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Multiplicatively Trigonometric Convex Functions for Hermite–Hadamard-Type Inequalities

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Abstract

A novel category of convex functions, termed multiplicatively trigonometric convex functions, are introduced in this paper. We explore their algebraic characteristics and establish connections between such functions and other forms of convex functions. We even show that these functions are symmetric with respect to their components. Furthermore, we prove the Hermite–Hadamard inequality for the mentioned category of functions. In addition, we present new structures of the Hermite–Hadamard inequality within the framework of multiplicative integrals. By broadening these inequalities, the purpose is to reveal some properties and relations that help the advancement of more robust mathematical techniques.

Keywords: convex function; multiplicative trigonometric convexity; multiplicative calculus; integral inequalities

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1. Introduction

The concept of convexity has a significant importance in both engineering and mathematics, as it forms a foundational framework for understanding a vast range of phenomena. Moreover, convex sets and functions are integral to this framework, as they simplify complex processes and enable the optimization of intricate mathematical models. These problems become easier to solve due to the presence of a single global minimum, making them crucial in optimization tasks. Beyond its theoretical significance, convexity finds practical applications in control theory, economics, and optimization, among other disciplines. Engineers often rely on convexity to enhance system stability and mitigate efficiency challenges.

For instance, in control theory, convexity ensures that systems are designed to endure varying conditions. Similarly, in economics, it facilitates the analysis of market behaviors and consumer preferences, aiding in resource allocation decisions to maximize efficiency. Another fascinating aspect of convexity lies in its connection to integral inequalities, which represent a rich area of research. Integral inequalities are frequently employed in mathematical analysis to establish bounds on certain integral values. These inequalities, like those of the Hermite–Hadamard (H–H) type that bound the average value of a function with the convexity property, are closely tied to the study of convex functions (see [1–8]).

Multiplicative calculus, often referred to as non-Newtonian calculus, introduces an alternative approach to integration and differentiation. It is in the basis of some principles of arithmetic addition via division for integration and multiplication for differentiation. Such a branch of calculus prepares a broader framework compared to Newtonian calculus, which has dominated mathematical theory since its introduction by Newton and Leibniz in the 17th century. One of the earliest explorations in this field was conducted by Grossman, whose work in the 1970s brought significant attention to the concept of multiplicative calculus (see [9]). Grossman's contributions marked a paradigm shift from traditional calculus, offering a novel perspective on integral and differential operations.

Although multiplicative calculus (such as the geometric or bigeometric multiplicative calculus) is less well-known than its Newtonian counterpart, it possesses a distinct methodology [10,11]. Its relatively limited range of applications, primarily focused on positive values, has contributed to its lesser popularity. However, despite this, multiplicative calculus has served as the foundation for numerous intriguing discoveries and applications across various domains. For instance, a fundamental theorem in this area was introduced by Bashirov [12], with a more advanced form of the concept later extended by Riza et al. in [13].

Some integral inequalities in the context of multiplicative calculus have been fundamentally explored in other studies. For instance, Ali et al. introduced H-H-type inequalities based on the existing rules of multiplicative calculus [14]. Du and Peng expanded the concept in [15] by employing multiplicative Riemann–Liouville integrals to establish H–H inequalities in a fractional context. In [16], Frioui et al. derived parametrized multiplicative integral inequalities. Additionally, in [17], Ali et al. demonstrated generalized fractional integrals in the setting of multiplicative calculus. In [18], Peng et al. developed fractional variants of H–H-type inequalities.

Moreover, Du and Long leveraged the multiplicative Riemann-Liouville integral operators in [19] to derive integral inequalities multi-parameterically. Young researchers are encouraged to consult the referenced works, including the research in [20–24], for a more comprehensive understanding of the applications and advancements in multiplicative calculus. These studies provide both a detailed historical perspective on the field's development and an overview of its potential interdisciplinary applications.

Our research focuses on exploring novel integral inequalities, particularly for multiplicatively trigonometric convex functions within the realms of multiplicative calculus. While the well-established H–H inequalities have been extended and generalized by various researchers in multiplicative calculus, the specific case of multiplicatively trigonometric convex functions remains largely unexplored. Previous studies on integral inequalities in multiplicative calculus have primarily addressed different types of convex functions, leaving a gap in understanding of the properties and applications of multiplicatively trigonometric convex functions. Recognizing this gap, we aim to contribute to the advancement of multiplicative calculus by formulating new integral inequalities centered on multiplicatively trigonometric convex functions. More precisely, for the first time, we define the multiplicatively trigonometric convex functions, and then prove some important properties of these functions in the context of the geometric multiplicative calculus. Moreover, after the establishment of the multiplicative H–H inequalities under the multiplicatively trigonometric convex functions, we extract some other applicable multiplicative inequalities.

2. Preliminaries

Throughout this study, $I \subseteq \mathbb{R}$ is an interval.

Definition 1 ([25]). *The function $\bar{\Psi} : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if the following inequality holds for all $\nu_1, \nu_2 \in I$ and $\omega \in [0,1]$:*

$$\bar{\Psi}(\omega\nu_1 + (1-\omega)\nu_2) \leq \omega\bar{\Psi}(\nu_1) + (1-\omega)\bar{\Psi}(\nu_2).$$

The most renowned inequality associated with the integral mean of a convex function is the H–H inequality [25,26], which is stated below:

Let $\bar{\Psi} : I \to \mathbb{R}$ be a convex function. Then the inequality

$$\bar{\Psi}\left(\frac{\nu_1+\nu_2}{2}\right) \leq \frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2} \bar{\Psi}(\varkappa)d\varkappa \leq \frac{\bar{\Psi}(\nu_1)+\bar{\Psi}(\nu_2)}{2}$$

holds.

In [27], Kadakal introduced the class of trigonometrically convex functions and derived the H–H inequalities for such a family of functions.

Definition 2 ([27]). *A non-negative function $\bar{\Psi} : I \to \mathbb{R}$ is said to be trigonometrically convex if*

$$\bar{\Psi}(\omega\nu_1 + (1-\omega)\nu_2) \leq \sin\frac{\pi\omega}{2}\bar{\Psi}(\nu_1) + \cos\frac{\pi\omega}{2}\bar{\Psi}(\nu_2),$$

for all $\nu_1, \nu_2 \in I$ and $\omega \in [0,1]$.

Theorem 1 ([27]). *Let $\bar{\Psi} : [\nu_1, \nu_2] \to \mathbb{R}$ be a trigonometrically convex function. If $\nu_1 < \nu_2$ and $\bar{\Psi} \in L[\nu_1, \nu_2]$, then*

$$\bar{\Psi}\left(\frac{\nu_1+\nu_2}{2}\right) \leq \frac{\sqrt{2}}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2} \bar{\Psi}(\varkappa)d\varkappa \leq \frac{2\sqrt{2}}{\pi}[\bar{\Psi}(\nu_1)+\bar{\Psi}(\nu_2)].$$

Definition 3 ([28]). *A function $\bar{\Psi} : I \to (0,\infty)$ is said to be multiplicatively or log-convex if*

$$\bar{\Psi}(\omega\nu_1 + (1-\omega)\nu_2) \leq [\bar{\Psi}(\nu_1)]^{\omega}[\bar{\Psi}(\nu_2)]^{1-\omega},$$

for all $\nu_1, \nu_2 \in I$ and $\omega \in [0,1]$.

