

Article

Hyers–Ulam–Rassias Stability of Generalized Quadratic Functional Equation on Non-Archimedean Normed Space over p -Adic Numbers

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Abstract

We investigate the Hyers–Ulam–Rassias stability of a generalized quadratic functional equation of the asymmetric four-function form $F(x + y) + G(x - y) = L(x) + M(y)$, where F , G , L , and M are unknown mappings. This study is conducted within the framework of non-Archimedean normed spaces over the p -adic numbers. Our approach employs a separation technique, analyzing the even and odd parts of the functions to establish stability results. We show that all four functions are approximated by a combination of a quadratic function and two additive functions.

Keywords: stability; quadratic-additive functional equation; non-Archimedean; p -adic numbers



Academic Editor: Francisco Martinez

Received: 29 August 2025

Revised: 26 September 2025

Accepted: 2 October 2025

Published: 4 October 2025

Citation: Tongsomporn, J.; Aksornthong, N. Hyers–Ulam–Rassias Stability of Generalized Quadratic Functional Equation on Non-Archimedean Normed Space over p -Adic Numbers. *Symmetry* **2025**, *17*, 1651. <https://doi.org/10.3390/sym17101651>

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1. Introduction

The study of functional equations has been an active research area for many decades, with stability problems constituting one of its most prominent branches. The stability problem was originally introduced by Ulam (1940) [1] in connection with approximate homomorphisms of groups. Specifically, let G_1 be a group and let G_2 be a metric group equipped with the metric $d(\cdot, \cdot)$. The question is whether, for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever a mapping $h: G_1 \rightarrow G_2$ satisfies

$$d(h(xy), h(x)h(y)) < \delta$$

for all $x, y \in G_1$, one can find a homomorphism $H: G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$.

Hyers (1941) [2] established what is now referred to as Hyers–Ulam stability for the Cauchy additive equation. More precisely, let X, Y be Banach spaces and consider a mapping $f: X \rightarrow Y$ that satisfies

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon$$

for all x, y in X . Then, for each $x \in X$, the limit

$$a(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$$

exists, and the function $a: X \rightarrow Y$ defines the unique additive mapping such that

$$\|f(x) - a(x)\| \leq \epsilon$$

for all x in X .

Later, Rassias (1978) [3] generalized Hyers' theorem by allowing unbounded Cauchy differences, thereby introducing the celebrated Hyers–Ulam–Rassias stability. He proved the following theorem:

Let $f: E_1 \rightarrow E_2$ be a function between Banach spaces. Suppose that f satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p), \quad x, y \in E_1,$$

for some constant $\theta \geq 0$ and some exponent p with $0 \leq p < 1$. Then, there is a unique additive function $A: E_1 \rightarrow E_2$ such that

$$\|f(x) - A(x)\| \leq \left(\frac{2\theta}{2-2^p}\right) \|x\|^p, \quad x \in E_1.$$

If we further assume that $f(tx)$ is continuous in t for every fixed $x \in E_1$, then the mapping A is linear.

The well-known quadratic functional equation, commonly referred to as the Jordan–von Neumann equation, is given by

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad (1)$$

The solutions of this equation define quadratic mappings and exhibit a strong connection with inner product spaces. In their classical result, Jordan and von Neumann (1935) [4] proved that a normed vector space is an inner product space precisely when its norm fulfills the parallelogram identity

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

for all x and y in the space. Each such relation is naturally referred to as a quadratic functional equation. In particular, any solution of the classical quadratic Equation (1) is regarded as a quadratic function. Quadratic functional equations play an important role in various fields, including functional analysis, information theory, and geometry, and their stability has been the subject of extensive investigation in both Archimedean and non-Archimedean settings.

In the Archimedean setting, Skof (1983) [5] was the first to investigate the Hyers–Ulam stability of the quadratic functional equation. He showed that if a function $f: E_1 \rightarrow E_2$ satisfies

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \delta, \quad x, y \in E_1,$$

for some $\delta \geq 0$, then the limit

$$Q(x) = \lim_{n \rightarrow \infty} 2^{-2n} f(2^n x)$$

exists for every $x \in E_1$. Moreover, Q defines the unique quadratic mapping such that

$$\|f(x) - Q(x)\| \leq \frac{\delta}{2}, \quad x \in E_1.$$

Subsequently, in 1995, Borelli and Forti [6] established stability results for functional equations in which the error bound explicitly depends on the variables x and y . Specifically,

let G be an abelian group and E a Banach space, and consider a mapping $f: G \rightarrow E$ with $f(0) = 0$ that satisfies

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \phi(x, y), \quad x, y \in G.$$

Suppose that for each $x \in G$, at least one of the series $\sum_{i=1}^{\infty} 2^{-2i}\phi(2^{i-1}x, 2^{i-1}x)$ and $\sum_{i=1}^{\infty} 2^{2(i-1)}\phi(2^{-i}x, 2^{-i}x)$ is convergent, and denote its sum by $\Phi(x)$. Furthermore, if either $2^{-2i}\phi(2^{i-1}x, 2^{i-1}y) \rightarrow 0$ or $2^{2(i-1)}\phi(2^{-i}x, 2^{-i}y) \rightarrow 0$ as $i \rightarrow \infty$, then there is a unique quadratic function $Q: G \rightarrow E$ such that

$$\|f(x) - Q(x)\| \leq \Phi(x), \quad x \in G.$$

An intuitive generalization of the quadratic functional equation is its Pexiderized variant, in which the single function is replaced by two separate functions. In 1992, Czerwik [7] established a version of the Hyers–Ulam–Rassias stability for this Pexider-type quadratic equation. Specifically, let G be an abelian group divisible by an integer number $k \geq 2$, and let E denote a Banach space. Consider two functions $f, g: G \rightarrow E$ that satisfy the condition

$$\|f(x+y) + f(x-y) - 2g(x) - 2g(y)\| \leq \phi(x, y), \quad x, y \in G,$$

where $\phi: G^2 \rightarrow \mathbb{R}^+$ is a given function. Assume further that the series $\sum_{i=1}^{\infty} k^{2i}\phi(mk^{-i}x, k^{-i}x)$ and $\sum_{i=1}^{\infty} k^{2i}\phi(k^{-i}x, 0)$ are convergent for all $x \in G$ and each integer m in the range $1, \dots, k-1$. Moreover, assume that

$$\liminf_{i \rightarrow \infty} k^{2i}\phi(k^{-i}x, k^{-i}y) = 0, \quad x, y \in G.$$

