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An Exhaustive Analysis of the OR-Product of Soft Sets: A Symmetry Perspective

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Abstract

This paper provides a theoretical investigation of the OR-product ($\vee$-product) in soft set theory, an operation of central importance for handling uncertainty in decision-making. A comprehensive algebraic analysis is carried out with respect to various types of subsets and equalities, with particular emphasis on M-subset and M-equality, which represent the strictest forms of subsethood and equality. This framework reveals intrinsic algebraic symmetries, particularly in commutativity, associativity, and idempotency, which enrich the structural understanding of soft set theory. In addition, certain missing results on OR-products in the literature are completed, and our findings are systematically compared with existing ones, ensuring a more rigorous theoretical framework. A central contribution of this study is the demonstration that the collection of all soft sets over a universe, equipped with a restricted/extended intersection and the OR-product, forms a commutative hemiring with identity under soft L-equality. This structural result situates the OR-product within one of the most fundamental algebraic frameworks, connecting soft set theory with broader areas of algebra. To illustrate its practical relevance, the int-uni decision-making method on the OR-product is applied to a pilot recruitment case, showing how theoretical insights can support fair and transparent multi-criteria decision-making under uncertainty. From an applied perspective, these findings embody a form of symmetry in decision-making, ensuring fairness and balanced evaluation among multiple decision-makers. By bridging abstract algebraic development with concrete decision-making applications, the results affirm the dual significance of the OR-product—strengthening the theoretical framework of soft set theory while also providing a viable methodology for applied decision-making contexts.

Keywords: soft set; OR-product; soft subsets; soft equal relations; hemiring

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1. Introduction

Most of the problems encountered in everyday life are vague rather than precise. In recent years, scientists and engineers have shown increasing interest in modeling vagueness, as many problems in economics, engineering, environmental science, social science, and medicine involve data containing various forms of uncertainty. To address such challenges, several theoretical frameworks have been developed, including Probability Theory, Fuzzy Set Theory [1], Intuitionistic Fuzzy Set Theory, Vague Set Theory, Interval Mathematics, and Rough Set Theory. Molodtsov [2] observed in 1999 that these frameworks have inherent limitations. He further noted that these limitations may arise from the inadequacy of the parameterization tools employed in these theories.

In this context, Molodtsov's Soft Set ($\mathcal{SS}$) Theory is markedly distinct from these other approaches. Because it imposes no restrictions on approximate descriptions, $\mathcal{SS}$ is highly useful, broadly applicable, and adaptable. Following Maji et al. [3], who applied $\mathcal{SS}$ theory to a decision-making problem, the theory has been extensively employed in addressing real-life decision-making problems involving uncertainty [4–19]. Since then, $\mathcal{SS}$ theory and its various extensions have attracted considerable attention and have been widely and effectively applied to decision-making problems [19–33].

Research on $\mathcal{SS}$ theory has progressed along two main directions: its application to real-life decision-making problems and the development of its theoretical foundations. In this regard, numerous scholars have investigated the principles of $\mathcal{SS}$ theory in recent years. Maji et al. [34] conducted a comprehensive theoretical study of $\mathcal{SS}$s, including soft subsets, soft equality, and $\mathcal{SS}$ operations. Pei and Miao [35] redefined certain concepts within $\mathcal{SS}$ theory, while several fundamental properties were further emphasized in [36–38]. Ali et al. [39] introduced novel operations on $\mathcal{SS}$s. Subsequent studies [40–54] identified various conceptual misconceptions in the literature and proposed improved methods to address them. Restricted and extended intersection, union, and difference—key operations in $\mathcal{SS}$ theory—were systematically redefined and studied in detail in [55–57], where earlier contributions were integrated, missing results were completed, and corrected theorems were provided. These works [55–57] significantly advanced the theory by filling a critical gap, serving as a valuable resource for newcomers to the field, and offering insights for future research on $\mathcal{SS}$s. As a result, research on $\mathcal{SS}$s has substantially expanded in recent years [58–72].

Soft subsets and soft equal relations are fundamental concepts within the framework of $\mathcal{SS}$ theory. Maji et al. [34] introduced a rigorous formulation of soft subsets, which was later extended by Pei and Miao [35] and Feng et al. [37]. Qin and Hong [73] developed congruence relations on $\mathcal{SS}$s, thereby enriching the structural foundations of the theory. Jun and Yang [74] explored a broader class of soft subsets, which we refer to as J-soft equal relations, a terminology adopted here for consistency. Building upon this line of research, Liu et al. [75] provided a concise yet influential note inspired by the innovations of Jun and Yang. Feng and Li [76] further advanced this work by examining soft subsets in relation to soft product operations. Unlike the preliminary study in [75], Feng and Li [76] focused on diverse forms of soft subsets and conducted a detailed algebraic analysis of soft product operations. Their work offered a systematic theoretical treatment of the AND-product and OR-product, filling gaps in the literature and clarifying their algebraic properties with respect to specific notions of soft subsethood and equality. Additional progress on soft subsets and equal relations can be found in [77–81]. In particular, Ali et al. [81] introduced generalized finite relaxed soft equality and generalized finite relaxed unions and intersections, thereby providing a more robust and comprehensive framework for generalization. Complementing these advances, Chen et al. [69] proposed a novel ES-structure for $\mathcal{SS}$s based on soft J-subsets, which resolved deficiencies in earlier structures and yielded an entirely new algebraic system of $\mathcal{SS}$s with the structure of a distributive lattice.

Çağman and Enginoğlu [8] introduced four distinct types of products in soft set theory: the AND-product, OR-product, AND–NOT-product, and OR–NOR-product. They also proposed uni–int operators and a uni–int decision function for the AND-product, thereby formulating a uni–int decision-making method that reduces a set to its relevant subset based on the parameters specified by decision-makers. Feng et al. [10] extended this approach by introducing newly defined notions and developing several innovative schemes and algorithms to address soft decision-making problems more effectively. From a different perspective, Sezgin et al. [82] undertook a rigorous theoretical investigation of the AND-product ($\wedge$-product), analyzing its complete algebraic properties in relation to soft F-subsets and soft M-equality. Their work not only resolved certain incomplete results concerning the AND-product in the literature but also systematically compared the newly established properties with previously obtained findings, thereby clarifying and strengthening the theoretical foundations of soft set theory.

The central objective of this paper is to deliver a systematic and rigorous exploration of the OR-product, which has long served as a cornerstone in decision-making studies, and to extend its theoretical foundations by demonstrating its applicability to real decision-making scenarios, thereby highlighting the dual strength of the contribution in advancing both algebraic theory and practical methodology. Although certain algebraic properties of the OR-product have previously been examined in the literature by several researchers [34,39,48,74–76] in connection with various notions of soft subsets and soft equalities, this paper undertakes a comprehensive investigation of its full range of algebraic properties, with particular emphasis on M-subset and M-equality—the strictest forms of subsets and equalities—and systematically compares these findings with earlier results. Furthermore, we establish that the collection of all soft sets over a universe, equipped with restricted/extended intersection and the OR-product, constitutes a commutative hemiring with identity under soft L-equality by rigorously analyzing the distributive properties of the OR-product over selected soft set operations. This structural characterization is not merely a routine observation; rather, it significantly enriches the algebraic foundations of soft set theory by embedding the OR-product within one of its most central algebraic frameworks. To underscore the practical significance of these theoretical advancements, we further employ the int-uni decision-making method on the OR-product in a real-world pilot recruitment scenario. In this application, multiple decision-makers evaluate candidates against distinct elimination criteria, and the proposed approach ensures a fair, transparent, and mathematically sound process for narrowing selections under uncertainty. The results thus reinforce the dual significance of the OR-product, affirming it as both a mathematically rigorous construct and a practically applicable tool, thereby deepening the theoretical framework of soft set theory while simultaneously extending its relevance to real-world decision-making problems. Moreover, as symmetry underlies many algebraic structures, examining the OR-product through a symmetry-oriented perspective clarifies its structural and decision-theoretic relevance.

The remainder of this paper is organized as follows: Section 2 reviews the basic concepts of $\mathcal{SS}$ theory. Section 3 presents a comprehensive study of the OR-product and its algebraic properties with respect to various types of soft subsets and equalities while systematically comparing the results with those previously obtained in the literature. Section 4 focuses on the distributive properties of the OR-product over other $\mathcal{SS}$ operations, thereby revealing the underlying algebraic structures formed by combining the OR-product with other fundamental operations. In Section 5, the int-uni decision-making method of Çağman and Enginoğlu [8] is applied to the OR-product in the context of a pilot recruitment case study, demonstrating its ability to provide a fair, transparent, and mathematically rigorous framework for real-world decision-making under uncertainty. Finally, Section 6 offers concluding remarks and outlines potential directions for future research.

2. Materials and Methods

Definition 1 ([1]). *Let U be the universal set, E be the parameter set ($\mathcal{PS}$), $P(U)$ be the power set of U and $\mathbb{L} \subseteq E$. A pair $(F, \mathbb{L})$ is called a soft set ($\mathcal{SS}$) over U where F is a set-valued function such that $F : \mathbb{L} \to P(U)$.*

Henceforth, let $S_E(U)$ denote the collections of all the $\mathcal{SS}$s defined over U, $S_{\mathbb{L}}(U)$ denote the collection of all $\mathcal{SS}$s over U with the fixed $\mathcal{PS}$ $\mathbb{L}$, where $\mathbb{L} \subseteq E$. That is, while in $S_{\mathbb{L}}(U)$, each soft set has $\mathbb{L}$ as its $\mathcal{PS}$; whereas in $S_E(U)$, the $\mathcal{PS}$ may vary.

Definition 2 ([40]). *Let (F,$\mathbb{L}$) be an $\mathcal{SS}$. (F,$\mathbb{L}$) is called a relative null $\mathcal{SS}$ with respect to the $\mathcal{PS}$ $\mathbb{L}$, denoted by $\varnothing_{\mathbb{L}}$, if $F(\sigma) = \varnothing$ for all $\sigma \in \mathbb{L}$.(F, $\mathbb{L}$) is called a relative whole $\mathcal{SS}$ with respect to the $\mathcal{PS}$ $\mathbb{L}$, denoted by $U_{\mathbb{L}}$, if $F(\sigma) = U$ for all $\sigma \in \mathbb{L}$. The relative whole $\mathcal{SS}$ U_E with respect to E is called the absolute $\mathcal{SS}$.*

We denote by $\varnothing_{\varnothing}$ the unique $\mathcal{SS}$ with an empty $\mathcal{PS}$, called the empty $\mathcal{SS}$. It should be noted that $\varnothing_{\varnothing}$ and $\varnothing_{\mathbb{L}}$ are distinct $\mathcal{SS}$s [41]. Unless stated otherwise, we consider $\mathcal{SS}$s with non-empty $\mathcal{PS}$ in what follows.

Maji et al. [34] first proposed the notion of a soft subset, referred to here as a soft M-subset—as defined below.

Definition 3 ([34]). *Let (Ђ,$\mathcal{V}$) and (Q,$\mathcal{U}$) be two $\mathcal{SS}$s. (Ђ,$\mathcal{V}$) is called a soft M-subset of (Q,$\mathcal{U}$), denoted by (Ђ, $\mathcal{V}$) $\widetilde{\subseteq}_M$(Q,$\mathcal{U}$), if $\mathcal{V} \subseteq \mathcal{U}$ and Ђ(ʓ) = Q(ʓ), for all ʓ∈$\mathcal{V}$. (Ђ,$\mathcal{V}$) and (Q,$\mathcal{U}$) are said to be soft M-equal, denoted by (Ђ,W) $=_M$(Q,$\mathcal{U}$), if (Ђ,$\mathcal{V}$) $\widetilde{\subseteq}_M$(Q,$\mathcal{U}$) and (Q,$\mathcal{U}$) $\widetilde{\subseteq}_M$(Ђ,$\mathcal{V}$).*

Definition 4 ([35]). *Let (Ђ,$\mathcal{V}$) and (Q,$\mathcal{U}$) be two $\mathcal{SS}$s. (Ђ,$\mathcal{V}$) is called a soft F-subset of (Q,$\mathcal{U}$), denoted by (Ђ,$\mathcal{V}$)$\widetilde{\subseteq}_F$(Q,$\mathcal{U}$), if $\mathcal{V} \subseteq \mathcal{U}$ and Ђ(ʓ) $\subseteq$ Q(ʓ), for all ʓ∈$\mathcal{V}$. (Ђ,$\mathcal{V}$) and (Q,$\mathcal{U}$) are said to be soft F-equal, denoted (Ђ,W) $=_F$(Q,$\mathcal{U}$), if (Ђ,$\mathcal{V}$)$\widetilde{\subseteq}_F$(Q,$\mathcal{U}$) and (Q,$\mathcal{U}$)$\widetilde{\subseteq}_F$(Ђ,$\mathcal{V}$).*

It should be noted that the definitions of a soft F-subset and soft F-equality were initially introduced by Pei and Miao [36], although several papers incorrectly attribute these definitions to Feng et al. [37]. For this reason, the notation "F" refers to Feng.

Proposition 1 ([74]). *Let (Ђ,$\mathcal{V}$) and (L,$\mathcal{U}$) be $\mathcal{SS}$s. Then, (Ђ,$\mathcal{V}$) $=_M$ (L,$\mathcal{U}$) if and only if (Ђ,$\mathcal{V}$) $=_F$(L,$\mathcal{U}$).*

If two $\mathcal{SS}$s satisfy such soft equivalence, then they are essentially identical, as they have the same set of parameters and approximate functions [75]. Thus, (Ђ, $\mathcal{V}$) $=_M$(L,$\mathcal{U}$) means that (Ђ,$\mathcal{V}$) = (L,$\mathcal{U}$), where $\mathcal{V}$ and $\mathcal{U}$ are fixed subsets of U.

Definition 5 ([74]). *Let (Ђ,$\mathcal{V}$) and (Q,$\mathcal{U}$) be two $\mathcal{SS}$s. (Ђ,$\mathcal{V}$) is called a soft J-subset of (Q,$\mathcal{U}$), denoted by (Ђ,$\mathcal{V}$) $\widetilde{\subseteq}_J$(Q,$\mathcal{U}$), if for all ʓ∈$\mathcal{V}$, there exists ƙ∈ $\mathcal{U}$ such that Ђ(ʓ) ⊆Q(ƙ). (Ђ,$\mathcal{V}$) and (Q,$\mathcal{U}$) are said to be soft J-equal, denoted (Ђ,W) $=_J$(Q,$\mathcal{U}$), if (Ђ,W) $\widetilde{\subseteq}_J$(Q,$\mathcal{U}$) and (Q,$\mathcal{U}$) $\widetilde{\subseteq}_J$(Ђ,$\mathcal{V}$).*

Proposition 2 ([75,76]). *Let (Ђ,$\mathcal{V}$) and (Q,$\mathcal{U}$) be two $\mathcal{SS}$s. Then, (Ђ,$\mathcal{V}$) $\widetilde{\subseteq}_M$(Q,$\mathcal{U}$) ⇒ (Ђ,$\mathcal{V}$) $\widetilde{\subseteq}_F$(Q,$\mathcal{U}$) ⇒(Ђ,$\mathcal{V}$) $\widetilde{\subseteq}_J$(Q,$\mathcal{U}$); however, the converse may not be true.*

Definition 6 ([75]). *Let (Ђ,$\mathcal{V}$) and (Q,$\mathcal{U}$) be two $\mathcal{SS}$s. (Ђ,$\mathcal{V}$) is called a soft L-subset of (Q,$\mathcal{U}$), denoted by (Ђ,$\mathcal{V}$)$\widetilde{\subseteq}_L$(Q,$\mathcal{U}$), if for all ʓ∈$\mathcal{V}$, there exists ƙ∈$\mathcal{U}$ such that Ђ(ʓ) = Q(ƙ). (Ђ,$\mathcal{V}$) and (Q,$\mathcal{U}$) are said to be soft L-equal, denoted (Ђ,$\mathcal{V}$)$=_L$(Q, $\mathcal{U}$), if (Ђ, $\mathcal{V}$) $\widetilde{\subseteq}_L$(Q, $\mathcal{U}$) and (Q, $\mathcal{U}$) $\widetilde{\subseteq}_L$(Ђ, $\mathcal{V}$).*

Proposition 3 ([75]). *Let (Ђ,$\mathcal{V}$) and (Q,$\mathcal{U}$) be two $\mathcal{SS}$s. Then, (Ђ,$\mathcal{V}$)$\widetilde{\subseteq}_M$(Q,$\mathcal{U}$) ⇒ (Ђ,$\mathcal{V}$) $\widetilde{\subseteq}_L$(Q,$\mathcal{U}$) ⇒ (Ђ,$\mathcal{V}$) $\widetilde{\subseteq}_J$(Q,$\mathcal{U}$) and (Ђ,$\mathcal{V}$) $=_M$ (Q,$\mathcal{U}$) ⇒ (Ђ,$\mathcal{V}$) $=_L$ (Q,$\mathcal{U}$) ⇒ (Ђ,$\mathcal{V}$) $=_J$ (Q,$\mathcal{U}$) . However, the converse may be true.*

Remark 1 ([75]). *Soft J-equality is the weakest among the considered equality relations, while soft M-equality (and thus soft F-equality) is the strongest. The soft L-equality lies somewhere in between.*

For further discussion on various types of soft equalities, see References [73–81].

