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The Sharp Coefficients and Hankel Determinants for a Novel Class $\mathcal{R}_{LP}$

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Abstract

Let $\mathcal{R}_{LP}$ denote a newly introduced subclass of bounded turning functions. The primary aim of this study is to investigate the sharp bounds of the coefficients of $|d_2|, |d_3|, |d_4|, |d_5|$, as well as to establish precise estimates for the second- and third-order Hankel determinants $H_{2,1}, H_{2,2}, H_{2,3}$, and $H_{3,1}$ for functions belonging to this class. The coefficient bounds and Hankel determinant estimates derived herein are all shown to be sharp.

Keywords: sharp coefficients; Hankel determinant; Schwartz functions; bounded turning functions; extremal problems

MSC: 30C45; 30C80

1. Introduction

Complex analysis is a fundamental discipline within the mathematical sciences, extensively studied due to its broad applicability in areas such as engineering, electronics, and nonlinear integrable systems theory, among others. Notably, the theory of geometric functions forms an intriguing research direction within complex analysis, focusing on the exploration of geometric properties and characteristics of analytic functions.

Within geometric function theory, the precise estimation of coefficients and the analysis of Hankel determinants for analytic and univalent functions have constituted a central research area for decades. Although these concepts are distinct, they are intimately associated, offering crucial information about the structure and attributes of functions analytic in the open unit disk. Sharp coefficients pertain to the most restrictive admissible bounds for the Taylor coefficients of a given function class, ensuring that no uniformly smaller bound exists. The Hankel determinant, constructed from these coefficients, provides a quantitative measure of functional complexity and finds extensive use in exploring univalence, starlikeness, and convexity properties of analytic functions.

Recent advancements in the field have generated increasing interest in extending classical results to more specialized subclasses of analytic functions. These extensions frequently involve functions characterized by q-calculus, fractional operators, or subordination to particular analytic functions that map into geometrically significant or symmetric domains. For instance, investigations have addressed starlike functions associated with symmetric domains, including the Booth lemniscate, limaçon curves, and domains bounded by trigonometric or hyperbolic curves [1,2]. Such studies not only generalize foundational results but also provide new insights into the intricate relationship between coefficient bounds and the geometric attributes of image domains for analytic mappings.

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Let $g(\zeta)$ be holomorphic in the unit disk $\mathbb{U} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ normalized such that $g(0) = g'(0) - 1 = 0$ and let $\mathcal{H}$ denote the class of all such functions $g(\zeta)$. Then, for $g(\zeta) \in \mathcal{H}$, the expansion has the form:

$$g(\varepsilon) = \varepsilon + \sum_{k=2}^{\infty} d_k \varepsilon^k \quad (\varepsilon \in \mathbb{U}). \tag{1}$$

For given $i, k \in \mathbb{N} = \{1, 2, 3 \ldots\}$, the Hankel determinant $H_{i,k}(g)$ was introduced by Pommerenke [3,4] for a function $g \in \mathcal{S}$ with a power series expansion (1) as follows:

$$H_{i,k}(g) = \begin{vmatrix} d_k & d_{k+1} & \cdots & d_{k+i-1} \\ d_{k+1} & d_{k+2} & \cdots & d_{k+i} \\ \vdots & \vdots & \vdots & \vdots \\ d_{k+i-1} & d_{k+i} & \cdots & d_{k+2i-2} \end{vmatrix}.$$

For $d_1 = 1$, and some particular values of k and i, we get the second- and third- order Hankel determinants as:

$$\mathcal{H}_{2,1}(g) = -d_2^2 + d_3, \tag{2}$$

$$\mathcal{H}_{2,2}(g) = d_2 d_4 - d_3^2, \tag{3}$$

$$\mathcal{H}_{2,3}(g) = -d_4^2 + d_5 d_3, \tag{4}$$

$$H_{3,1}(g) = -d_2^2 d_5 + 2 d_2 d_3 d_4 - d_4^2 - d_3^3 + d_3 d_5 \tag{5}$$

The Hankel matrix serves as a pivotal analytical tool in pure mathematics and numerous applied fields, with far-reaching applications in Markov process theory, nonstationary signal processing within the context of the Hamburger moment problem, and partition functions represented by Jacobi weights, among others [5–7]. The initial sharp bounds for the third-order Hankel determinant for convex functions were established in [8], subsequently motivating extensive related research [9,10].

In [10], a subclass of starlike functions, denoted as $\mathcal{S}_{LP}^*$, was defined using the subordination technique, where sharp upper bounds were obtained for the first four Taylor coefficients as well as for the second- and third-order Hankel determinants.

$$\mathcal{S}_{LP}^* = \left\{ g \in \mathcal{H} : \frac{\varepsilon g'(\varepsilon)}{g(\varepsilon)} \prec \sqrt{1 + sin\varepsilon} (\varepsilon \in \mathbb{U}) \right\}.$$

Motivated by the approach for $\mathcal{S}_{LP}^*$, we introduce a novel subclass of bounded turning functions as follows:

$$\mathcal{R}_{LP} = \left\{ g \in \mathcal{H} : g'(\varepsilon) \prec \sqrt{1 + sin\varepsilon}, (\varepsilon \in \mathbb{U}) \right\}.$$

This condition is geometrically distinct from that of $\mathcal{S}_{LP}^*$ and leads to a different set of coefficient relations and challenges. The primary aim is to perform a complete analysis of the early coefficient and Hankel determinant problems for this new class.

This paper introduces and investigates a novel subclass of analytic functions with bounded turning, denoted $\mathcal{R}_{LP}$, defined by the subordination condition $g'(\varepsilon) \prec \sqrt{1 + sin\varepsilon}$. The primary objective is to derive the sharp bounds for the initial Taylor-Maclaurin co-

efficients $|d_n|$ for $n = 2, 3, 4, 5$. Furthermore, we establish precise and sharp estimates for several Hankel determinants, specifically the second-order determinants $H_{2,1}$, $H_{2,2}$, $H_{2,3}$, and the third-order determinant $H_{3,1}$.The methodology involves transforming the subordination condition into a problem of optimizing functionals over the Carathéodory class of functions with positive real part. By employing a series of established lemmas and intricate inequalities, we perform a detailed and systematic analysis to obtain the optimal bounds. For each result, we explicitly construct extremal functions that demonstrate the sharpness of our inequalities, confirming that the derived bounds cannot be improved.