Under this condition, there exists a unique quadratic function $Q: G \rightarrow E$ such that for every $x \in G$ the following inequalities hold:

$$\|f(x) - f(0) - Q(x)\| \leq k^{-2} \sum_{m=1}^{k-1} \sum_{i=1}^{\infty} (k-m)k^{2i}\Phi(mk^{-i}x, k^{-i}x)$$

and

$$\|g(x) - g(0) - 2Q(x)\| \leq k^{-2} \sum_{m=1}^{k-1} \sum_{i=1}^{\infty} (k-m)k^{2i}\Psi((mk^{-i}x, k^{-i}x),$$

where the auxiliary functions Φ and Ψ are defined by $\Phi(x, y) = \phi(x, y) + \phi(x, 0) + \phi(y, 0) + \phi(0, 0)$ and $\Psi(x, y) = 2\phi(x, y) + \phi(x+y, 0) + \phi(x-y, 0)$.

A notable advancement in this area was provided by Jung (2000) [8], who studied the stability properties of the Pexider-type quadratic functional equation

$$f_1(x+y) + f_2(x-y) = 2f_3(x) + f_4(y),$$

within the setting of Banach normed spaces. By employing fixed-point techniques, he was able to extend the classical Hyers–Ulam–Rassias stability results. More precisely, the theorem asserts the following: Let E_1 be a real normed space, E_2 a Banach space, and let an integer $k \geq 2$. If functions $f_1, f_2, f_3, f_4: E_1 \rightarrow E_2$ satisfy

$$\|f_1(x+y) + f_2(x-y) - f_3(x) - f_4(y)\| \leq \varphi(x, y), \quad \forall x, y \in E_1,$$

then there are a quadratic function $Q: E_1 \rightarrow E_2$ and additive functions $A_1, A_2: E_1 \rightarrow E_2$ such that for all $x \in E_1$,

$$\begin{aligned} & \|f_1(x) - Q(x) - A_1(x) - A_2(x) - f_1(0)\| \\ & \leq \frac{1}{2k^2} \hat{\Phi}_k(x, x) + \frac{1}{2k} \hat{\Phi}'_k(x, x) + \frac{1}{2k} \hat{\Phi}''_k(x, x) + 3\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \frac{5}{2}\varphi(x, 0) + \frac{11}{2}\varphi(0, 0), \end{aligned}$$

$$\begin{aligned} & \|f_2(x) - Q(x) - A_1(x) + A_2(x) - f_2(0)\| \\ & \leq \frac{1}{2k^2} \hat{\Phi}_k(x, x) + \frac{1}{2k} \hat{\Phi}'_k(x, x) + \frac{1}{2k} \hat{\Phi}''_k(x, x) + 3\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \frac{5}{2}\varphi(x, 0) + \frac{7}{2}\varphi(0, 0), \end{aligned}$$

$$\|f_3(x) - 2Q(x) - A_1(x) - f_3(0)\| \leq \frac{1}{k^2} \hat{\Phi}_k(x, x) + \frac{1}{k} \hat{\Phi}'_k(x, x) + 2\varphi(x, 0) + 2\varphi(0, 0),$$

and

$$\|f_4(x) - 2Q(x) - A_2(x) - f_4(0)\| \leq \frac{1}{k^2} \hat{\Phi}_k(x, x) + \frac{1}{k} \hat{\Phi}'_k(x, x)$$

where $\hat{\Phi}_k$, $\hat{\Phi}'_k$, and $\hat{\Phi}''_k$ are auxiliary control functions expressed as series derived from φ .

In parallel, increasing attention has been devoted to non-Archimedean normed spaces, whose geometry is governed by the strong triangle inequality

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}.$$

Such spaces naturally arise in p -adic analysis and exhibit convergence properties distinct from Banach spaces, often simplifying stability arguments and leading to sharper results. The relevant definitions will be presented in detail in the preliminary section. Beyond classical normed and ultrametric frameworks, stability problems have also been extended to fuzzy structures. Fuzzy mathematics, introduced by Zadeh (1965) [9] and Goguen (1967) [10], provides a natural language for imprecision. In this direction, Mirmostafaei and Moslehian (2009) [11] established stability results of additive mapping in non-Archimedean fuzzy normed spaces. Later in 2016, Eghbali [12] studied the stability equation

$$f(x + y) + f(x - y) = 2g(x) + 2h(y)$$

in non-Archimedean fuzzy normed spaces. After that, El-Fassi and Kabbaj (2017) [13] studied the generalized orthogonal stability of the Pexiderized quadratic functional equation in modular spaces, thereby extending stability theory to settings where both orthogonality and modular structure play essential roles. In 2020, Ciepliński [14] examined the stability of a general functional equation in four variables as follows

$$\begin{aligned} & f(a_1(x_1 + x_2), b_1(y_1 + y_2)) + f(a_2(x_1 + x_2), b_2(y_1 - y_2)) \\ & + f(a_3(x_1 - x_2), b_3(y_1 + y_2)) + f(a_4(x_1 - x_2), b_4(y_1 - y_2)) \\ & = C_{11}f(x_1, y_1) + C_{12}f(x_1, y_2) + C_{21}f(x_2, y_1) + C_{22}f(x_2, y_2) \end{aligned}$$

thereby further underscoring the versatility of the non-Archimedean approach.