Definition 7 ([39]). *Let (Ђ,Ѷ) be an $\mathcal{SS}$. The relative complement of an $\mathcal{SS}$ $(\text{Ђ}, \text{Ѷ})$, denoted by $(\text{Ђ}, \text{Ѷ})^r$, is defined by $(\text{Ђ}, \text{Ѷ})^r = (\text{Ђ}^r, \text{Ѷ})$, where $\text{Ђ}^r : \text{Ѷ} \to P(U)$ is a mapping given by $(\text{Ђ}, \text{Ѷ})^r = U \backslash \text{Ђ}(\rho) = \left(\text{Ђ}(\rho)\right)' = \text{Ђ}'(\rho)$ for all $\rho \in \text{Ѷ}$.*

Definition 8 ([34]). *Let (Ђ,Ѷ) and (Q,И) be two $\mathcal{SS}$s. The AND-product (∧-product) of the $\mathcal{SS}$s (Ђ,Ѷ) and (Q,И) is an $\mathcal{SS}$ defined by (Ђ,Ѷ)∧(Q,И) = (S,ѶxИ), where S(ρ,σ) = Ђ(ρ)∩Q(σ) for all (ρ,σ)∈ѶxИ.*

Definition 9 ([34]). *Let (Ђ,Ѷ) and (Q,И) be two $\mathcal{SS}$s. The OR-product (∨-product) of the $\mathcal{SS}$s (Ђ,Ѷ) and (Q,И) is an $\mathcal{SS}$ defined by (Ђ,Ѷ)∨(Q,И) = (S,ѶxИ), where S(ρ,σ) = Ђ(ρ)∪Q(σ) for all (ρ,σ)∈Ѷx И.*

Definition 10 ([55]). *Let $(\text{Ђ}, \text{Ѷ})$ and $(Q, \text{И})$ be $\mathcal{SS}$s. The restricted intersection of $(\text{Ђ}, \text{Ѷ})$ and $(Q, \text{И})$, denoted by $(\text{Ђ}, \text{Ѷ}) \cap_R (Q, \text{И})$, is defined as $(\text{Ђ}, \text{Ѷ}) \cap_R (Q, \text{И})$ =(Y,S), where $S = \text{Ѷ} \cap \text{И}$. Here, if $\text{Ѷ} \cap \text{И} \neq \emptyset$, then $Y(ȥ) = \text{Ђ}(ȥ) \cap Q(\tau)$, for all $ȥ \in S$, and if $\text{Ѷ} \cap \text{И} = \emptyset$, then $(\text{Ђ}, \text{Ѷ}) \cap_R (Q, \text{И}) = \emptyset_\emptyset$.*

Definition 11 ([55]). *Let $(\text{Ђ}, \text{Ѷ})$ and $(Q, \text{И})$ be $\mathcal{SS}$s . The restricted union of $(\text{Ђ}, \text{Ѷ})$ and $(Q, \text{И})$, denoted by $(\text{Ђ}, \text{Ѷ}) \cup_R (Q, \text{И})$, is defined as $(\text{Ђ}, \text{Ѷ}) \cup_R (Q, \text{И})$ = (Y,S), where $S = \text{Ѷ} \cap \text{И}$. Here, if $\text{Ѷ} \cap \text{И} \neq \emptyset$, then $Y(ȥ) = \text{Ђ}(ȥ) \cup Q(\tau)$, for all $ȥ \in S$, and if $\text{Ѷ} \cap \text{И} = \emptyset$, then $(\text{Ђ}, \text{Ѷ}) \cap_R (Q, \text{И}) = \emptyset_\emptyset$.*

Definition 12 ([39]). *Let $(\text{Ђ}, \text{Ѷ})$ and $(Q, \text{И})$ be $\mathcal{SS}$s . The extended intersection of $(\text{Ђ}, \text{Ѷ})$ and $(Q, \text{И})$ is the $\mathcal{SS}$ (Y,S), denoted by $(\text{Ђ}, \text{Ѷ}) \cap_\varepsilon (Q, \text{И}) = (Y, S)$, where $S = \text{Ѷ} \cup \text{И}$, and for all $ȥ \in S$,*

$$Y(ȥ) = \begin{cases} \text{Ђ}(ȥ), & ȥ \in \text{Ѷ} \backslash \text{И} \\ Q(ȥ), & ȥ \in \text{И} \backslash \text{Ѷ} \\ \text{Ђ}(ȥ) \cap Q(ȥ), & ȥ \in \text{Ѷ} \cap \text{И} \end{cases}$$

Definition 13 ([34]). *Let $(\text{Ђ}, \text{Ѷ})$ and $(Q, \text{И})$ be $\mathcal{SS}$s . The extended union of $(\text{Ђ}, \text{Ѷ})$ and $(Q, \text{И})$ is the $\mathcal{SS}$ (Y,S), denoted by $(\text{Ђ}, \text{Ѷ}) \cup_\varepsilon (\text{Ђ}, \text{И}) = (Y, S)$, where $S = \text{Ѷ} \cup \text{И}$, and for all $ȥ \in S$,*

$$Y(ȥ) = \begin{cases} \text{Ђ}(ȥ), & ȥ \in \text{Ѷ} \backslash \text{И} \\ Q(ȥ), & ȥ \in \text{И} \backslash \text{Ѷ} \\ \text{Ђ}(ȥ) \cup Q(ȥ), & ȥ \in \text{Ѷ} \cap \text{И} \end{cases}$$

Definition 14 ([83]). *Let $(\text{Ђ}, \text{Ѷ})$ and $(Q, \text{И})$ be $\mathcal{SS}$s . The soft binary piecewise intersection operation of $(\text{Ђ}, \text{Ѷ})$ and $(Q, \text{И})$ is the $\mathcal{SS}$ (Y,Ѷ), denoted by $(\text{Ђ}, \text{Ѷ}) \,\widetilde{\cap}\, (Q, \text{И}) = (Y, \text{Ѷ})$, where for all ȥ∈Ѷ,*

$$Y(ȥ) = \begin{cases} \text{Ђ}(ȥ), & ȥ \in \text{Ѷ} \backslash \text{И} \\ \text{Ђ}(ȥ) \cap Q(ȥ), & ȥ \in \text{Ѷ} \cap \text{И} \end{cases}$$

Definition 15 ([83]). *Let* $(Ƀ, \mathbb{V})$ *and* $(Q, И)$ *be* $\mathcal{SS}$s *. The soft binary piecewise union operation of* $(Ƀ, \mathbb{V})$ *and* $(Q, И)$ *is the* $\mathcal{SS}$ $(Y, \mathbb{V})$*, denoted by* $(Ƀ, \mathbb{V}) \tilde{\cup} (Q, И) = (Y, \mathbb{V})$*, where for all* $ϛ \in \mathbb{V}$*,*

$$Y(ϛ) = \begin{cases} Ƀ(ϛ), & ϛ \in \mathbb{V} \setminus И \\ Ƀ(ϛ) \cup Q(ϛ), & ϛ \in \mathbb{V} \cap И \end{cases}$$

3. Results

This section presents a comprehensive investigation of the OR-product in the context of algebraic properties, particularly with respect to various types of soft subsets and equalities—most notably, M-subsets and M-equality. Furthermore, the results are compared with those previously established in the literature [34,39,40,48,75,76]. The following results can be viewed as algebraic symmetries of the OR-product, reflecting the balance, regularity, and harmony of operations within soft set theory.

Proposition 4. $S_E(U)$ *is closed under OR-product. That is, if* $(\mathcal{Q},\mathbb{L})$ *and* $(ጣ, И)$ *are two* $\mathcal{SS}$s *over* U*, then their OR-product* $(\mathcal{Q},\mathbb{L}) \wedge (ጣ, И)$ *is also a* $\mathcal{SS}$ *over* U*.*

Proposition 5. $S_{\mathbb{L}}(U)$ *is not closed under OR-product.*

Proof. Let $(\mathcal{Q},\mathbb{L})$ and $(ጣ,\mathbb{L}) \in S_{\mathbb{L}}(U)$. Then, $(\mathcal{Q},\mathbb{L}) \vee (ጣ,\mathbb{L}) \in S_{\mathbb{L}x\mathbb{L}}(U)$; that is, $(\mathcal{Q},\mathbb{L}) \vee (ጣ,\mathbb{L}) \notin S_{\mathbb{L}}(U)$. □

Example 1. *Let* $E = \{e_1, e_2, e_3, e_4\}$ *be the* $\mathcal{PS}$*,* $\mathbb{L} = \{e_1, e_3\}$ *be the subset of* E*,* $U = \{ч_1, ч_2, ч_3, ч_4\}$ *be the universal set, and* $(\mathcal{Q},\mathbb{L})$ *and* $(ጣ,\mathbb{L})$ *be* $\mathcal{SS}$s *over* U *such that*

$$(\mathcal{Q}, \mathbb{L}) = \{(e_1, \emptyset), (e_3, U)\}, (ጣ, \mathbb{L}) = \{(e_1, \{ч_3\}), (e_3, \emptyset)\}$$

Let $(\mathcal{Q},\mathbb{L}) \vee (ጣ,\mathbb{L}) = (ዸ, \mathbb{L}x\mathbb{L})$*. Thereby,*

$$(ዸ, x) = \{((e_1, e_1), \{ч_3\}), ((e_1, e_3), \emptyset), ((e_3, e_1), U), ((e_1, e_1), U)\}$$

It is observed that $(ዸ, \mathbb{L}x\mathbb{L}) \in S_{\mathbb{L}x\mathbb{L}}$*, which implies that* $S_{\mathbb{L}}(U)$ *is not closed under OR-product.*

Note 1. *Maji* et al. [34] *proposed that the associative law holds for the OR-product as regards soft M-equality (and, consequently, soft F-equality). However, in* [39]*, it was demonstrated that*

$$(\mathcal{Q}, \mathbb{L}) \vee ((ጣ, И) \vee (\mathcal{P}, ω)) \neq_M ((\mathcal{Q}, \mathbb{L}) \vee (ጣ, И)) \vee (\mathcal{P}, ω)$$

since, from a set-theoretic perspective, $\mathbb{L}\ x(И\ x\ ω) \neq (\mathbb{L}xИ)xω$*. That is, the associativity fails under soft M-equality due to the non-associativity of the Cartesian product of parameter sets. As shown in* [75]*, the associative law for the OR-product holds only in the sense of soft L-equality, rather than soft M-equality.*

Proposition 6 ([75]). $(\mathcal{Q},\mathbb{L}) \vee ((ጣ,И) \vee (\mathcal{P},ω)) =_L ((\mathcal{Q},\mathbb{L}) \vee (ጣ,И)) \vee (\mathcal{P},ω)$ *(Generalized Soft Associative Laws).*

Example 2. *Let* $E = \{e_1, e_2, e_3, e_4\}$ *be the* $\mathcal{PS}$*,* $\mathcal{T} = \{e_2, e_3\}$*,* $\mathcal{W} = \{e_1\}$ *and* $\mathcal{L} = \{e_4\}$ *be the subsets of* E*,* $U = \{ч_1, ч_2, ч_3, ч_4, ч_5\}$ *be the universal set, and* $(₹, \mathcal{T})$*,* $(\mathcal{O}, \mathcal{W})$ *and* $(ϫ, \mathcal{L})$ *be* $\mathcal{SS}$s *over* U *such that*

$$(₹,\mathcal{T}) = \{(\mathfrak{e}_2,\{ч_3,ч_4\}),\ (\mathfrak{e}_3,\{ч_1\})\},(\mathcal{O},\mathcal{W}) = \{(\mathfrak{e}_1,\varnothing)\},(\Sigma,\mathcal{L}) = \{(\mathfrak{e}_4,\{ч_1,ч_3,ч_5\})\}$$

We show that $(₹,\mathcal{T}) \vee [(\mathcal{O},\mathcal{W}) \vee (\Sigma,\mathcal{L})] \neq_M [(₹,\mathcal{T}) \vee (\mathcal{O},\mathcal{W})] \vee (\Sigma,\mathcal{L})$ *and* $(₹,\mathcal{T}) \vee [(\mathcal{O},\mathcal{W}) \vee (\Sigma,\mathcal{L})] =_L [(₹,\mathcal{T}) \vee (\mathcal{O},\mathcal{W})] \vee (\Sigma,\mathcal{L})$. *Since* $\mathcal{T}x(\mathcal{W}x\mathcal{L}) \neq (\mathcal{T}x\mathcal{W})x\mathcal{L}$ *from a set-theoretic point of view* $(₹,\mathcal{T}) \vee [(\mathcal{O},\mathcal{W}) \vee (\Sigma,\mathcal{L})] \neq_M [(₹,\mathcal{T}) \vee (\mathcal{O},\mathcal{W})] \vee (\Sigma,\mathcal{L})$. *Let* $(\mathcal{O},\mathcal{W}) \vee (\Sigma,\mathcal{L}) = (\mho,\mathcal{W}x\mathcal{L})$, *then* $(\mho,\mathcal{W}x\mathcal{L}) = \{((\mathfrak{e}_1,\mathfrak{e}_4),\ \{ч_1,ч_3,ч_5\})\}$. *Let* $(₹,\mathcal{T}) \vee (\mho,\mathcal{W}x\mathcal{L}) = (\mathfrak{X},\mathcal{T}x(\mathcal{W}x\mathcal{L}))$. *Thus,*

$$(\mathfrak{X},\mathcal{T}x(\mathcal{W}x\mathcal{L})) = \{((\mathfrak{e}_2,(\mathfrak{e}_1,\mathfrak{e}_4)),\ \{ч_1,ч_3,ч_4,ч_5\}),((\mathfrak{e}_3,(\mathfrak{e}_1,\mathfrak{e}_4)),\{ч_1,ч_3,ч_5\})\}$$

Assume that $(₹,\mathcal{T}) \vee (\mathcal{O},\mathcal{W}) = (\mathfrak{C},\mathcal{T}x\mathcal{W})$. *Thereby,* $(\mathfrak{C},\mathcal{T}x\mathcal{W}) = \{((\mathfrak{e}_2,\mathfrak{e}_1),\{ч_3,ч_4\}),((\mathfrak{e}_3,\mathfrak{e}_1),\{ч_1\})\}$. *Suppose that* $(\mathfrak{C},\mathcal{T}x\mathcal{W}) \vee (\Sigma,\mathcal{L}) = (\wp,(\mathcal{T}x\mathcal{W})x\mathcal{L})$. *Therefore,*

$$(\wp,(\mathcal{T}x\mathcal{W})x\mathcal{L}) = \{(((\mathfrak{e}_2,\mathfrak{e}_1),\mathfrak{e}_4),\{ч_1,ч_3,ч_4,ч_5\}),(((\mathfrak{e}_3,\mathfrak{e}_1),\mathfrak{e}_4),\{ч_1,ч_3,ч_5\})\}$$

It is observed that $(\mathfrak{X},\mathcal{T}x(\mathcal{W}x\mathcal{L})) =_L (\wp,(\mathcal{T}x\mathcal{W})x\mathcal{L})$.

Note 2. *By Proposition 4 and Proposition 6, it can be deduced that the algebraic structure* $(S_E(U),\vee)$ *forms a semigroup only in the sense of L-soft equality, not in the sense of M-soft equality. Moreover, since the OR-product is not closed in* $S_{\mathbb{L}}(U)$, *it follows from Proposition 5 and Example 1 that the structure* $(S_{\mathbb{L}}(U),\vee)$ *cannot be a semigroup, even in the sense of the soft L-equality.*

Proposition 7 ([76]). *Let* $(\mathcal{Q},\mathbb{L})$ *and* $(\mathcal{P},\mathcal{U})$ *be two* $\mathcal{SS}$*s. Then,* $(\mathcal{Q},\mathbb{L})\vee(\mathcal{P},\mathcal{U})=_L(\mathcal{P},\mathcal{U})\vee(\mathcal{Q},\mathbb{L})$ *(Generalized soft commutative laws).*

Proposition 8. *Let* $(\mathcal{Q},\mathbb{L})$,$(\mathcal{P},\mathbb{L})$ *and* $(\mathcal{P},\mathcal{U})$ *be* $\mathcal{SS}$*s. Then,* $(\mathcal{Q},\mathbb{L})\vee(\mathcal{P},\mathcal{U}) \neq_M (\mathcal{P},\mathcal{U})\vee(\mathcal{Q},\mathbb{L})$, *moreover* $(\mathcal{Q},\mathbb{L})\vee(\mathcal{P},\mathbb{L}) \neq_M (\mathcal{P},\mathbb{L})\vee(\mathcal{Q},\mathbb{L})$.