In [14], Ali et al. extracted the related H–H inequality under the multiplicatively convex functions.

Theorem 2 ([14]). *Let the positive function $\bar{\Psi}$ be multiplicatively convex on $[\nu_1, \nu_2]$. Then*

$$\bar{\Psi}\left(\frac{\nu_1+\nu_2}{2}\right) \leq \left(\int_{\nu_1}^{\nu_2} (\bar{\Psi}(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}} \leq G(\bar{\Psi}(\nu_1), \bar{\Psi}(\nu_2)),$$

so that the symbol $G(\cdot,\cdot)$ denotes the geometric mean.

Definition 4 ([12]). *On the basis of the bigeometric calculus, the multiplicative derivative of $\bar{\Psi} : \mathbb{R} \to \mathbb{R}^+$ is formulated as*

$$\frac{d^*\bar{\Psi}}{d\varkappa} = \bar{\Psi}^*(\varkappa) = \lim_{k\to 0}\left(\frac{\bar{\Psi}(\varkappa+k)}{\bar{\Psi}(\varkappa)}\right)^{\frac{1}{k}}.$$

On the basis of the geometric calculus, if $\bar{\Psi}$ is positive and differentiable at ω, in this case, $\bar{\Psi}^*$ exists and one can write the following relation between $\bar{\Psi}^*$ and the standard derivative $\bar{\Psi}'$:

$$\bar{\Psi}^*(\varkappa) = e^{(\ln \bar{\Psi}(\varkappa))'} = e^{\frac{\bar{\Psi}'(\varkappa)}{\bar{\Psi}(\varkappa)}}.$$

Furthermore, the multiplicative integral or * integral is represented by $\int_{\nu_1}^{\nu_2} (\bar{\Psi}(\varkappa))^{\mathrm{d}\varkappa}$, while $\int_{\nu_1}^{\nu_2} \bar{\Psi}(\varkappa)\, \mathrm{d}\varkappa$ is the ordinary integral. It is due to this fact that, since the sum of the terms of the product is applied to define the Riemann integral of $\bar{\Psi}$ on $[\nu_1, \nu_2]$, the product of terms raised to certain powers is applied to define *integral of $\bar{\Psi}$ on $[\nu_1, \nu_2]$.

The following properties hold for *differentiable functions.

Theorem 3 ([12]). *Let $\bar{\Psi}_1$ and $\bar{\Psi}_2$ be *differentiable. For every constant α, $\alpha\bar{\Psi}_1$, $\bar{\Psi}_1\bar{\Psi}_2$, $\bar{\Psi}_1 + \bar{\Psi}_2$, $\bar{\Psi}_1/\bar{\Psi}_2$ and $\bar{\Psi}_1^{\bar{\Psi}_2}$ are *differentiable too, and*

1. $(\alpha\bar{\Psi}_1)^*(\varkappa) = \bar{\Psi}_1^*(\varkappa)$,
2. $(\bar{\Psi}_1\bar{\Psi}_2)^*(\varkappa) = \bar{\Psi}_1^*(\varkappa)\bar{\Psi}_2^*(\varkappa)$,
3. $(\bar{\Psi}_1 + \bar{\Psi}_2)^*(\varkappa) = \bar{\Psi}_1^*(\varkappa)^{\frac{\bar{\Psi}_1(\varkappa)}{\bar{\Psi}_1(\varkappa)+\bar{\Psi}_2(\varkappa)}} \bar{\Psi}_2^*(\varkappa)^{\frac{\bar{\Psi}_2(\varkappa)}{\bar{\Psi}_1(\varkappa)+\bar{\Psi}_2(\varkappa)}}$,
4. $\left(\frac{\bar{\Psi}_1}{\bar{\Psi}_2}\right)^*(\varkappa) = \frac{\bar{\Psi}_1^*(\varkappa)}{\bar{\Psi}_2^*(\varkappa)}$,
5. $\left(\bar{\Psi}_1^{\bar{\Psi}_2}\right)^*(\varkappa) = \bar{\Psi}_1^*(\varkappa)^{\bar{\Psi}_2(\varkappa)}\bar{\Psi}_1(\varkappa)^{\bar{\Psi}_2'(\varkappa)}$.

This proposition shows a logical relation between *integral and the Riemann integral.

Proposition 1 ([12]). *Let the positive function $\bar{\Psi}$ be Riemann integrable on $[\nu_1, \nu_2]$. Then, $\bar{\Psi}$ is * integrable on $[\nu_1, \nu_2]$, and*

$$\int_{\nu_1}^{\nu_2} (\bar{\Psi}(\varkappa))^{\mathrm{d}\varkappa} = e^{\int_{\nu_1}^{\nu_2} \ln \bar{\Psi}(\varkappa)\mathrm{d}\varkappa}.$$

Proposition 2 ([12]). *Let $\bar{\Psi}$, $\bar{\Psi}_1$, and $\bar{\Psi}_2$ be positive and Riemann integrable on $[\nu_1, \nu_2]$. Then*

1. $\int_{\nu_1}^{\nu_2} ((\bar{\Psi}(\varkappa))^p)^{d\varkappa} = \int_{\nu_1}^{\nu_2} \left((\bar{\Psi}(\varkappa))^{d\varkappa}\right)^p$,
2. $\int_{\nu_1}^{\nu_2} (\bar{\Psi}(\varkappa)\bar{\Psi}_2(\varkappa))^{d\varkappa} = \int_{\nu_1}^{\nu_2} (\bar{\Psi}(\varkappa))^{d\varkappa} \cdot \int_{\nu_1}^{\nu_2} (\bar{\Psi}_2(\varkappa))^{d\varkappa}$,
3. $\int_{\nu_1}^{\nu_2} \left(\frac{\bar{\Psi}(\varkappa)}{\bar{\Psi}_2(\varkappa)}\right)^{d\varkappa} = \frac{\int_{\nu_1}^{\nu_2} (\bar{\Psi}(\varkappa))^{d\varkappa}}{\int_{\nu_1}^{\nu_2} (\bar{\Psi}_2(\varkappa))^{d\varkappa}}$,
4. $\int_{\nu_1}^{\nu_2} (\bar{\Psi}(\varkappa))^{d\varkappa} = \int_{\nu_1}^{\nu} (\bar{\Psi}(\varkappa))^{d\varkappa} \cdot \int_{\nu}^{\nu_2} (\bar{\Psi}(\varkappa))^{d\varkappa}, \nu_1 \leq \nu \leq \nu_2$,
5. $\int_{\nu_1}^{\nu_1} (\bar{\Psi}(\varkappa))^{d\varkappa} = 1$ *and* $\int_{\nu_1}^{\nu_2} (\bar{\Psi}(\varkappa))^{d\varkappa} = \left(\int_{\nu_2}^{\nu_1} (\bar{\Psi}(\varkappa))^{d\varkappa}\right)^{-1}$.