In a related contribution, Schwaiger (2020) [15] explored the interplay between the completion of normed spaces over non-Archimedean fields and the stability properties of the Cauchy functional equation, demonstrating that the process of completing a space has a direct impact on stability behavior in ultrametric contexts. The principal result can be stated as follows: Let $(S, +)$ be a commutative semigroup, and let X be a normed space over $(\mathbb{Q}, |\cdot|_p)$ with completion X^c . For any $\epsilon > 0$, if functions $f, g, h: S \rightarrow X$ satisfy

$\|f(x+y) - g(x) - h(y)\| \leq \epsilon$ for all $x, y \in S$, then there exist functions $f_1, g_1, h_1: S \rightarrow X^c$ such that

$$f_1(x+y) - g_1(x) - h_1(y) = 0,$$

and the approximations satisfy

$$\|f_1(x+y) - f(x+y)\| \leq (48p+3)\epsilon$$

and

$$\|g_1(x) - g(x)\|, \|h_1(x) - h(x)\| \leq (24p+1)\epsilon.$$

Later, Bettencourt and Mendes (2021) [16] established the stability for the equation of the quadratic type

$$f(x+y+z) + f(x) + f(y) + f(z) = f(x+y) + f(y+z) + f(z+x)$$

where f is a function from an abelian group to a non-Archimedean Banach space.

In recent work, Tamilvanan et al. (2023) [17] proposed a novel class of generalized mixed-type functional equations combining quadratic and additive components. They analyzed the Ulam-type stability of these equations within the framework of non-Archimedean fuzzy ϕ -2-normed spaces and non-Archimedean Banach spaces, employing both direct methods and fixed-point techniques. Subsequently, in 2025, Elumalai and Sangeetha [18] examined the Hyers–Ulam stability of mixed quadratic-cubic functional equations in non-Archimedean 2-normed spaces, utilizing the fixed-point approach.

More recently, Ciepliński (2025) [19] presented a comprehensive survey of Ulam stability in non-Archimedean spaces, covering Cauchy, Jensen, quadratic, and Pexiderized equations. He showed that many Banach space results admit precise analogues in strong triangle settings, but with additional technical advantages. The theorem states the following: Let G be a commutative group, and let X be a complete non-Archimedean normed space over a non-Archimedean field of characteristic not equal to 2. Consider a function $\varphi: G^{2n} \rightarrow [0, \infty)$ such that, for every $(x_{11}, x_{12}, \dots, x_{n1}, x_{n2}) \in G^{2n}$,

$$\lim_{k \rightarrow \infty} \frac{\varphi(2^k x_{11}, 2^k x_{12}, \dots, 2^k x_{n1}, 2^k x_{n2})}{|4|^{nk}} = 0$$

and the limit

$$\lim_{k \rightarrow \infty} \max \left\{ \frac{\varphi(2^j x_{11}, 2^j x_{12}, \dots, 2^j x_{n1}, 2^j x_{n2})}{|4|^{nj}} : 0 \leq j < k \right\},$$

exists, denoted by $\psi(x_{11}, \dots, x_{n1})$. Let $f: G^n \rightarrow X$ be a mapping satisfying $f(x_{11}, \dots, x_{n1}) = 0$ whenever at least one component x_{i1} equals zero, and

$$\left\| \sum_{i_1, \dots, i_n \in \{-1, 1\}} f(x_{11} + i_1 x_{12}, \dots, x_{n1} + i_n x_{n2}) - 2^n \sum_{j_1, \dots, j_n \in \{1, 2\}} f(x_{1j_1}, \dots, x_{nj_n}) \right\| \leq \varphi(x_{11}, x_{12}, \dots, x_{n1}, x_{n2}), (x_{11}, x_{12}, \dots, x_{n1}, x_{n2}) \in G^{2n}$$

Under these conditions, there exists a mapping $Q: G^n \rightarrow X$ satisfying

$$\sum_{i_1, \dots, i_n \in \{-1, 1\}} Q(x_{11} + i_1 x_{12}, \dots, x_{n1} + i_n x_{n2}) = 2^n \sum_{j_1, \dots, j_n \in \{1, 2\}} Q(x_{1j_1}, \dots, x_{nj_n}),$$

and

$$\|f(x_{11}, \dots, x_{n1}) - Q(x_{11}, \dots, x_{n1})\| \leq \frac{1}{|4|^n} \psi(x_{11}, \dots, x_{n1}), (x_{11}, \dots, x_{n1}) \in G_n.$$

Furthermore, if for every $(x_{11}, \dots, x_{n1}) \in G_n$,

$$\lim_{l \rightarrow \infty} \lim_{k \rightarrow \infty} \max \left\{ \frac{\varphi(2^j x_{11}, 2^j x_{11}, \dots, 2^j x_{n1}, 2^j x_{n1})}{|4|^{nj}} : l \leq j < k + l \right\} = 0,$$

then Q is the unique mapping satisfying the above functional equation and the corresponding inequality.

Motivated by Jung's results on Pexiderized quadratic equations [8] and Ciepliński's recent non-Archimedean approach [19], we aimed to obtain sharper stability results for a quadratic functional equation involving four functions in the non-Archimedean framework. Specifically, the purpose of this paper is to investigate the Hyers–Ulam–Rassias stability of the generalized asymmetric quadratic functional equation

$$F(x + y) + G(x - y) = L(x) + M(y),$$

where F, G, L, M are unknown mappings, in non-Archimedean normed spaces over the p -adic numbers.

2. Preliminary

In this section, we recall the basic concepts, notations, and lemmas that will be used throughout the paper.

Definition 1. An abelian group S is k -divisible if it is equipped with a function $\dot{\cdot}_k: S \rightarrow S$ such that $k \dot{\cdot}_k x = x$ for all $x \in S$.

More reading on divisible groups can be found on Ref. [20].

Definition 2. A function Q from an abelian group S into an abelian group G is quadratic if for any $x, y \in S$,

$$Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y).$$

Proposition 1. If Q is quadratic, then for any $n \in \mathbb{N}_0$ and any $x \in \text{dom}(Q)$,

$$Q(nx) = n^2 Q(x).$$

Proof. Clearly, the claim holds for $n = 0, 1$. Then, for $n \geq 1$,

$$\begin{aligned} Q((n+1)x) &= 2Q(nx) + 2Q(x) - Q((n-1)x) \\ &= 2n^2 Q(x) + 2Q(x) - (n-1)^2 Q(x) \\ &= (n+1)^2 Q(x) \end{aligned}$$

by induction. $\square$

Definition 3. A function A from an abelian group S into an abelian group G is additive if for any $x, y \in S$,

$$A(x + y) = A(x) + A(y).$$

Definition 4 ([21]). A non-Archimedean normed space is a vector space V over a valued field $(\mathbb{K}, |\cdot|)$ equipped with a map $\|\cdot\|: V \rightarrow [0, \infty)$ such that for all $x, y \in V$ and $\alpha \in \mathbb{K}$,

1. $\|x\| = 0$ iff $x = 0$;
2. $\|\alpha x\| = |\alpha| \|x\|$;
3. $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ (strong triangle inequality).