Proof. Since $\mathbb{L}x\mathcal{U} \neq \mathcal{U}x\mathbb{L}$, it is evident that $(\mathcal{Q},\mathbb{L})\vee(\mathcal{P},\mathcal{U})\neq_M(\mathcal{P},\mathcal{U})\vee(\mathcal{Q},\mathbb{L})$. Suppose that $(\varsigma,\hat{u})\in\mathbb{L}x\mathbb{L}$ such that $\varsigma \neq \hat{u}$ and let $(\mathcal{Q},\mathbb{L})\vee(\mathcal{P},\mathbb{L}) = (H,\mathbb{L}x\mathbb{L})$, where $H(\varsigma,\hat{u}) = \mathcal{Q}(\varsigma)\cup\mathcal{P}(\hat{u})$ for all $(\varsigma,\hat{u})\in\mathbb{L}x\mathbb{L}$ and $(\mathcal{P},\mathbb{L})\vee(\mathcal{Q},\mathbb{L}) = (K,\mathbb{L}x\mathbb{L})$, where $K(\varsigma,\hat{u}) = \mathcal{P}(\varsigma)\cup\mathcal{Q}(\hat{u})$ for all $(\varsigma,\hat{u})\in\mathbb{L}x\mathbb{L}$. Since $\mathcal{Q}(\varsigma)\cup\mathcal{P}(\hat{u})$ is not necessarily equal to $\mathcal{P}(\varsigma)\cup\mathcal{Q}(\hat{u})$, it follows that $(\mathcal{Q},\mathbb{L})\vee(\mathcal{P},\mathbb{L})\neq_M(\mathcal{P},\mathbb{L})\vee(\mathcal{Q},\mathbb{L})$. □

Example 3. *Let* $E = \{\mathfrak{e}_1,\mathfrak{e}_2,\mathfrak{e}_3\}$ *be the* $\mathcal{PS}$, $\mathcal{T} = \{\mathfrak{e}_1,\mathfrak{e}_2\}$ *be the subset of E,* $U = \{ч_1,ч_2,ч_3,ч_4\}$ *be the universal set, and* $(₹,\mathcal{T})$ *and* $(\mathfrak{F},\mathcal{T})$ *be* $\mathcal{SS}$*s over U such that*

$$(₹,\mathcal{T}) = \{(\mathfrak{e}_1,\{ч_1,ч_4\}),\ (\mathfrak{e}_2,\varnothing)\},(\mathfrak{F},\mathcal{T}) = \{(\mathfrak{e}_1,\{ч_3\}),(\mathfrak{e}_2,\{ч_2\})\}$$

Let $(₹,\mathcal{T}) \vee (\mathfrak{F},\mathcal{T}) = (\mathfrak{C},\mathcal{T}x\mathcal{T})$. *Thereby,*

$$(\mathfrak{C},\mathcal{T}x\mathcal{T}) = \{((\mathfrak{e}_1,\mathfrak{e}_1),\ \{ч_1,ч_3,ч_4\}),((\mathfrak{e}_1,\mathfrak{e}_2),\{ч_1,ч_2,ч_4\}),\ ((\mathfrak{e}_2,\mathfrak{e}_1),\{ч_3\}),((\mathfrak{e}_2,\mathfrak{e}_2),\{ч_2\})\}$$

Let $(\mathfrak{F},\mathcal{T}) \vee (₹,\mathcal{T}) = (\pitchfork,\mathcal{T}x\mathcal{T})$. *Thus,*

$$(\pitchfork,\mathcal{T}x\mathcal{T}) = \{((\mathfrak{e}_1,\mathfrak{e}_1),\ \{ч_1,ч_3,ч_4\}),((\mathfrak{e}_1,\mathfrak{e}_2),\{ч_3\}),\ ((\mathfrak{e}_2,\mathfrak{e}_1),\{ч_1,ч_2,ч_4\}),((\mathfrak{e}_2,\mathfrak{e}_2),\{ч_2\})\}$$

It is observed that $(\mathfrak{C}, \mathcal{T}x\mathcal{T}) \neq_M (\mathfrak{n}, \mathcal{T}x\mathcal{T})$*, which implies that OR-product is not commutative in the sense of M-equality, and even the* $\mathcal{SS}$*s involved have the same* $\mathcal{PS}$*s.*

Proposition 9. *Let* $(\mathcal{Q},\mathbb{L})$ *and* $(\mathfrak{N},\mathcal{U})$ *be two* $\mathcal{SS}$*s. If* $(\mathcal{Q},\mathbb{L}) =_M (\mathfrak{N},\mathcal{U})$*, then* $(\mathcal{Q},\mathbb{L}) \vee (\mathfrak{N},\mathcal{U}) =_M (\mathfrak{N},\mathcal{U}) \vee (\mathcal{Q},\mathbb{L})$*.*

Example 4 illustrates that Proposition 9 cannot be reversed in general, that is, $(\mathcal{Q},\mathbb{L}) \vee (\mathfrak{N},\mathcal{U}) =_M (\mathfrak{N},\mathcal{U}) \vee (\mathcal{Q},\mathbb{L})$ does not imply that $(\mathcal{Q},\mathbb{L}) =_M (\mathfrak{N},\mathcal{U})$.

Example 4. *Let* $E = \{e_1, e_2, e_3, e_4\}$ *be the* $\mathcal{PS}$*,* $\mathbb{L} = \mathcal{U} = \{e_1, e_3\}$ *be the subsets of E and* $U = \{u_1, u_2, u_3, u_4, u_5\}$ *be the universal set. Let* $(\mathcal{Q}, \mathbb{L})$ *and* $(\mathfrak{N},\mathcal{U})$ *be* $\mathcal{SS}$*s defined as follows:*

$$(\mathcal{Q},\mathbb{L}) = \{(e_1, \{u_2, u_5\}), (e_3, \{u_2, u_5\})\}, (\mathfrak{N},\mathcal{U}) = \{(e_1, \{u_1, u_3, u_4\}), (e_3, \{u_1, u_2, u_3, u_4\})\}$$

Then,

$$(\mathcal{Q},\mathbb{L}) \vee (\mathfrak{N},\mathcal{U}) =_M \{((e_1, e_1), U), ((e_1, e_3), U), ((e_3, e_1), U), ((e_3, e_3), U)\}.$$

$$(\mathfrak{N},\mathcal{U}) \vee (\mathcal{Q},\mathbb{L}) =_M \{((e_1, e_1), U), ((e_1, e_3), U), ((e_3, e_1), U), ((e_3, e_3), U)\}.$$

It is observed that $(\mathcal{Q},\mathbb{L}) \vee (\mathfrak{N},\mathcal{U}) =_M (\mathfrak{N},\mathcal{U}) \vee (\mathcal{Q},\mathbb{L})$*; however,* $(\mathcal{Q},\mathbb{L}) \neq_M (\mathfrak{N},\mathcal{U})$*.*

Proposition 10 ([76]). *Let* $(\mathcal{Q},\mathbb{L})$ *be an* $\mathcal{SS}$*. Then,* $(\mathcal{Q},\mathbb{L}) \vee U_{\mathcal{U}} =_L U_{\mathcal{U}} \vee (\mathcal{Q}, \mathbb{L}) =_L U_{\mathcal{U}}$.

Note 3. *By Proposition 10, it follows that* $U_{\mathcal{U}}$ *commutes with any* $\mathcal{SS}$ *whose* $\mathcal{PS}$ *is* $\mathcal{U}$ *under OR-product as regards soft L-equality. In addition,* $U_{\mathcal{U}}$ *serves as the absorbing element of OR-product in* $S_{\mathbb{L}}(U)$ *as regards L-equality.*

Proposition 11. *Let* $(\mathcal{Q},\mathbb{L})$ *be an* $\mathcal{SS}$*. Then,* $(\mathcal{Q},\mathbb{L}) \vee U_E =_L U_E \vee (\mathcal{Q}, \mathbb{L}) =_L U_E$.

Note 4. *By Proposition 11, we deduce that* U_E *commutes with any* $\mathcal{SS}$ *under OR-product, and* U_E *is the absorbing element for OR-product in* $S_E(U)$ *as regards L-equality.*

Proposition 12. *Let* $(\mathcal{Q},\mathbb{L})$ *be an* $\mathcal{SS}$*. Then,* $(\mathcal{Q},\mathbb{L}) \vee U_{\mathbb{L}} =_M U_{\mathbb{L}} \vee (\mathcal{Q},\mathbb{L}) =_M U_{\mathbb{L}x\mathbb{L}}$.

Proof. Let $U_{\mathbb{L}} = (S,\mathbb{L})$, where $S(\varsigma) = U$ for all $\varsigma \in \mathbb{L}$. Then, $(\mathcal{Q},\mathbb{L}) \vee U_{\mathbb{L}} =_M (\mathcal{Q},\mathbb{L}) \vee (S,\mathbb{L}) =_M (H,\mathbb{L}x\mathbb{L})$, where $H(\varsigma,\omega) = \mathcal{Q}(\varsigma) \cup S(\omega) = \mathcal{Q}(\varsigma) \cup U = U$ for all $(\varsigma,\omega) \in \mathbb{L}x\mathbb{L}$. Hence, $(H,\mathbb{L}x\mathbb{L}) =_M U_{\mathbb{L}x\mathbb{L}}$.

Let $U_{\mathbb{L}} \vee (\mathcal{Q},\mathbb{L}) =_M (S,\mathbb{L}) \vee (\mathcal{Q}, \mathbb{L}) =_M (K,\mathbb{L}x\mathbb{L})$, where $K(\varsigma,\omega) = S(\varsigma) \cup \mathcal{Q}(\omega) = U \cup \mathcal{Q}(\omega) = U$ for all $(\varsigma,\omega) \in \mathbb{L}x\mathbb{L}$. Hence, $(K,\mathbb{L}x\mathbb{L}) =_M U_{\mathbb{L}x\mathbb{L}}$. □

Remark 2. *Proposition 12 demonstrates that, although* $U_{\mathbb{L}}$ *commutes with any* $\mathcal{SS}$ *whose* $\mathcal{PS}$ *is* $\mathbb{L}$ *under OR-product as regards soft M-equality as well;* $U_{\mathbb{L}}$ *is not the absorbing element for OR-product in* $S_{\mathbb{L}}(U)$ *when considered under M-equality.*

Proposition 13. *Let* $(\mathcal{Q},\mathbb{L})$ *and* $(\mathfrak{N},\mathcal{U})$ *be two* $\mathcal{SS}$*s. If* $(\mathcal{Q},\mathbb{L}) = U_{\mathbb{L}}$ *or* $(\mathfrak{N},\mathcal{U}) = U_{\mathcal{U}}$*, then* $(\mathcal{Q},\mathbb{L}) \vee (\mathfrak{N},\mathcal{U})$ *needs not be soft M-equal to* $(\mathfrak{N},\mathcal{U}) \vee (\mathcal{Q},\mathbb{L})$*.*

Proof. Without loss of generality, let $(\mathcal{\eta},\mathcal{N}) =_M U_{\mathcal{N}}$. Then, $(\mathcal{Q},\mathbb{L})\vee(\eta,\mathcal{N}) =_M (\mathcal{Q},\mathbb{L}) \vee U_{\mathcal{N}} =_M U_{\mathbb{L}x\mathcal{N}}$, $(\eta,\mathcal{N})\vee(\mathcal{Q},\mathbb{L}) = U_{\mathcal{N}}\vee(\mathcal{Q},\mathbb{L}) =_M U_{\mathcal{N}x}$. Since $U_{\mathbb{L}xB} \neq_M U_{\mathcal{N}x}$, $(\mathcal{Q},\mathbb{L}) \vee (\eta,\mathcal{N}) \neq_M (\eta,\mathcal{N}) \vee (\mathcal{Q},\mathbb{L})$.□

Example 5. *Let* $E = \{e_1, e_2, e_3, e_4\}$ *be the* $\mathcal{PS}$, $\mathcal{T} = \{e_2, e_4\}$ *and* $\mathcal{W} = \{e_2\}$ *be the subsets of* E, $U = \{u_1, u_2, u_3, u_4\}$ *be the universal set, and* $(₹,\mathcal{T})$ *and* $(\mathfrak{F},\mathcal{W})$ *be* $\mathcal{SS}$*s over* U *such that*

$$(₹,\mathcal{T}) = \{(e_2, U), (e_4, U)\} = U_{\mathcal{T}}, \ (\mathfrak{F},\mathcal{W}) = \{(e_1, \{u_3, u_4\})\}$$

Let $(₹,\mathcal{T}) \vee (\mathfrak{F},\mathcal{W}) = (\mathfrak{C},\mathcal{T}x\mathcal{W})$, *where*

$$(\mathfrak{C},\mathcal{T}x\mathcal{W}) = \{((e_2,e_1),U),((e_4,e_2),U)\}$$

Let $(\mathfrak{F},\mathcal{W}) \vee (₹,\mathcal{T}) = (\wp,\mathcal{W}x\mathcal{T})$, *where*

$$(\wp,\mathcal{W}x\mathcal{T}) = \{((e_2,e_2),U),((e_2,e_4),U)\}$$

It is observed that $(\mathfrak{C},\mathcal{T}x\mathcal{W}) =_M (\wp,\mathcal{W}x\mathcal{T})$.

Proposition 14 ([76]). *Let* $(\mathcal{Q},\mathbb{L})$ *be an* $\mathcal{SS}$. *Then,* $(\mathcal{Q},\mathbb{L})\vee\emptyset_{\mathbb{L}} =_L \emptyset_{\mathbb{L}} \vee (\mathcal{Q},\mathbb{L}) =_L (\mathcal{Q},\mathbb{L})$.

Proposition 15. *Let* $(\mathcal{Q},\mathbb{L})$ *be an* $\mathcal{SS}$. *Then,* $(\mathcal{Q},\mathbb{L})\vee\emptyset_E =_L \emptyset_E \vee (\mathcal{Q},\mathbb{L}) =_L (\mathcal{Q},\mathbb{L})$.

Note 5. *Propositions 14 and 15 show that* $\emptyset_{\mathbb{L}}$*commutes with any* $\mathcal{SS}$ *with* $\mathcal{PS}$ $\mathbb{L}$ *under OR-product and* $\emptyset_{\mathbb{L}}$*is the identity element for OR-product in* $S_{\mathbb{L}}(U)$ *under OR-product as regards soft L-equality. Moreover,* $\emptyset_E$*commutes with any* $\mathcal{SS}$ *under OR-product, and* $\emptyset_E$*is the identity element for OR-product in* $S_E(U)$ *as regards L-equality, not M-equality.*

Example 6. *Let* $E = \{e_1,e_2,e_3\}$ *be the* $\mathcal{PS}$, $\mathbb{L} = \{e_1, e_3\}$ *be the subset of* E *and* $U = \{u_1,u_2,u_3,u_4,u_5\}$ *be the universal set. Let* $(\mathcal{Q},\mathbb{L})$ *and* (η,E) *be* $\mathcal{SS}$*s defined as follows:*

$$(\mathcal{Q},\mathbb{L}) \ \{(e_1, \{u_1, u_3\}), (e_3, \{u_2, u_4\})\}, \ (\eta, E) \ \{(e_1,\emptyset), (e_2,\emptyset), (e_3,\emptyset)\} \ \emptyset_E$$

Then,

$$(\mathcal{Q},\mathbb{L}) \vee \emptyset_E = \{((e_1,e_1), \{u_1,u_3\}), ((e_1,e_2), \{u_1,u_3\}), ((e_1,e_3), \{u_1,u_3\}), ((e_3,e_1), \{u_2,u_4\}), ((e_3,e_2), \{u_2,u_4\}), ((e_3,e_3), \{u_2,u_4\}))\}$$

$$\emptyset_E\vee(\mathcal{Q},\mathbb{L}) \ \{((e_1,e_1), \{u_1,u_3\}), ((e_1,e_3), \{u_2,u_4\}), ((e_2,e_1), \{u_1,u_3\}), ((e_2,e_3), \{u_2,u_4\}), ((e_3,e_1), \{u_1,u_3\}), ((e_3,e_3), \{u_2,u_4))\}$$

It is observed that $(\mathcal{Q},\mathbb{L})\vee\emptyset_E =_L \emptyset_E \vee (\mathcal{Q},\mathbb{L}) =_L (\mathcal{Q},\mathbb{L})$; *however,* $(\mathcal{Q},\mathbb{L})\vee\emptyset_E \neq_M (\mathcal{Q},\mathbb{L})$ *and* $\emptyset_E\vee(\mathcal{Q},\mathbb{L})\neq_M(\mathcal{Q},\mathbb{L})$.

Proposition 16. *Let* $(\mathcal{Q},\mathbb{L})$ *and* $(\eta,\mathbb{L})$ *be* $\mathcal{SS}$*s. If either* $(\mathcal{Q},\mathbb{L}) =_M \emptyset_{\mathbb{L}}$ *or* $(\eta,\mathbb{L}) =_M \emptyset_{\mathbb{L}}$, *then it does not necessarily follow that* $(\mathcal{Q},\mathbb{L})\vee(\eta,\mathbb{L}) =_M (\eta,\mathbb{L})\vee(\mathcal{Q},\mathbb{L})$.

Example 7. *Let* $E = \{e_1,e_2,e_3,e_4\}$ *be the* $\mathcal{PS}$, $\mathbb{L} = \{e_1, e_3\}$ *be the subset of* E *and* $U = \{u_1,u_2,u_3,u_4,u_5\}$ *be the universal set. Let* $(\mathcal{Q},\mathbb{L})$ *and* $(\eta,\mathbb{L})$ *be* $\mathcal{SS}$*s defined as follows:*

$$(\mathcal{Q},\mathbb{L}) = \{(e_1,\emptyset), (e_3,\emptyset)\} = \emptyset_{\mathbb{L}}, \ (\eta,\mathcal{N}) = \{(e_1, \{u_1, u_4\}), (e_3, \{u_3, u_4\})\}$$

Then,

$$(\mathcal{Q},\mathbb{L})\vee(\eta,\mathbb{L})=\{((e_1,e_1),\{u_1,u_4\}),\ ((e_1,e_3),\ \{u_3,u_4\}),\ ((e_3,e_1),\{u_1,u_4\}),((e_3,e_3),\{u_3,u_4\})\}$$

$$(\eta,\mathbb{L})\vee(\mathcal{Q},\mathbb{L})=\{((e_1,e_1),\{u_1,u_4\}),\ ((e_1,e_3),\ \{u_1,u_4\}),\ ((e_3,e_1),\{u_3,u_4\}),((e_3,e_3),\{u_3,u_4\})\}$$

It is observed that $\emptyset_{\mathbb{L}}\vee(\eta,\mathbb{L})\neq_M(\eta,\mathbb{L})\vee\emptyset_{\mathbb{L}}$.