In [29], Khan and Budak give the following lemma, which we need in the sequel:

Lemma 1. *Let $\bar{\Psi} : I^\circ \subseteq \mathbb{R} \to \mathbb{R}^+$ be multiplicative differentiable and $\nu_1, \nu_2 \in I^\circ$ with $\nu_1 < \nu_2$. If $\bar{\Psi}^*$ is multiplicative integrable on $[\nu_1, \nu_2]$, then*

$$\frac{\sqrt{\bar{\Psi}(\nu_1)\bar{\Psi}(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2} (\bar{\Psi}(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}} = \left[\int_0^1 \left([\bar{\Psi}^*(\omega\nu_1 + (1-\omega)\nu_2)]^{1-2\omega}\right)^{dw}\right]^{\frac{\nu_2-\nu_1}{2}}.$$

3. Multiplicatively Trigonometric Convex Functions

In this section, we begin by presenting the category of multiplicatively trigonometric convex functions and exploring some of their fundamental algebraic characteristics.

Definition 5. *A positive function $\bar{\Psi} : I \to \mathbb{R}$ is said to be multiplicatively trigonometric convex (in short, MTC) if $\forall\, \varkappa, y \in I$ and $\omega \in [0, 1]$,*

$$\bar{\Psi}(\omega\varkappa + (1-\omega)y) \leq (\bar{\Psi}(\varkappa))^{\sin\frac{\pi\omega}{2}} \cdot (\bar{\Psi}(y))^{\cos\frac{\pi\omega}{2}}.$$

MTC-functions are a variant of convexity where the usual convex combination ω and $1-\omega$ is replaced by trigonometric functions of ω. Also, in this definition, the weights are nonlinear and trigonometric.

Moreover, for symmetry, we might consider how the function behaves when we swap $\varkappa$ and y. This is symmetric to the original definition if we swap $\varkappa$ and y and replace ω with $1-\omega$. Therefore, the definition is symmetric with respect to $\varkappa$ and y.

Remark 1. *Every MTC-$\bar{\Psi}$-function is a multiplicative P-function. In other words, for each $\varkappa, y \in I$ and $\omega \in [0, 1]$, one has*

$$\bar{\Psi}(\omega\varkappa + (1-\omega)y) \leq (\bar{\Psi}(\varkappa))^{\sin\frac{\pi\omega}{2}} \cdot (\bar{\Psi}(y))^{\cos\frac{\pi\omega}{2}} \leq \bar{\Psi}(\varkappa)\bar{\Psi}(y).$$

Example 1. *Positive constant functions are MTC due to the fact that $\sin\frac{\pi\omega}{2} + \cos\frac{\pi\omega}{2} \geq 1$ for all $\omega \in [0, 1]$.*

Proposition 3. *Every multiplicative convex function is an MTC-function.*

Proof. Since the cardinal sine function $\frac{\sin\varkappa}{\varkappa}$ is decreasing on $\left[0, \frac{\pi}{2}\right]$, it follows that $\omega \leq \sin\frac{\pi\omega}{2} \leq \frac{\pi\omega}{2}$ for all $\omega \in [0, 1]$. Then $1-\omega \leq \cos\frac{\pi\omega}{2} \leq \frac{\pi}{2}(1-\omega)$ for all $\omega \in [0, 1]$. □

Theorem 4. *Let $\bar{\Psi}, \bar{\Phi} : [\varkappa, y] \to \mathbb{R}$. If $\bar{\Psi}$ and $\bar{\Phi}$ are MTC-functions, then $\bar{\Psi} \cdot \bar{\Phi}$ is an MTC-function.*

Proof. Let $\bar{\Psi}$ and $\bar{\Phi}$ be MTC-functions. Then

$$\begin{aligned}
&(\bar{\Psi} \cdot \bar{\Phi})(\omega\varkappa + (1-\omega)y) \\
&\quad = \bar{\Psi}(\omega\varkappa + (1-\omega)y) \cdot \bar{\Phi}(\omega\varkappa + (1-\omega)y) \\
&\quad \leq (\bar{\Psi}(\varkappa))^{\sin\frac{\pi\omega}{2}} (\bar{\Psi}(y))^{\cos\frac{\pi\omega}{2}} \cdot (\bar{\Phi}(\varkappa))^{\sin\frac{\pi\omega}{2}} (\bar{\Phi}(y))^{\cos\frac{\pi\omega}{2}} \\
&\quad = (\bar{\Psi}(\varkappa) \cdot \bar{\Phi}(\varkappa))^{\sin\frac{\pi\omega}{2}} (\bar{\Psi}(y) \cdot \bar{\Phi}(y))^{\cos\frac{\pi\omega}{2}} \\
&\quad = ((\bar{\Psi} \cdot \bar{\Phi})(\varkappa))^{\sin\frac{\pi\omega}{2}} \cdot ((\bar{\Psi} \cdot \bar{\Phi})(y))^{\cos\frac{\pi\omega}{2}},
\end{aligned}$$

and the proof is concluded. □

It is notable that a function $\bar{\Psi}$ is an MTC-function if $\ln(\bar{\Psi})$ is a trigonometrically convex. This means that

$$\ln\left(\bar{\Psi}(\omega\varkappa + (1-\omega)y)\right) \leq \sin\frac{\pi\omega}{2} \ln\left(\bar{\Psi}(\varkappa)\right) + \cos\frac{\pi\omega}{2} \ln\left(\bar{\Psi}(y)\right),$$

for all $\varkappa, y \in I$ and $\omega \in [0,1]$.

Theorem 5. *Let $\bar{\Psi}, \bar{\Phi} : [\varkappa, y] \to \mathbb{R}$. If $\bar{\Psi}$ and $\bar{\Phi}$ are MTC-functions with $\bar{\Phi} \neq 0$, then $\frac{\bar{\Psi}}{\bar{\Phi}}$ is an MTC-function.*

Proof. To prove this, we need to prove that $\ln\left(\frac{\bar{\Psi}(\varkappa)}{\bar{\Phi}(\varkappa)}\right) = \ln\left(\bar{\Psi}(\varkappa)\right) - \ln\left(\bar{\Phi}(\varkappa)\right)$ is a trigonometrically convex function. For simplicity, we set $H^*(\varkappa) = \ln\left(\frac{\bar{\Psi}(\varkappa)}{\bar{\Phi}(\varkappa)}\right)$ and $\bar{\Psi}^*(\varkappa) = \ln\left(\bar{\Psi}(\varkappa)\right)$ and $\bar{\Phi}^*(\varkappa) = \ln\left(\bar{\Phi}(\varkappa)\right)$. In this case, we assume that

$$H^*(\varkappa) = \bar{\Psi}^*(\varkappa) - \bar{\Phi}(\varkappa)^*.$$

Since $\bar{\Psi}$ and $\bar{\Phi}$ are MTC-functions, $\bar{\Psi}^*(\varkappa)$ and $\bar{\Phi}^*(\varkappa)$ are trigonometrically convex functions. Now, we have

$$\begin{aligned}
H^*(\omega\varkappa + (1-\omega)y) &= \bar{\Psi}^*(\omega\varkappa + (1-\omega)y) - \bar{\Phi}^*(\omega\varkappa + (1-\omega)y) \\
&\leq \left[\sin\frac{\pi\omega}{2}\bar{\Psi}^*(\varkappa) + \cos\frac{\pi\omega}{2}\bar{\Psi}^*(y)\right] \\
&- \left[\sin\frac{\pi\omega}{2}\bar{\Phi}^*(\varkappa) + \cos\frac{\pi\omega}{2}\bar{\Phi}^*(y)\right] \\
&= \sin\frac{\pi\omega}{2}\left[\bar{\Psi}^*(\varkappa) - \bar{\Phi}^*(\varkappa)\right] + \cos\frac{\pi\omega}{2}\left[\bar{\Psi}^*(y) - \bar{\Phi}^*(y)\right] \\
&= \sin\frac{\pi\omega}{2}H^*(\varkappa) + \cos\frac{\pi\omega}{2}H^*(y).
\end{aligned}$$