Definition 5 ([21]). Let p be a prime number. Define the p -adic norm, $|\cdot|_p$, on $\mathbb{Q}$ by

$$\left| p^n \frac{a}{b} \right|_p = p^{-n}$$

if $p \nmid a$ and $p \nmid b$, and define $|0|_p = 0$.

Proposition 2 ([21]). $(\mathbb{Q}, |\cdot|_p)$ is not complete.

Notation 1. Let $\mathbb{Q}_p$ denote the completion of $(\mathbb{Q}, |\cdot|_p)$.

Proposition 3 ([21]). $\mathbb{Q}_p$ satisfies the strong triangle inequality,

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}.$$

Moreover, any normed space over $\mathbb{Q}_p$ is non-Archimedean, i.e., it also satisfies the strong triangle inequality,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}.$$

Throughout this paper, let p be a prime. Let S be a $2p$ -divisible abelian group. Let X be a normed space over the field $\mathbb{Q}_p$, with completion X^c . Let $\varphi: S^2 \rightarrow [0, \infty)$ be a function satisfying

1. $\varphi(y, x) = \varphi(x, y)$;
2. $\varphi(x, -y) = \varphi(x, y)$.

3. Main Theorems

We are now ready to turn to our main results. After recalling some preliminaries on non-Archimedean normed spaces, we establish new stability theorems for the generalized quadratic functional equation in p -adic contexts.

Theorem 1. Assume further that

1. $\lim_{n \rightarrow \infty} \frac{1}{p^{2n}} \varphi_{2, \max} \left(\frac{x}{p^n}, \frac{y}{p^n} \right) = 0$, where

$$\varphi_{2, \max}(x, y) = \max\{\varphi(x, y), \varphi(x + y, 0), \varphi(x - y, 0), \varphi(x, 0), \varphi(0, 0), \varphi\left(\frac{x+y}{2}, \frac{x+y}{2}\right), \varphi\left(\frac{x-y}{2}, \frac{x-y}{2}\right)\},$$

2. $\Phi_2(x) := \sum_{n=0}^{\infty} \frac{1}{p^{2n}} \varphi_{2, p} \left(\frac{x}{p^{n+1}} \right) < \infty$, where

$$\varphi_{2, k}(x) = \max_{0 \leq n \leq m \wedge n+m \leq k} \left\{ \varphi(nx, mx), \varphi\left(\frac{mx}{2}, \frac{mx}{2}\right) \right\}.$$

Let $f: S \rightarrow X$ be such that $f(0) = 0$ and

$$\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \varphi_{2, \max}(x, y). \quad (2)$$

Then, for any $x \in S$, the sequence $\left(p^{2n}f\left(\frac{x}{p^n}\right)\right)$ is Cauchy. Hence, we may define $Q: S \rightarrow X^c$ by

$$Q(x) = \lim_{n \rightarrow \infty} p^{2n}f\left(\frac{x}{p^n}\right).$$

It follows that Q is quadratic and satisfies

$$\|f(x) - Q(x)\| \leq \Phi_2(x).$$

Moreover, if Q' is a quadratic function such that there is $M > 0$ with

$$\|f(x) - Q'(x)\| \leq M\Phi_2(x)$$

for all $x \in S$, then $Q' = Q$.

Proof. Note that for any $x \in S$, for any $m, k \in \mathbb{N}$, $\varphi_{2,\max}(kx, x) \leq \varphi_{2,k+1}(x)$ and

$$\lim_{n \rightarrow \infty} \frac{1}{p^{2n}} \max_{0 \leq i < m} \left\{ \frac{1}{p^{2i}} \varphi_{2,p}\left(\frac{x}{p^{n+i+1}}\right) \right\} = 0.$$

First, we claim that $k \in \mathbb{N}$, $\|f(kx) - k^2f(x)\| \leq \varphi_{2,k}(x)$ by induction on k . Clearly, the claim holds for $k = 1$, and for $k = 0$, the left-hand side is 0. For $k \geq 1$, consider replacing x by kx and y by x in the inequality (2). Then, we have

$$\begin{aligned} \|f((k+1)x) - (k+1)^2f(x)\| &= \| [f((k+1)x) + f((k-1)x) - 2f(kx) - 2f(x)] \\ &\quad - [f((k-1)x) - (k-1)^2f(x)] + 2[f(kx) - k^2f(x)] \| \\ &\leq \max\{\varphi_{2,k+1}(x), \varphi_{2,k-1}(x), \varphi_{2,k}(x)\} \\ &= \varphi_{2,k+1}(x). \end{aligned}$$

Let $x \in S$ be fixed. Then, for any $m \in \mathbb{N}$,

$$\begin{aligned} \left\| p^{2n}f\left(\frac{x}{p^n}\right) - p^{2n+2m}f\left(\frac{x}{p^{n+m}}\right) \right\| &= \left\| p^{2n} \left[f\left(\frac{x}{p^n}\right) - p^{2m}f\left(\frac{x}{p^{n+m}}\right) \right] \right\| \\ &= \frac{1}{p^{2n}} \left\| \left[f\left(\frac{x}{p^n}\right) - p^2f\left(\frac{x}{p^{n+1}}\right) \right] + \dots \right. \\ &\quad \left. + 2^{m-1} \left[f\left(\frac{x}{p^{n+m-1}}\right) - p^2f\left(\frac{x}{p^{n+m}}\right) \right] \right\| \\ &\leq \frac{1}{p^{2n}} \max_{0 \leq i < m} \left\{ \frac{1}{p^{2i}} \varphi_{2,p}\left(\frac{x}{p^{n+i+1}}\right) \right\} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