After this introduction, Section 2 lists the necessary lemmas required for the proofs. Section 3 presents our main results: sharp bounds for $|d_2|$ to $|d_5|$ (Theorem 1), $H_{2,1}$ (Theorem 2), $H_{2,2}$ (Theorem 3), $H_{2,3}$ (Theorem 4), and $H_{3,1}$ (Theorem 5). Each theorem is followed by a detailed proof, outlining the methodological steps and the construction of extremal functions to prove sharpness. The paper concludes with acknowledgments and references.

2. Set of Lemmas

Let $\mathcal{P}$ denote the family of functions p holomorphic in $\mathbb{U}$ such that $Rep(\varepsilon) > 0$, with the following power series expansion:

$$p(\varepsilon) = p_1\varepsilon + p_2\varepsilon^2 + p_3\varepsilon^3 + \dots \quad (\varepsilon \in \mathbb{U}). \tag{6}$$

To establish the main results, we require the following lemmas.

Lemma 1 ([11]). *If $p \in \mathcal{P}$ is of the form (6), then*

$$|p_m| \leq 2 \quad m \geq 1 \tag{7}$$
$$|p_{m+k} - \eta p_m p_k| \leq 2, \quad 0 \leq \eta \leq 1. \tag{8}$$

Lemma 2 ([12]). *If $p \in \mathcal{P}$ is of the form (6). Then for $0 \leq L \leq 1$ and $L(2L-1) \leq T \leq L$ the following inequality hold true:*

$$|Tp_1^3 - 2Lp_1p_2 + p_3| \leq 2. \tag{9}$$

Lemma 3 ([13]). *Let ζ, v, k and μ satisfying the conditions $0 < \mu < 1$, $0 < \zeta < 1$ and*

$$8\mu(1-\mu)[(\zeta v - 2k)^2 + (\zeta(\mu+\zeta) - v)^2 + \zeta(1-\zeta)(v - 2\mu\zeta)^2] \leq 4\zeta^2(1-\zeta)^2\mu(1-\mu)$$

then for $p \in \mathcal{P}$, given in (6), we have

$$|-p_4 - \frac{3}{2}vp_1^2p_2 + 2\zeta p_1p_3 + \mu p_2^2 + kp_1^4| \leq 2. \tag{10}$$

Lemma 4 ([14,15]). *If $p \in \mathcal{P}$ is of the form (6), then*

$$2p_2 = p_1^2 + \xi(4 - p_1^2), \tag{11}$$
$$4p_3 = p_1^3 + 2p_1\xi(4 - p_1^2) - p_1\xi^2(4 - p_1^2) + 2(4 - p_1^2)(1 - |\xi|^2)t, \tag{12}$$
$$8p_4 = p_1^4 + (4 - p_1^2)\xi[p_1^2(\xi^2 - 3\xi + 3) + 4\xi] - 4(4 - p_1^2)(1 - |\xi|^2) \Big[p_1(\xi - 1)t + \overline{\xi}t^2 - (1 - |t|^2)\varrho\Big]. \tag{13}$$

for some ξ, t, ϱ with $|\xi| \leq 1$, $|t| \leq 1$ and $|\varrho| \leq 1$.

3. Main Results

Theorem 1. *If $g \in \mathcal{R}_{LP}$, then*

$$|d_2| \leq \frac{1}{4}, \quad |d_3| \leq \frac{1}{6}, \quad |d_4| \leq \frac{1}{8}, \quad |d_5| \leq \frac{1}{10}.$$

These bounds are sharp.

$$g_1(\varepsilon) = \int_0^{\varepsilon} (\sqrt{1+sint})dt = \varepsilon + \frac{1}{4}\varepsilon^2 + \cdots, \tag{14}$$

$$g_2(\varepsilon) = \int_0^{\varepsilon} (\sqrt{1+sint^2})dt = \varepsilon + \frac{1}{6}\varepsilon^3 + \cdots, \tag{15}$$

$$g_3(\varepsilon) = \int_0^{\varepsilon} (\sqrt{1+sint^3})dt = \varepsilon + \frac{1}{8}\varepsilon^4 + \cdots, \tag{16}$$

$$g_4(\varepsilon) = \int_0^{\varepsilon} (\sqrt{1+sint^4})dt = \varepsilon + \frac{1}{10}\varepsilon^5 + \cdots. \tag{17}$$

Proof. Let $g \in \mathcal{R}_{LP}$. There exists a Schwartz function $w(\varepsilon)$ with $w(0) = 0$ and $|w(\varepsilon)| < 1$, such that

$$g'(\varepsilon) = \sqrt{1+sin(w(\varepsilon))}.$$

Define the auxiliary function:

$$p(\varepsilon) = \frac{1+w(\varepsilon)}{1-w(\varepsilon)} = 1 + p_1\varepsilon + p_2\varepsilon^2 + p_3\varepsilon^3 + p_4\varepsilon^4 + \cdots.$$

Clearly, we have $p(\varepsilon) \in \mathcal{P}$ and

$$\begin{aligned} w(\varepsilon) = \frac{p(\varepsilon)-1}{p(\varepsilon)+1} &= \frac{p_1\varepsilon + p_2\varepsilon^2 + p_3\varepsilon^3 + p_4\varepsilon^4 + \cdots}{2 + p_1\varepsilon + p_2\varepsilon^2 + p_3\varepsilon^3 + p_4\varepsilon^4 + \cdots} \\ &= \frac{1}{2}p_1\varepsilon + (-\frac{1}{4}p_1^2 + \frac{1}{2}p_2)\varepsilon^2 + (-\frac{1}{2}p_1p_2 + \frac{1}{8}p_1^3 + \frac{1}{2}p_3)\varepsilon^3 \\ &+ (\frac{1}{2}p_4 + \frac{3}{8}p_1^2p_2 - \frac{1}{2}p_1p_3 - \frac{1}{16}p_1^4 - \frac{1}{4}p_2^2)\varepsilon^4 + \cdots. \end{aligned}$$

As

$$\sqrt{1+sin(\varepsilon)} = 1 + \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 - \frac{1}{48}\varepsilon^3 + \frac{1}{384}\varepsilon^4 + \cdots. \tag{18}$$

