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The One-Fault Dimension-Balanced Hamiltonian Problem in Toroidal Mesh Graphs

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Abstract: Finding a Hamiltonian cycle in a graph $G = (V, E)$ is a well-known problem. The challenge of finding a Hamiltonian cycle that avoids these faults when faulty vertices or edges are present has been extensively studied. When the edge set of G is partitioned into k dimensions, the problem of dimension-balanced Hamiltonian cycles arises, where the Hamiltonian cycle uses approximately the same number of edges from each dimension (differing by at most one). This paper studies whether a dimension-balanced Hamiltonian cycle (DBH) exists in toroidal mesh graphs $T_{m,n}$ when a single vertex or edge is faulty, called the one-fault DBH problem. We establish that $T_{m,n}$ is one-fault DBH, except in the following cases: (1) both m and n are even; (2) one of m and n is 3, while the other satisfies mod 4 = 3 and is greater than 6; (3) one of m and n is odd, while the other satisfies mod 4 = 2. Additionally, this paper resolves a conjecture from prior literature, thereby providing a complete solution to the DBP problem on $T_{m,n}$.

Keywords: toroidal mesh graph; Hamiltonian; dimension-balanced; faulty vertex; faulty edge

1. Introduction

With the widespread use of networks today, we can represent the topological structure of the network with graphs, where vertices represent devices, and edges between vertices represent communication between those devices. The connection paths between vertices in the graph can show the network's operational efficiency, where a Hamiltonian cycle can connect all the vertices and return to the original vertex, ensuring the network remains smooth. The Hamilton problem is a famous problem in interconnection networks [1–5]. However, when some vertices or edges fail, the Hamiltonian cycle is disrupted, causing data transmission issues. This inconvenience requires new routes for data transmission to be ensured in the network. Therefore, scholars have further discussed Hamiltonian cycles, particularly whether Hamiltonian cycles still exist in network graphs when there are faulty vertices or edges [6–15].

Given a graph $G = (V, E)$ whose edge set $E(G)$ is partitioned into k dimensions ($E = E_1 \cup E_2 \cup \ldots \cup E_k$), a cycle C in G for i with $1 \leq i \leq k$ has edges in i-dimension $E_i(C)$, defined as $E(C) \cap E_i$. A cycle C is called a *dimension-balanced cycle* (DBC for short) if $||E_i(C)| - |E_j(C)|| \leq 1$ for all i and j with $1 \leq i < j \leq k$. If C is also a Hamiltonian cycle (passing through all vertices in the graph), it is called a *dimension-balanced Hamiltonian cycle* (DBH for short). If G contains a DBC of every length between 3 and $|V|$, G is called *dimension-balanced pancyclic* (DBP for short). If G contains a DBC of every length between a and $|V(G)|$, G is called *dimension-balanced a-pancyclic* (a-DBP). If for any vertex x in G there exists a DBC containing

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x whose length can be any integer between a and $|V(G)|$, G is called *dimension-balanced a-vertex-pancyclic* (a-DBVP). Moreover, a cycle C is called a *weakly dimension-balanced cycle* (WDBC for short) if $||E_i(C)| - |E_j(C)|| \leq 3$ for all i and j with $1 \leq i < j \leq k$. If C is also a Hamiltonian cycle, it is called a *weakly dimension-balanced Hamiltonian cycle* (WDBH for short). Similarly, if G contains a WDBC of every length between 3 and $|V|$, G is called *weakly dimension-balanced pancyclic* (WDBP for short).

A cycle C_n with n vertices forms a *toroidal mesh graph* $T_{m,n}$ when taking the Cartesian product of C_m and C_n. It is defined as follows: vertex set $V(T_{m,n}) = \{(x, y) \mid 0 \leq x \leq m-1; 0 \leq y \leq n-1\}$, and edge set $E(T_{m,n}) = \{(x_1, y_1)(x_2, y_2) \mid x_1 = x_2$ and $|y_1 - y_2| = (1$ or $n-1)$ or $|x_1 - x_2| = (1$ or $m-1)$ and $y_1 = y_2\}$. Figure 1 shows an example of $T_{4,3}$. The toroidal mesh graph is an important interconnection network graph. We partition its edges into two sets: horizontal edges and vertical edges. Thus, $E(T_{m,n}) = E_1 \cup E_2$, where $E_1 = \{(i, j)(i+1, j) \mid 0 \leq i \leq m-2; 0 \leq j \leq n-1\} \cup \{(m-1, j)(0, j) \mid 0 \leq j \leq n-1\}$ and $E_2 = \{(i, j)(i, j+1) \mid 0 \leq i \leq m-1; 0 \leq j \leq n-2\} \cup \{(i, n-1)(i, 0) \mid 0 \leq i \leq m-1\}$. This paper's research is based on this partition.

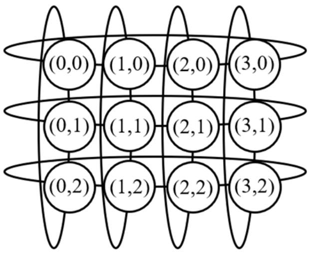

Figure 1. $T_{4,3}$.