This implies that $\left(p^{2n}f\left(\frac{x}{p^n}\right)\right)$ is Cauchy. We can see that $Q: x \mapsto \lim_{n \rightarrow \infty} p^{2n}f\left(\frac{x}{p^n}\right)$ is quadratic since

$$\begin{aligned} &\left\| p^{2n}f\left(\frac{x+y}{p^n}\right) + p^{2n}f\left(\frac{x-y}{p^n}\right) - 2p^{2n}f\left(\frac{x}{p^n}\right) - 2p^{2n}f\left(\frac{y}{p^n}\right) \right\| \\ &= \left\| p^{2n} \left[f\left(\frac{x+y}{p^n}\right) + f\left(\frac{x-y}{p^n}\right) - 2f\left(\frac{x}{p^n}\right) - 2f\left(\frac{y}{p^n}\right) \right] \right\| \\ &\leq \frac{1}{p^{2n}} \varphi_{2,\max}\left(\frac{x}{p^n}, \frac{y}{p^n}\right) \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Consider

$$\left\| f(x) - p^{2n} f\left(\frac{x}{p^n}\right) \right\| = \max_{0 \leq i < n} \left\{ \frac{1}{p^{2i}} \varphi_{2,p}\left(\frac{x}{p^i}\right) \right\} \leq \Phi_2(x).$$

By taking $n \rightarrow \infty$, we have $\|f(x) - Q(x)\| \leq \Phi_2(x)$. For the uniqueness part,

$$\begin{aligned} \|Q(x) - Q'(x)\| &= \left| p^{2n} \right|_p \left\| Q\left(\frac{x}{p^n}\right) - Q'\left(\frac{x}{p^n}\right) \right\| \\ &\leq \frac{1}{p^n} \max \left\{ \left\| f\left(\frac{x}{p^n}\right) - Q\left(\frac{x}{p^n}\right) \right\|, \left\| f\left(\frac{x}{p^n}\right) - Q'\left(\frac{x}{p^n}\right) \right\| \right\} \\ &= \frac{1}{p^n} \max \left\{ \Phi_2\left(\frac{x}{p^n}\right), M\Phi_2\left(\frac{x}{p^n}\right) \right\} \\ &\leq \frac{1}{p^n} (M+1) \Phi_2\left(\frac{x}{p^n}\right) \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

since

$$\begin{aligned} \frac{1}{p^n} \Phi_2\left(\frac{x}{p^n}\right) &= \frac{1}{p^n} \sum_{i=0}^{\infty} \frac{1}{p^{2i}} \varphi_{2,p}\left(\frac{x}{p^{n+i+1}}\right) \\ &= \sum_{i=n}^{\infty} \frac{1}{p^{2i}} \varphi_{2,p}\left(\frac{x}{p^{i+1}}\right), \end{aligned}$$

which is the tail of $\Phi_2(x) = \sum_{n=0}^{\infty} \frac{1}{p^{2n}} \varphi_{2,p}\left(\frac{x}{p^{n+1}}\right) < \infty$. $\square$

If we take $\varphi(x, y)$ to be a constant ϵ , then $\Phi_2(x) = \frac{p^2 \epsilon}{p^2 - 1}$.

Corollary 1. Let $f: S \rightarrow X$ and $\epsilon \geq 0$ be such that $f(0) = 0$ and

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \epsilon.$$

Then, there is a quadratic function $Q: S \rightarrow X^c$ such that

$$\|f(x) - Q(x)\| \leq \frac{p^2 \epsilon}{p^2 - 1}.$$

Moreover, such a quadratic function is unique.

Theorem 2. Assume further that

1. $\lim_{n \rightarrow \infty} \frac{1}{p^n} \varphi_{1,\max}\left(\frac{x}{p^n}, \frac{y}{p^n}\right) = 0$, where

$$\varphi_{1,\max}(x, y) = \max\{\varphi(x, y), \varphi(x, 0), \varphi(y, 0), \varphi(0, 0)\},$$

2. $\Phi_1(x) := \sum_{n=0}^{\infty} \frac{1}{p^n} \varphi_{1,p}\left(\frac{x}{p^{n+1}}\right) < \infty$, where

$$\varphi_{1,k}(x) = \max_{0 \leq n \leq m \wedge n+m \leq k} \varphi(nx, mx).$$

Let $f: S \rightarrow X$ be such that $f(0) = 0$ and

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi_{1,\max}(x, y). \quad (3)$$

Then, for any $x \in S$, the sequence $\left(p^n f\left(\frac{x}{p^n}\right)\right)$ is Cauchy. Hence, we may define $A: S \rightarrow X^c$ by

$$A(x) = \lim_{n \rightarrow \infty} p^n f\left(\frac{x}{p^n}\right).$$

It follows that A is additive and satisfies

$$\|f(x) - A(x)\| \leq \Phi_1(x),$$

Moreover, if A' is an additive function such that there is $M > 0$ with

$$\|f(x) - A'(x)\| \leq M\Phi_1(x)$$

for all $x \in S$, then $A' = A$.

Proof. First, we claim that $k \in \mathbb{N}$, $\|f(kx) - kf(x)\| \leq \varphi_{1,k}(x)$ by induction on k . Clearly, the claim holds for $k = 1$. For $k \geq 1$, consider replacing x by kx and y by x in the inequality (3). Then, we have

$$\begin{aligned} \|f((k+1)x) - (k+1)f(x)\| &= \| [f((k+1)x) - f(kx) - f(x)] + [f(kx) - kf(x)] \| \\ &\leq \max\{\varphi_{1,k+1}(x), \varphi_{1,k}(x)\} \\ &= \varphi_{1,k+1}(x). \end{aligned}$$

Let $x \in S$ be fixed. Then, for any $m \in \mathbb{N}$,

$$\begin{aligned} \left\| p^n f\left(\frac{x}{p^n}\right) - p^{n+m} f\left(\frac{x}{p^{n+m}}\right) \right\| &= \left\| p^n \left[f\left(\frac{x}{p^n}\right) - p^m f\left(\frac{x}{p^{n+m}}\right) \right] \right\| \\ &\leq \frac{1}{p^n} \max_{0 \leq i < m} \left\{ \frac{1}{p^i} \varphi_{1,p}\left(\frac{x}{p^{n+i+1}}\right) \right\} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