Proposition 17. *Let* $(\mathcal{Q},\mathbb{L})$ *and* $(\eta,\mathbb{L})$ *be* $\mathcal{SS}$*s. If one of the* $\mathcal{SS}$*s is* $\emptyset_{\mathbb{L}}$*, then* $(\mathcal{Q},\mathbb{L})\vee(\eta,\mathbb{L}) =_M(\eta,\mathbb{L})\vee(\mathcal{Q},\mathbb{L})$ *if and only if the other* $\mathcal{SS}$ *is a constant function (CF), that is, a* $\mathcal{SS}$ *whose approximate value is the same subset of U for every parameter in* $\mathbb{L}$*.*

Proof. Without loss of generality, let $(\mathcal{Q},\mathbb{L})$ and $(\eta,\mathbb{L})$ be $\mathcal{SS}$s such that $(\mathcal{Q},\mathbb{L})=_M\emptyset_{\mathbb{L}}$.

Necessity*: Let* $(\mathcal{Q},\mathbb{L})\vee(\eta,\mathbb{L})=_M(\eta,\mathbb{L})\vee(\mathcal{Q},\mathbb{L})$, that is, $\emptyset_{\mathbb{L}}\vee(\eta,\mathbb{L})=_M(\eta,\mathbb{L})\vee\emptyset_{\mathbb{L}}$. Suppose that $(\varsigma,\breve{u})\in\mathbb{L}x\mathbb{L}$ such that $\varsigma=\breve{u}$. Then, $\emptyset_{\mathbb{L}}\vee(\eta,\mathbb{L})=_M(\eta,\mathbb{L})\vee\emptyset_{\mathbb{L}}$ implies that $\emptyset\cup\eta(\breve{u})=\eta(\varsigma)\cup\emptyset$, where $\varsigma=\breve{u}$. We observe that $\eta(\varsigma)=\eta(\varsigma)$ is already satisfied for all $(\varsigma,\breve{u})\in\mathbb{L}x\mathbb{L}$ such that $\varsigma=\breve{u}$. Thus, the condition holds automatically in this case and provides no constraint on η

Let $(\varsigma,\breve{u})\in\mathbb{L}x\mathbb{L}$ such that $\varsigma\neq\breve{u}$. Suppose $\emptyset_{\mathbb{L}}\vee(\eta,\mathbb{L})=_M(\eta,\mathbb{L})\vee\emptyset_{\mathbb{L}}$. Then, by the definition of the OR-product, this implies that $\emptyset\cup\eta(\breve{u})=\eta(\varsigma)\cup\emptyset$, which simplifies to $\eta(\varsigma)=\eta(\breve{u})$. Since this must hold for all for all $(\varsigma,\breve{u})\in\mathbb{L}x\mathbb{L}$ with $\varsigma\neq\breve{u}$, it follow that η assigns the same subset of U to every parameter in $\mathbb{L}$, that is, η is an CF.

Sufficiency: Let $(\mathcal{Q},\mathbb{L})=_M\emptyset_{\mathbb{L}}$ and η be an CF. Let $(\mathcal{Q},\mathbb{L})\vee(\eta,\mathbb{L})=_M(H,\mathbb{L}x\mathbb{L})$, where $H(\varsigma,\breve{u})=\mathcal{Q}(\varsigma)\cup\eta(\breve{u})=\emptyset\cup\eta(\breve{u})=\eta(\breve{u})$ for all $(\varsigma,\breve{u})\in\mathbb{L}x\mathbb{L}$. Let $(\eta,\mathbb{L})\vee(\mathcal{Q},\mathbb{L})=(W,\mathbb{L}x\mathbb{L})$, where $W(\varsigma,\breve{u})=\eta(\varsigma)\cup\mathcal{Q}(\breve{u})=\eta(\varsigma)\cup\emptyset=\eta(\varsigma)$, for all $(\varsigma,\breve{u})\in\mathbb{L}x\mathbb{L}$. Since η is an CF, it follows that $\eta(\varsigma)=\eta(\breve{u})$ for all $\varsigma,\breve{u}\in\mathbb{L}$. Hence, $H(\varsigma,\breve{u})=W(\varsigma,\breve{u})$ for all $(\varsigma,\breve{u})\in\mathbb{L}x\mathbb{L}$, which implies that $(H,\mathbb{L}x\mathbb{L})=_M(W,\mathbb{L}x\mathbb{L})$. Thus, $(\mathcal{Q},\mathbb{L})\vee(\eta,\mathbb{L})=_M(\eta,\mathbb{L})\vee(\mathcal{Q},\mathbb{L})$. □

Proposition 18. *Let* $(\mathcal{Q},\mathbb{L})$ *be an* $\mathcal{SS}$*. Then,* $\emptyset_\emptyset\vee(\mathcal{Q},\mathbb{L})=_M(\mathcal{Q},\mathbb{L})\vee\emptyset_\emptyset=_M\emptyset_\emptyset$*.*

Proof. Since $\emptyset x\mathbb{L}=\mathbb{L}x\emptyset=\emptyset$, and $\emptyset_\emptyset$ is the unique $\mathcal{SS}$ with an empty $\mathcal{PS}$, the result follows immediately. □

Note 6. *Proposition 18 shows that* $\emptyset_\emptyset$*commutes with any* $\mathcal{SS}$ *under the OR-product in* $S_E(U)$*, and that* $\emptyset_\emptyset$*is the absorbing element of OR-product in* $S_E(U)$ *with respect to soft M-equality, soft L-equality and soft J-equality. Therefore, combining Propositions 11 and 18, we observe that both* $\emptyset_\emptyset$*and* U_E*are the absorbing elements for OR-product in* $S_E(U)$ *as regards L-equality. However, it is well-known that a magma can have at most one absorbing element. That is, within a binary operation on a set, two distinct absorbing elements cannot coexist. Hence, it is not possible that for* $(S_E(U),\vee)$ *to have two different absorption elements under soft M-equality. In fact,* $\emptyset_\emptyset$*is the unique absorbing element for the OR-product in* $S_E(U)$ *in the sense of soft M-equality.*

Corollary 1. *Let* $(\mathcal{Q},\mathbb{L})$ *and* $(\eta,И)$ *be* $\mathcal{SS}$*s. Then,* $(\mathcal{Q},\mathbb{L})\vee(\eta,И)=_M(\eta,И)\vee(\mathcal{Q},\mathbb{L})$ *if and only if* $\mathbb{L}=И$ *and* $\mathcal{Q}(\varsigma)\cup\eta(\breve{u})=\eta(\varsigma)\cup\mathcal{Q}(\breve{u})$*, for all* $(\varsigma,\breve{u})\in\mathbb{L}x\ \mathbb{L}$ *such that* $\varsigma\neq\breve{u}$*.*

Example 8. *Let* $E=\{e_1,e_2,e_3,e_4\}$ *be the* $\mathcal{PS}$*,* $\mathbb{L}=\{e_1,e_3\}$ *be the subset of E and* $U=\{u_1,u_2,u_3,u_4\}$ *be the universe set. Let* $(\mathcal{Q},\mathbb{L})$ *and* $(\eta,\mathbb{L})$ *be the* $\mathcal{SS}$*s defined as follows:*

$$(\mathcal{Q},\mathbb{L})=\{(e_1,\{u_1,u_2\}),\ (e_3,\{u_1,u_2,u_4\})\}$$

$$(Ƞ, 𝕃) = \{(℮_1, \{ч_1, ч_4\}), (℮_3, \{ч_2, ч_4\})\}$$

Then,

$$(𝒬,𝕃)\vee(Ƞ,𝕃) = \{((℮_1,℮_1), \{ч_1, ч_2, ч_4\}), ((℮_1,℮_3), \{ч_1, ч_2, ч_4\}), ((℮_3,℮_1), \{ч_1, ч_2, ч_4\}), ((℮_3,℮_3), \{ч_1, ч_2, ч_4\})\}$$

$$(Ƞ,𝕃)\vee(𝒬,𝕃) = \{((℮_1,℮_1), \{ч_1, ч_2, ч_4\}), ((℮_1,℮_3), \{ч_1, ч_2, ч_4\}), ((℮_3,℮_1), \{ч_1, ч_2, ч_4\}), ((℮_3,℮_3), \{ч_1, ч_2, ч_4\})\}.$$

It is observed that $(𝒬,𝕃)\vee(Ƞ,𝕃) =_M (Ƞ,𝕃)\vee(𝒬,𝕃)$, *provided that for all* $ϛ \neq ŭ$, *the condition* $𝒬(ϛ)\cup Ƞ(ŭ) = Ƞ(ϛ)\cup 𝒬(ŭ)$ *holds.*

Proposition 19. *Let* $(𝒬,𝕃)$ *and* $(Ƞ,И)$ *be* $\mathcal{SS}$*s.* $(𝒬,𝕃)\vee(Ƞ,И) =_M \emptyset_{𝕃xB}$ *if and only if* $(𝒬, 𝕃) =_M \emptyset_{𝕃}$ *and* $(Ƞ, И) =_M \emptyset_{И}$.

Proof. Necessity: Let $(𝒬,𝕃)\vee(Ƞ,И) =_M (Ꝑ,𝕃xИ)$, where $Ꝑ(ϛ,ŭ) = 𝒬(ϛ)\cup Ƞ(ŭ)$, for all $(ϛ,ŭ)\in 𝕃xИ$. Let $\emptyset_{𝕃xB} =_M (T,𝕃xИ)$, where $T(ϛ,ŭ) = \emptyset$ for all $(ϛ,ŭ)\in 𝕃xИ$. Since, $Ꝑ(ϛ,ŭ) = T(ϛ,ŭ) = 𝒬(ϛ)\cup Ƞ(ŭ) = \emptyset$, *then* $𝒬(ϛ) = \emptyset$ for all $ϛ\in 𝕃$, and $Ƞ(ŭ) = \emptyset$ for all $ŭ\in И$. Hence, $(𝒬, 𝕃) =_M \emptyset_{𝕃}$ and $(Ƞ, И) =_M \emptyset_{И}$. Sufficiency: It is obvious. □

Proposition 20. *Let* $(𝒬,𝕃)$ *be a* $\mathcal{SS}$*. Then,* $(𝒬,𝕃)\vee(Ƞ,И) =_M \emptyset_{\emptyset}$ *if and only if* $(𝒬, 𝕃) =_M \emptyset_{\emptyset}$ *or* $(Ƞ, И) = \emptyset_{\emptyset}$.

Proposition 21. *Let* $(𝒬,𝕃)$, $(Ƞ,И)$, *and* $(Ꝑ,Ꞡ)$ *be* $\mathcal{SS}$*s. If* $(𝒬,𝕃) \tilde{\subseteq}_F (Ƞ,И)$, *then* $(𝒬, 𝕃)\vee(Ꝑ,Ꞡ) \tilde{\subseteq}_F (Ƞ,И)\vee(Ꝑ,Ꞡ)$ *and* $(Ꝑ,Ꞡ)\vee(𝒬, 𝕃) \tilde{\subseteq}_F (Ꝑ, Ꞡ)\vee(Ƞ, И)$.

Proof. Let $(𝒬,𝕃)\tilde{\subseteq}_F(Ƞ,И)$. Then, $𝕃\subseteq И$ and hence, $𝕃xꞠ\subseteq ИxꞠ$. Moreover, since $(𝒬,𝕃) \tilde{\subseteq}_F (Ƞ,И)$, then $𝒬(ϛ)\subseteq Ƞ(ϛ)$ for all $ϛ\in 𝕃$. Thus, $𝒬(ϛ)\cup Ꝑ(ŭ)\subseteq Ƞ(ϛ)\cup Ꝑ(ŭ)$ for all $(\tau, \omega) \in 𝕃xꞠ$. Thus, $(𝒬, 𝕃)\vee(Ꝑ,Ꞡ) \tilde{\subseteq}_F (Ƞ,И)\vee(Ꝑ,Ꞡ)$. The proof of $(Ꝑ,Ꞡ)\vee(𝒬, 𝕃)\tilde{\subseteq}_F(Ꝑ,Ꞡ)\vee(Ƞ,И)$ is analogous and therefore omitted. □

Proposition 22. *Let* $(𝒬,𝕃)$, $(Ƞ,И)$, *and* $(Ꝑ,Ꞡ)$ *be* $\mathcal{SS}$*s. If* $(𝒬,𝕃)\vee(Ꝑ,Ꞡ)\tilde{\subseteq}_F(Ƞ,И)\vee(Ꝑ,Ꞡ)$ *and* $(Ꝑ,Ꞡ)\neq_F \emptyset_{\emptyset}$, *then* $(𝒬,𝕃) \tilde{\subseteq}_F (Ƞ,И)$.

Proof. Let $(𝒬,𝕃)\vee(Ꝑ,Ꞡ)\tilde{\subseteq}_F(Ƞ,И)\vee(Ꝑ,Ꞡ)$ and $(Ꝑ,Ꞡ)\neq_F \emptyset_{\emptyset}$. Then, $𝕃xꞠ\subseteq ИxꞠ$ and $Ꞡ \neq \emptyset$. Hence, $𝕃\subseteq И$. Let $(𝒬,𝕃)\vee(Ꝑ,Ꞡ) = (Ƀ,𝕃xꞠ)$, where $Ƀ(ϛ,ŭ) = 𝒬(ϛ)\cup Ꝑ(ŭ)$ and $(Ƞ,И)\vee(Ꝑ,Ꞡ) = (Y,ИxꞠ)$, where $Y(ϛ,ŭ) = Ƞ(ϛ)\cup Ꝑ(ŭ)$, for all $(ϛ,ŭ)\in 𝕃xꞠ$. Since $𝕃\subseteq И$, $ϛ\in 𝕃$ implies that $ϛ\in И$. By assumption, since $(Ƀ,𝕃xꞠ)\tilde{\subseteq}_F(Y,ИxꞠ)$, then $Ƀ(ϛ,ŭ)\subseteq Y(ϛ,ŭ)$ for all $(ϛ,ŭ)\in 𝕃xꞠ$. Thus, $Ƀ(ϛ,ŭ) = 𝒬(ϛ)\cup Ꝑ(ŭ) \subseteq Ƞ(ϛ)\cup Ꝑ(ŭ) = Y(ϛ,ŭ)$, implying that $𝒬(ϛ)\subseteq Ƞ(ϛ)$ for all $ϛ\in 𝕃$. Thus, $(𝒬,𝕃)\tilde{\subseteq}_F(Ƞ,И)$. □

Note 7. *In* [69], *it was shown that if* $(𝒬,𝕃)$, $(Ƞ,И)$, $(Ꝑ,Ꞡ)$ *and* $(Ƀ,D)$ *are* $\mathcal{SS}$*s such that* $(𝒬,𝕃)\tilde{\subseteq}_J(Ƞ,И)$ *and* $(Ꝑ,Ꞡ)\tilde{\subseteq}_J(Ƀ,D)$, *then then it follows that* $(𝒬, 𝕃)\vee(Ꝑ,Ꞡ)\tilde{\subseteq}_J(Ƞ,И)\vee(Ƀ,D)$. *However, Proposition 23 demonstrates that this property also holds in the context of F-subsets, which are strictly stronger than J-subsets. This implies that the distributive behavior of the OR-product is preserved under a stricter inclusion relation.*

Proposition 23. *Let* $(𝒬,𝕃)$, $(Ƞ,И)$, $(Ꝑ,Ꞡ)$ *and* $(Ƀ,D)$ *be* $\mathcal{SS}$*s. If* $(𝒬,𝕃)\tilde{\subseteq}_F(Ƞ,И)$ *and* $(Ꝑ,Ꞡ)\tilde{\subseteq}_F(Ƀ,D)$, *then* $(𝒬, 𝕃)\vee(Ꝑ,Ꞡ) \tilde{\subseteq}_F (Ƞ,И)\vee(Ƀ,D)$ *and* $(Ꝑ,Ꞡ)\vee(𝒬, 𝕃)\tilde{\subseteq}_F(Ƀ,D)\vee(Ƞ,И)$.