The dimension-balanced Hamiltonian problem was first proposed in 2012 [16], and originated from the 3D reconstruction problem using Gray codes [17]. It has been widely explored since then [18–21]: the authors of [16] first studied the DBH problem on $T_{m,n}$, and the DBP problem on $T_{m,n}$ was solved [18]. Subsequently, the extension problem known as the WDBH problem was raised, and the WDBH problem on $T_{m,n}$ was solved in [19], and then, the WDBP problem was also studied [20]. In addition, [21] conducted research on the DBH problem on a 3-Dimensional Toroidal Mesh Graph $T_{m,n,r}$.

The design of the network-on-chip (NoC) has become a significant research focus in recent years. Congestion in the NoC can lead to a decline in network performance. Therefore, the efficient selection of paths and the development of effective strategies to address congestion are critical for optimizing NoC performance. This topic has attracted considerable attention in recent studies [22,23]. Through abstraction, some NoCs can be represented as a toroidal mesh graph. A dimension-balanced Hamiltonian cycle on a toroidal mesh graph provides a foundation for designing simple algorithms with low communication costs, effectively mitigating congestion. In addition to being used to assist in the design of NoC, toroidal mesh graphs are also very helpful for research in fields such as cryptography and graphics. The dimension-balanced Hamiltonian cycle problem in toroidal mesh graphs has been discussed in [16]. The question of whether a dimension-balanced Hamiltonian cycle exists for toroidal mesh graphs $T_{m,n}$ has been thoroughly discussed. This paper focuses on the problem of dimension-balanced Hamiltonian cycles in toroidal mesh graphs with a faulty point f or faulty edge e.

A graph G is called *k-node Hamiltonian* if it remains Hamiltonian after removing any k-nodes; a graph G is called *k-edge* (or say *k-link*) *Hamiltonian* if it remains Hamiltonian after removing any k edges (links); G is called *k-fault Hamiltonian* if it remains Hamiltonian after removing any k-nodes and/or edges [5,6]. In recent years, many researchers have studied the Hamiltonian problem on several topologies with faulty nodes or edges, like

Cartesian product graphs [8,11], locally twisted cubes [10], augmented cubes [7,12], hypercube graphs [13], the basic WK-recursive pyramid [14], and so on. This gives us strong motivation to study DBC problems with faulty nodes or edges.

Therefore, we have the following definitions which define a dimension-balanced Hamiltonian on a graph G with faulty nodes and/or edges. A graph G is called *k-node dimension-balanced Hamiltonian* (*k-node* DBH for short) if it has a DBH after removing any k-nodes; a graph G is called *k-edge dimension-balanced Hamiltonian* (*k-edge* DBH for short) if it remains Hamiltonian after removing any k edges; a graph G is called *k-fault dimension-balanced Hamiltonian* (*k-fault* DBH for short) if it remains Hamiltonian after removing any k-nodes and/or edges.

This paper mainly discusses under what conditions a toroidal mesh graph is one-fault dimension-balanced Hamiltonian. The main contributions of this paper are as follows:

- We completely solve the one-fault dimension-balanced Hamiltonian problem on the toroidal mesh graph $T_{m,n}$.
- We prove the three conjectures in [18]: when m and $n \geq 5$ are both odd numbers, for $k = \lfloor 1(mn-1) / 4 \rfloor$, there is a DBC of length 4k in $T_{m,n}$; therefore, $T_{m,n}$ is (2 max$\{m, n\}$ $-$ 1)-DBP and (2 max$\{m, n\}$ $-$ 1)-DBVP.

The rest of this paper is organized as follows. In Section 2, the problem of finding dimension-balanced Hamiltonian cycles in $T_{m,n} - f$ is discussed. According to the parity of m and n, there are three subsections: both m and n are even; both m and n are odd; one of m and n is even and the other is odd. Section 3 illustrate the problem in $T_{m,n} - e$ and presents our conclusions.

2. Main Research Results

Due to the complexity of the faulty vertex problem, this section mainly discusses it. Note that since the toroidal mesh graph is vertex-symmetric [24], the faulty vertex can be assumed to be any position in the graph. Finding a dimension-balanced Hamiltonian cycle in the graph with the faulty vertex in any position suffices. This section discusses the problem based on the parity of m and n.

2.1. Both m and n Are Even

Theorem 1. *When both m and n are even, $T_{m,n} - f$ does not have a dimension-balanced Hamiltonian cycle, where f is any vertex in the graph.*

Proof of Theorem 1. Since both m and n are even, the vertices can be divided into black and white vertices in an alternating manner (see Figure 2 as an example), ensuring black vertices are only connected to white vertices and vice versa. Thus, $T_{m,n}$ can be considered a bipartite graph. The cycle in a bipartite graph must have an even number of vertices. When both m and n are even, $mn - 1$ is odd. Thus, $T_{m,n} - f$ does not have a Hamiltonian cycle, and therefore, it does not have a dimension-balanced Hamiltonian cycle. □

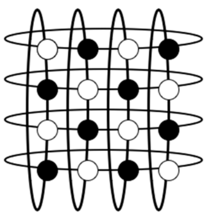

Figure 2. Viewing $T_{m,n}$ as a bipartite graph.