This implies that the sequence $\left(p^n f\left(\frac{x}{p^n}\right)\right)$ is Cauchy. We can see that $A: x \mapsto \lim_{n \rightarrow \infty} p^n f\left(\frac{x}{p^n}\right)$ is additive since

$$\begin{aligned} &\left\| p^n f\left(\frac{x+y}{p^n}\right) - p^n f\left(\frac{x}{p^n}\right) - p^n f\left(\frac{y}{p^n}\right) \right\| \\ &\leq \frac{1}{p^n} \varphi_{1,\max}\left(\frac{x}{p^n}, \frac{y}{p^n}\right) \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Consider

$$\begin{aligned} \left\| f(x) - p^n f\left(\frac{x}{p^n}\right) \right\| &= \max_{0 \leq i < n} \left\{ \frac{1}{p^i} \varphi_{1,p}\left(\frac{x}{p^i}\right) \right\} \\ &\leq \Phi_1(x). \end{aligned}$$

By taking $n \rightarrow \infty$, we have $\|f(x) - A(x)\| \leq \Phi_1(x)$. For the uniqueness part,

$$\begin{aligned}\|A(x) - A'(x)\| &= \frac{1}{p^n} \left\| A\left(\frac{x}{p^n}\right) - A'\left(\frac{x}{p^n}\right) \right\| \\ &= \frac{1}{p^n} \max \left\{ \Phi_1\left(\frac{x}{p^n}\right), M\Phi_1\left(\frac{x}{p^n}\right) \right\} \\ &\leq \frac{1}{p^n} (M+1)\Phi_1\left(\frac{x}{p^n}\right) \\ &\rightarrow 0 \text{ as } n \rightarrow \infty\end{aligned}$$

since

$$\begin{aligned}\frac{1}{p^n} \Phi_1\left(\frac{x}{p^n}\right) &= \frac{1}{p^n} \sum_{i=0}^{\infty} \frac{1}{p^i} \varphi_{1,p}\left(\frac{x}{p^{n+i+1}}\right) \\ &= \sum_{i=n}^{\infty} \frac{1}{p^i} \varphi_{1,p}\left(\frac{x}{p^{i+1}}\right),\end{aligned}$$

which is the tail of $\Phi_1(x) = \sum_{n=0}^{\infty} \frac{1}{p^n} \varphi_{1,p}\left(\frac{x}{p^{n+1}}\right) < \infty$. $\square$

Similar to the quadratic case, if $\varphi(x, y)$ is a constant ϵ , then $\Phi_1(x) = \frac{p\epsilon}{p-1}$.

Corollary 1. Let $f: S \rightarrow X$ and $\epsilon \geq 0$ be such that $f(0) = 0$ and

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon.$$

Then, there is an additive function $A: S \rightarrow X^c$ such that

$$\|f(x) - A(x)\| \leq \frac{p\epsilon}{p-1}.$$

Moreover, such an additive function is unique.

Now, we shall use Theorems 1 and 2 to prove a new result regarding the stability of a generalized quadratic functional equation with four functions in non-Archimedean normed space over p -adic numbers by considering the even and odd parts of each function.

Theorem 3. Suppose that φ satisfies the conditions on Theorems 1 and 2. Let $\mathcal{F}, \mathcal{G}, \mathcal{L}$, and $\mathcal{M}$ be functions from S into X such that

$$\|\mathcal{F}(x+y) + \mathcal{G}(x-y) - \mathcal{L}(x) - \mathcal{M}(y)\| \leq \varphi(x, y).$$

Then, there are a quadratic function $Q: S \rightarrow X^c$ and two additive functions $A_F, A_G: S \rightarrow X^c$ such that

$$\|\mathcal{F}(x) - \mathcal{F}(0) - Q(x) - A_F(x)\| \leq \Phi_{\max}(x),$$

$$\|\mathcal{G}(x) - \mathcal{G}(0) - Q(x) - A_G(x)\| \leq \Phi_{\max}(x),$$

$$\|\mathcal{L}(x) - \mathcal{L}(0) - 2Q(x) - A_F(x) - A_G(x)\| \leq \Phi_{\max}(x),$$

and

$$\|\mathcal{M}(x) - \mathcal{M}(0) - 2Q(x) - A_F(x) + A_G(x)\| \leq \Phi_{\max}(x),$$

where

$$\Phi_{\max}(x) = \max\{\varphi_{1,p}(x), \varphi_{2,p}(x), \Phi_1(x), \Phi_2(x)\}.$$

Moreover, such quadratic and additive functions are unique.

Proof. For simplicity, assume that $p > 2$. By defining $F(x) := \mathcal{F}(x) - \mathcal{F}(0)$, we see that we now also have $F(0) = 0$. Similarly, define G, L, M for $\mathcal{G}, \mathcal{L}, \mathcal{M}$, respectively. Therefore, by the strong triangle inequality,

$$\|F(x+y) + G(x-y) - L(x) - M(y)\| \leq \max\{\varphi(x, y), \varphi(0, 0)\}. \quad (4)$$

To approximate each function on the left-hand side, we shall analyze the even and odd parts of the function separately. For each function $f: S \rightarrow X$, we define the even part of f as

$$f_e = \frac{f(x) + f(-x)}{2},$$

and the odd part of f as

$$f_o = \frac{f(x) - f(-x)}{2}.$$

Then, $f = f_e + f_o$. Note that $f_e(-x) = f_e(x)$ and $f_o(-x) = -f_o(x)$.

First, we shall derive an inequality involving only M_e on the left-hand side. Then, we can approximate M_e using a quadratic function. After that, we use the same function to approximate the even part of other functions.