Putting the value of $w(\varepsilon)$ in Equation (18), we get

$$\begin{aligned} \sqrt{1+sin(w(\varepsilon))} = 1 + \frac{1}{4}p_1\varepsilon + (-\frac{5}{32}p_1^2 + \frac{1}{4}p_2)\varepsilon^2 + (\frac{1}{4}p_3 - \frac{5}{16}p_1p_2 + \frac{35}{384}p_1^3)\varepsilon^3 \\ + (-\frac{311}{6144}p_1^4 - \frac{5}{32}p_2^2 - \frac{5}{16}p_1p_3 + \frac{35}{128}p_1^2p_2 + \frac{1}{4}p_4)\varepsilon^4 + \cdots. \end{aligned} \tag{19}$$

From (1), we achieve

$$g'(\varepsilon) = 1 + 2d_2\varepsilon + 3d_3\varepsilon^2 + 4d_4\varepsilon^3 + 5d_5\varepsilon^4 + \cdots. \tag{20}$$

Comparing (19) and (20), we have

$$d_2 = \frac{1}{8}p_1, \tag{21}$$

$$d_3 = \frac{1}{12}(p_2 - \frac{5}{8}p_1^2), \tag{22}$$

$$d_4 = \frac{1}{16}p_3 - \frac{5}{64}p_1p_2 + \frac{35}{1536}p_1^3, \tag{23}$$

$$a_5 = \frac{1}{20}p_4 - \frac{5}{80}p_1p_3 - \frac{5}{160}p_2^2 + \frac{35}{640}p_1^2p_2 - \frac{311}{30720}p_1^4, \tag{24}$$

From (21) and (22), and applying Lemma 1, we yield $|d_2| \leq \frac{1}{4}$ and $|d_3| \leq \frac{1}{6}$.

From (23), we can write

$$|d_4| = \frac{1}{16}|p_3 - 2(\frac{5}{8})p_1p_2 + \frac{35}{96}p_1^3|.$$

By (9), we get $L = \frac{5}{8} = 0.625$ and $T = \frac{35}{96} = 0.3646$. Clearly, $T \leq L$, and calculation also shows $L(2L-1) = \frac{5}{32} = 0.15625 \leq T$. Then, by Lemma 2, we have

$$|d_4| \leq \frac{1}{8}.$$

From (24), we have

$$|d_5| = \frac{1}{20}\left|\frac{311}{1536}p_1^4 + \frac{5}{8}p_2^2 + 2(\frac{5}{8})p_3p_1 - \frac{3}{2}(\frac{35}{48})p_2p_1^2 - p_4\right|.$$

Comparing the right-hand side with (10), we have $K = \frac{311}{1536}, \mu = \frac{5}{8}, \zeta = \frac{5}{8}$ and $v = \frac{35}{48}$. It can be seen that $0 < \zeta < 1$ and $0 < \mu < 1$ and calculations show

$$\begin{aligned} &8 \times \frac{5}{8}(1-\frac{5}{8})[(\frac{5}{8} \times \frac{35}{48} - \frac{311}{768})^2 + (\frac{5}{8}(\frac{5}{8} + \frac{5}{8}) - \frac{35}{48})^2] + \frac{5}{8}(1-\frac{5}{8})(\frac{35}{48} - \frac{5}{4} \times \frac{5}{8})^2 \\ &= \frac{49815}{4718592} = 0.0105572\ldots \leq 4(\frac{5}{8})^2(\frac{3}{8})^2 \times \frac{5}{8} \times \frac{3}{8} = \frac{3375}{65536} = 0.051498\ldots. \end{aligned}$$

Then, by Lemma 3, we have

$$|d_5| \leq \frac{1}{10}.$$

□

Theorem 2. *If $g \in \mathcal{R}_{LP}$, then*

$$|H_{2,1}| \leq \frac{1}{6}.$$

The bound is sharp. The sharpness of the inequality can be obtained by the $g_2(\varepsilon)$ given in (15).

Proof. Let $g \in \mathcal{R}_{LP}$. From (21), (22), and (2), we get

$$|d_3 - d_2^2| = \frac{1}{12}|p_2 - \frac{13}{16}p_1^2|. \tag{25}$$

From Lemma 1, we have

$$|H_{2,1}| \leq \frac{1}{6}.$$

□

Theorem 3. *If* $g \in \mathcal{R}_{LP}$, *then*

$$|H_{2,2}| \leq \frac{1}{36}.$$

The bound is sharp. The sharpness of the inequality can be obtained by (15).

Proof. Let $g \in \mathcal{R}_{LP}$. From (21), (22), (23), and (3), we have

$$|d_2d_4 - d_3^2| = \frac{1}{36864}|288p_1p_3 - 40p_1^2p_2 + 5p_1^4 - 256p_2^2|.$$

Using Lemma 4, we have

$$|d_2d_4 - d_3^2| = \frac{1}{36864}|-7p_1^4 - 4p_1^2(4 - P_1^2)\xi - 72p_1^2(4 - p_1^2)\xi^2 + 144p_1(4 - p_1^2)(1 - |\xi|^2)t - 64(4 - p_1^2)^2\xi^2|.$$

Let $|p_1| = p \in [0,2]$, $|\xi| = s \in [0,1]$ and $|t| \leq 1$, and by using triangular inequality, we yield

$$\begin{aligned}|d_2d_4 - d_3^2| &\leq \frac{1}{36864}[7p^4 + 4p^2(4 - P^2)s + 72p^2(4 - p^2)s^2 + 144p(4 - p^2)(1 - s^2) + 64(4 - p^2)^2s^2] \\ &= \frac{1}{36864}\alpha(p,s).\end{aligned}$$

For $s \in [0,1]$, $p \in [0,2]$, we get

$$\frac{\partial \alpha}{\partial s} = 4p^2(4 - p^2) + 16(4 - p^2)(p^2 - 18p + 32)s \geq 0,$$

which shows that the maximum of $\alpha(p,s)$ occurs at $s = 1$. Therefore,

$$|d_2d_4 - d_3^2| \leq \frac{1}{36864}\alpha_(p,1) = \frac{1}{36864}(-5p^4 - 208p^2 + 1024) \leq \frac{1024}{36864} = \frac{1}{36}.$$

□

Theorem 4. *If* $g \in \mathcal{R}_{LP}$, *then*

$$|H_{2,3}(g)| = |d_5d_3 - d_4^2| \leq \frac{1}{64}$$

The bound is sharp. The sharpness of the inequality can be obtained by (16).