2.2. Both m and n Are Odd

This section discusses whether $T_{m,n} - f$ has a dimension-balanced Hamiltonian cycle when both m and n are odd. The proof will be given through two theorems, with all cases classified based on the remainders of m and n divided by 8. A detailed classification is shown in Table 1, where p and q are non-negative integers. Since $T_{m,n}$ is vertex-symmetric, without loss of generality, we say $n = 3$ when m or n equals 3. First, we introduce a useful lemma.

Table 1. Classification of cases when both m and n are odd.

m	m, n Are Odd				
n	**3**	**5 + 8p**	**7 + 8p**	**9 + 8p**	**11+ 8p**
3	Thm. 2 (a)	Thm. 2 (b)	Cor. 1	Thm. 2 (c)	Cor. 1
5 + 8q	Thm. 2 (b)	Thm. 3 (a)	Thm. 3 (b)	Thm. 3 (c)	Thm. 3 (d)
7 + 8q	Cor. 1	Thm. 3 (b)	Thm. 3 (e)	Thm. 3 (f)	Thm. 3 (g)
9 + 8q	Thm. 2 (c)	Thm. 3 (c)	Thm. 3 (f)	Thm. 3 (h)	Thm. 3 (i)
11 + 8q	Cor. 1	Thm. 3 (d)	Thm. 3 (g)	Thm. 3 (i)	Thm. 3 (j)

Lemma 1 [18]**.** *If m is odd, $T_{m,3}$ does not embed every 4k-DBC for k with $3 < k < \lfloor 3m/4 \rfloor$.*

Corollary 1. *When $m \equiv 3 \pmod 4$ and $m \geq 7$, $T_{m,3} - f$ does not have a dimension-balanced Hamiltonian cycle, where f is any vertex in the graph.*

Proof of Corollary 1. Let $m = 7 + 4p$, where p is a non-negative integer. The number of vertices of $T_{m,3} - f$ is $20 + 12p = 4(5 + 3p)$. According to Lemma 1, we know that $T_{m,3}$ does not have a DBC with a length of $4(5 + 3p)$. So, $T_{m,3} - f$ does not have a DBH, where f is any vertex in the graph. □

In the following figures, the red dot represent faulty vertex f in each figure; the blue dashed lines represent edges that will be removed when forming a larger image, and the red lines represent newly added edges, for convenience.

Theorem 2. *When $m = 3$ or $m \equiv 1 \pmod 4$, $T_{m,3} - f$ has a dimension-balanced Hamiltonian cycle, where f is any vertex in the graph.*

Proof of Theorem 2. The following will be divided into three cases to prove this theorem.

Case (a). *When $m = 3$: It can be seen from Figure 3 that $T_{m,3} - f$ has a dimension-balanced Hamiltonian cycle. In the figure, the red node denotes the faulty vertex. In the subsequent figures, the red nodes have the same meaning.*

Case (b). *When $m = 5 + 8p$, $n = 3$: Figure 4 shows how to form a DBH in $T_{5+8p,3} - f$ from one DBH in $T_{5,3} - f$ and p DBHs in $T_{8,3}$. In this case, if we delete the edge set $\{(0, 2)(4, 2)\} \cup \{(5 + 8x, 2)(12 + 8x, 2) \mid 0 \leq x < p\}$ and add $\{(0, 2)(4 + 8p, 2)\} \cup \{(4 + 8x, 2)(5 + 8x, 2) \mid 0 \leq x < p\}$, we obtain the Hamiltonian cycle C on$T_{5+8p,3} - f$, where $|E_1(C)| = 7 + 12p = |E_2(C)|$. Therefore, $||E_1(C)| - |E_2(C)|| = 0$, and cycle C satisfies the condition of dimension balance, so C is a DBH on $T_{5+8p,3} - f$. In Figure 4, the blue dotted lines represent the edges that need to be deleted when merging the graphs, and the red lines represent the edges that need to be added when merging the graphs. In the subsequent figures, the blue dotted lines and the red lines have the same meaning.*

Case (c). *When $m = 9 + 8p$, $n = 3$: Figure 5 shows how to form a DBH in $T_{9+8p,3} - f$ from one DBH in $T_{9,3} - f$ and p DBHs in $T_{8,3}$. In this case, if we delete the edge set $\{(0, 2)(8, 2)\} \cup \{(9 + 8x,$*

2)(16 + 8x, 2) | *0* $\leq$ *x* < *p} and add {(0, 2)(8 + 8p, 2)}* $\cup$ *{(8 + 8x, 2), (9 + 8x, 2)* | *0* $\leq$ *x* < *p}, we obtain the Hamiltonian cycle C on* $T_{9+8p,3} - f$*, where* $|E_1(C)| = 13 + 12p = |E_2(C)|$*. Therefore,* $||E_1(C)| - |E_2(C)|| = 0$*, and cycle C satisfies the condition of dimension balance, so C is a DBH on* $T_{9+8p,3} - f$*.* □

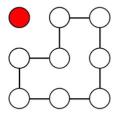

Figure 3. There is a DBH in $T_{3,3} - f$.