Proof. Let $(\mathcal{Q},\mathbb{L})\widetilde{\subseteq}_F(\mathcal{T},\mathcal{N})$ and $(\mathcal{P},\mathcal{G})\widetilde{\subseteq}_F(\mathcal{B},D)$. Then, $\mathbb{L}\subseteq \mathcal{N}$ and $\mathcal{G}\subseteq D$. Hence, $\mathbb{L}x\mathcal{G}\subseteq \mathcal{N}xD$. By assumption, $\mathcal{Q}(\varsigma)\subseteq\mathcal{T}(\varsigma)$ for all $\varsigma\in\mathbb{L}$ and $\mathcal{P}(\hbar)\subseteq\mathcal{B}(\hbar)$ for all $\hbar\in\mathcal{G}$. Thus, $\mathcal{Q}(\varsigma)\cup\mathcal{P}(\hbar)\subseteq\mathcal{T}(\varsigma)\cup\mathcal{P}(\hbar)$ for all $(\tau,\hbar)\in\mathbb{L}x\mathcal{G}$. Thus, $(\mathcal{Q},\mathbb{L})\vee(\mathcal{P},\mathcal{G})\widetilde{\subseteq}_F(\mathcal{T},\mathcal{N})\vee(\mathcal{B},D)$. The proof of $(\mathcal{P},\mathcal{G})\vee(\mathcal{Q},\mathbb{L})\widetilde{\subseteq}_F(\mathcal{B},D)\vee(\mathcal{T},\mathcal{N})$ is analogous and therefore omitted. □

Proposition 24. *Let $(\mathcal{Q},\mathbb{L})$, $(\mathcal{T},\mathcal{N})$, $(\mathcal{P},\mathcal{G})$ and $(\mathcal{B},D)$ be $\mathcal{SS}$s. Then, $[(\mathcal{Q},\mathbb{L})\vee(\mathcal{T},\mathcal{N})]\cup_R[(\mathcal{P},\mathcal{G})\vee(\mathcal{B},D)]=_M[(\mathcal{Q},\mathbb{L})\cup_R(\mathcal{P},\mathcal{G})]\vee[(\mathcal{T},\mathcal{N})\cup_R(\mathcal{B},D)]$.*

Proposition 25. *Let $(\mathcal{Q},\mathbb{L})$ and $(\mathcal{T},\mathcal{N})$ be $\mathcal{SS}$s. Then, $\emptyset_{\mathbb{L}xB}\widetilde{\subseteq}_F(\mathcal{Q},\mathbb{L})\vee(\mathcal{T},\mathcal{N})$, $\emptyset_{\mathcal{N}x\mathbb{L}}\widetilde{\subseteq}_F(\mathcal{T},\mathcal{N})\vee(\mathcal{Q},\mathbb{L})$, $(\mathcal{Q},\mathbb{L})\vee(\mathcal{T},\mathcal{N})\widetilde{\subseteq}_F U_{\mathbb{L}xB}$ and $(\mathcal{T},\mathcal{N})\vee(\mathcal{Q},\mathbb{L})\widetilde{\subseteq}_F U_{\mathcal{N}x\mathbb{L}}$.*

Proposition 26 ([76]). *Let $(\mathcal{Q},\mathbb{L})$ be an $\mathcal{SS}$. Then, in general, $(\mathcal{Q},\mathbb{L})\vee(\mathcal{Q},\mathbb{L})\neq_J(\mathcal{Q},\mathbb{L})$; although it holds that $(\mathcal{Q},\mathbb{L})\widetilde{\subseteq}_L(\mathcal{Q},\mathbb{L})\vee(\Psi,\mathbb{L})$. Moreover, if $(\mathcal{Q},\mathbb{L})$ is a sublattice $\mathcal{SS}$, then $(\mathcal{Q},\mathbb{L})\vee(\mathcal{Q},\mathbb{L})=_L(\mathcal{Q},\mathbb{L})$.*

Note 8. *Proposition 26 demonstrates that the OR-product is not idempotent with respect to soft J-equality. However, under certain conditions, it is idempotent in the sense of soft L-subset.*

Regarding soft M-equality, we have the following results:

Proposition 27. *Let $(\mathcal{Q},\mathbb{L})$ be an $\mathcal{SS}$. Then, $(\mathcal{Q},\mathbb{L})\vee(\mathcal{Q},\mathbb{L})\neq_M(\mathcal{Q},\mathbb{L})$.*

Proof. OR-product is not idempotent under OR-product as regards soft M-equality since $\mathbb{L}x\mathbb{L}\neq\mathbb{L}$. □

Proposition 28 ([40]). *Let $(\mathcal{Q},\mathbb{L})$ and $(\mathcal{T},\mathcal{N})$ be $\mathcal{SS}$s. Then, $(\mathcal{Q},\mathbb{L})\wedge(\mathcal{T},\mathcal{N})^r=_M(\mathcal{Q},\mathbb{L})^r\vee(\mathcal{T},\mathcal{N})^r$.*

Note 9. *In [34], it was proposed that the AND-product distributes over the OR-product, and vice versa, with respect to soft M-equality. However, [39] demonstrated that these assertions do not hold due to the inequality of the $\mathcal{PS}$s of the $\mathcal{SS}$s on both sides of the distributive laws. Similarly, in [74], it was suggested that the AND-product distributes over the OR-product, and* vice versa, *with respect to soft J-equality; yet counterexamples provided in [75,76] disproved these claims. Finally, the correct formulations of the soft distributive laws were established as follows:*

Proposition 29 ([75,76]). *Let $(\mathcal{Q},\mathbb{L})$, $(\mathcal{T},\mathcal{N})$ and $(\mathcal{P},\mathcal{G})$ be $\mathcal{SS}$s. Then,*

(i) $(\mathcal{Q},\mathbb{L})\wedge((\mathcal{T},\mathcal{N})\vee(\mathcal{P},\mathcal{G}))\widetilde{\subseteq}_L((\mathcal{Q},\mathbb{L})\wedge(\mathcal{T},\mathcal{N}))\vee((\mathcal{Q},\mathbb{L})\wedge(\mathcal{P},\mathcal{G}))$.

(ii) $(\mathcal{Q},\mathbb{L})\vee((\mathcal{T},\mathcal{N})\wedge(\mathcal{P},\mathcal{G}))\widetilde{\subseteq}_L((\mathcal{Q},\mathbb{L})\vee(\mathcal{T},\mathcal{N}))\wedge((\mathcal{Q},\mathbb{L})\vee(\mathcal{P},\mathcal{G}))$.

(iii) $((\mathcal{Q},\mathbb{L})\wedge(\mathcal{T},\mathcal{N}))\vee(\mathcal{P},\mathcal{G})\widetilde{\subseteq}_L((\mathcal{Q},\mathbb{L})\vee(\mathcal{P},\mathcal{G}))\wedge(\mathcal{T},\mathcal{N})\vee(\mathcal{P},\mathcal{G}))$.

(iv) $((\mathcal{Q},\mathbb{L})\vee(\mathcal{T},\mathcal{N}))\wedge(\mathcal{P},\mathcal{G})\widetilde{\subseteq}_L((\mathcal{Q},\mathbb{L})\wedge(\mathcal{P},\mathcal{G}))\vee(\mathcal{T},\mathcal{N})\wedge(\mathcal{P},\mathcal{G}))$.

Proposition 30 ([82]). *Let $(\mathcal{Q},\mathbb{L})$ and $(\mathcal{T},\mathcal{N})$ be $\mathcal{SS}$s. Then, $(\mathcal{Q},\mathbb{L})\wedge(\mathcal{T},\mathcal{N})\widetilde{\subseteq}_F(\mathcal{Q},\mathbb{L})\vee(\mathcal{T},\mathcal{N})$.*

Note 10. *In ([82]), it is demonstrated by a counterexample that $(\mathcal{Q},\mathbb{L})\wedge(\mathcal{T},\mathcal{N})=_M(\mathcal{Q},\mathbb{L})\vee(\mathcal{T},\mathcal{N})$ does not imply $(\mathcal{Q},\mathbb{L})=_M(\mathcal{T},\mathcal{N})$.*

Proposition 31 ([82]). *Let $(\mathcal{Q},\mathbb{L})$ and $(\mathcal{T},\mathcal{N})$ be $\mathcal{SS}$s. $(\mathcal{Q},\mathbb{L})\wedge(\mathcal{T},\mathcal{N})=_M(\mathcal{Q},\mathbb{L})\vee(\mathcal{T},\mathcal{N})$ if and only if $\mathcal{Q}$ and $\mathcal{T}$ are the same CFs.*

Corollary 2 ([82]). *Let $(\mathcal{Q},\mathbb{L})$ be an $\mathcal{SS}$. $(\mathcal{Q},\mathbb{L})\wedge(\mathcal{Q},\mathbb{L})=_M(\mathcal{Q},\mathbb{L})\vee(\mathcal{Q},\mathbb{L})$ if and only if $\mathcal{Q}$ is an CF.*

4. Distributions

In this section, we establish the distributive properties of the OR-product over the restricted, extended, and soft binary piecewise operations of $\mathcal{SS}$, respectively.

4.1. Distributions of OR-Product over Restricted $\mathcal{SS}$ Operations

In this subsection, we examine the distributive properties of the OR-product over restricted $\mathcal{SS}$ operations. It is important to note that in [48], only the left distributive laws of the OR-product over restricted $\mathcal{SS}$ operations were established, and the condition where the intersection of the $\mathcal{PS}$s is empty was overlooked. In contrast, here we address the distributions of the OR-product over restricted $\mathcal{SS}$ operations while explicitly considering and including these previously omitted cases.

Theorem 1. *Let (Ձ,IL),(Ꝭ,И), and (Ꝑ,GƆ) be $\mathcal{SS}$s. Then,*

(1) (Ձ,IL)∨[(Ꝭ,И)$\cup_R$(Ꝑ,GƆ)])]$=_M$[(Ձ,IL)∨(Ꝭ,И)]$\cup_R$[(Ձ,IL)∨(Ꝑ,GƆ)].

(2) (Ձ,IL)∨[(Ꝭ,И)$\cap_R$(Ꝑ,GƆ)])]$=_M$[(Ձ,IL)∨(Ꝭ,И)]$\cap_R$[(Ձ,IL)∨(Ꝑ,GƆ)].

(3) [(Ձ,IL)$\cap_R$(Ꝭ,И)]∨(Ꝑ,GƆ))]$=_M$[(Ձ,IL)∨H,GƆ)]$\cap_R$[(Ꝭ,И)∨(Ꝑ,GƆ)].

(4) [(Ձ,IL)$\cup_R$(Ꝭ,И)]∨(Ꝑ,GƆ))]$=_M$[(Ձ,IL)∨(Ꝑ,GƆ)]$\cup_R$[(Ꝭ,И)∨(Ꝑ,GƆ)].

Proof . (1) The $\mathcal{PS}$ of the LHS is *ILx*(И∩GƆ), and the $\mathcal{PS}$ of the RHS is (*ILx*И)∩ (*ILx*GƆ), where *ILx*(И∩GƆ) = (*ILx*И)∩ (*ILx*GƆ). Assume that (Ꝭ,И)$\cup_R$ (Ꝑ,GƆ)$=_M$(*K*, И∩GƆ), where *K*(ŭ) = Ꝭ(ŭ)∪Ꝑ(ŭ) for all ŭ∈И∩GƆ. Let (Ձ,IL)∨(*K*, И∩GƆ)$=_M$(*Z*,*ILx*(И∩GƆ)), where *Z*(ʓ,ŭ) = Ձ(ʓ)∪*K*(ŭ) for all (ʓ,ŭ)∈ *ILx*(И∩GƆ). Hence, for all (ʓ,ŭ)∈ *ILx*(И∩GƆ),

$$Z(ʓ, ŭ) = Ձ(ʓ)\cup K(ŭ) = Ձ(ʓ)\cup(Ꝭ(ŭ) \cup Ꝑ(ŭ))$$

Let (Ձ,IL)∨(Ꝭ,И)$=_M$(*M*,*ILx*И), where *M*(ʓ,ŭ) = Ձ(ʓ)∪Ꝭ(ŭ) for all (ʓ,ŭ)∈*ILx*И and (Ձ,IL)∨ (Ꝑ,GƆ)$=_M$(*N*,*ILx*GƆ), where *N*(ʓ,ŭ) = Ձ(ʓ)∪Ꝑ(ŭ) for all (ʓ,ŭ)∈*ILx*GƆ. Let (*M*,*ILx*И) $\cup_R$(*N*,*ILx*GƆ) $=_M$(Ƀ,(*ILx*И)∪(*ILx*GƆ)), where Ƀ(ʓ,ŭ) = *M*(ʓ,ŭ)∪*N*(ʓ,ŭ) for all (ʓ,ŭ)∈ (*ILx*И)∩(*ILx*GƆ). Thus, for all (ʓ,ŭ)∈ (*ILx*И)∩(*ILx*GƆ),

$$Ƀ(ʓ, ŭ) = (ʓ, ŭ)\cup N(ʓ, ŭ) = Ձ(ʓ)\cup Ꝭ(ŭ)]\cup[Ձ(ʓ) \cup Ꝑ(ŭ)]$$

Note that if И∩GƆ = ∅, then *ILx*(И∩GƆ) = (*ILx*И)∩(*ILx*GƆ) = ∅, which implies that both sides equal $\emptyset_\emptyset$. Therefore, in all cases, *Z* = Ƀ, and the proof is complete.

(3) The $\mathcal{PS}$ of the LHS *is* (IL∩И)*x*GƆ, the $\mathcal{PS}$ of the RHS is (*ILx*GƆ)∩(И*x*GƆ), where (IL∩И)*x*GƆ = (*ILx*GƆ)∩(И*x*GƆ). Let (Ձ,IL)$\cap_R$(Ꝭ,И)$=_M$(*K*,IL∩И), where *K*(ʓ) = Ձ(ʓ)∩Ꝭ(ʓ) for all ʓ∈IL∩И and (*K*,IL∩И)∨(Ꝑ,GƆ)$=_M$(*S*,(IL∩И)*x*GƆ), where *S*(ʓ,ŭ) = *K*(ʓ)∪Ꝑ(ŭ) for all (ʓ,ŭ)∈(IL∩И) *x*GƆ. Hence, for all (ʓ,ŭ)∈(IL∩И)*x*GƆ,

$$L(ʓ,\ ŭ) = [Ձ(ʓ)\cap Ꝭ(ʓ)]\cup Ꝑ(ŭ) = [Ձ(ʓ)\cup Ꝑ(ŭ)] \cap [Ꝭ(ʓ)\cup Ꝑ(ŭ)]$$

Let (Ձ,IL)∨(Ꝑ,GƆ)$=_M$(*M*,*ILx*GƆ), where *M*(ʓ,ŭ) = Ձ(ʓ)∪Ꝑ(ŭ) for all (ʓ,ŭ)∈*ILx*GƆ and (Ꝭ,И)∨ (Ꝑ,GƆ) = (*J*, И*x*GƆ), where *J*(ʓ,ŭ) = Ꝭ(ʓ)∪Ꝑ(ŭ) for all (ʓ,ŭ)∈И*x*GƆ. *Let* (*M*,*ILx*GƆ)$\cap_R$(*J*, И*x*GƆ) $=_M$(Ƀ,(*ILx*GƆ)∩(И*x*GƆ)), where Ƀ(ʓ,ŭ) = *M*(ʓ,ŭ)∩*J*(ʓ,ŭ) for all (ʓ,ŭ)(*ILx*GƆ)∩(И*x*GƆ). Thus, for all (ʓ, ŭ)∈ (*ILx*GƆ)∩(И*x*GƆ),

$$Ƀ(ʓ,\ ŭ) = M(ʓ, ŭ)\cap J(ʓ, ŭ) = [Ձ(ʓ)\cup Ꝑ(ŭ)]\cap[Ꝭ(ʓ) \cup Ꝑ(ŭ)]$$

Note that if $IL\cap И = \emptyset$, then $(IL\cap И)xGЭ = (ILxGЭ)\cap(ИxGЭ) = \emptyset$, which implies that both sides equal $\emptyset_{\emptyset}$. Thus, in all cases, $L = Ƀ$, and the proof is complete. □

Theorem 2. *$(S_E(U), \cap_R, \vee)$ is a commutative hemiring with identity $\emptyset_E$ as regards soft L-equality (and hence J-equality).*

Proof. Let $(F, \mathbb{L}), (G, И) \in S_E(U)$. It is known that $(S_E(U), \cap_R)$ is a commutative monoid with identity U_E [41,55]. Hence, $(S_E(U), \cap_R)$ is a semigroup. Moreover, $(S_E(U), \vee)$ is a semigroup in the sense of soft L-equality (and therefore also in the sense of soft J-equality). The OR-product distributes over restricted intersection from both sides. Therefore, $(S_E(U), \cap_R, \vee)$ forms a semiring in the sense of soft L-equality (and consequently J-equality). Since the restricted intersection $\cap_R$ is commutative in $S_E(U)$, and for any $(F, \mathbb{L}) \in S_E(U)$, $(F, \mathbb{L}) \cap_R U_E =_M U_E \cap_R (F, \mathbb{L}) =_M (F, \mathbb{L})$ [41,55] and $(F, \mathbb{L}) \vee U_E =_L U_E \vee (F, \mathbb{L}) =_L U_E$, it follows that U_E is the zero element of $(S_E(U), \cap_R, \vee)$. Therefore, $(S_{\mathbb{L}}(U), \cap_R, \vee)$ is a hemiring as regards soft L-equality (and hence J-equality). Additionally, since $(F, \mathbb{L}) \vee (G, И) =_L (G, И) \vee (F, \mathbb{L})$ and $(F, \mathbb{L}) \vee \emptyset_E =_L \emptyset_E \vee (F, \mathbb{L}) =_L (F, \mathbb{L})$, $(S_{\mathbb{L}}(U), \cap_R, \vee)$ is a commutative hemiring with identity $\emptyset_E$ as regards soft L-equality (and hence J-equality). □

4.2. Distributions of OR-Product over Extended $\mathcal{SS}$ Operations

In this subsection, we examine the distributions of the OR-product over extended $\mathcal{SS}$ operations. It is worth noting that in [48], only the left distributive properties of the OR-product over extended $\mathcal{SS}$ operations were established, and some parts of the proof were incomplete. In contrast, here we provide a comprehensive treatment of the distributions of the OR-product over restricted $\mathcal{SS}$ operations, ensuring that no parts of the proof are omitted.