Proof. Let $g \in \mathcal{R}_{LP}$. From (4), (22), (23), and (24), we achieve

$$\begin{aligned}H_{2,3}(g) = \frac{1}{11796480}&(-1552p_1^4p_2 + 4800p_1^3p_3 + 95p_1^6 + 960p_1^2p_2^2 + 49152p_2p_4 + 53760p_1p_2p_3 \\ &-30720p_1^2p_4 - 30720p_2^3 - 46080p_3^2).\end{aligned} \tag{26}$$

Let $\chi = 4 - p^2$, $p_1 = p \in [0,2]$ in (11), (12), and (13), we have

$$
\begin{aligned}
-1552p^4p_2 &= -776p^6 - 776p^4\chi\xi,\\
4800p^3p_3 &= 1200p^6 + 2400p^4\chi\xi - 1200p^4\chi\xi^2 + 2400p^3\chi\left(1-|\xi|^2\right)t,\\
960p^2p_2^2 &= 240p^6 + 480p^4\chi\xi + 240p^2\chi^2\xi^2,\\
-30720p^2p_4 &= -3840p^4\chi\xi^3 + 15360p^2\chi\bar{\xi}t^2\left(1-|\xi|^2\right) + 15360p^3\chi\xi\left(1-|\xi|^2\right)t + 11520p^4\chi\xi^2\\
&\quad -15360p^2\chi\left(1-|\xi|^2\right)\left(1-|t|^2\right)\varrho - 15360p^3\chi\left(1-|\xi|^2\right)t - 11520p^4\chi\xi\\
&\quad -3840p^6 - 15360p^2\chi\xi^2,\\
53760pp_2p_3 &= -6720p^2\chi^2\xi^3 - 6720p^4\chi\xi^2 + 13440p\xi\chi^2\left(1-|\xi|^2\right)t + 13440p^2\chi^2\xi^2 + 13440p^3\chi\left(1-|\xi|^2\right)t\\
&+ 6720p^6 + 20160p^4\chi\xi,\\
-30720p_2^3 &= -3840\chi^3\xi^3 - 11520p^2\chi^2\xi^2 - 11520p^4\chi\xi - 3840p^6,\\
49152p_2p_4 &= 3072p^6 + 3072p^4\chi\xi^3 - 9216p^4\chi\xi^2 + 12288p^4\chi\xi + 12288p^2\chi\xi^2 - 12288p^3\chi\xi(1-|\xi|^2)t\\
&\quad +12288p^3\chi(1-|\xi|^2)t - 12288p^2\chi(1-|\xi|^2)\bar{\xi}t^2 + 12288p^2\chi(1-|\xi|^2)(1-|t|^2)\varrho\\
&\quad +3072p^2\chi^2\xi^4 - 9216p^2\chi^2\xi^3 + 9216p^2\chi^2\xi^2 + 12288\chi^2\xi^3 + 12288p\chi^2\xi t(1-|\xi|^2)\\
&\quad -12288\chi^2(1-|\xi|^2)\xi\bar{\xi}t^2 + 12288\chi^2\xi(1-|\xi|^2)(1-|t|^2)\varrho - 12288p\chi^2\xi^2t(1-|\xi|^2),\\
-46080p_3^2 &= -2880p^6 - 11520p^4\chi\xi - 11520p^2\chi^2\xi^2 + 5760p^4\chi\xi^2 - 11520p^3\chi(1-|\xi|^2)t + 11520p^2\chi^2\xi^3\\
&\quad -23040p\xi\chi^2(1-|\xi|^2)t - 2880p^2\chi^2\xi^4 + 11520p\chi^2\xi^2(1-|\xi|^2)t - 11520\chi^2t^2(1-|\xi|^2)^2.
\end{aligned}
$$

Substituting these expressions into (26), we obtain

$$
\begin{aligned}
H_{2,3}(g) &= \frac{1}{11796480}\Big[-9p^6 + 12288\chi^2\xi^3 + 192p^2\chi^2\xi^4 + 144p^4\chi\xi^2 - 144p^2\chi^2\xi^2 - 4416p^2\chi^2\xi^3 - 768p^4\chi\xi^3\\
&\quad -3072p^2\chi\xi^2 - 3840\chi^3\xi^3 + 2688p\chi^2\xi(1-|\xi|^2)t + 3072p^3\xi(1-|\xi|^2)\chi t + 3072p^2(1-|\xi|^2)\chi\bar{\xi}t^2\\
&\quad -3072p^2(1-|\xi|^2)(1-|t|^2)\chi\varrho - 768p\chi^2\xi^2(1-|\xi|^2)t - 12288\chi^2(1-|\xi|^2)\xi\bar{\xi}t^2 - 8\chi\xi p^4\\
&\quad +12288\chi^2\xi(1-|\xi|^2)(1-|t|^2)\varrho - 11520\chi^2t^2(1-|\xi|^2)^2 + 1248\chi p^3(1-|\xi|^2)t\Big].
\end{aligned}
$$

Thus, we achieve

$$
H_{2,3}(g) = \frac{1}{11796480}\left(\bigvee_1(p,\xi) + \bigvee_2(p,\xi)t + \bigvee_3(p,\xi)t^2 + \Upsilon_1(p,\xi,t)\varrho\right),
$$

where

$$
\begin{aligned}
\bigvee_1(p,\xi) &= -9p^6 + \left(4-p^2\right)\Big[\left(4-p^2\right)\left(-144p^2\xi^2 - 576p^2\xi^3 - 3072\xi^3 + 192p^2\xi^4\right) - 8p^4\xi - 768p^4\xi^3\\
&\quad +16p^2(9p^2-192)\xi^2\Big],\\
\bigvee_2(p,\xi) &= \left(4-p^2\right)\left(1-|\xi|^2\right)\Big[\left(2688p\xi - 768p\xi^2\right)\left(4-p^2\right) + 1248p^3 + 3072p^3\xi\Big],\\
\bigvee_3(p,\xi) &= \left(4-p^2\right)\left(1-|\xi|^2\right)\Big[\left(-768|\xi|^2 - 11520\right)\left(4-p^2\right) + 3072\bar{\xi}p^2\Big],\\
\Upsilon_1(p,\xi,t) &= \left(4-p^2\right)\left(1-|\xi|^2\right)\left(1-|t|^2\right)\Big[12288\xi\left(4-p^2\right) - 3072p^2\Big].
\end{aligned}
$$