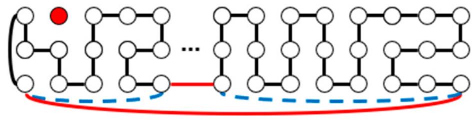

Figure 4. There is a DBH in $T_{5+8p,3} - f$.

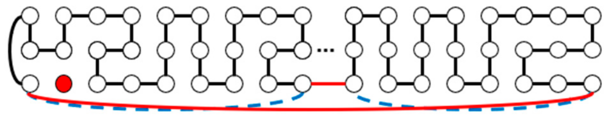

Figure 5. There is a DBH in $T_{9+8p,3} - f$.

Theorem 3. *When m and n are both odd integers greater than 4,* $T_{m,n} - f$ *has a dimension-balanced Hamiltonian cycle, where f is any vertex in the graph.*

Proof of Theorem 3. The following will be divided into ten cases for discussion based on the remainder of *m* and *n* divided by eight.

Case (a). *When m = 5 + 8p, n = 5 + 8q: Firstly, as Figure 6 shows, if we put p HCs (Hamiltonian cycles) of* $T_{8,5}$ *on the right hand side of the DBH of* $T_{5,5} - f$*, delete the edge set {(4 + 8x, 3)(4 + 8x, 4)* | *0* $\leq$ *x* < *p}* $\cup$ *{(5 + 8x, 3)(5 + 8x, 4)* | *0* $\leq$ *x* < *p}, and add {(4 + 8x, 3)(5 + 8x, 3)* | *0* $\leq$ *x* < *p}* $\cup$ *{(4 + 8x, 4)(5 + 8x, 4)* | *0* $\leq$ *x* < *p}, we obtain a DBH* C_1 *of* $T_{5+8p,5} - f$*, where* $|E_1(C_1)| = 12 + 20p = |E_2(C_1)|$*. On the other hand, as Figure 7 shows, if we put p HCs of* $T_{8,8}$ *on the right hand side of the HC of* $T_{5,8}$*, delete the edge set {(4 + 8x, 6)(4 + 8x, 7)* | *0* $\leq$ *x* < *p}* $\cup$ *{(5 + 8x, 6)(5 + 8x, 7)* | *0* $\leq$ *x* < *p}, and add {(4 + 8x, 6)(5 + 8x, 6)* | *0* $\leq$ *x* < *p}* $\cup$ *{(4 + 8x, 7)(5 + 8x, 7)* | *0* $\leq$ *x* < *p}, an HC* C_2 *of* $T_{5+8p,8}$ *is obtained, where* $|E_1(C_2)| = 22 + 32p$*, and* $|E_2(C_2)| = 18 + 32p$*. Lastly, if we put q* C_2*s under* C_1 *then delete the edge set {(0, 4 + 8x)(1, 4 + 8x)* | *0* $\leq$ *x* < *q }* $\cup$ *{(0, 5 + 8x)(1, 5 + 8x)* | *0* $\leq$ *x* < *q} and add the edge set {(0, 4 + 8x)(0, 5 + 8x)* | *0* $\leq$ *x* < *q}* $\cup$ *{(1, 4 + 8x)(1, 5 + 8x)* | *0* $\leq$ *x* < *q}, there is an HC* C_3 *of* $T_{5+8p,5+8q} - f$ *(see Figure 8 for an illustration), where* $|E_1(C_3)| = 12 + 20p + q(20 + 32p) = |E_2(C_3)|$*; therefore,* C_3 *is a DBH of* $T_{5+8p,5+8q} - f$*. In Figure 8 and the subsequent figures, similarly to the previous theorem, the red node denotes the faulty vertex, the blue dotted lines represent the edges that need to be deleted when merging the graphs, and the red lines represent the edges that need to be added when merging the graphs.*

Case (b). *When m = 7 + 8p, n = 5 + 8q: In this case, similarly to case (a), a DBH for* $T_{7+8p,5+8q} - f$ *will be constructed. Please refer to Figure 9 for the DBH of* $T_{7,5} - f$*, DBH of* $T_{8,5}$*, DBH of* $T_{7,8}$*, HC of* $T_{8,8}$ *and construction method. The obtained HC C on* $T_{7+8p,5+8q} - f$ *has* $|E_1(C)| = 17 + 20p + q(28 + 32p) = |E_2(C)|$*, so C is a DBH on* $T_{7+8p,5+8q} - f$*. Since* $T_{m,n}$ *has symmetry, by transposing this figure, a DBH on* $T_{5+8p,7+8q} - f$ *can be obtained. Every subsequent case when m and n are not equal has this property, which will not be described again.*