Theorem 3. *Let $(Ձ,IL)$, $(Պ,И)$, and $(Ք,GЭ)$ be $\mathcal{SS}$s. Then,*

a. $(Ձ,IL)\vee[(Պ,И)\cap_\varepsilon(Ք,GЭ)])]=_M[(Ձ,IL)\vee(Պ,И)]\cap_\varepsilon[(Ձ,IL)\vee(Ք,GЭ)]$ [39].
b. $(Ձ,IL)\vee[(Պ,И)\cup_\varepsilon(Ք,GЭ)])]=_M[(Ձ,IL)\vee(Պ,И)]\cup_\varepsilon[(Ձ,IL)\vee(Ք,GЭ)]$ [39].
c. $[(Ձ,IL)\cup_\varepsilon(Պ,И)]\vee(Ք,GЭ))]=_M[(Ձ,IL)\vee(H,GЭ)]\cup_\varepsilon[(Պ,И)\vee(Ք,GЭ)]$.
d. $[(Ձ,IL)\cap_\varepsilon(Պ,И)]\vee(Ք,GЭ))]=_M[(Ձ,IL)\vee(Ք,GЭ)]\cap_\varepsilon[(Պ,И)\vee(Ք,GЭ)]$.

Proof. (a) The $\mathcal{PS}$ of the LHS is $ILx(И\cup GЭ)$, the $\mathcal{PS}$ of the RHS is $(ILxИ)\cup(ILxGЭ)$, where $ILx(И\cup GЭ) = (ILxИ)\cup(ILxGЭ)$. Let $(Պ,И)\cap_\varepsilon(Ք,GЭ)=_M(M,И\cup GЭ)$, where

$$M(ȗ) = \begin{cases} Պ(ȗ), & ȗ \in И\setminus GЭ \\ Ք(ȗ), & ȗ \in GЭ\setminus И \\ Պ(ȗ) \cap Ք(ȗ), & ȗ \in И \cap GЭ \end{cases}$$

for all $ȗ\in И\cup GЭ$, and let $(Ձ,IL)\vee(M,И\cup GЭ)=_M(S,ILx(И\cup GЭ))$, where $S(ⵣ,ȗ) = Ձ(ⵣ)\cup M(ȗ)$ for all $(ⵣ,ȗ)\in ILx(И\cup GЭ)$. (Note here that $ⵣ\in IL$ and $ȗ\in И\cup GЭ$). Hence,

$$S(ⵣ, ȗ) = \begin{cases} Ձ(ⵣ) \cup Պ(ȗ), & (ⵣ, ȗ) \in \mathbb{L}x(И\setminus GЭ) \\ Ձ(ⵣ) \cup Ք(ȗ), & (ⵣ, ȗ) \in \mathbb{L}x(GЭ\setminus И) \\ Ձ(ⵣ) \cup [Պ(ȗ) \cap Ք(ȗ)], & (ⵣ, ȗ) \in \mathbb{L}x(И \cap GЭ) \end{cases}$$

Let (Ձ,𝕃)∨(Ⴄ,И)$=_M$(K,𝕃xИ), where K(ʓ,ũ) = Ձ(ʓ)∪Ⴄ(ũ) for all (ʓ,ũ)∈𝕃xИ and (Ձ,𝕃)∨(Ꝑ,Ꮹ)$=_M$(Ѣ,𝕃xᏱ), where Ѣ(ʓ,ũ) = Ձ(ʓ)∪Ꝑ(ũ) for all (ʓ,ũ)∈𝕃xᏱ. Suppose that (K,𝕃xИ) $\cap_\varepsilon$ (Ѣ,𝕃xᏱ)$=_M$(Υ,(𝕃xИ)∪(𝕃xᏱ)), where

$$Υ(ʓ,ũ)=\begin{cases} K(ʓ,ũ), & (ʓ,ũ)\in(𝕃xИ)\backslash(𝕃xᏱ) \\ Ѣ(ʓ,ũ), & (ʓ,ũ)\in(𝕃xᏱ)\backslash(𝕃xИ) \\ K(ʓ,ũ)\cap Ѣ(ʓ,ũ), & (ʓ,ũ)\in(𝕃xИ)\cap(𝕃xᏱ)\end{cases}$$

for all (ʓ,ũ)∈(𝕃xИ)∪(𝕃xᏱ). Hence,

$$Υ(ʓ,ũ)=\begin{cases} Ձ(ʓ)\cup Ⴄ(ũ), & (ʓ,ũ)\in(𝕃xИ)\backslash(𝕃xᏱ) \\ Ձ(ʓ)\cup Ꝑ(ũ), & (ʓ,ũ)\in(𝕃xᏱ)\backslash(𝕃xИ) \\ \{Ձ(ʓ)\cup Ⴄ(ũ)]\cap[Ձ(ʓ)\cup Ꝑ(ũ)], & (ʓ,ũ)\in(𝕃xИ)\cap(𝕃xᏱ)\end{cases}$$

Here, if И = Ᏹ $=\emptyset$, then И∪Ᏹ $=\emptyset$, which implies that both sides are equal $\emptyset_\emptyset$. Thus, in all cases, N = Υ, and the proof is complete.

(c) The $\mathcal{PS}$ of the LHS is (𝕃∪И)xᏱ, the $\mathcal{PS}$ of the RHS is (𝕃xᏱ)∪(ИxᏱ), where (𝕃∪И)xᏱ = (𝕃xᏱ)∪(ИxᏱ). Let (Ձ,𝕃)$\cup_\varepsilon$(Ⴄ,И)$=_M$(M,𝕃∪И), where for *all* ʓ∈𝕃∪И,

$$M(ʓ)=\begin{cases} Ձ(ʓ), & ʓ\in 𝕃\backslash И \\ Ⴄ(ʓ), & ʓ\in И\backslash 𝕃 \\ Ձ(ʓ)\cup Ⴄ(ʓ), & ʓ\in 𝕃\cap И\end{cases}$$

Assume that (M,𝕃∪И)∨(Ꝑ,Ᏹ)$=_M$(S,(𝕃∪И)xᏱ), where S(ʓ,ũ)=M(ʓ)∪Ꝑ(ũ) for all (ʓ,ũ)∈(𝕃∪И)xᏱ. (Here ʓ∈𝕃∪И and ũ∈Ᏹ). Hence,

$$S(ʓ,ũ)=\begin{cases} Ձ(ʓ)\cup Ꝑ(ũ), & (ʓ,ũ)\in(𝕃\backslash И)xᏱ \\ Ⴄ(ʓ)\cup Ꝑ(ũ), & (ʓ,ũ)\in(И\backslash 𝕃)xᏱ \\ [Ձ(ʓ)\cup Ⴄ(ʓ)]\cup Ꝑ(ũ), & (ʓ,ũ)\in(𝕃\cap И)xᏱ\end{cases}$$

Let (Ձ,𝕃)∨(Ꝑ,Ᏹ)$=_M$(K,𝕃xᏱ), where K(ʓ,ũ) = Ձ(ʓ)∪Ꝑ(ũ) for all (ʓ,ũ)∈(𝕃xᏱ). Let (Ⴄ,И)∨(Ꝑ,Ᏹ)$=_M$(Ѣ,ИxᏱ), where Ѣ(ʓ,ũ) = Ⴄ(ʓ)∪Ꝑ(ũ). Assume that (K,𝕃xᏱ) $\cup_\varepsilon$(Ѣ,ИxᏱ) $=_M$(Υ,(𝕃xᏱ)∪(ИxᏱ)), where

$$Υ(ʓ,ũ)=\begin{cases} K(ʓ,ũ), & (ʓ,ũ)\in(𝕃xᏱ)\backslash(ИxᏱ) \\ Ѣ(ʓ,ũ), & (ʓ,ũ)\in(ИxᏱ)\backslash(𝕃xᏱ) \\ K(ʓ,ũ)\cup Ѣ(ʓ,ũ), & (ʓ,ũ)\in(𝕃xᏱ)\cap(ИxᏱ)\end{cases}$$

Therefore,

$$Υ(ʓ,ũ)=\begin{cases} Ձ(ʓ)\cup Ꝑ(ũ), & (ʓ,ũ)\in(𝕃xᏱ)\backslash(ИxᏱ) \\ Ⴄ(ʓ)\cup Ꝑ(ũ), & (ʓ,ũ)\in(ИxᏱ)(𝕃xᏱ) \\ [Ձ(ʓ)\cup Ꝑ(ũ)]\cup[Ⴄ(ʓ)\cup Ꝑ(ũ)], & (ʓ,ũ)\in(𝕃xᏱ)\cap(ИxᏱ)\end{cases}$$

Here, if 𝕃 = И $=\emptyset$, then 𝕃∪И $=\emptyset$, which implies that both sides are equal $\emptyset_\emptyset$. Thus, in all circumstances, N = Υ, and the proof is complete. □

Theorem 4. *(S_E(U),$\cap_\varepsilon$, ∨) is a commutative hemiring with identity $\emptyset_E$ as regards soft L-equality (and hence J-equality).*

Proof. Let (F, 𝕃), (G, И) ∈ S_E(U). It is known that (S_E(U),$\cap_\varepsilon$) is a commutative monoid with identity $\emptyset_\emptyset$ [41,55]. Hence, (S_E(U),$\cap_\varepsilon$) is a semigroup. Moreover, (S_E(U),∨) is a semigroup in the sense of soft L-equality (hence J-equality). The OR-product distributes over extended intersection from both sides. Therefore, (S_E(U),$\cap_\varepsilon$, ∨) is a semiring. Since the extended intersection $\cap_{\varepsilon}$ is commutative in $S_E(U)$, and for any $(\mathrm{F}, \mathbb{L}) \in \mathrm{S_E}\ (\mathrm{U})$, $(F, \mathbb{L}) \cap_{\varepsilon} \emptyset_{\emptyset} =_M \emptyset_{\emptyset} \cap_{\varepsilon} (F, \mathbb{L}) =_M (F, \mathbb{L})$ by [41,55] and $(F, \mathbb{L}) \vee \emptyset_{\emptyset} =_M \emptyset_{\emptyset} \vee (F, \mathbb{L}) =_M \emptyset_{\emptyset}$, it follows that $\emptyset_{\emptyset}$ is the zero element of $(S_E(U),\cap_{\varepsilon}, \vee)$. Therefore, $(S_E(U),\cap_{\varepsilon}, \vee)$ is a hemiring as regards soft L-equality (hence J-equality). In addition, since $(F, \mathbb{L}) \vee (G, \text{И}) =_L (G, \text{И}) \vee (F, \mathbb{L})$ and $(F, \mathbb{L}) \vee \emptyset_E =_L \emptyset_E \vee (F, \mathbb{L}) =_L (F, \mathbb{L})$, $(S_E(U),\cap_{\varepsilon}, \vee)$ forms a commutative hemiring with identity $\emptyset_E$ as regards soft L-equality (and hence J-equality). □

4.3. Distributions of OR-Product over Soft Binary Piecewise Operation

In this subsection, we investigate the distributions of OR-product over soft binary piecewise operations.

Theorem 5. *Let (Ձ,ɪʟ),(Պ,И), and (Ꝑ,Ꮹ) be $\mathcal{SS}$s. Then,*

(1) $(\text{Ձ},\mathbb{L})\vee[(\text{Պ},\text{И})\widetilde{\cap}(\text{Ꝑ},\text{Ꮹ})]=_M[(\text{Ձ},\mathbb{L})\vee(\text{Պ},\text{И})]\widetilde{\cap}[(\text{Ձ},\mathbb{L})\vee(\text{Ꝑ},\text{Ꮹ})]$.

(2) $(\text{Ձ},\mathbb{L})\vee[(\text{Պ},\text{И})\widetilde{\cup}(\text{Ꝑ},\text{Ꮹ})]=_M[(\text{Ձ},\mathbb{L})\vee(\text{Պ},\text{И})]\widetilde{\cup}[(\text{Ձ},\mathbb{L})\vee(\text{Ꝑ},\text{Ꮹ})]$.

(3) $[(\text{Ձ},\mathbb{L})\widetilde{\cup}(\text{Պ},\text{И})]\vee(\text{Ꝑ},\text{Ꮹ})=_M[(\text{Ձ},\mathbb{L})\vee(\text{Ꝑ},\text{Ꮹ})]\widetilde{\cup}[(\text{Պ},\text{И})\vee(\text{Ꝑ},\text{Ꮹ})]$.

(4) $[(\text{Ձ},\mathbb{L})\widetilde{\cap}(\text{Պ},\text{И})]\vee(\text{Ꝑ},\text{Ꮹ})=_M[(\text{Ձ},\mathbb{L})\vee(\text{Ꝑ},\text{Ꮹ})]\widetilde{\cap}[(\text{Պ},\text{И})\vee(\text{Ꝑ},\text{Ꮹ})]$.

Proof. (1) The $\mathcal{PS}$ of the $\mathcal{SS}$s of both sides is $\mathbb{L}x\text{И}$. Let $(\text{Պ},\text{И}) \widetilde{\cap} (\text{Ꝑ},\text{Ꮹ})=_M(T,\text{И})$, where

$$T(\text{ŭ}) = \begin{cases} \text{Պ}(\text{ŭ}), & \text{ŭ} \in \text{И}\backslash\text{Ꮹ} \\ \text{Պ}(\text{ŭ}) \cap \text{Ꝑ}(\text{ŭ}), & \text{ŭ} \in \text{И} \cap \text{Ꮹ} \end{cases}$$

for all ŭ∈И. Let $(\text{Ձ},\mathbb{L})\vee(T,\text{И})=_M(S,\mathbb{L}x\text{И})$, where $S(\text{ȥ},\text{ŭ}) = \text{Ձ}(\text{ȥ})\cup T(\text{ŭ})$ for all $(\text{ȥ},\text{ŭ})\in\mathbb{L}x\text{И}$. Hence,

$$S(\text{ȥ}, \text{ŭ}) = \begin{cases} \text{Ձ}(\text{ȥ}) \cup \text{Պ}(\text{ŭ}), & (\text{ȥ}, \text{ŭ}) \in \mathbb{L}x(\text{И}\backslash\text{Ꮹ}) \\ \text{Ձ}(\text{ȥ}) \cup [\text{Պ}(\text{ŭ}) \cap \text{Ꝑ}(\text{ŭ})], & (\text{ȥ}, \text{ŭ}) \in \mathbb{L}x(\text{И} \cap \text{Ꮹ}) \end{cases}$$

for all $(\text{ȥ},\text{ŭ})\in\mathbb{L}x\ \text{И}$.

Let $(\text{Ձ},\mathbb{L})\vee(\text{Պ},\text{И})=_M(K,\mathbb{L}x\text{И})$, where $K(\text{ȥ},\text{ŭ}) = \text{Ձ}(\text{ȥ})\cup\text{Պ}(\text{ŭ})$ for all $(\text{ȥ},\text{ŭ})\in\mathbb{L}x\text{И}$, and let $(\text{Ձ},\mathbb{L})\vee(\text{Ꝑ},\text{Ꮹ})=_M(\text{Ƀ},\mathbb{L}x\text{Ꮹ})$, where $\text{Ƀ}(\text{ȥ},\text{ŭ}) = \text{Ձ}(\text{ȥ})\cup\text{Ꝑ}(\text{ŭ})$ for all $(\text{ȥ},\text{ŭ})\in\mathbb{L}x\text{Ꮹ}$. Assume that $(K,\mathbb{L}x\text{И})\widetilde{\cap}(\text{Ƀ},\mathbb{L}x\text{Ꮹ}) = (R,(\mathbb{L}x\text{И})$, where

$$R(\text{ȥ}, \text{ŭ}) = \begin{cases} K(\text{ȥ}, \text{ŭ}), & (\text{ȥ}, \text{ŭ}) \in (\mathbb{L}x\text{И})\backslash(\mathbb{L}x\text{Ꮹ}) \\ K(\text{ȥ}, \text{ŭ}) \cap \text{Ƀ}(\text{ȥ}, \text{ŭ}), & (\text{ȥ}, \text{ŭ}) \in (\mathbb{L}x\text{И}) \cap (\mathbb{L}x\text{Ꮹ}) \end{cases}$$

for all $(\text{ȥ},\text{ŭ})\in\mathbb{L}x\text{И}$. Hence,

$$R(\text{ȥ}, \text{ŭ}) = \begin{cases} \text{Ձ}(\text{ȥ}) \cup \text{Պ}(\text{ŭ}), & (\text{ȥ}, \text{ŭ}) \in (\mathbb{L}x\text{И})\backslash(\mathbb{L}x\text{Ꮹ}) \\ \{\text{Ձ}(\text{ȥ}) \cup \text{Պ}(\text{ŭ})] \cap [\text{Ձ}(\text{ȥ}) \cup \text{Ꝑ}(\text{ŭ})], & (\text{ȥ}, \text{ŭ}) \in (\mathbb{L}x\text{И}) \cap (\mathbb{L}x\text{Ꮹ}) \end{cases}$$

for all $(\text{ȥ},\text{ŭ})\in\mathbb{L}x\text{И}$. Here, if $\mathbb{L} = \text{И} = \emptyset$, then $\mathbb{L}x\text{И} = \emptyset$, which implies that both sides equal $\emptyset_{\emptyset}$. Thus, in all circumstances, $S = R$, and the proof is complete.