Let $|p| = p$ $|\xi| = s$, $|t| = t$ and $|\rho| \leq 1$, then

$$\begin{aligned} |H_{3,1}(g)| &\leq \frac{1}{11796480}\left(\left|\bigvee_1(p,\xi)\right| + \left|\bigvee_2(p,\xi)\right|t + \left|\bigvee_3(p,\xi)\right|t^2 + |\Upsilon_1(p,\xi,t)|\right) \\ &\leq \frac{1}{11796480}\Gamma_1(p,s,t), \end{aligned} \tag{27}$$

where

$$\Gamma_1(p,s,t) = \tau_1(p,s) + \tau_2(p,s)t + \tau_3(p,s)t^2 + \tau_4(p,s)(1-t^2), \tag{28}$$

with

$$\begin{aligned} \tau_1(p,s) &= 9p^6 + \left(4-p^2\right)\Big[\left(4-p^2\right)\left(144p^2s^2 + 576p^2s^3 + 3072s^3 + 192p^2s^4\right) + 8p^4s + 768p^4s^3 \\ &\quad +16p^2(192-9p^2)s^2\Big], \\ \tau_2(p,s) &= \left(4-p^2\right)\left(1-s^2\right)\Big[\left(2688ps + 768ps^2\right)\left(4-p^2\right) + 1248p^3 + 3072p^3s\Big], \\ \tau_3(p,s) &= \left(1-s^2\right)\left(4-p^2\right)\Big[\left(768s^2 + 11520\right)\left(4-p^2\right) + 3072p^2s\Big] \end{aligned}$$

and

$$\tau_4(p,s) = \left(4-p^2\right)\left(1-s^2\right)\Big[12288s\left(4-p^2\right) + 3072p^2\Big].$$

In light of $(p,s) \in [0,2] \times [0,1]$, we observe that

$$\tau_3(p,s) \leq (4-p^2)(1-s^2)[(4-p^2)(11520 + 768s^2) + 3072p^2] = \varsigma_3(p,s).$$

Taking $\varsigma_j(p,s) = \tau_j(p,s)$ for $j = 1,2,4$ and

$$\Xi_1(p,s,t) = \varsigma_1(p,s) + \varsigma_2(p,s)t + \varsigma_3(p,s)t^2 + \varsigma_4(p,s)(1-t^2).$$

By partially differentiating $\Xi_1(p,s,t)$ with respect to t, we yield

$$\begin{aligned} \frac{\partial \Xi_1}{\partial t} &= \varsigma_2(p,s) + 2(\varsigma_3(p,s) - \varsigma_4(p,s))t \\ &= \varsigma_2(p,s) + 1536t(4-p^2)^2(1-s^2)(s^2 - 16s + 15) \geq 0, \end{aligned}$$

which confirms that $\Xi_1(p,s,t)$ is a decreasing function for $t \in [0,1]$. Therefore,

$$\begin{aligned} \Gamma_1(p,s,t) &\leq \Xi_1(p,s,t) \leq \Xi_1(p,s,1) = \varsigma_1(p,s) + \varsigma_2(p,s) + \varsigma_3(p,s) \\ &= 184320 - 79872p^2 + 4992p^3 + 8448p^4 - 1248p^5 + 9p^6 + (43008p - 9216p^3 + 32p^4 - 384p^5 \\ &\quad -8p^6)s + (-184320 + 12288p + 106752p^2 - 11136p^3 - 19392p^4 + 2016p^5 + 1056p^6)s^2 \\ &\quad +(49152 - 43008p - 15360p^2 + 9216p^3 + 1536p^4 + 384p^5 - 192p^6)s^3 + (-12288p - 9216p^2 \\ &\quad +6144p^3 + 4608p^4 - 768p^5 - 576p^6)s^4 = \Delta_1(p,s). \end{aligned}$$

The optimal points of Δ_1 satisfy

$$\begin{cases} \frac{\partial \Delta_1}{\partial p} = -159744p + 14976p^2 + 33792p^3 - 6240p^4 + 54p^5 + (43008 - 27648p^2 + 128p^3 - 1920p^4 - 48p^5)s \\ +(12288 + 213504p - 33408p^2 - 77568p^3 + 10080p^4 + 6336p^5)s^2 + (-43008 - 30720p + 27648p^2 + 6144p^3 \\ +1920p^4 - 1152p^5)s^3 + (-12288 - 18432p + 18432p^2 + 18432p^3 - 3840p^4 - 3456p^5)s^4 = 0, \\ \frac{\partial \Delta_1}{\partial s} = -8p^6 - 384p^5 + 32p^4 - 9216p^3 + 43008p + (-368640 + 24576p + 213504p^2 - 22272p^3 - 38784p^4 \\ +4032p^5 + 2112p^6)s + (147456 - 129024p - 46080p^2 + 27648p^3 + 4608p^4 + 1152p^5 - 576p^6)s^2 \\ +(-49152p - 36864p^2 + 24576p^3 + 18432p^4 - 3072p^5 - 2304p^6)s^3 = 0. \end{cases}$$