This shows that H^* is trigonometrically convex; i.e., the function $\ln\left(\frac{\bar{\Psi}(\varkappa)}{\bar{\Phi}(\varkappa)}\right)$ is trigonometrically convex. Therefore, the quotient function $\frac{\bar{\Psi}(\varkappa)}{\bar{\Phi}(\varkappa)}$ is an MTC-function. □

Theorem 6. *Let $\bar{\Psi} : I \to J \subseteq \mathbb{R}$ be convex and $\bar{\Phi} : J \to \mathbb{R}$ be an increasing MTC-function. Then $\bar{\Phi} \circ \bar{\Psi} : I \to \mathbb{R}$ is an MTC-function.*

Proof. For $\varkappa, y \in I$ and $\omega \in [0,1]$, one gets

$$\begin{aligned}
(\bar{\Phi} \circ \bar{\Psi})(\omega\varkappa + (1-\omega)y) &= \bar{\Phi}(\bar{\Psi}(\omega\varkappa + (1-\omega)y)) \\
&\leq \bar{\Phi}(\omega\bar{\Psi}(\varkappa) + (1-\omega)\bar{\Psi}(y)) \\
&\leq [\bar{\Phi}(\bar{\Psi}(\varkappa))]^{\sin\frac{\pi\omega}{2}} \cdot [\bar{\Phi}(\bar{\Psi}(y))]^{\cos\frac{\pi\omega}{2}} \\
&= [(\bar{\Phi} \circ \bar{\Psi})(\varkappa)]^{\sin\frac{\pi\omega}{2}} \cdot [(\bar{\Phi} \circ \bar{\Psi})(y)]^{\cos\frac{\pi\omega}{2}}.
\end{aligned}$$

The proof is completed. □

Theorem 7. *Let $\bar{\Psi}_i : [\varkappa, y] \to \mathbb{R}$ be an arbitrary category of MTC-functions and $\bar{\Psi}(\varkappa) = \sup_i \bar{\Psi}_i(\varkappa)$. If $J = \{r \in [\varkappa, y] \subseteq \mathbb{R} : \bar{\Psi}(r) < \infty\} \neq \emptyset$, then $\bar{\Psi}$ is an MTC-function on J.*

Proof. For all $\varkappa, y \in J$ and $\omega \in [0,1]$, one has

$$\begin{aligned}\bar{\Psi}(\omega\varkappa + (1-\omega)y) &= \sup_i \bar{\Psi}_i(\omega\varkappa + (1-\omega)y) \\ &\le \sup_i \left[(\bar{\Psi}_i(\varkappa))^{\sin\frac{\pi\omega}{2}} \cdot (\bar{\Psi}_i(y))^{\cos\frac{\pi\omega}{2}} \right] \\ &\le \sup_i (\bar{\Psi}_i(\varkappa))^{\sin\frac{\pi\omega}{2}} \cdot \sup_i (\bar{\Psi}_i(y))^{\cos\frac{\pi\omega}{2}} \\ &= (\bar{\Psi}(\varkappa))^{\sin\frac{\pi\omega}{2}} \cdot (\bar{\Psi}(y))^{\cos\frac{\pi\omega}{2}} < \infty.\end{aligned}$$

Thus, $\bar{\Psi}$ is an MTC-function on J. This concludes the proof. □

4. New Version of H–H Inequalities for MTC-Functions

This section derives the integral H–H inequalities for MTC-functions in the context of multiplicative calculus. We begin with the main theorem in this regard.

Theorem 8. *Let $\bar{\Psi} : [\nu_1, \nu_2] \to \mathbb{R}$ be an MTC-function on $[\nu_1, \nu_2]$. If $\bar{\Psi} \in L[\nu_1, \nu_2]$, then*

$$\bar{\Psi}\left(\frac{\nu_1 + \nu_2}{2}\right) \le \left(\int_{\nu_1}^{\nu_2} (\bar{\Psi}(\varkappa))^{d\varkappa}\right)^{\frac{\sqrt{2}}{\nu_2 - \nu_1}} \le [\bar{\Psi}(\nu_1)\bar{\Psi}(\nu_2)]^{\frac{2\sqrt{2}}{\pi}}. \tag{1}$$

Proof. Note that

$$\begin{aligned}&\ln \bar{\Psi}\left(\frac{\nu_1 + \nu_2}{2}\right) \\ &= \ln\left(\bar{\Psi}\left(\frac{\omega\nu_1 + (1-\omega)\nu_2 + \omega\nu_2 + (1-\omega)\nu_1}{2}\right)\right) \\ &= \ln\left(\bar{\Psi}\left(\frac{\omega\nu_1 + (1-\omega)\nu_2}{2} + \frac{\omega\nu_2 + (1-\omega)\nu_1}{2}\right)\right) \\ &\le \ln\left[\left(\bar{\Psi}\left(\frac{\omega\nu_1 + (1-\omega)\nu_2}{2}\right)\right)^{\sin\frac{\pi}{4}} \cdot \left(\bar{\Psi}\left(\frac{\omega\nu_2 + (1-\omega)\nu_1}{2}\right)\right)^{\cos\frac{\pi}{4}}\right] \\ &= \frac{\sqrt{2}}{2} \ln \bar{\Psi}\left(\frac{\omega\nu_1 + (1-\omega)\nu_2}{2}\right) + \frac{\sqrt{2}}{2} \ln \bar{\Psi}\left(\frac{\omega\nu_2 + (1-\omega)\nu_1}{2}\right).\end{aligned}$$

In the next step, we integrate from the latter inequality with respect to ω on $[0,1]$. Hence,

$$\begin{aligned}&\ln \bar{\Psi}\left(\frac{\nu_1 + \nu_2}{2}\right) \\ &\le \frac{\sqrt{2}}{2} \int_0^1 \ln \bar{\Psi}\left(\frac{\omega\nu_1 + (1-\omega)\nu_2}{2}\right) d\omega + \frac{\sqrt{2}}{2} \int_0^1 \ln\left(\frac{\omega\nu_2 + (1-\omega)\nu_1}{2}\right) d\omega \\ &= \frac{\sqrt{2}}{2}\left[\frac{1}{\nu_2 - \nu_1} \int_{\nu_1}^{\nu_2} \ln \bar{\Psi}(\varkappa) d\varkappa + \frac{1}{\nu_1 - \nu_2} \int_{\nu_2}^{\nu_1} \ln \bar{\Psi}(\varkappa) d\varkappa\right] \\ &= \frac{\sqrt{2}}{2}\left[\frac{1}{\nu_2 - \nu_1} \int_{\nu_1}^{\nu_2} \ln \bar{\Psi}(\varkappa) d\varkappa + \frac{1}{\nu_2 - \nu_1} \int_{\nu_1}^{\nu_2} \ln \bar{\Psi}(\varkappa) d\varkappa\right]\end{aligned}$$

$$= \frac{\sqrt{2}}{\nu_2 - \nu_1} \int_{\nu_1}^{\nu_2} \ln \bar{\Psi}(\varkappa) d\varkappa.$$