Case (c). *When m = 9 + 8p, n = 5 + 8q: Similarly to case (a), a DBH for* $T_{9+8p,5+8q} - f$ *will be constructed with a DBH of* $T_{9,5} - f$*, p HCs of* $T_{8,5}$*, q HCs of* $T_{9,8}$*, and pq HCs of* $T_{8,8}$*. Please refer* *to Figure 10. The constructed HC C has* $|E_1(C)| = 22 + 20p + q(36 + 32p) = |E_2(C)|$. *Therefore, C is a DBH of* $T_{9+8p,5+8q} - f$.

Case (d). *When* $m = 11 + 8p$, $n = 5 + 8q$: *Similarly to case (b), a DBH for* $T_{11+8p,5+8q} - f$ *will be constructed with a DBH of* $T_{11,5} - f$, *p DBHs of* $T_{8,5}$, *q DBHs of* $T_{11,8}$, *and pq HCs of* $T_{8,8}$. *Please refer to Figure 11. The constructed HC C has* $|E_1(C)| = 27 + 20p + q(44 + 32p) = |E_2(C)|$. *Therefore, C is a DBH of* $T_{11+8p,5+8q} - f$.

Case (e). *When* $m = 7 + 8p$, $n = 7 + 8q$: *Similarly to case (a), a DBH for* $T_{7+8p,7+8q} - f$ *will be constructed with a DBH of* $T_{7,7} - f$, *p HCs of* $T_{8,7}$, *q HCs of* $T_{7,8}$, *and pq HCs of* $T_{8,8}$. *Please refer to Figure 12. The constructed HC C has* $|E_1(C)| = 24 + 28p + q(28 + 32p) = |E_2(C)|$. *Therefore, C is a DBH of* $T_{7+8p,7+8q} - f$.

Case (f). *When* $m = 9 + 8p$, $n = 7 + 8q$: *Similarly to case (b), a DBH for* $T_{9+8p,7+8q} - f$ *will be constructed with a DBH of* $T_{9,7} - f$, *p HCs of* $T_{8,7}$, *q DBHs of* $T_{9,8}$, *and pq HCs of* $T_{8,8}$. *Please refer to Figure 13. The constructed HC C has* $|E_1(C)| = 31 + 28p + q(36 + 32p) = |E_2(C)|$. *Therefore, C is a DBH of* $T_{9+8p,7+8q} - f$.

Case (g). *When* $m = 11 + 8p$, $n = 7 + 8q$: *Similarly to case (a), a DBH for* $T_{11+8p,7+8q} - f$ *will be constructed with a DBH of* $T_{11,7} - f$, *p HCs of* $T_{8,7}$, *q HCs of* $T_{11,8}$, *and pq HCs of* $T_{8,8}$. *Please refer to Figure 14. The constructed HC C has* $|E_1(C)| = 38 + 28p + q(44 + 32p) = |E_2(C)|$. *Therefore, C is a DBH of* $T_{11+8p,7+8q} - f$.

Case (h). *When* $m = 9 + 8p$, $n = 9 + 8q$: *Similarly to case (a), a DBH for* $T_{9+8p,9+8q} - f$ *will be constructed with a DBH of* $T_{9,9} - f$, *p HCs of* $T_{8,9}$, *q HCs of* $T_{9,8}$, *and pq HCs of* $T_{8,8}$. *Please refer to Figure 15. The constructed HC C has* $|E_1(C)| = 40 + 36p + q(36 + 32p) = |E_2(C)|$. *Therefore, C is a DBH of* $T_{9+8p,9+8q} - f$.

Case (i). *When* $m = 11 + 8p$, $n = 9 + 8q$: *Similarly to case (b), a DBH for* $T_{11+8p,9+8q} - f$ *will be constructed with a DBH of* $T_{11,9} - f$, *p DBHs of* $T_{8,9}$, *q HCs of* $T_{11,8}$, *and pq HCs of* $T_{8,8}$. *Please refer to Figure 16. The constructed HC C has* $|E_1(C)| = 49 + 36p + q(44 + 32p) = |E_2(C)|$. *Therefore, C is a DBH of* $T_{11+8p,9+8q} - f$.

Case (j). *When* $m = 11 + 8p$, $n = 11 + 8q$: *Similarly to case (a), a DBH for* $T_{11+8p,11+8q} - f$ *will be constructed with a DBH of* $T_{11,11} - f$, *p HCs of* $T_{8,11}$, *q HCs of* $T_{11,8}$, *and pq HCs of* $T_{8,8}$. *Please refer to Figure 17. The constructed HC C has* $|E_1(C)| = 60 + 44p + q(44 + 32p) = |E_2(C)|$. *Therefore, C is a DBH of* $T_{11+8p,11+8q} - f$.