After computations, we get

$$\begin{cases} p_1 = 2, \\ s_1 = 1.3830\ldots, \end{cases} \begin{cases} p_2 = 2, \\ s_2 = -1.5582\ldots, \end{cases} \begin{cases} p_3 = 2, \\ s_3 = -0.8248\ldots, \end{cases} \begin{cases} p_4 = -2, \\ s_4 = 0.6692\ldots, \end{cases} \begin{cases} p_5 = -2, \\ s_5 = -0.9569\ldots, \end{cases}$$

$$\begin{cases} p_6 = -2, \\ s_6 = 0.0793\ldots, \end{cases} \begin{cases} p_7 = 0, \\ s_7 = 0, \end{cases} \begin{cases} p_8 = 5.0639\ldots, \\ s_8 = 0.0931\ldots, \end{cases} \begin{cases} p_9 = 110.1957\ldots, \\ s_9 = 0.0054\ldots, \end{cases} \begin{cases} p_{10} = -0.4967\ldots, \\ s_{10} = -16.1109\ldots, \end{cases}$$

$$\begin{cases} p_{11} = -1.0196\ldots, \\ s_{11} = 1.0578\ldots, \end{cases} \begin{cases} p_{12} = -1.7382\ldots, \\ s_{12} = 61.2047\ldots, \end{cases} \begin{cases} p_{13} = -1.9518\ldots, \\ s_{13} = 0.4220\ldots, \end{cases} \begin{cases} p_{14} = -2.1965\ldots, \\ s_{14} = -0.6427\ldots, \end{cases}$$

$$\begin{cases} p_{15} = -3.1462\ldots, \\ s_{15} = -1.4645\ldots. \end{cases}$$

(1) For $p = 2$,

$$\Delta_1(2, s) = 576.$$

(2) For $p = 0$,

$$\Delta_1(0, s) = 184320 - 184320s^2 + 49152s^3 \leq 184320.$$

(3) For $s = 1$,

$$\Delta_1(p, 1) = 49152 + 2304p^2 - 4768p^4 + 289p^6 = \delta_1(p) \leq \delta_1(0.4972\ldots) = 49435.$$

(4) For $s = 0$,

$$\Delta_1(p, 0) = 9p^6 - 1248p^5 + 8448p^4 + 4992p^3 - 79872p^2 + 184320 \leq 184320.$$

Thus, we have

$$|H_{2,3}| \leq \frac{184320}{11796480} = \frac{1}{64}.$$

□

Theorem 5. *If $g \in \mathcal{R}_{LP}$, then*

$$|H_{3,1}(g)| \leq \frac{1}{64}.$$

The bound is sharp. The sharpness of the inequality can be obtained by the $g_3(\varepsilon)$ given in (16).

Proof. Let $g \in \mathcal{R}_{LP}$. From (5), (21), (22), (23), and (24), we get

$$\begin{aligned} H_{3,1} &= \frac{1}{35389440}(207360p_1p_2p_3 + 960p_1^2p_2^2 - 6096p_1^4p_2 - 138240p_3^2 + 20160p_1^3p_3 - 112640p_2^3 \\ &\quad -119808p_1^2p_4 + 383p_1^6 + 147456p_2p_4). \end{aligned} \tag{29}$$

Let $p_1 = p \in [0,2]$ $\chi = 4 - p^2$ in (11), (12), and (13), we achieve

$$\begin{aligned} -6096p^4p_2 &= -3048p^6 - 3048p^4\xi\chi, \\ 20160p^3p_3 &= 5040p^6 + 10080\xi p^4\chi - 5040p^4\xi^2\chi + 10080p^3\chi t\left(1 - |\xi|^2\right), \\ 960p^2p_2^2 &= 480p^4\xi\chi + 240p^6 + 240p^2\xi^2\chi^2, \\ -119808p^2p_4 &= -14976p^4\chi\xi^3 + 59904\chi p^2\bar{\xi}\left(1 - |\xi|^2\right)t^2 + 59904\xi p^3\chi\left(1 - |\xi|^2\right)t + 44928p^4\xi^2\chi \\ &\quad -59904\chi p^2\left(1 - |t|^2\right)\left(1 - |\xi|^2\right)\varrho - 15360p^3\chi\left(1 - |\xi|^2\right)t - 44928\xi p^4\chi \\ &\quad -14976p^6 - 59904\chi p^2\xi^2, \\ 207360pp_2p_3 &= -25920\xi^3p^2\chi^2 - 25920p^4\xi^2\chi + 51840\chi^2p\xi\left(1 - |\xi|^2\right)t + 51840\xi^2\chi^2p^2 + 51840\chi p^3t\left(1 - |\xi|^2\right) \\ &+ 25920p^6 + 77760\xi p^4\chi, \\ -112640p_2^3 &= -14080\xi^3\chi^3 - 42240\xi^2p^2\chi^2 - 14080p^6 - 42240p^4\chi\xi, \\ 147456p_2p_4 &= 9216\chi p^4\xi^3 - 27648p^4\xi^2\chi + 36864\chi p^4\xi + 36864\chi p^2\xi^2 + 9216p^6 - 36864p^3\xi\chi(1 - |\xi|^2)t \\ &\quad +36864\chi p^3(1 - |\xi|^2)t - 36864\chi p^2(1 - |\xi|^2)\bar{\xi}t^2 + 36864p^2\chi(1 - |\xi|^2)(1 - |t|^2)\varrho \\ &\quad +9216p^2\xi^4\chi^2 - 27648\chi^2p^2\xi^3 + 27648\xi^2p^2\chi^2 + 36864\chi^2\xi^3 + 36864p\xi\chi^2t(1 - |\xi|^2) \\ &\quad -36864(1 - |\xi|^2)\xi\bar{\xi}\chi^2t^2 + 36864\chi^2(1 - |\xi|^2)(1 - |t|^2)\xi\varrho - 36864\chi^2p\xi^2t(1 - |\xi|^2), \\ -138240p_3^2 &= -34560\xi p^4\chi - 34560\chi^2p^2\xi^2 + 17280p^4\chi\xi^2 - 34560p^3(1 - |\xi|^2)\chi t - 8640p^6 + 34560p^2\xi^3\chi^2 \\ &\quad -69120\chi^2\xi p(1 - |\xi|^2)t - 8640p^2\xi^4\chi^2 + 34560\chi^2p\xi^2(1 - |\xi|^2)t - 34560t^2\chi^2(1 - |\xi|^2)^2. \end{aligned}$$