Thus,

$$\bar{\Psi}\left(\frac{\nu_1 + \nu_2}{2}\right) \leq e^{\left(\frac{\sqrt{2}}{\nu_2 - \nu_1} \int_{\nu_1}^{\nu_2} \ln \bar{\Psi}(\varkappa) d\varkappa\right)}$$

$$= \left(\int_{\nu_1}^{\nu_2} (\bar{\Psi}(\varkappa))^{d\varkappa}\right)^{\frac{\sqrt{2}}{\nu_2 - \nu_1}}.$$

Hence,

$$\bar{\Psi}\left(\frac{\nu_1 + \nu_2}{2}\right) \leq \left(\int_{\nu_1}^{\nu_2} (\bar{\Psi}(\varkappa))^{d\varkappa}\right)^{\frac{\sqrt{2}}{\nu_2 - \nu_1}}. \tag{2}$$

Consider the second inequality. We have

$$\left(\int_{\nu_1}^{\nu_2} (\bar{\Psi}(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2 - \nu_1}} = \left(e^{\left(\int_{\nu_1}^{\nu_2} \ln \bar{\Psi}(\varkappa) d\varkappa\right)}\right)^{\frac{1}{\nu_2 - \nu_1}}$$

$$= e^{\frac{1}{\nu_2 - \nu_1}\left(\int_{\nu_1}^{\nu_2} \ln \bar{\Psi}(\varkappa) d\varkappa\right)}$$

$$= e^{\left(\int_0^1 \ln \bar{\Psi}(\omega\nu_1 + (1-\omega)\nu_2) d\omega\right)}$$

$$\leq e^{\int_0^1 \ln\left[(\bar{\Psi}(\nu_1))^{\sin\frac{\pi\omega}{2}} . (\bar{\Psi}(\nu_2))^{\cos\frac{\pi\omega}{2}}\right] d\omega}$$

$$= e^{\int_0^1 \left[\sin\frac{\pi\omega}{2} \ln \bar{\Psi}(\nu_1) + \cos\frac{\pi\omega}{2} \ln \bar{\Psi}(\nu_2)\right] d\omega}$$

$$= e^{\ln[\bar{\Psi}(\nu_1)\bar{\Psi}(\nu_2)]^{\frac{2}{\pi}}}$$

$$= [\bar{\Psi}(\nu_1)\bar{\Psi}(\nu_2)]^{\frac{2}{\pi}}.$$

Hence,

$$\left(\int_{\nu_1}^{\nu_2} (\bar{\Psi}(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2 - \nu_1}} \leq [\bar{\Psi}(\nu_1)\bar{\Psi}(\nu_2)]^{\frac{2}{\pi}}. \tag{3}$$

Combining the inequalities (2) and (3), one has

$$\bar{\Psi}\left(\frac{\nu_1 + \nu_2}{2}\right) \leq \left(\int_{\nu_1}^{\nu_2} (\bar{\Psi}(\varkappa))^{d\varkappa}\right)^{\frac{\sqrt{2}}{\nu_2 - \nu_1}} \leq [\bar{\Psi}(\nu_1)\bar{\Psi}(\nu_2)]^{\frac{2\sqrt{2}}{\pi}}.$$

This inequality completes our proof. □

From this theorem, the following corollaries can be stated.

Corollary 1. *Let the functions* $\bar{\Psi}$ *and* $\bar{\Phi}$ *be MTC on* $[\nu_1, \nu_2]$. *Then*

$$\bar{\Psi}\left(\frac{\nu_1+\nu_2}{2}\right)\bar{\Phi}\left(\frac{\nu_1+\nu_2}{2}\right) \le \left(\int_{\nu_1}^{\nu_2}(\bar{\Psi}(\varkappa))^{d\varkappa}\int_{\nu_1}^{\nu_2}(\bar{\Phi}(\varkappa))^{d\varkappa}\right)^{\frac{\sqrt{2}}{\nu_2-\nu_1}}$$

$$\le [\bar{\Psi}(\nu_1)\bar{\Psi}(\nu_2)\bar{\Phi}(\nu_1)\bar{\Phi}(\nu_2)]^{\frac{2\sqrt{2}}{\pi}}.$$

Corollary 2. *Let the functions* $\bar{\Psi}$ *and* $\bar{\Phi}$ *be MTC on* $[\nu_1, \nu_2]$. *Then*

$$\frac{\bar{\Psi}\left(\frac{\nu_1+\nu_2}{2}\right)}{\bar{\Phi}\left(\frac{\nu_1+\nu_2}{2}\right)} \le \left(\frac{\int_{\nu_1}^{\nu_2}(\bar{\Psi}(\varkappa))^{d\varkappa}}{\int_{\nu_1}^{\nu_2}(\bar{\Phi}(\varkappa))^{d\varkappa}}\right)^{\frac{\sqrt{2}}{\nu_2-\nu_1}} \le \left(\frac{\bar{\Psi}(\nu_1)\bar{\Psi}(\nu_2)}{\bar{\Phi}(\nu_1)\bar{\Phi}(\nu_2)}\right)^{\frac{2\sqrt{2}}{\pi}}.$$

In the next two theorems, we prove two integral quotient inequalities.

Theorem 9. *Let the functions* $\bar{\Psi}$ *and* $\bar{\Phi}$ *be convex and MTC on* $[\nu_1, \nu_2]$, *respectively. Then*

$$\left(\frac{\int_{\nu_1}^{\nu_2}(\bar{\Psi}(\varkappa))^{d\varkappa}}{\int_{\nu_1}^{\nu_2}(\bar{\Phi}(\varkappa))^{d\varkappa}}\right)^{\frac{1}{\nu_2-\nu_1}} \le \frac{\left(\frac{(\bar{\Psi}(\nu_2))^{\bar{\Psi}(\nu_2)}}{(\bar{\Psi}(\nu_1))^{\bar{\Psi}(\nu_1)}}\right)^{\frac{1}{\bar{\Psi}(\nu_2)-\bar{\Psi}(\nu_1)}}}{e\cdot(\bar{\Phi}(\nu_1)\bar{\Phi}(\nu_2))^{\frac{2}{\pi}}}.$$

Proof. On the basis of the related properties for the MTC-functions, we may write

$$\left(\frac{\int_{\nu_1}^{\nu_2}(\bar{\Psi}(\varkappa))^{d\varkappa}}{\int_{\nu_1}^{\nu_2}(\bar{\Phi}(\varkappa))^{d\varkappa}}\right)^{\frac{1}{\nu_2-\nu_1}} = \left(\frac{e^{\left(\int_{\nu_1}^{\nu_2}\ln\bar{\Psi}(\varkappa)d\varkappa\right)}}{e^{\left(\int_{\nu_1}^{\nu_2}\ln\bar{\Phi}(\varkappa)d\varkappa\right)}}\right)^{\frac{1}{\nu_2-\nu_1}}$$

$$= \left(e^{\left(\int_{\nu_1}^{\nu_2}\ln\bar{\Psi}(\varkappa)d\varkappa-\int_{\nu_1}^{\nu_2}\ln\bar{\Phi}(\varkappa)d\varkappa\right)}\right)^{\frac{1}{\nu_2-\nu_1}}$$

$$= e^{\left(\int_0^1\ln\bar{\Psi}(\nu_2+\omega(\nu_1-\nu_2))d\omega-\int_0^1\ln\bar{\Phi}(\nu_2+\omega(\nu_1-\nu_2))d\omega\right)}$$

