modified conjugate residual method and nearest Kronecker product preconditioner for the generalized coupled Sylvester tensor equations. *Mathematics* **2022**, *10*, 1730. [\[CrossRef\]](#)
318. Varga, A. A numerically reliable approach to robust pole assignment for descriptor systems. *Futur. Gener. Comp. Syst.* **2012**, *19*, 1221–1230. [\[CrossRef\]](#)
319. Trinh, H.; Tran, T.D.; Nahavandi, S. Design of scalar functional observers of order less than $(v - 1)$. *Int. J. Control* **2006**, *79*, 1654–1659. [\[CrossRef\]](#)
320. Jin, L.; Yan, J.; Du, X.; Xiao, X.; Fu, D. RNN for solving time-variant generalized Sylvester equation with applications to robots and acoustic source localization. *IEEE Trans. Ind. Inform.* **2020**, *16*, 6359–6369. [\[CrossRef\]](#)
321. Kiran, N.U. Simultaneous triangularization of pseudo-differential systems. *J. Pseudo-Differ. Oper. Appl.* **2013**, *4*, 45–61.
322. Gu, D.; Zhang, D.; Liu, Q. Parametric control to permanent magnet synchronous motor via proportional plus integral feedback. *Trans. Inst. Meas. Control* **2021**, *43*, 925–932.
323. Song, W.; Jin, A. Observer-based model reference tracking control of the Markov jump system with partly unknown transition rates. *Appl. Sci.* **2023**, *13*, 914. [\[CrossRef\]](#)
324. Xu, R.; Wei, T.; Wei, Y.; Yan, H. UTV decomposition of dual matrices and its applications. *Comput. Appl. Math.* **2024**, *43*, 41. [\[CrossRef\]](#)
325. Radjavi, H.; Rosenthal, P. *Simultaneous Triangularization*; Springer: New York, NY, USA, 2000.
326. Taylor, M. *Pseudo Differential Operators*; Springer: Berlin/Heidelberg, Germany, 1974.
327. Mooring, B.W.; Roth, Z.S.; Driels, M.R. *Fundamentals of Manipulator Calibration*; Wiley: New York, NY, USA, 1991.
328. Shiu, Y.C.; Ahmad, S. Calibration of wrist-mounted robotic sensors by solving homogeneous transformation equations of the form $AX = XB$. *IEEE Trans. Robot. Autom.* **1989**, *5*, 16–29. [\[CrossRef\]](#)

329. Zhuang, H.; Roth, Z.S. Comments on “Calibration of wrist-mounted robotic sensors by solving homogeneous transformation equations of the form $AX = XB$ ”. *IEEE Trans. Robot. Autom.* **1991**, *7*, 877–878. [[CrossRef](#)]
330. Zhuang, H.; Roth, Z.S.; Sudhakar, R. Simultaneous robot/world and tool/flange calibration by solving homogeneous transformation equations of the form $AX = YB$. *IEEE Trans. Robot. Autom.* **1994**, *10*, 549–554. [[CrossRef](#)]
331. Qi, L. Standard dual quaternion optimization and its applications in hand-eye calibration and SLAM. *Commun. Appl. Math. Comput.* **2023**, *5*, 1469–1483. [[CrossRef](#)]
332. Li, A.; Wang, L.; Wu, D. Simultaneous robot-world and hand-eye calibration using dual-quaternions and Kronecker product. *Int. J. Phys. Sci.* **2010**, *5*, 1530–1536.
333. Chen, Z.; Ling, C.; Qi, L.; Yan, H. A regularization-patching dual quaternion optimization method for solving the hand-eye calibration problem. *J. Optim. Theory Appl.* **2024**, *200*, 1193–1215. [[CrossRef](#)]
334. Wang, X.; Huang, J.; Song, H. Simultaneous robot-world and hand-eye calibration based on a pair of dual equations. *Measurement* **2021**, *181*, 109623. [[CrossRef](#)]
335. Daniilidis, K. Hand-eye calibration using dual quaternions. *Int. J. Robot. Res.* **1999**, *18*, 286–298. [[CrossRef](#)]
336. Wang, J.; Qu, D.; Xu, F. A new hybrid calibration method for extrinsic camera parameters and hand-eye transformation. In Proceedings of the IEEE International Conference on Mechatronics and Automation, Niagara Falls, ON, Canada, 29 July–1 August 2005; Volume 4, pp. 1981–1985.
337. Ernst, F.; Richter, L.; Matthäus, L.; Martens, V.; Bruder, R.; Schlaefter, A.; Schweikard, A. Non-orthogonal tool/flange and robot/world calibration. *Int. J. Med. Robotics Comput. Assist. Surg.* **2012**, *8*, 407–420; Erratum in *Int. J. Med. Robotics Comput. Assist. Surg.* **2017**, *13*, e1883. [[CrossRef](#)]
338. Shah, M. Solving the robot-world/hand-eye calibration problem using the Kronecker product. *J. Mech. Robot.* **2013**, *5*, 031007. [[CrossRef](#)]
339. Wu, L.; Ren, H. Finding the kinematic base frame of a robot by hand-eye calibration using 3D position data. *IEEE Trans. Autom. Sci. Eng.* **2017**, *14*, 314–324. [[CrossRef](#)]
340. Tan, N.; Gu, X.; Ren, H. Simultaneous robot-world, sensor-tip, kinematics calibration of an underactuated robotic hand with soft fingers. *IEEE Access* **2018**, *6*, 22705–22715. [[CrossRef](#)]
341. Shah, M.; Bostelman, R.; Legowik, S.; Hong, T. Calibration of mobile manipulators using 2D positional features. *Measurement* **2018**, *124*, 322–328. [[CrossRef](#)]
342. Condurache, D.; Ciureanu, I.A. A novel solution for $AX = YB$ sensor calibration problem using dual Lie algebra. In Proceedings of the 2019 6th International Conference on Control, Decision and Information Technologies (CoDIT'19), Paris, France, 23–26 April 2019; pp. 302–307.
343. Wu, J.; Liu, M.; Zhu, Y.; Zou, Z.; Dai, Z.M.; Zhang, C.; Jiang, Y.; Li, C. Globally optimal symbolic hand-eye calibration. *IEEE-ASME Trans. Mechatron.* **2021**, *26*, 1369–1379. [[CrossRef](#)]
344. Pan, J.; Fu, Z.; Yue, H.; Lei, X.; Li, M.; Chen, X. Toward simultaneous coordinate calibrations of $AX = YB$ problem by the LMI-SDP optimization. *IEEE Trans. Autom. Sci. Eng.* **2023**, *20*, 2445–2453. [[CrossRef](#)]
345. Ha, J. Probabilistic framework for hand-eye and robot-world calibration $AX = YB$. *IEEE Trans. Robot.* **2023**, *39*, 1196–1211. [[CrossRef](#)]
346. Wang, X.; Song, H. One-step solving the robot-world and hand-eye calibration based on the principle of transference. *J. Mech. Robot.* **2024**, *17*, 031014.
347. Xie, L.M.; Wang, Q.W. Some novel results on a classical system of matrix equations over the dual quaternion algebra. *Filomat* **2025**, *39*, 1477–1490. [[CrossRef](#)]

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.