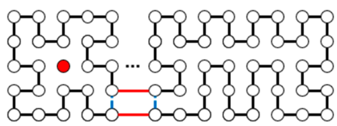

Figure 6. The DBH C_1 of $T_{5+8p,5} - f$.

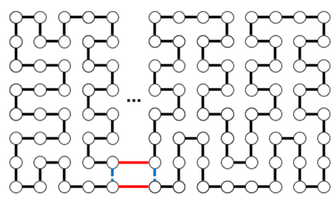

Figure 7. The HC C_2 of $T_{5+8p,8}$.

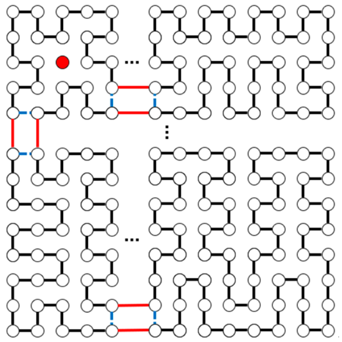

Figure 8. The DBH C_3 of $T_{5+8p,5+8q} - f$.

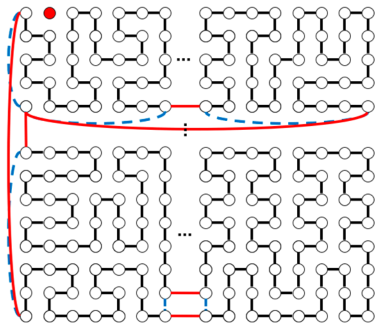

Figure 9. The DBH of $T_{7+8p,5+8q} - f$.

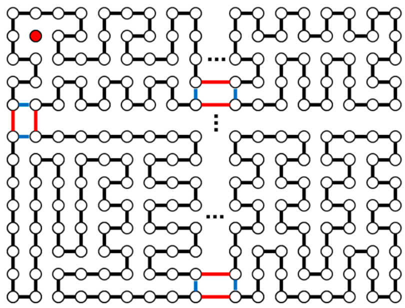

Figure 10. The DBH of $T_{9+8p,5+8q} - f$.

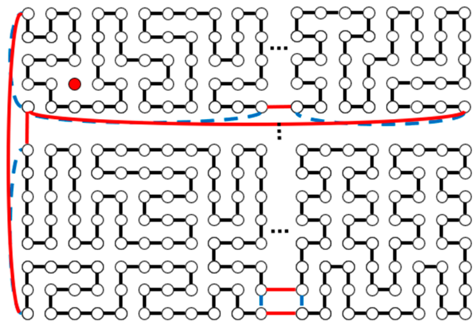

Figure 11. The DBH of $T_{11+8p,5+8q} - f$.

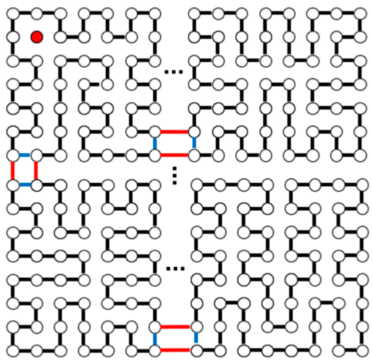

Figure 12. The DBH of $T_{7+8p,7+8q} - f$.

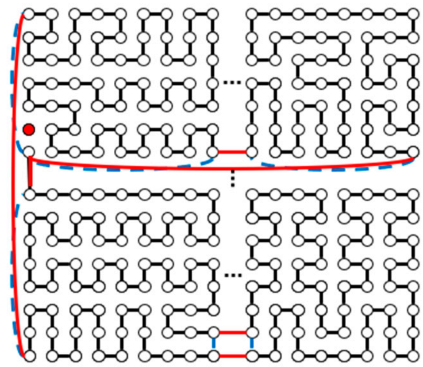

Figure 13. The DBH of $T_{9+8p,7+8q} - f$.

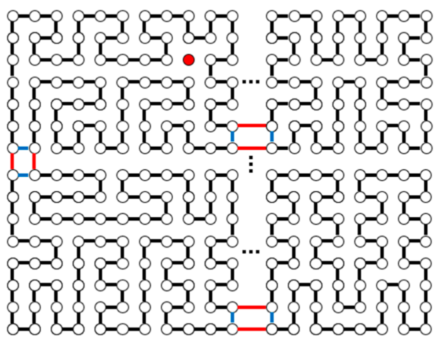

Figure 14. The DBH of $T_{11+8p,7+8q} - f$.

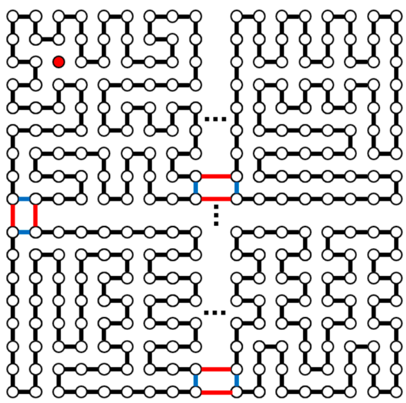

Figure 15. The DBH of $T_{9+8p,9+8q} - f$.