Substituting these expressions into (26), we obtain

$$\begin{aligned} H_{3,1}(g) &= \frac{1}{35389440}\Big[576p^2\chi^2\xi^4 + 55p^6 + 36864\chi^2\xi^3 + 3600\chi p^4\xi^2 + 2928\chi^2\xi^2p^2 - 19008\xi^3p^2\chi^2 - 5760\chi p^4\xi^3 \\ &\quad -23040p^2\xi^2\chi - 14080\chi^3\xi^3 + 19584\chi^2p(1 - |\xi|^2)\xi t + 23040(1 - |\xi|^2)\xi p^3\chi t + 23040p^2(1 - |\xi|^2)\bar{\xi}\chi t^2 \\ &\quad -23040p^2(1 - |t|^2)\chi(1 - |\xi|^2)\varrho - 23040p\chi^2t\xi^2(1 - |\xi|^2) - 36864\xi\bar{\xi}\chi^2(1 - |\xi|^2)t^2 + 408\xi p^4\chi \\ &\quad +36864\chi^2(1 - |t|^2)\xi(1 - |\xi|^2)\varrho - 34560\chi^2(1 - |\xi|^2)^2t^2 + 4320p^3\chi(1 - |\xi|^2)t\Big]. \end{aligned}$$

It is seen that we can write $H_{3,1}(g)$ in the form of

$$H_{3,1}(g) = \frac{1}{35389440}\Big[\bigvee_4(p,\xi) + \bigvee_5(p,\xi)t + \bigvee_6(p,\xi)t^2 + \Upsilon_2(p,\xi,t)\varrho\Big],$$

where $\xi, t, \varrho \in \bar{\mathbb{U}}$, and

$$\begin{aligned} \bigvee_4(p,\xi) &= 55p^6 + (4 - p^2)\Big[(576p^2\xi^4 - 19456\xi^3 - 4928p^2\xi^3 + 2928p^2\xi^2)(4 - p^2) + 408p^4\xi - 5760p^4\xi^3 \\ &\quad +720p^2(5p^2 - 32)\xi^2\Big] \end{aligned}$$

$$\begin{aligned}
\bigvee_{5}(p,\xi) &= \left(4-p^2\right)\left(1-|\xi|^2\right)\Big[\left(19584p\xi-2304\xi^2p\right)\left(4-p^2\right)+4320p^3+23040p^3\xi\Big],\\
\bigvee_{6}(p,\xi) &= \left(4-p^2\right)\left(1-|\xi|^2\right)\Big[\left(-2304|\xi|^2-34560\right)\left(4-p^2\right)+23040\bar{\xi}p^2\Big],\\
\Upsilon_2(p,\xi,t) &= \left(1-|t|^2\right)\left(4-p^2\right)\left(1-|\xi|^2\right)\Big[36864\left(4-p^2\right)\xi-23040p^2\Big].
\end{aligned}$$

Let $|t|=t$, $|\xi|=s$ and $|\rho|\leq 1$, then we have

$$\begin{aligned}
|H_{3,1}(g)| &\leq \frac{1}{35389440}\left(\left|\bigvee_{4}(p,\xi)\right|+\left|\bigvee_{5}(p,\xi)\right|t+\left|\bigvee_{6}(p,\xi)\right|t^2+|\Upsilon_2(p,\xi,t)|\right)\\
&\leq \frac{1}{35389440}\Gamma_2(p,s,t), \qquad (30)
\end{aligned}$$

where

$$\Gamma_2(p,s,t)=\tau_5(p,s)+\tau_6(p,s)t+\tau_7(p,s)t^2+\tau_8(p,s)(1-t^2), \qquad (31)$$

with

$$\begin{aligned}
\tau_5(p,s) &= 55p^6+\left(4-p^2\right)\Big[\left(4-p^2\right)\left(2928s^2p^2+4928s^3p^2+19456s^3+576s^4p^2\right)+408p^4s+5760p^4s^3\\
&\quad +720p^2(32-5p^2)s^2\Big],\\
\tau_6(p,s) &= \left(4-p^2\right)\left(1-s^2\right)\Big[\left(19584sp+2304ps^2\right)\left(4-p^2\right)+4320p^3+23040sp^3\Big],\\
\tau_7(p,s) &= \left(1-s^2\right)\left(4-p^2\right)\Big[\left(2304s^2+34560\right)\left(4-p^2\right)+23040sp^2\Big]
\end{aligned}$$

and

$$\tau_8(p,s) = \left(1-s^2\right)\left(4-p^2\right)\Big[36864s\left(4-p^2\right)+23040p^2\Big].$$

In light of $(p,s)\in[0,2]\times[0,1]$, we observe that

$$\tau_7(p,s)\leq(4-p^2)(1-s^2)[(4-p^2)(34560+2304s^2)+23040p^2]=\lambda_7(p,s).$$

Taking $\lambda_j(p,s)=\tau_j(p,s)$ for $j=5,6,8$ and

$$\Xi_2(p,s,t)=\lambda_5(p,s)+\lambda_6(p,s)t+\lambda_7(p,s)t^2+\lambda_8(p,s)(1-t^2).$$

By partially differentiating $\Xi_2(p,s,t)$ with respect to t, we have

$$\begin{aligned}
\frac{\partial\Xi_2}{\partial t} &= \lambda_6(p,s)+2(\lambda_7(p,s)-\lambda_8(p,s))t\\
&= \lambda_6(p,s)+4608(s^2-16s+15)(4-p^2)^2(1-s^2)t\geq 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
\Gamma_2(p,s,t) &\leq \Xi_2(p,s,t)\leq\Xi_2(p,s,1)=\lambda_5(p,s)+\lambda_6(p,s)+\lambda_7(p,s)\\
&= 552960-184320p^2+17280p^3+11520p^4-4320p^5+55p^6+(313344p-64512p^3+1632p^4\\
&- 3456p^5-408p^6)s+(-516096+36864p+304896p^2-35712p^3-70080p^4+6624p^5+6528p^6)s^2\\
&+ (311296-313344p-76800p^2+64512p^3+3072p^4+3456p^5-832p^6)s^3+(-36864-36864p\\
&+ 27648p^2+18432p^3-6912p^4-2304p^5+576p^6)s^4=\Delta_2(p,s).
\end{aligned}$$