Replacing x with $-x$ and y with $-y$ in (4), we have

$$\|F(-(x+y)) + G(-(x-y)) - L(-x) - M(-y)\| \leq \max\{\varphi(x, y), \varphi(0, 0)\}. \quad (5)$$

By (4) and (5), using strong triangle inequality,

$$\|2\|_p \|F_e(x+y) + G_e(x-y) - L_e(x) - M_e(y)\| \leq \max\{\varphi(x, y), \varphi(0, 0)\}.$$

Since $p > 2$, $\|2\|_p = 1$; thus,

$$\|F_e(x+y) + G_e(x-y) - L_e(x) - M_e(y)\| \leq \max\{\varphi(x, y), \varphi(0, 0)\}. \quad (6)$$

Substituting $y = 0$ in (6), we get

$$\|F_e(x) + G_e(x) - L_e(x)\| \leq \max\{\varphi(x, 0), \varphi(0, 0)\}. \quad (7)$$

Substituting $x = 0$ and $y = x$ in (6),

$$\|F_e(x) + G_e(x) - M_e(x)\| \leq \max\{\varphi(x, 0), \varphi(0, 0)\}. \quad (8)$$

By (7) and (8), we obtain

$$\|L_e(x) - M_e(x)\| \leq \max\{\varphi(x, 0), \varphi(0, 0)\}. \quad (9)$$

Substituting $y = x$ in (6),

$$\|F_e(2x) - L_e(x) - M_e(x)\| \leq \max\{\varphi(x, x), \varphi(0, 0)\}. \quad (10)$$

Substituting $y = -x$ in (6),

$$\|G_e(2x) - L_e(x) - M_e(x)\| \leq \max\{\varphi(x, x), \varphi(0, 0)\}. \quad (11)$$

By (10) and (11), and replacing x by $\frac{x}{2}$,

$$\|F_e(x) - G_e(x)\| \leq \max\left\{\varphi\left(\frac{x}{2}, \frac{x}{2}\right), \varphi(0, 0)\right\}. \quad (12)$$

By (6), (9), and (12), replacing x with $x + y$,

$$\begin{aligned} \|G_e(x + y) + G_e(x - y) - M_e(x) - M_e(y)\| \\ \leq \max\left\{\varphi(x, y), \varphi(x, 0), \varphi\left(\frac{x + y}{2}, \frac{x + y}{2}\right), \varphi(0, 0)\right\}. \end{aligned} \quad (13)$$

Substituting $y = 0$ in (13), we get

$$\|2G_e(x) - M_e(x)\| \leq \max\left\{\varphi(x, 0), \varphi\left(\frac{x}{2}, \frac{x}{2}\right), \varphi(0, 0)\right\}. \quad (14)$$

By (13) and (14), replacing x with $x + y$, and (14), replacing x with $x - y$,

$$\begin{aligned} \|M_e(x + y) + M_e(x - y) - 2M_e(x) - 2M_e(y)\| \\ \leq \max\{\varphi(x, y), \varphi(x + y, 0), \varphi(x - y, 0), \varphi(x, 0), \\ \varphi(0, 0), \varphi\left(\frac{x + y}{2}, \frac{x + y}{2}\right), \varphi\left(\frac{x - y}{2}, \frac{x - y}{2}\right)\} \\ = \varphi_{2, \max}(x, y). \end{aligned}$$

By Theorem 1, there is a unique quadratic function Q such that

$$\|M_e(x) - 2Q(x)\| \leq \Phi_2(x). \quad (15)$$

By (9) and (15), we have

$$\|L_e(x) - 2Q(x)\| \leq \max\{\Phi_2(x), \varphi(x, 0), \varphi(0, 0)\} \leq \Phi_{\max}(x).$$

By (14) and (15), we get

$$\begin{aligned} \|G_e(x) - Q(x)\| &= \|2G_e(x) - 2Q(x)\| \\ &\leq \max\left\{\Phi_2(x), \varphi(x, 0), \varphi\left(\frac{x}{2}, \frac{x}{2}\right), \varphi(0, 0)\right\} \\ &\leq \Phi_{\max}(x). \end{aligned} \quad (16)$$

By (12) and (16), we obtain

$$\|F_e(x) - Q(x)\| \leq \Phi_{\max}(x).$$

Now, we shall work on the odd part.

By (4) and (5), we have

$$\|F_o(x + y) + G_o(x - y) - L_o(x) - M_o(y)\| \leq \max\{\varphi(x, y), \varphi(0, 0)\}. \quad (17)$$

Substituting $y = 0$ in (17), we obtain

$$\|F_o(x) + G_o(x) - L_o(x)\| \leq \max\{\varphi(x, 0), \varphi(0, 0)\}. \quad (18)$$

Substituting $x = 0$ and $y = x$ in (17),

$$\|F_o(x) - G_o(x) - M_o(x)\| \leq \max\{\varphi(x, 0), \varphi(0, 0)\}. \quad (19)$$

By (17)–(19), we have

$$\begin{aligned} \|F_o(x+y) - F_o(x) - F_o(y) + G_o(x-y) - G_o(x) + G_o(y)\| \\ \leq \max\{\varphi(x, y), \varphi(x, 0), \varphi(y, 0), \varphi(0, 0)\}. \end{aligned} \quad (20)$$

Replacing x by y and y by x in (20),

$$\begin{aligned} \|F_o(x+y) - F_o(x) - F_o(y) - G_o(x-y) + G_o(x) - G_o(y)\| \\ \leq \max\{\varphi(x, y), \varphi(x, 0), \varphi(y, 0), \varphi(0, 0)\}. \end{aligned} \quad (21)$$

By (20) and (21), we obtain

$$\begin{aligned} \|F_o(x+y) - F_o(x) - F_o(y)\| &= \|2(F_o(x+y) - F_o(x) - F_o(y))\| \\ &\leq \max\{\varphi(x, y), \varphi(x, 0), \varphi(y, 0), \varphi(0, 0)\} \\ &= \varphi_{1,\max}(x, y). \end{aligned}$$

Similarly,

$$\|G_o(x+y) - G_o(x) - G_o(y)\| \leq \varphi_{1,\max}(x, y).$$

By Theorem 2, there are unique additive functions A_F, A_G such that

$$\|F_o(x) - A_F(x)\| \leq \Phi_1(x), \quad (22)$$

and

$$\|G_o(x) - A_G(x)\| \leq \Phi_1(x). \quad (23)$$

By (18), (22), and (23), we get

$$\|L_o(x) - A_F(x) - A_G(x)\| \leq \max\{\Phi_1(x), \varphi(x, 0), \varphi(0, 0)\} \leq \Phi_{\max}(x).$$

By (19), (22), and (23), we obtain

$$\|M_o(x) - A_F(x) + A_G(x)\| \leq \Phi_{\max}(x).$$

The proof is completed by adding each respective even and odd part. Note that the proof also works for $p = 2$ since a scalar multiple of a quadratic/additive function is also quadratic/additive. $\square$

If we take $\varphi(x, y)$ to be a constant ϵ , then $\Phi_{\max}(x) = \Phi_1(x) = \frac{p\epsilon}{p-1}$. Thus, we conclude the following corollary.