$$\le e^{\left(\int_0^1\ln(\bar{\Psi}(\nu_2)+\omega(\bar{\Psi}(\nu_1)-\bar{\Psi}(\nu_2)))d\omega-\int_0^1\ln\left((\bar{\Phi}(\nu_1))^{\sin\frac{\pi\omega}{2}}\cdot(\bar{\Phi}(\nu_2))^{\cos\frac{\pi\omega}{2}}\right)d\omega\right)}$$

$$= e^{\ln\left(\frac{(\bar{\Psi}(\nu_2))^{\bar{\Psi}(\nu_2)}}{(\bar{\Psi}(\nu_1))^{\bar{\Psi}(\nu_1)}}\right)^{\frac{1}{\bar{\Psi}(\nu_2)-\bar{\Psi}(\nu_1)}}-1-\ln(\bar{\Phi}(\nu_1)\bar{\Phi}(\nu_2))^{\frac{2}{\pi}}}$$

$$= \frac{\left(\frac{(\bar{\Psi}(\nu_2))^{\bar{\Psi}(\nu_2)}}{(\bar{\Psi}(\nu_1))^{\bar{\Psi}(\nu_1)}}\right)^{\frac{1}{\bar{\Psi}(\nu_2)-\bar{\Psi}(\nu_1)}}}{e\cdot(\bar{\Phi}(\nu_1)\bar{\Phi}(\nu_2))^{\frac{2}{\pi}}},$$

and this is the end of the proof. □

Theorem 10. *Let the functions $\bar{\Psi}$ and $\bar{\Phi}$ be MTC and convex $[\nu_1, \nu_2]$, respectively. Then*

$$\left(\frac{\int_{\nu_1}^{\nu_2}(\bar{\Psi}(\varkappa))^{d\varkappa}}{\int_{\nu_1}^{\nu_2}(\bar{\Phi}(\varkappa))^{d\varkappa}}\right)^{\frac{1}{\nu_2-\nu_1}} \leq \frac{e\cdot(\bar{\Psi}(\nu_1)\bar{\Psi}(\nu_2))^{\frac{2}{\pi}}}{\left(\frac{(\bar{\Phi}(\nu_2))^{\bar{\Phi}(\nu_2)}}{(\bar{\Phi}(\nu_1))^{\bar{\Phi}(\nu_1)}}\right)^{\frac{1}{\bar{\Phi}(\nu_2)-\bar{\Phi}(\nu_1)}}}.$$

Proof. On the basis of the related properties for the MTC-functions and convex functions, we may write

$$\left(\frac{\int_{\nu_1}^{\nu_2}(\bar{\Psi}(\varkappa))^{d\varkappa}}{\int_{\nu_1}^{\nu_2}(\bar{\Phi}(\varkappa))^{d\varkappa}}\right)^{\frac{1}{\nu_2-\nu_1}} = \left(\frac{e^{\left(\int_{\nu_1}^{\nu_2}\ln\bar{\Psi}(\varkappa)d\varkappa\right)}}{e^{\left(\int_{\nu_1}^{\nu_2}\ln\bar{\Phi}(\varkappa)d\varkappa\right)}}\right)^{\frac{1}{\nu_2-\nu_1}}$$

$$= \left(e^{\left(\int_{\nu_1}^{\nu_2}\ln\bar{\Psi}(\varkappa)d\varkappa - \int_{\nu_1}^{\nu_2}\ln\bar{\Phi}(\varkappa)d\varkappa\right)}\right)^{\frac{1}{\nu_2-\nu_1}}$$

$$= e^{\left(\int_0^1 \ln\bar{\Psi}(\nu_2+\omega(\nu_1-\nu_2))d\omega - \int_0^1 \ln\bar{\Phi}(\nu_2+\omega(\nu_1-\nu_2))d\omega\right)}$$

$$\leq e^{\left(\int_0^1 \ln\left((\bar{\Psi}(\nu_1))^{\sin\frac{\pi\omega}{2}}\cdot(\bar{\Psi}(\nu_2))^{\cos\frac{\pi\omega}{2}}\right)d\omega - \int_0^1 \ln(\bar{\Phi}(\nu_2)+\omega(\bar{\Phi}(\nu_1)-\bar{\Phi}(\nu_2)))d\omega\right)}$$

$$= e^{\ln(\bar{\Psi}(\nu_1)\bar{\Psi}(\nu_2))^{\frac{2}{\pi}} - \ln\left(\frac{(\bar{\Phi}(\nu_2))^{\bar{\Phi}(\nu_2)}}{(\bar{\Phi}(\nu_1))^{\bar{\Phi}(\nu_1)}}\right)^{\frac{1}{\bar{\Phi}(\nu_2)-\bar{\Phi}(\nu_1)}} + 1}$$

$$= \frac{e\cdot(\bar{\Psi}(\nu_1)\bar{\Psi}(\nu_2))^{\frac{2}{\pi}}}{\left(\frac{(\bar{\Phi}(\nu_2))^{\bar{\Phi}(\nu_2)}}{(\bar{\Phi}(\nu_1))^{\bar{\Phi}(\nu_1)}}\right)^{\frac{1}{\bar{\Phi}(\nu_2)-\bar{\Phi}(\nu_1)}}}.$$

The proof is complete now. □

Theorem 11. *Let the functions $\bar{\Psi}$ and $\bar{\Phi}$ be convex and MTC $[\nu_1, \nu_2]$, respectively. Then*

$$\left(\int_{\nu_1}^{\nu_2}(\bar{\Psi}(\varkappa))^{d\varkappa}\int_{\nu_1}^{\nu_2}(\bar{\Phi}(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}} \leq \frac{\left(\frac{(\bar{\Psi}(\nu_2))^{\bar{\Psi}(\nu_2)}}{(\bar{\Psi}(\nu_1))^{\bar{\Psi}(\nu_1)}}\right)^{\frac{1}{\bar{\Psi}(\nu_2)-\bar{\Psi}(\nu_1)}}(\bar{\Phi}(\nu_1)\bar{\Phi}(\nu_2))^{\frac{2}{\pi}}}{e}.$$

Proof. By the hypotheses, we have

$$\left(\int_{\nu_1}^{\nu_2}(\bar{\Psi}(\varkappa))^{d\varkappa}\int_{\nu_1}^{\nu_2}(\bar{\Phi}(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}$$

$$= \left(e^{\int_{\nu_1}^{\nu_2} \ln \bar{\Psi}(\varkappa) d\varkappa} . e^{\int_{\nu_1}^{\nu_2} \ln \bar{\Phi}(\varkappa) d\varkappa} \right)^{\frac{1}{\nu_2 - \nu_1}}$$

$$= \left(e^{\int_{\nu_1}^{\nu_2} \ln \bar{\Psi}(\varkappa) d\varkappa + \int_{\nu_1}^{\nu_2} \ln \bar{\Phi}(\varkappa) d\varkappa} \right)^{\frac{1}{\nu_2 - \nu_1}}$$

$$= e^{\left(\int_0^1 \ln \bar{\Psi}(\nu_2 + \omega(\nu_1 - \nu_2)) d\omega + \int_0^1 \ln \bar{\Phi}(\nu_2 + \omega(\nu_1 - \nu_2)) d\omega \right)}$$