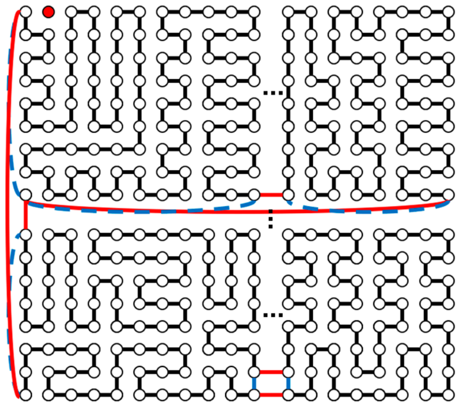

Figure 16. The DBH of $T_{11+8p,9+8q} - f$.

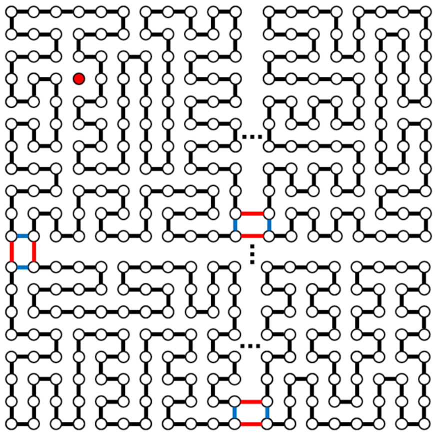

Figure 17. The DBH of $T_{11+8p,11+8q} - f$.

From the analysis of the above ten cases, it can be seen that when m and n are both odd numbers, even if there is a faulty vertex f, a DBH can be created in $T_{m,n} - f$, so the proof is completed. □

2.3. One of m and n Is Even and the Other Is Odd

This section will discuss whether $T_{m,n} - f$ still has a dimension-balanced Hamiltonian cycle when one of m and n is even and the other is odd. The following theorem is the main analytical result of this section, which discusses all possibilities of the remainder after dividing m by 8. Please refer to Table 2 for specific classification cases, where p and q are non-negative integers. Since $T_{m,n}$ is equivalent to $C_m \times C_n$ and isomorphic to $C_n \times C_m$, it can be assumed without loss of generality that m is an even number and n is an odd number. When n is an even number and m is an odd number, the corresponding DBH can be obtained by transposing the DBH obtained in the theorem (horizontally versus vertically).

Table 2. Classification of cases when one of m and n is even and the other is odd.

m	***m* Is Even and *n* Is Odd**			
n	**4 + 8*p***	**6 + 8*p***	**8 + 8*p***	**10 + 8*p***
3 + 2*q*	Thm. 4 (a)	Thm. 4 (b)	Thm. 4 (c)	Thm. 4 (d)

For convenience, Y_m is defined as the graph generated by the Cartesian product of C_m and P_2, which is called a *prism graph*, where P_n is the path of n vertices. Compared to $T_{m,n}$, the vertex set is the same and the edge set is the edge that lacks "crossing" edges in the vertical part (in fact, the vertical part only has two vertices, and the original definition is that there is an edge connecting the two vertices). It is defined as follows: vertex set $V(Y_m)$ = {(x, y) ∣ $0 \leq x \leq m-1$; $0 \leq y \leq 1$}; edge set $E(Y_m)$ = {(x_1, y_1)(x_2, y_2) ∣ $x_1 = x_2$ and ∣$y_1 - y_2$∣ = 1, or ∣$x_1 - x_2$∣ = (1 or $m - 1$) and $y_1 = y_2$}. On the other hand, Y_n^T represents the graph generated by the Cartesian product of P_2 and C_m, and its vertex set $V(Y_n^T)$ = {(x, y) ∣ $0 \leq x \leq 1$; $0 \leq y \leq m-1$}, and its edge set $E(Y_n^T)$ = {(x_1, y_1)(x_2, y_2) ∣ $x_1 = x_2$ and ∣$y_1 - y_2$∣ = (1 or $n - 1$), or ∣$x_1 - x_2$∣ = 1 and $y_1 = y_2$}.

Theorem 4. *When one of m and n is even and the other is odd integer, both are greater than 2, $T_{m,\ n} - f$ has a dimension-balanced Hamiltonian cycle, where f is any vertex in the graph.*

Proof of Theorem 4. Without loss of generality, it is assumed that m is an even number and n is an odd number. Since $m, n \geq 3$, we start from the graph of $T_{4,3}$ to find the dimension-balanced Hamiltonian cycle when there is a faulty vertex. As shown in Table 2, four cases are discussed based on the remainder of m divided by eight.