Corollary 2. Let $\mathcal{F}, \mathcal{G}, \mathcal{L}$, and $\mathcal{M}$ be functions from S into X , and let $\epsilon \geq 0$ be such that

$$\|\mathcal{F}(x+y) + \mathcal{G}(x-y) - \mathcal{L}(x) - \mathcal{M}(y)\| \leq \epsilon.$$

Then, there are a quadratic function $Q: S \rightarrow X^c$ and two additive functions $A_F, A_G: S \rightarrow X^c$ such that

$$\|\mathcal{F}(x) - \mathcal{F}(0) - Q(x) - A_F(x)\| \leq \frac{p\epsilon}{p-1},$$

$$\|\mathcal{G}(x) - \mathcal{G}(0) - Q(x) - A_G(x)\| \leq \frac{p\epsilon}{p-1},$$

$$\|\mathcal{L}(x) - \mathcal{L}(0) - 2Q(x) - A_F(x) - A_G(x)\| \leq \frac{p\epsilon}{p-1},$$

and

$$\| \mathcal{M}(x) - \mathcal{M}(0) - 2Q(x) - A_F(x) + A_G(x) \| \leq \frac{p\epsilon}{p-1}.$$

Moreover, these quadratic and additive functions are unique.

We shall give an example of an application of Theorem 3.

Example 1. Let $X = S$ be any field extension of $\mathbb{Q}_p$, and fix $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{Q}_p$. Let

$$\mathcal{F}(x) = \alpha_1 x^2 + \beta_1 x + \gamma_1$$

and

$$\mathcal{G}(x) = \alpha_2 x^2 + \beta_2 x + \gamma_2.$$

Let $\mathcal{L}$ and $\mathcal{M}$ be functions from X into X , and let $\epsilon \geq 0$ be such that

$$\| \mathcal{F}(x+y) + \mathcal{G}(x-y) - \mathcal{L}(x) - \mathcal{M}(y) \| \leq \epsilon.$$

Then, $\alpha_1 = \alpha_2$,

$$\| \mathcal{L}(x) - \mathcal{L}(0) - 2\alpha_1 x^2 - (\beta_1 + \beta_2)x \| \leq \frac{p\epsilon}{p-1},$$

and

$$\| \mathcal{M}(x) - \mathcal{M}(0) - 2\alpha_1 x^2 - (\beta_1 - \beta_2)x \| \leq \frac{p\epsilon}{p-1}.$$

Proof. Using the same F, G, L , and M notations as in the proof of Theorem 3, we have $F_e(x) = \alpha_1 x^2$, $G_e(x) = \alpha_2 x^2$, $F_o(x) = \beta_1 x$, and $G_o(x) = \beta_2 x$. Then, by the proof of Theorem 3, there are quadratic $Q: X \rightarrow X^c$ and additive $A_F, A_G: X \rightarrow X^c$ such that

$$\max\{\| \alpha_1 x^2 - Q(x) \|, \| \alpha_2 x^2 - Q(x) \|, \| L_e(x) - 2Q(x) \|, \| M_e(x) - 2Q(x) \| \} \leq \frac{p\epsilon}{p-1}$$

and

$$\max\{\| \beta_1 x - A_F(x) \|, \| \beta_2 x - A_G(x) \|, \| L_o(x) - A_F(x) - A_G(x) \|, \| M_o(x) - A_F(x) + A_G(x) \| \} \leq \frac{p\epsilon}{p-1}.$$

By the strong triangle inequality, $\| \alpha_1 - \alpha_2 \| \cdot \| x \|^2 \leq \frac{p\epsilon}{p-1}$. Considering (p^{-n}) is a sequence in $\mathbb{Q}_p \subseteq X$ with $\lim_{n \rightarrow \infty} \| p^{-n} \| = \infty$. Thus, $\alpha_1 = \alpha_2$. Again, by the strong triangle inequality,

$$\| L_e(x) - 2\alpha_1 x^2 \| \leq \max\{\| L_e(x) - 2Q(x) \|, \| Q(x) - \alpha_1 x^2 \| \} \leq \frac{p\epsilon}{p-1}.$$

The same argument holds for the remaining cases. The proof is completed by adding each respective even and odd part. $\square$

For more concrete example, we can take X to be the algebraic closure of $\mathbb{Q}_p$ and pick any $\alpha_i, \beta_i, \gamma_i$ in $\mathbb{Q}_p$. For example, if we would like to pick an element outside $\mathbb{Q}$, let a_n 's be the base- p decimal digits of π ; then, $\sum_{n=1}^{\infty} a_n p^n$ is in $\mathbb{Q}_p \setminus \mathbb{Q}$.

4. Discussion and Conclusions

We have established the Hyers–Ulam–Rassias stability of an asymmetric generalized quadratic functional equation with four unknown functions in non-Archimedean normed spaces over p -adic numbers. Using the ultrametric property and separating functions into even and odd parts, we extend previous results and show that all four functions can be approximated by a combination of a quadratic function and two additive functions. Moreover, by the ultrametric property, compared to the previous results, we achieve a more precise approximation. This provides new insights into the structural behavior of solutions and advances the study of the stability of functional equations in the p -adic setting. The separating technique might be useful for studies in other frameworks such as fuzzy normed space. For further study in a non-Archimedean framework, the underlining field can be changed to another non-trivial non-Archimedean field, which is not a field extension of $\mathbb{Q}_p$.

Author Contributions: Conceptualization, J.T. and N.A.; methodology, J.T. and N.A.; formal analysis, J.T. and N.A.; investigation, J.T. and N.A.; writing—original draft preparation, N.A.; writing—review and editing, J.T. and N.A.; visualization, J.T.; supervision, J.T.; project administration, N.A.; funding acquisition, N.A. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by Walailak University under the New Researcher Development scheme WU68207 and WU68221.

Data Availability Statement: No new data were created or analyzed in this study.

Acknowledgments: The authors would like to express their sincere gratitude to the anonymous reviewers for their careful reading of the manuscript and for providing valuable comments and suggestions that greatly improved the clarity and quality of this work.

Conflicts of Interest: The authors declare no conflicts of interest.

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