$$\leq e^{\left(\int_0^1 \ln(\bar{\Psi}(\nu_2) + \omega(\bar{\Psi}(\nu_1) - \bar{\Psi}(\nu_2))) d\omega - \int_0^1 \ln\left((\bar{\Phi}(\nu_1))^{\sin \frac{\pi\omega}{2}} . (\bar{\Phi}(\nu_2))^{\cos \frac{\pi\omega}{2}} \right) d\omega \right)}$$

$$= e^{\ln \left(\frac{(\bar{\Psi}(\nu_2))^{\bar{\Psi}(\nu_2)}}{(\bar{\Psi}(\nu_1))^{\bar{\Psi}(\nu_1)}} \right)^{\frac{1}{\bar{\Psi}(\nu_2) - \bar{\Psi}(\nu_1)}} - 1 + \ln(\bar{\Phi}(\nu_1)\bar{\Phi}(\nu_2))^{\frac{2}{\pi}}}$$

$$= \frac{\left(\frac{(\bar{\Psi}(\nu_2))^{\bar{\Psi}(\nu_2)}}{(\bar{\Psi}(\nu_1))^{\bar{\Psi}(\nu_1)}} \right)^{\frac{1}{\bar{\Psi}(\nu_2) - \bar{\Psi}(\nu_1)}} (\bar{\Phi}(\nu_1)\bar{\Phi}(\nu_2))^{\frac{2}{\pi}}}{e}.$$

This is the desired right-hand side in our proof. □

5. Some New Inequalities for MTC-Functions

In this section, we now establish some new integral inequalities for MTC-functions in the context of multiplicative calculus.

Note that the following integrals will be used in this section:

$$\int_0^1 \sin \frac{\pi\omega}{2} d\omega = \int_0^1 \cos \frac{\pi\omega}{2} d\omega = \frac{2}{\pi},$$

$$\int_0^1 |1 - 2\omega| \sin \frac{\pi\omega}{2} d\omega = \int_0^1 |1 - 2\omega| \cos \frac{\pi\omega}{2} d\omega = \frac{2}{\pi^2}\left(\pi - 4\sqrt{2} + 4\right),$$

$$\int_0^1 |1 - 2\omega|^p d\omega = \frac{1}{p+1}.$$

Theorem 12. *Let $\bar{\Psi} : I^\circ \subseteq \mathbb{R} \to \mathbb{R}^+$ be multiplicative differentiable and $\nu_1, \nu_2 \in I^\circ$ with $\nu_1 < \nu_2$. If $\bar{\Psi}$ is increasing on $[\nu_1, \nu_2]$ and $\bar{\Psi}^*$ is an MTC-function on $[\nu_1, \nu_2]$, then*

$$\left| \frac{\sqrt{\bar{\Psi}(\nu_1)\bar{\Psi}(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2} (\bar{\Psi}(\varkappa))^{d\varkappa} \right)^{\frac{1}{\nu_2 - \nu_1}}} \right| \leq (\bar{\Psi}^*(\nu_1)\bar{\Psi}^*(\nu_2))^{\frac{\pi - 4\sqrt{2} + 4}{\pi^2}(\nu_2 - \nu_1)}.$$

Proof. Using the conclusion of Lemma 1, one gets

$$\left|\frac{\sqrt{\bar{\Psi}(\nu_1)\bar{\Psi}(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2}(\bar{\Psi}(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}}\right| \leq \left|\int_0^1\left([\bar{\Psi}^*(\omega\nu_1+(1-\omega)\nu_2)]^{1-2\omega}\right)^{dw}\right|^{\frac{\nu_2-\nu_1}{2}}$$

$$\leq e^{\frac{\nu_2-\nu_1}{2}\int_0^1\left|\ln(\bar{\Psi}^*(\omega\nu_1+(1-\omega)\nu_2))^{1-2\omega}\right|dw}$$

$$= e^{\frac{\nu_2-\nu_1}{2}\int_0^1|1-2\omega||\ln\bar{\Psi}^*(\omega\nu_1+(1-\omega)\nu_2)|dw}. \tag{4}$$

Since the function $\bar{\Psi}^*$ is MTC, one gets

$$\int_0^1|1-2\omega|\ln\bar{\Psi}^*(\omega\nu_1+(1-\omega)\nu_2)dw$$

$$\leq \int_0^1|1-2\omega|\left((\bar{\Psi}^*(\nu_1))^{\sin\frac{\pi\omega}{2}}.(\bar{\Psi}^*(\nu_2))^{\cos\frac{\pi\omega}{2}}\right)d\omega$$

$$= \ln\bar{\Psi}^*(\nu_1)\int_0^1|1-2\omega|\sin\frac{\pi\omega}{2}d\omega + \ln\bar{\Psi}^*(\nu_2)\int_0^1|1-2\omega|\cos\frac{\pi\omega}{2}d\omega$$

$$= \frac{2}{\pi^2}\left(\pi-4\sqrt{2}+4\right)(\ln\bar{\Psi}^*(\nu_1)+\ln\bar{\Psi}^*(\nu_2)). \tag{5}$$

If we substitute Inequality (5) into (4), it becomes

$$\left|\frac{\sqrt{\bar{\Psi}(\nu_1)\bar{\Psi}(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2}(\bar{\Psi}(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}}\right| \leq e^{\frac{\nu_2-\nu_1}{2}\int_0^1\left|\ln(\bar{\Psi}^*(\omega\nu_1+(1-\omega)\nu_2))^{1-2\omega}\right|dw}$$

$$\leq e^{\frac{\nu_2-\nu_1}{2}\cdot\frac{2}{\pi^2}\left(\pi-4\sqrt{2}+4\right)(\ln\bar{\Psi}^*(\nu_1)+\ln\bar{\Psi}^*(\nu_2))}$$

$$= (\bar{\Psi}^*(\nu_1)\bar{\Psi}^*(\nu_2))^{\frac{\pi-4\sqrt{2}+4}{\pi^2}(\nu_2-\nu_1)}.$$

So, the proof is completed. □

Theorem 13. *Let $\bar{\Psi} : I^\circ \subset \mathbb{R} \to \mathbb{R}^+$ be multiplicative differentiable and $\nu_1, \nu_2 \in I^\circ$ with $\nu_1 < \nu_2$. If $\bar{\Psi}$ is increasing on $[\nu_1, \nu_2]$ and $(\ln\bar{\Psi}^*)^q$ is trigonometrically convex on $[\nu_1, \nu_2]$, then one has*

$$\left|\frac{\sqrt{\bar{\Psi}(\nu_1)\bar{\Psi}(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2}(\bar{\Psi}(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}}\right| \leq (\bar{\Psi}^*(\nu_1)\bar{\Psi}^*(\nu_2))^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\pi}\right)^{\frac{1}{q}}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Applying Lemma 1 and Hölder's inequality, it follows that

$$\left|\frac{\sqrt{\bar{\Psi}(\nu_1)\bar{\Psi}(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2}(\bar{\Psi}(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}}\right| \le \left|\int_0^1\left([\bar{\Psi}^*(\omega\nu_1+(1-\omega)\nu_2)]^{1-2\omega}\right)^{dw}\right|^{\frac{\nu_2-\nu_1}{2}}$$

$$\le e^{\frac{\nu_2-\nu_1}{2}\int_0^1|1-2\omega||\ln\bar{\Psi}^*(\omega\nu_1+(1-\omega)\nu_2)|dw}$$

$$\le e^{\frac{\nu_2-\nu_1}{2}\left(\int_0^1|1-2\omega|^p d\omega\right)^{\frac{1}{p}}\left(\int_0^1(\ln(\bar{\Psi}^*(\omega\nu_1+(1-\omega)\nu_2)))^q dw\right)^{\frac{1}{q}}}. \tag{6}$$