The optimal points of Δ_2 satisfy

$$\begin{cases} \frac{\partial \Delta_2}{\partial p} = -368640p + 51840p^2 + 46080p^3 - 21600p^4 + 330p^5 + (313344 - 193536p^2 + 6528p^3 - 17280p^4 - 2448p^5)s \\ +(36864 + 609792p - 107136p^2 - 280320p^3 + 33120p^4 + 39168p^5)s^2 + (-313344 - 153600p + 193536p^2 + 12288p^3 \\ +17280p^4 - 4992p^5)s^3 + (-36864 + 55296p + 55296p^2 - 27648p^3 - 11520p^4 + 3456p^5)s^4 = 0, \\ \frac{\partial \Delta_2}{\partial s} = -408p^6 - 3456p^5 + 1632p^4 - 64512p^3 + 313344p + (-1032192 + 73728p + 609792p^2 - 71424p^3 \\ -140160p^4 + 13248p^5 + 13056p^6)s + (933888 - 940032p - 230400p^2 + 193536p^3 + 9216p^4 + 10368p^5 - 2496p^6)s^2 \\ +(-147456 - 147456p + 110592p^2 + 73728p^3 - 27648p^4 - 9216p^5 + 2304p^6)s^3 = 0. \end{cases}$$

After computations, we get

$$\begin{cases} p_1 = 2, \\ s_1 = 1.3264\ldots, \end{cases} \begin{cases} p_2 = 2, \\ s_2 = -1.7446\ldots, \end{cases} \begin{cases} p_3 = 2, \\ s_3 = -0.5818\ldots, \end{cases} \begin{cases} p_4 = -2, \\ s_4 = 0.6031\ldots, \end{cases} \begin{cases} p_5 = -2, \\ s_5 = -0.8901\ldots, \end{cases}$$

$$\begin{cases} p_6 = -2, \\ s_6 = 0.3703\ldots, \end{cases} \begin{cases} p_7 = 0, \\ s_7 = 0, \end{cases} \begin{cases} p_8 = 2.5990\ldots, \\ s_8 = 9.9401\ldots, \end{cases} \begin{cases} p_9 = 73.7241\ldots, \\ s_9 = 0.0346\ldots, \end{cases} \begin{cases} p_{10} = -0.3492\ldots, \\ s_{10} = 0.9567\ldots, \end{cases}$$

$$\begin{cases} p_{11} = -1.4612\ldots, \\ s_{11} = -36.5653\ldots, \end{cases} \begin{cases} p_{12} = -1.9795\ldots, \\ s_{12} = 0.4924\ldots, \end{cases} \begin{cases} p_{13} = -2.5160\ldots, \\ s_{13} = -0.4322\ldots. \end{cases}$$

(5) For $p = 2$,

$$\Delta_2(2, s) = 3520.$$

(6) For $p = 0$,

$$\Delta_2(0, s) = 552960 - 516096s^2 + 311296s^3 - 36864s^4 \leq 552960.$$

(7) For $s = 1$,

$$\Delta_2(p, 1) = 311296 + 71424p^2 - 60768p^4 + 5919p^6 = \delta_2(p) \leq \delta_2(0.8058\ldots) = 333670.$$

(8) For $s = 0$,

$$\Delta_2(p, 0) = 55p^6 - 4320p^5 + 11520p^4 + 17280p^3 - 184320p^2 + 552960 \leq 552960.$$

Thus, we have

$$|H_{3,1}| \leq \frac{552960}{35389440} = \frac{1}{64}.$$

□

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References

1. Khan, M.G.; Mashwani, W.K.; Shi, L.; Araci, S.; Ahmad, B.; Khan, B. Hankel inequalities for bounded turning functions in the domain of cosine Hyperbolic function. *AIMS Math.* **2023**, *8*, 21993–22008. [CrossRef]
2. Wang, Z.G.; Arif, M.; Liu, Z.H.; Zainab, S.; Fayyaz, R.; Ihsan, M.; Shutaywi, M. Sharp Bounds On Hankel Determinants for Certain Subclass of Starlike Functions. *J. Appl. Anal. Comput.* **2022**, *13*, 860–873. [CrossRef] [PubMed]
3. Pommerenke, C. On the coefficients and Hankel determinants of univalent functions. *J. Lond. Math. Soc.* **1966**, *1*, 111–122. [CrossRef]
4. Pommerenke, C. On the Hankel determinants of univalent functions. *Mathematika* **1967**, *14*, 108–112. [CrossRef]
5. Răducanu, D. On Coefficient Estimates for a Certain Class of Analytic Functions. *Mathematics* **2022**, *11*, 12. [CrossRef]
6. Orhan, H.; Toklu, E.; Kadıoğlu, E. Second Hankel determinant for certain subclasses of bi-univalent functions involving Chebyshev polynomials. *Turk. J. Math.* **2018**, *42*, 1927–1940. [CrossRef]
7. Tang, H.; Arif, M.; Abbas, M.; Tawfiq, F.M.; Malik, S.N. Analysis of Coefficient-Related Problems for Starlike Functions with Symmetric Points Connected with a Three-Leaf-Shaped Domain. *Symmetry* **2023**, *15*, 1837. [CrossRef]
8. Kowalczyk, B.; Lecko, A.; Sim, Y.J. The sharp bound for the Hankel determinant of the third kind for convex functions. *Aust. Math. Soc.* **2018**, *97*, 435–445. [CrossRef]
9. Wen, C.; Li, Z.; Guo, D. Some Results on Coefficient Estimate Problems for Four-Leaf-Type Bounded Turning Functions. *Mathematics* **2024**, *12*, 1875. [CrossRef]
10. Liu, D.; Ahmad, A.; Ikhlas, H.; Hussain, S.; Noor, S.; Tang, H. On Sharp Coefficients and Hankel Determinants for a Novel Class of Analytic Functions. *Axioms* **2025**, *14*, 191. [CrossRef]
11. Pommerenke, C. *Univalent Functions*; Vandenhoeck and Ruprecht: Gottingen, Germany, 1975.
12. Libera, R.J.; Zlotkiewicz, E. Coefficient bounds for the inverse of a function with derivative in *P*. *Proc. Am. Math. Soc.* **1983**, *87*, 251–257. [CrossRef]
13. Ravichandran, V.; Verma, V.S. Bound for the fifth coefficient of certain starlike functions. *Comptes Rendus Math.* **2015**, *353*, 505–510. [CrossRef]
14. Rlibera, R.J.; Zlotkiewicz, E.J. Early coefficients of the inverse of a regular convex function. *Proc. Am. Math. Soc.* **1982**, *85*, 225–230. [CrossRef]
15. Kwon, O.S.; Lecho, A.; Sim, Y.J. On the fourth coefficient of functions in the Carathéodory class. *Comput. Methods Funct. Theory* **2018**, *18*, 307–314. [CrossRef]

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