(3) The $\mathcal{PS}$ of the $\mathcal{SS}$ of both sides is $\mathbb{L}x\text{Ꮹ}$. Let $(\text{Ձ},\mathbb{L})\widetilde{\cup}(\text{Պ},\text{И})=_M(T,\mathbb{L})$ where

$$T(\text{ȥ}) = \begin{cases} \text{Ձ}(\text{ȥ}), & \text{ȥ} \in \mathbb{L}\backslash\text{И} \\ \text{Ձ}(\text{ȥ}) \cup \text{Պ}(\text{ȥ}), & \text{ȥ} \in \mathbb{L} \cap \text{И} \end{cases}$$

for all $\varsigma \in \mathbb{L}$. Let $(M,\mathbb{L}) \vee (\wp, \mho) =_M (S,(\mathbb{L}x\mho))$, where $S(\varsigma,\mathfrak{u}) = T(\varsigma) \cup \wp(\mathfrak{u})$, for all $(\varsigma,\mathfrak{u}) \in \mathbb{L}x\mho$. Hence,

$$S(\varsigma,\mathfrak{u}) = \begin{cases} \mathfrak{Q}(\varsigma) \cup \wp(\mathfrak{u}), & (\varsigma,\mathfrak{u}) \in (\mathbb{L} \backslash \mathcal{N})x\mho \\ [\mathfrak{Q}(\varsigma) \cup \eta(\varsigma)] \cup \wp(\mathfrak{u}), & (\varsigma,\mathfrak{u}) \in (\mathbb{L} \cap \mathcal{N})x\mho \end{cases}$$

for all $(\varsigma,\mathfrak{u}) \in \mathbb{L}x\mho$.

Let $(\mathfrak{Q},\mathbb{L}) \vee (\wp,\mho) =_M (K,\mathbb{L}x\mho)$, where $K(\varsigma,\mathfrak{u}) = \mathfrak{Q}(\varsigma) \cup \wp(\mathfrak{u})$, for all $(\varsigma,\mathfrak{u}) \in \mathbb{L}x\mho$ and let $(\eta,\mathcal{N}) \vee (\wp,\mho) =_M (\mathcal{B},\mathcal{N}x\mho)$, where $\mathcal{B}(\varsigma,\mathfrak{u}) = \eta(\varsigma) \cup \wp(\mathfrak{u})$, for all $(\varsigma,\mathfrak{u}) \in \mathbb{L}x\mho$. Assume that $(K,\mathbb{L}x\mho)\, \widetilde{\cup} (\mathcal{B},\mathcal{N}x\mho) =_M (R,\mathbb{L}x\mho)$, where

$$R(\varsigma,\mathfrak{u}) = \begin{cases} K(\varsigma,\mathfrak{u}), & (\varsigma,\mathfrak{u}) \in (\mathbb{L}x\mho) \backslash (\mathcal{N}x\mho) \\ K(\varsigma,\mathfrak{u}) \cup \mathcal{B}(\varsigma,\mathfrak{u}), & (\varsigma,\mathfrak{u}) \in (\mathbb{L}x\mho) \cap (\mathcal{N}x\mho) \end{cases}$$

for all $(\varsigma,\mathfrak{u}) \in \mathbb{L}x\mho$. Thus,

$$R(\varsigma,\mathfrak{u}) = \begin{cases} \mathfrak{Q}(\varsigma) \cup \wp(\mathfrak{u}), & (\varsigma,\mathfrak{u}) \in (\mathbb{L}x\mho) \backslash (\mathcal{N}x\mho) \\ [\mathfrak{Q}(\varsigma) \cup \wp(\mathfrak{u})] \cup [\eta(\varsigma) \cup \wp(\mathfrak{u})], & (\varsigma,\mathfrak{u}) \in (\mathbb{L}x\mho) \cap (\mathcal{N}x\mho) \end{cases}$$

Here, if $\mathbb{L} = \mho = \emptyset$, then $\mathbb{L}x\mho = \emptyset$, which implies that both sides equal $\emptyset_\emptyset$. Thus, in all circumstances, $S = R$, and the proof is complete. □

Note 11. *Since soft binary piecewise operations are non-associative operations in $S_E(U)$ [83], they do not form a semigroup in $S_E(U)$. Additionally, because the OR-product is not closed in$S_{\mathbb{L}}(U)$, the structure $(S_{\mathbb{L}}(U),V)$ cannot be a semigroup even under soft L-equality. Hence, the OR-product combined with soft binary piecewise operation cannot form any algebraic structure in either in $S_E(U)$ or in $S_{\mathbb{L}}(U)$.*

5. Int-uni Decision-Making Method Applied to OR-Product

In this section, the int-uni operator and int-uni decision function, as defined by Çağman and Enginoğlu [8], are applied for the OR-product to develop an int-uni decision-making method.

Throughout this section, all OR-products of the $\mathcal{SS}$s over U are assumed to be contained in the set $V(U)$. The approximation function of the OR-product of $(₹,\mathcal{T})$ and $(\mathfrak{F},\mathcal{W})$, denoted by $₹_{\mathcal{T}} V \mathfrak{F}_{\mathcal{W}}$, is defined as follows:

$$₹_{\mathcal{T}} V \mathfrak{F}_{\mathcal{W}} : \mathcal{T}x\mathcal{W} \to P(U),$$

where $₹_{\mathcal{T}} V \mathfrak{F}_{\mathcal{W}}(t,w) = ₹(t) \cup \mathfrak{F}(w)$ for all $(t,w) \in \mathcal{T}\text{x}\mathcal{W}$.

Definition 16 ([8]). *Let $(₹,\mathcal{T})$ and $(\mathfrak{F},\mathcal{W})$ be $\mathcal{SS}$ over U. Then, int-uni operators for OR-product, denoted by $int_t - uni_w$ and $int_w - uni_t$ are defined respectively as*

$$int_t - uni_w : V(U) \to P(U),$$

$$int_t - uni_w(₹_{\mathcal{T}} V \mathfrak{F}_{\mathcal{W}}) = \bigcap_{t \in \mathcal{T}} (\bigcup_{w \in \mathcal{W}} (₹_{\mathcal{T}} V \mathfrak{F}_{\mathcal{W}}(t,w)))$$

$$int_w - uni_t : V(U) \to P(U),$$

$$int_w - uni_t(₹_{\mathcal{T}} V \mathfrak{F}_{\mathcal{W}}) = \bigcap_{w \in \mathcal{W}} (\bigcup_{t \in \mathcal{T}} (₹_{\mathcal{T}} V \mathfrak{F}_{\mathcal{W}}(t,w)))$$

Definition 17 ([8]). *Let $(₹, \mathcal{T})V(\mathfrak{F}, \mathcal{W}) \in V(U)$. Then, int-uni decision function for OR-product, denoted by int-uni are defined by*

$$int\text{-}uni: V(U) \rightarrow P(U),$$

$$int\text{-}uni(₹_{\mathcal{T}} V \mathfrak{F}_{\mathcal{W}}) = [int_{t} - uni_{w}(₹_{\mathcal{T}} V \mathfrak{F}_{\mathcal{W}})] \cup [int_{w} - uni_{t}(₹_{\mathcal{T}} V \mathfrak{F}_{\mathcal{W}})]$$

The values int-uni($₹_{\mathcal{T}} V \mathfrak{F}_{\mathcal{W}}$), called int-uni decision set of $₹_{\mathcal{T}} V \mathfrak{F}_{\mathcal{W}}$, is a subset of U.

Assume that a set of parameters and a set of options are given. The int-uni decision-making method, structured as follows, is then employed to select a collection of optimal options tailored to the problem at hand.

Step 1: Select feasible subsets from the collection of parameters.

Step 2: Construct the soft sets ($\mathcal{SS}$s) corresponding to each selected parameter subset.

Step 3: Compute the OR-product of the constructed soft sets.

Step 4: Determine the product using the int-uni decision function.

We are now ready to demonstrate how soft set theory can be applied to the int-uni decision-making problem using the OR-product.

Example 9. *The pilot recruitment process comprises multiple stages, including interviews, psychotechnical assessments, simulator evaluations, and medical examinations. Critical parameters influencing the overall evaluation include candidates' English language proficiency, technical knowledge, competence in mathematics and physics, visual memory, and social skills. To establish a young and dynamic team of pilots, an airline company has initiated a structured recruitment program. Due to the high volume of applications, the company has implemented a two-stage evaluation framework designed to ensure both efficiency and fairness. This process is jointly administered by Mr. Ahmet from the Human Resources department and Mr. Mehmet, a member of the board of directors.*

Stage One: Candidate Filtering

The primary objective of the first stage is to reduce the candidate pool to a manageable size. To this end, the company applies the int–uni decision-making method, a systematic and mathematically rigorous approach for filtering applicants. Mr. Ahmet and Mr. Mehmet evaluate candidates based on their interview performance and examination outcomes, with special attention to eliminating those who fail to demonstrate essential qualities necessary for professional pilot training. The evaluation emphasizes not only overall performance but also specifically targets candidates whose deficiencies in key parameters make them unsuitable for further consideration. This decision-making framework is operationalized through the int–uni method applied to the OR-product of soft sets, ensuring consistency, transparency, and analytical rigor in candidate elimination. During this process, Mr. Ahmet and Mr. Mehmet focus on the parameters they absolutely want to see in candidates to be eliminated. Using the int-uni decision-making method on the OR-product allows them to make well-founded and objective decisions.

Stage Two: Comprehensive Evaluation and Training

Candidates who pass the initial evaluation are invited to participate in a more comprehensive interview process in the second stage. Those who qualify are then enrolled in an intensive training program designed to prepare them for the professional responsibilities of aviation. Upon successful completion of this training, candidates are formally recognized as qualified pilots and integrated into the company's professional pilot team.

Let $U = \{\eth_1, \eth_2, \ldots, \eth_{21}\}$ *denote the universal set of all candidates whose applications have been validated for the pilot recruitment process. Let the set of parameters for identifying candidates to be eliminated be*

$$E = \{\text{ꞷ}_1, \text{ꞷ}_2, \ldots, \text{ꞷ}_8\}$$

where each parameter ꞷ_i, $i = 1,\ 2, \ldots, 8$ *corresponds to:*

ꞷ_1: *"Not confident and self-disciplined"*
ꞷ_2: *"Insufficient mathematical knowledge and creative ability"*
ꞷ_3: *"Lacking adequate English speaking proficiency"*
ꞷ_4: *"Lacking adequate situational awareness"*
ꞷ_5: *"Not having leadership spirit and the ability to work as part of a team"*
ꞷ_6: *"Poor ability to understand technical information"*
ꞷ_7: *"Unable to remain calm under pressure"*
ꞷ_8: *"Having poor communication skills"*

To address the pilot selection problem effectively, we apply the int-uni decision-making method based on the OR-product operation of soft sets. This approach systematically integrates the evaluations of both decision-makers, Mr. Ahmet and Mr. Mehmet, considering the parameters they deem critical for candidate elimination. The method ensures a transparent, rigorous, and fair selection process by filtering out candidates who lack essential qualifications.

The solution procedure is structured as follows:

Step 1. Determining the sets of parameters

The decision-makers select the parameters that represent the characteristics they absolutely do want in the pilot candidates who will be eliminated. These sets are defined as follows:

For Mr. Ahmet ($\mathcal{T}$)*:* $\mathcal{T} = \{\text{ꞷ}_1, \text{ꞷ}_3, \text{ꞷ}_6\}$, *meaning Mr. Ahmet does not want a pilot who is not confident and self-disciplined, who lacks adequate English speaking proficiency, and who has a poor ability to understand technical information.*

For Mr. Mehmet $(\mathcal{W})$: $\mathcal{W} = \{\text{ꞷ}_2, \text{ꞷ}_5, \text{ꞷ}_7\}$, *meaning Mr. Mehmet does not want a pilot who has insufficient mathematical knowledge and creative ability, who does not have the leadership spirit and the ability to work as part of a team, and who is unable to remain calm under pressure.*

These parameters represent undesirable qualities that make a pilot unsuitable for selection, and therefore serve as criteria for their elimination.

Step 2. Constructing the $\mathcal{SS}$*s using the* $\mathcal{PS}$*s defined in Step 1*

In the first stage, the decision makers conduct in-depth interviews with the candidates. Following these interviews, each candidate is systematically evaluated against the predetermined objectives and constraints, represented by the two designated parameter sets, $\mathcal{T}$ *and* $\mathcal{W}$.

Mr. Ahmet and Mr. Mehmet then construct their respective soft sets ($\mathcal{SS}$*s) by identifying and assessing the specific parameters they consider essential for the elimination of candidates. Based on the parameter sets defined in Step 1, the decision-makers build the corresponding soft sets, denoted by* $(\mathfrak{Q}, \mathcal{T})$ *and* $(\text{б}, \mathcal{W})$*, respectively.*

$$\begin{gathered}(\mathfrak{Q}, \mathcal{T}) = \{(\text{ꞷ}_1, \{\eth_1, \eth_5, \eth_7, \eth_8, \eth_9,\ \eth_{11}, \eth_{12}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{17}, \eth_{18}, \eth_{19}, \eth_{20}, \eth_{21}\}), \\ (\text{ꞷ}_3, \{\eth_1, \eth_2, \eth_5, \eth_7, \eth_8,\ \eth_9, \eth_{12}, \eth_{13}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{19}, \eth_{20}\}), \\ ((\text{ꞷ}_6, \{\eth_2, \eth_5, \eth_6, \eth_7, \eth_8,\ \eth_9, \eth_{11}, \eth_{12}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{18}, \eth_{21}\})\},\end{gathered}$$

$$\begin{gathered}(\text{б}, \mathcal{W}) = \{((\text{ꞷ}_2, \{\eth_2, \eth_6, \eth_7, \eth_8, \eth_9,\ \eth_{11}, \eth_{12}, \eth_{13}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{18}, \eth_{19}, \eth_{20}\})), \\ (\text{ꞷ}_5, \{\eth_2, \eth_4, \eth_5, \eth_6, \eth_7,\ \eth_8, \eth_9, \eth_{11}, \eth_{12}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{17}, \eth_{18}, \eth_{20}\}), \\ (\text{ꞷ}_7, \{\eth_1, \eth_2, \eth_4, \eth_5, \eth_7,\ \eth_8, \eth_9, \eth_{11}, \eth_{12}, \eth_{13}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{16}, \eth_{18}, \eth_{19}, \eth_{20})\}\}.\end{gathered}$$

$(\mathfrak{Q}, \mathcal{T})$ *denotes the soft set constructed by Mr. Ahmet, representing the group of pilot candidates to be eliminated based on the undesirable parameters contained in* $\mathcal{T}$. *Similarly,* $(\text{б}, \mathcal{W})$

denotes the soft set constructed by Mr. Mehmet, representing the group of candidates to be eliminated due to undesirable parameters contained in $\mathcal{W}$. It is important to emphasize that, within this framework, the decision makers' primary objective is not the direct selection of candidates but rather their systematic elimination—an approach necessitated by the exceptionally high volume of applications received.