The trigonometrical convexity of $(\ln\bar{\Psi}^*)^q$ implies that

$$\int_0^1(\ln(\bar{\Psi}^*(\omega\nu_1+(1-\omega)\nu_2)))^q dw$$

$$\le \int_0^1\left[\sin\frac{\pi\omega}{2}(\ln\bar{\Psi}^*(\nu_1))^q+\cos\frac{\pi\omega}{2}(\ln\bar{\Psi}^*(\nu_2))^q\right]dw$$

$$= \frac{2\left((\ln\bar{\Psi}^*(\nu_1))^q+(\ln\bar{\Psi}^*(\nu_2))^q\right)}{\pi}. \tag{7}$$

By combining Inequalities (6) and (7), one gets

$$\left|\frac{\sqrt{\bar{\Psi}(\nu_1)\bar{\Psi}(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2}(\bar{\Psi}(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}}\right| \le e^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\pi}\right)^{\frac{1}{q}}\left((\ln\bar{\Psi}^*(\nu_1))^q+(\ln\bar{\Psi}^*(\nu_2))^q\right)^{\frac{1}{q}}}$$

$$\le e^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\pi}\right)^{\frac{1}{q}}(\ln(\bar{\Psi}^*(\nu_1)\bar{\Psi}^*(\nu_2)))}$$

$$= (\bar{\Psi}^*(\nu_1)\bar{\Psi}^*(\nu_2))^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\pi}\right)^{\frac{1}{q}}},$$

completing the proof. □

Theorem 14. *Let $\bar{\Psi}: I^\circ \subset \mathbb{R} \to \mathbb{R}^+$ be multiplicative differentiable and $\nu_1, \nu_2 \in I^\circ$ so that $\nu_1 < \nu_2$. If $\bar{\Psi}$ is increasing on $[\nu_1, \nu_2]$ and $(\ln\bar{\Psi}^*)^q$, $q \ge 1$, is trigonometrically convex on $[\nu_1, \nu_2]$, then*

$$\left|\frac{\sqrt{\bar{\Psi}(\nu_1)\bar{\Psi}(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2}(\bar{\Psi}(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}}\right| \le (\bar{\Psi}^*(\nu_1)\bar{\Psi}^*(\nu_2))^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{\pi-4\sqrt{2}+4}{\pi^2}\right)^{\frac{1}{q}}}.$$

Proof. At first, we suppose that $q > 1$. In accordance with the conclusion of Lemma 1, and by the trigonometrical convexity of $(\ln\bar{\Psi}^*)^q$, and by the power mean inequality, one has

$$\left|\frac{\sqrt{\bar{\Psi}(\nu_1)\bar{\Psi}(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2}(\bar{\Psi}(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}}\right|$$

$$\leq\left|\int_0^1\left([\bar{\Psi}^*(\omega\nu_1+(1-\omega)\nu_2)]^{1-2\omega}\right)^{dw}\right|^{\frac{\nu_2-\nu_1}{2}}$$

$$\leq e^{\frac{\nu_2-\nu_1}{2}\int_0^1|1-2\omega||\ln\bar{\Psi}^*(\omega\nu_1+(1-\omega)\nu_2)|dw}$$

$$\leq e^{\frac{\nu_2-\nu_1}{2}\left(\int_0^1|1-2\omega|^p d\omega\right)^{1-\frac{1}{q}}\left(\int_0^1|1-2\omega||\ln(\bar{\Psi}^*(\omega\nu_1+(1-\omega)\nu_2))|^q dw\right)^{\frac{1}{q}}}$$

$$\leq e^{\frac{\nu_2-\nu_1}{2}\left(\int_0^1|1-2\omega|^p d\omega\right)^{1-\frac{1}{q}}\left(\int_0^1|1-2\omega|\left[\sin\frac{\pi\omega}{2}(\ln\bar{\Psi}^*(\nu_1))^q+\cos\frac{\pi\omega}{2}(\ln\bar{\Psi}^*(\nu_2))^q\right]dw\right)^{\frac{1}{q}}}$$

$$= e^{\frac{\nu_2-\nu_1}{2}\left(\int_0^1|1-2\omega|^p d\omega\right)^{1-\frac{1}{q}}\left((\ln\bar{\Psi}^*(\nu_1))^q\int_0^1|1-2\omega|\sin\frac{\pi\omega}{2}dw+(\ln\bar{\Psi}^*(\nu_2))^q\int_0^1|1-2\omega|\cos\frac{\pi\omega}{2}dw\right)^{\frac{1}{q}}}.$$

Since, $\int_0^1|1-2\omega|d\omega=\frac{1}{2}$ and

$$\int_0^1|1-2\omega|\sin\frac{\pi\omega}{2}d\omega=\int_0^1|1-2\omega|\cos\frac{\pi\omega}{2}d\omega=\frac{2}{\pi^2}\left(\pi-4\sqrt{2}+4\right),$$

it follows that

$$\left|\frac{\sqrt{\bar{\Psi}(\nu_1)\bar{\Psi}(\nu_2)}}{\left(\int_{\nu_1}^{\nu_2}(\bar{\Psi}(\varkappa))^{d\varkappa}\right)^{\frac{1}{\nu_2-\nu_1}}}\right|$$

$$\leq e^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\frac{2}{\pi^2}\left(\pi-4\sqrt{2}+4\right)\left((\ln\bar{\Psi}^*(\nu_1))^q+(\ln\bar{\Psi}^*(\nu_2))^q\right)\right\}^{\frac{1}{q}}}$$

$$= e^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}.2^{\frac{1}{q}}\left\{\frac{1}{\pi^2}\left(\pi-4\sqrt{2}+4\right)\left((\ln\bar{\Psi}^*(\nu_1))^q+(\ln\bar{\Psi}^*(\nu_2))^q\right)\right\}^{\frac{1}{q}}}$$

$$= e^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}.\left(\frac{\pi-4\sqrt{2}+4}{\pi^2}\right)^{\frac{1}{q}}(\ln\bar{\Psi}^*(\nu_1)\ln\bar{\Psi}^*(\nu_2))}$$

$$=(\bar{\Psi}^*(\nu_1)\bar{\Psi}^*(\nu_2))^{\frac{\nu_2-\nu_1}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{\pi-4\sqrt{2}+4}{\pi^2}\right)^{\frac{1}{q}}}.$$

For the case $q=1$, we immediately obtain the desired result by using the estimates given in the proof of Theorem 13. So, the proof is completed. □

6. Conclusions

In this study, we introduced the class of symmetric MTC-functions and presented their associated properties. For these MTC-functions and other types of convex functions, we extracted some logical connections. The novel structure of the well-established H–H-inequality is demonstrated. Our approach may offer further applications within the scope of convexity theory. Expanding these findings to include other forms of convexities discussed in the literature would be an interesting path for future research. By employing this newly defined class of convexities, new forms of integral inequalities can be formulated. Our next aim to extend this research is to study MTC-functions in the context of the existing integrals in quantum calculus.

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