Step 3. Determine the OR-product of $\mathcal{SS}$s:

$$\Omega_{\mathcal{T}} V \Omega_{\mathcal{W}} = \left\{ \begin{array}{c} \left((\omega_1, \omega_2), \left\{ \begin{array}{c} \eth_1, \eth_2, \eth_5, \eth_6, \eth_7, \eth_8, \eth_9, \eth_{11}, \eth_{12}, \eth_{13}, \eth_{14}, \\ \eth_{15}, \eth_{16}, \eth_{17}, \eth_{18}, \eth_{19}, \eth_{20}, \eth_{21} \end{array} \right\} \right), \\ \left((\omega_1, \omega_5), \left\{ \begin{array}{c} \eth_1, \eth_2, \eth_4, \eth_5, \eth_6, \eth_7, \eth_8, \eth_9, \eth_{11}, \\ \eth_{12}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{17}, \eth_{18}, \eth_{19}, \eth_{20}, \eth_{21} \end{array} \right\} \right), \\ \left((\omega_1, \omega_7), \left\{ \begin{array}{c} \eth_1, \eth_2, \eth_4, \eth_5, \eth_7, \eth_8, \eth_9, \eth_{11}, \eth_{12}, \eth_{13}, \\ \eth_{14}, \eth_{15}, \eth_{16}, \eth_{17}, \eth_{18}, \eth_{19}, \eth_{20}, \eth_{21} \end{array} \right\} \right), \\ \left((\omega_3, \omega_2), \left\{ \begin{array}{c} \eth_1, \eth_2, \eth_5, \eth_6, \eth_7, \eth_8, \eth_9, \eth_{11}, \\ \eth_{12}, \eth_{13}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{18}, \eth_{19}, \eth_{20} \end{array} \right\} \right), \\ \left((\omega_3, \omega_5), \left\{ \begin{array}{c} \eth_1, \eth_2, \eth_4, \eth_5, \eth_6, \eth_7, \eth_8, \eth_9, \eth_{11}, \eth_{12}, \\ \eth_{13}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{17}, \eth_{18}, \eth_{19}, \eth_{20} \end{array} \right\} \right), \\ \left((\omega_3, \omega_7), \left\{ \begin{array}{c} \eth_1, \eth_2, \eth_4, \eth_5, \eth_7, \eth_8, \eth_9, \eth_{11}, \eth_{12}, \\ \eth_{13}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{18}, \eth_{19}, \eth_{20} \end{array} \right\} \right), \\ \left((\omega_6, \omega_2), \left\{ \begin{array}{c} \eth_2, \eth_5, \eth_6, \eth_7, \eth_8, \eth_9, \eth_{11}, \eth_{12}, \eth_{13}, \\ \eth_{14}, \eth_{15}, \eth_{16}, \eth_{18}, \eth_{19}, \eth_{20}, \eth_{21} \end{array} \right\} \right), \\ \left((\omega_6, \omega_5), \left\{ \begin{array}{c} \eth_2, \eth_4, \eth_5, \eth_6, \eth_7, \eth_8, \eth_9, \eth_{11}, \\ \eth_{12}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{17}, \eth_{18}, \eth_{20}, \eth_{21} \end{array} \right\} \right), \\ \left((\omega_6, \omega_7), \left\{ \begin{array}{c} \eth_1, \eth_2, \eth_4, \eth_5, \eth_6, \eth_7, \eth_8, \eth_9, \eth_{11}, \eth_{12}, \\ \eth_{13}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{18}, \eth_{19}, \eth_{20}, \eth_{21} \end{array} \right\} \right) \end{array} \right\},$$

Step 4. Determine the set of int-uni $(\Omega_{\mathcal{T}} V \sigma_{\mathcal{W}})$:

$$int_t - uni_w(\Omega_{\mathcal{T}} V \sigma_{\mathcal{W}}) = \bigcap_{t \in \mathcal{T}} \left(\bigcup_{w \in \mathcal{W}} ((\Omega_{\mathcal{T}} V \sigma_{\mathcal{W}})(t, w)) \right)$$

We first determine $\cup_{w \in \mathcal{W}}((\Omega_{\mathcal{T}} V \sigma_{\mathcal{W}})(t, w))$:

$$\begin{array}{c} (\Omega_{\mathcal{T}} V \sigma_{\mathcal{W}})(\omega_1, \omega_2) \cup (\Omega_{\mathcal{T}} V \sigma_{\mathcal{W}})(\omega_1, \omega_5) \cup (\Omega_{\mathcal{T}} V \sigma_{\mathcal{W}})(\omega_1, \omega_7) \\ = \{\eth_1, \eth_2, \eth_5, \eth_6, \eth_7, \eth_8, \eth_9, \eth_{11}, \eth_{12}, \eth_{13}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{17}, \eth_{18}, \eth_{19}, \eth_{20}, \eth_{21}\} \\ \cup \{\eth_1, \eth_2, \eth_4, \eth_5, \eth_6, \eth_7, \eth_8, \eth_9, \eth_{11}, \eth_{12}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{17}, \eth_{18}, \eth_{19}, \eth_{20}, \eth_{21}\} \\ \cup \{\eth_1, \eth_2, \eth_4, \eth_5, \eth_7, \eth_8, \eth_9, \eth_{11}, \eth_{12}, \eth_{13}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{17}, \eth_{18}, \eth_{19}, \eth_{20}, \eth_{21}\} \\ = \{\eth_1, \eth_2, \eth_4, \eth_5, \eth_6, \eth_7, \eth_8, \eth_9, \eth_{11}, \eth_{12}, \eth_{13}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{17}, \eth_{18}, \eth_{19}, \eth_{20}, \eth_{21}\} \end{array}$$

$$\begin{array}{c} (\Omega_{\mathcal{T}} V \sigma_{\mathcal{W}})(\omega_3, \omega_2) \cup (\Omega_{\mathcal{T}} V \sigma_{\mathcal{W}})(\omega_3, \omega_5) \cup (\Omega_{\mathcal{T}} V \sigma_{\mathcal{W}})(\omega_3, \omega_7) \\ = \{\eth_1, \eth_2, \eth_5, \eth_6, \eth_7, \eth_8, \eth_9, \eth_{11}, \eth_{12}, \eth_{13}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{18}, \eth_{19}, \eth_{20}\} \\ \cup \{\eth_1, \eth_2, \eth_4, \eth_5, \eth_6, \eth_7, \eth_8, \eth_9, \eth_{11}, \eth_{12}, \eth_{13}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{17}, \eth_{18}, \eth_{19}, \eth_{20}\} \\ \cup \{\eth_1, \eth_2, \eth_4, \eth_5, \eth_7, \eth_8, \eth_9, \eth_{11}, \eth_{12}, \eth_{13}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{18}, \eth_{19}, \eth_{20}\} \\ = \{\eth_1, \eth_2, \eth_4, \eth_5, \eth_6, \eth_7, \eth_8, \eth_9, \eth_{11}, \eth_{12}, \eth_{13}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{17}, \eth_{18}, \eth_{19}, \eth_{20}\} \end{array}$$

$$\begin{array}{c} (\Omega_{\mathcal{T}} V \sigma_{\mathcal{W}})(\omega_6, \omega_2) \cup (\Omega_{\mathcal{T}} V \sigma_{\mathcal{W}})(\omega_6, \omega_5) \cup (\Omega_{\mathcal{T}} V \sigma_{\mathcal{W}})(\omega_6, \omega_7) \\ = \{\eth_2, \eth_5, \eth_6, \eth_7, \eth_8, \eth_9, \eth_{11}, \eth_{12}, \eth_{13}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{18}, \eth_{19}, \eth_{20}, \eth_{21}\} \\ \cup \{\eth_2, \eth_4, \eth_5, \eth_6, \eth_7, \eth_8, \eth_9, \eth_{11}, \eth_{12}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{17}, \eth_{18}, \eth_{20}, \eth_{21}\} \\ \cup \{\eth_1, \eth_2, \eth_4, \eth_5, \eth_6, \eth_7, \eth_8, \eth_9, \eth_{11}, \eth_{12}, \eth_{13}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{18}, \eth_{19}, \eth_{20}, \eth_{21}\} \\ = \{\eth_1, \eth_2, \eth_4, \eth_5, \eth_6, \eth_7, \eth_8, \eth_9, \eth_{11}, \eth_{12}, \eth_{13}, \eth_{14}, \eth_{15}, \eth_{16}, \eth_{17}, \eth_{18}, \eth_{19}, \eth_{20}, \eth_{21}\} \end{array}$$

Thus,

$$
\begin{gathered}
(int_{t}-uni_{w})(\Omega_{\mathcal{T}}V\Omega_{\mathcal{W}})=\bigcap_{t\in\mathcal{T}}(\bigcup_{w\in\mathcal{W}}((\Omega_{\mathcal{T}}V\text{б}_{\mathcal{W}})(t,w)))=\\
\{\eth_1,\eth_2,\eth_4,\eth_5,\eth_6,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{13},\eth_{14},\eth_{15},\eth_{16},\eth_{17},\eth_{18},\eth_{19},\eth_{20},\eth_{21}\}\\
\cap\{\eth_1,\eth_2,\eth_4,\eth_5,\eth_6,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{13},\eth_{14},\eth_{15},\eth_{16},\eth_{17},\eth_{18},\eth_{19},\eth_{20}\}\\
\cap\{\eth_1,\eth_2,\eth_4,\eth_5,\eth_6,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{13},\eth_{14},\eth_{15},\eth_{16},\eth_{17},\eth_{18},\eth_{19},\eth_{20},\eth_{21}\}\\
=\{\eth_1,\eth_2,\eth_4,\eth_5,\eth_6,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{13},\eth_{14},\eth_{15},\eth_{16},\eth_{17},\eth_{18},\eth_{19},\eth_{20},\eth_{21}\}
\end{gathered}
$$

is obtained.

$$(int_{w}-uni_{t})(\Omega_{\mathcal{T}}V\text{б}_{\mathcal{W}})=\bigcap_{w\in\mathcal{W}}\left(\bigcup_{t\in\mathcal{T}}((\Omega_{\mathcal{T}}V\text{б}_{\mathcal{W}})(t,w))\right)$$

We first determine $\bigcup_{t\in\mathcal{T}}((\Omega_{\mathcal{T}}V\Omega\text{б}_{\mathcal{W}})(t,w))$*:*

$$
\begin{gathered}
(\Omega_{\mathcal{T}}V\text{б}_{\mathcal{W}})(\omega_1,\omega_2)\cup(\Omega_{\mathcal{T}}V\text{б}_{\mathcal{W}})(\omega_3,\omega_2)\cup(\Omega_{\mathcal{T}}V\text{б}_{\mathcal{W}})(\omega_6,\omega_2)\\
=\{\eth_1,\eth_2,\eth_5,\eth_6,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{13},\eth_{14},\eth_{15},\eth_{16},\eth_{17},\eth_{18},\eth_{19},\eth_{20},\eth_{21}\}\\
\cup\{\eth_1,\eth_2,\eth_5,\eth_6,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{13},\eth_{14},\eth_{15},\eth_{16},\eth_{18},\eth_{19},\eth_{20}\}\\
\cup\{\eth_2,\eth_5,\eth_6,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{13},\eth_{14},\eth_{15},\eth_{16},\eth_{18},\eth_{19},\eth_{20},\eth_{21}\}\\
=\{\eth_1,\eth_2,\eth_5,\eth_6,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{13},\eth_{14},\eth_{15},\eth_{16},\eth_{17},\eth_{18},\eth_{19},\eth_{20},\eth_{21}\}
\end{gathered}
$$

$$
\begin{gathered}
(\Omega_{\mathcal{T}}V\text{б}_{\mathcal{W}})(\omega_1,\omega_5)\cup(\Omega_{\mathcal{T}}V\text{б}_{\mathcal{W}})(\omega_3,\omega_5)\cup(\Omega_{\mathcal{T}}V\text{б}_{\mathcal{W}})(\omega_6,\omega_5)\\
=\{\eth_1,\eth_2,\eth_4,\eth_5,\eth_6,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{14},\eth_{15},\eth_{16},\eth_{17},\eth_{18},\eth_{19},\eth_{20},\eth_{21}\}\\
\cup\{\eth_1,\eth_2,\eth_4,\eth_5,\eth_6,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{13},\eth_{14},\eth_{15},\eth_{16},\eth_{17},\eth_{18},\eth_{19},\eth_{20}\}\\
\cup\{\eth_2,\eth_4,\eth_5,\eth_6,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{14},\eth_{15},\eth_{16},\eth_{17},\eth_{18},\eth_{20},\eth_{21}\}\\
=\{\eth_1,\eth_2,\eth_4,\eth_5,\eth_6,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{13},\eth_{14},\eth_{15},\eth_{16},\eth_{17},\eth_{18},\eth_{19},\eth_{20},\eth_{21}\}
\end{gathered}
$$

$$
\begin{gathered}
(\Omega_{\mathcal{T}}V\text{б}_{\mathcal{W}})(\omega_1,\omega_7)\cup(\Omega_{\mathcal{T}}V\text{б}_{\mathcal{W}})(\omega_3,\omega_7)\cup(\Omega_{\mathcal{T}}V\text{б}_{\mathcal{W}})(\omega_6,\omega_7)\\
=\{\eth_1,\eth_2,\eth_4,\eth_5,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{13},\eth_{14},\eth_{15},\eth_{16},\eth_{17},\eth_{18},\eth_{19},\eth_{20},\eth_{21}\}\\
\cup\{\eth_1,\eth_2,\eth_4,\eth_5,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{13},\eth_{14},\eth_{15},\eth_{16},\eth_{18},\eth_{19},\eth_{20}\}\\
\cup\{\eth_1,\eth_2,\eth_4,\eth_5,\eth_6,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{13},\eth_{14},\eth_{15},\eth_{16},\eth_{18},\eth_{19},\eth_{20},\eth_{21}\}\\
=\{\eth_1,\eth_2,\eth_4,\eth_5,\eth_6,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{13},\eth_{14},\eth_{15},\eth_{16},\eth_{17},\eth_{18},\eth_{19},\eth_{20},\eth_{21}\}
\end{gathered}
$$

is obtained. Therefore,

$$
\begin{gathered}
(int_{w}-uni_{t})(\Omega_{\mathcal{T}}V\text{б}_{\mathcal{W}})=\bigcap_{w\in\mathcal{W}}(\cup_{t\in\mathcal{T}}((\Omega_{\mathcal{T}}V\text{б}_{\mathcal{W}})(t,w)))=\\
\cap\{\eth_1,\eth_2,\eth_5,\eth_6,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{13},\eth_{14},\eth_{15},\eth_{16},\eth_{17},\eth_{18},\eth_{19},\eth_{20},\eth_{21}\}\\
\cap\{\eth_1,\eth_2,\eth_4,\eth_5,\eth_6,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{13},\eth_{14},\eth_{15},\eth_{16},\eth_{18},\eth_{19},\eth_{20},\eth_{21}\}\\
\cap\{\eth_1,\eth_2,\eth_4,\eth_5,\eth_6,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{13},\eth_{14},\eth_{15},\eth_{16},\eth_{18},\eth_{19},\eth_{20},\eth_{21}\}\\
=\{\eth_1,\eth_2,\eth_4,\eth_5,\eth_6,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{13},\eth_{14},\eth_{15},\eth_{16},\eth_{18},\eth_{19},\eth_{20},\eth_{21}\}
\end{gathered}
$$

Thus,

$$
\begin{gathered}
int\text{-}uni(\Omega_{\mathcal{T}}V\text{б}_{\mathcal{W}})=[int_{t}-uni_{w}\,(\Omega_{\mathcal{T}}V\text{б}_{\mathcal{W}})]\cup[int_{w}-uni_{t}\,(\Omega_{\mathcal{T}}V\text{б}_{\mathcal{W}})]\\
=\{\eth_1,\eth_2,\eth_4,\eth_5,\eth_6,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{13},\eth_{14},\eth_{15},\eth_{16},\eth_{17},\eth_{18},\eth_{19},\eth_{20},\eth_{21}\}\\
\cup\{\eth_1,\eth_2,\eth_4,\eth_5,\eth_6,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{13},\eth_{14},\eth_{15},\eth_{16},\eth_{17},\eth_{18},\eth_{19},\eth_{20},\eth_{21}\}\\
=\{\eth_1,\eth_2,\eth_4,\eth_5,\eth_6,\eth_7,\eth_8,\eth_9,\eth_{11},\eth_{12},\eth_{13},\eth_{14},\eth_{15},\eth_{16},\eth_{17},\eth_{18},\eth_{19},\eth_{20},\eth_{21}\}
\end{gathered}
$$

Thus, in the airline company's pilot recruitment process, out of the 21 candidates, 19 were eliminated during the first stage. The remaining candidates, $\{\eth_3,\eth_{10}\}$ *advanced to the second stage, where they will undergo a comprehensive training program. Candidates who successfully complete this program will be formally admitted to the company's professional pilot team. The outcome of this first stage reflects the set of options that simultaneously satisfy the elimination criteria established by both Mr. Ahmet and Mr. Mehmet.*

By employing the int–uni decision-making method on the OR-product, the process systematically integrates the evaluators' distinct preferences, ensuring that only the candidates meeting both decision-makers' requirements remain. This procedure embodies a symmetry of evaluation, where decision-makers participate on equal footing and candidates are judged under balanced and transparent rules. This methodological framework not only guarantees fairness and transparency in candidate selection but also demonstrates the practical applicability of algebraic soft set operations to real-world decision-making scenarios. Unlike conventional decision-making approaches, which often treat evaluation parameters in a rigid or isolated manner, the proposed framework offers a mathematically consistent yet flexible tool for handling multi-criteria decisions under uncertainty.

6. Conclusions

This study delivers the first rigorous and exhaustive algebraic formalization of the OR-product, elevating it from a partially explored concept to a cornerstone of soft set theory with far-reaching theoretical and practical significance. While previous works focused on limited aspects of its behavior with respect to specific soft subsets and equalities, our research systematically establishes its full framework under M-subset and M-equality. By proving that the class of all soft sets over a universe, endowed with a restricted/extended intersection and the OR-product, constitutes a commutative hemiring with identity under soft L-equality, we uncover a deep algebraic structure that situates the OR-product at the core of modern algebra, linking it naturally to automata theory, formal languages, and algebraic logic. This algebraic formulation highlights intrinsic structural symmetries that reinforce the central role of the OR-product within the symmetric architecture of modern algebra.

Beyond its theoretical contributions, the integration of the OR-product with the int–uni decision-making method demonstrates its practical utility in real-world scenarios. In a pilot recruitment study, multiple decision-makers applied distinct elimination criteria, and the proposed framework ensured a transparent, equitable, and mathematically rigorous selection process under uncertainty. This dual emphasis highlights the OR-product as both a mathematically robust construct and a valuable tool for applied decision-making.

Taken together, these results position the OR-product at the nexus of abstract algebra and operational decision science, filling a critical gap in the literature by bridging foundational mathematics with practical methodology. Beyond bridging theory and practice, our findings emphasize the dual manifestation of symmetry: algebraic symmetry in formal operations and decision-theoretic symmetry in ensuring equitable and transparent outcomes. Future research may explore the distributive properties of other soft products, their integration with multi-criteria and AI-driven decision frameworks, and their potential synergy with emerging paradigms such as rough–fuzzy hybrid models and granular computing, paving the way for both theoretical innovation and real-world impact.

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