Case (a). *$m = 4 + 8p$, $n = 3 + 2q$: Firstly, please refer to Figure 18. If we put p DBHs of $T_{8,3}$ on the right hand side of the DBH of $T_{4,3} - f$, delete the edge set {(0, 2)(3, 2)} ∪ {(4 + 8x, 2)(11 + 8x, 2) ∣ $0 \leq x < p$}, and add {(0, 2)(3 + 8p, 2)} ∪ {(3 + 8x, 2)(4 + 8x, 2) ∣ $0 \leq x < p$}, we obtain a DBH C_1 of $T_{4+8p,3} - f$, where ∣$E_1(C_1)$∣ = 6 + 12p, ∣$E_2(C_1)$∣ = 5 + 12p. Again, in Figure 18 and the subsequent figures, the red node denotes the faulty vertex, the blue dotted lines represent the edges that need to be deleted when merging the graphs, and the red lines represent the edges that need to be added when merging the graphs.*

Next, as Figure 19 shows, if we put p DBHs of Y_8 on the right hand side of the DBH of Y_4, delete the edge set {(0, 4)(3, 4)} ∪ {(4 + 8x, 4)(11 + 8x, 4) ∣ $0 \leq x < p$}, and add {(0, 4)(3 + 8p, 4)} ∪ {(3 + 8x, 4), (4 + 8x, 4) ∣ $0 \leq x < p$}, a DBH C_2 of Y_{4+8p} is obtained, where ∣$E_1(C_2)$∣ = 4 + 8p = ∣$E_2(C_2)$∣.

Lastly, if we put q C_2s under C_1 then delete the edge set {(0, 0)(0, 2)} ∪ {(0, 3 + 2x)(0, 4 + 2x) ∣ $0 \leq x < q$} and add the edge set {(0, 0)(0, 2 + 2q)} ∪ {(0, 2 + 2x), (0, 3 + 2x) ∣ $0 \leq x < q$}, there is an HC C_3 of $T_{4+8p,3+2q} - f$. As Figure 20 shows, after the calculation, ∣$E_1(C_3)$∣ = 6 + 12p + q(4 + 8p), ∣$E_2(C_3)$∣ = 5 + 12p + q(4 + 8p); therefore, C_3 is a DBH of $T_{4+8p,3+2q} - f$ because ∣ ∣$E_1(C_3)$∣ − ∣$E_2(C_3)$∣ ∣ = 1.

Case (b). *$m = 6 + 8p$, $n = 3 + 2q$: Similarly to case (a), a DBH for $T_{6+8p,3+2q} - f$ will be constructed with a DBH of $T_{6,3} - f$, the same p DBHs of $T_{8,3}$, q DBHs of Y_6, and the same pq DBHs of Y_8. Please refer to Figure 22. Place p DBHs of $T_{8,3}$ on the right side of the DBH of $T_{6,3} - f$. After deleting and connecting the appropriate edges, the DBH C_1 at $T_{6+8p,3}$ is obtained. Then, place p DBHs of Y_8 to the right of the DBH of Y_6, and after deleting and connecting the appropriate edges, the DBH C_2 at $T_{6+8p,8}$ can be obtained. Finally, place q pieces of C_2s under C_1, delete and connect the appropriate edges, and obtain the HC C_3 at $T_{6+8p,3+2q}$. Since ∣$E_1(C_3)$∣ = 8 + 12p + q(6 + 8p), ∣$E_2(C_3)$∣ = 9 + 12p + q(6 + 8p). Therefore ∣ ∣$E_1(C_3)$∣ − ∣$E_2(C_3)$∣ ∣ = 1, C_3 is a DBH on $T_{6+8p,3+2q} - f$.*

Case (c). *$m = 8 + 8p$, $n = 3 + 2q$: Similarly to case (a), a DBH for $T_{8+8p,3+2q} - f$ will be constructed with a DBH of $T_{8,3} - f$, the same p DBHs of $T_{8,3}$, and the same q + pq DBHs of Y_8. Please refer to Figure 21, which shows how to connect these DBHs to form an HC C of $T_{8+8p,3+2q}$. Since ∣$E_1(C)$∣ = 12 + 12p + q(8 + 8p), ∣$E_2(C)$∣ = 11 + 12p + q(8 + 8p). Therefore ∣ ∣$E_1(C)$∣ − ∣$E_2(C)$∣ ∣ = 1, C is a DBH on $T_{8+8p,3+2q} - f$.*

Case (d). *$m = 10 + 8p$, $n = 3 + 2q$: Similarly to case (a), a DBH for $T_{10+8p,3+2q} - f$ will be constructed with a DBH of $T_{10,3} - f$, the same p DBHs of $T_{8,3}$, q DBHs of Y_{10}, and the same pq DBHs of Y_8. Please refer to Figure 23, which shows how to connect these DBHs to form an HC C of*

$T_{10+8p,3+2q}$. Since $|E_1(C)| = 14 + 12p + q(10 + 8p)$, $|E_2(C)| = 15 + 12p + q(10 + 8p)$. Therefore $||E_1(C)| - |E_2(C)|| = 1$, C is a DBH on $T_{10+8p,3+2q} - f$.

From the analysis of the above four cases, it can be known that for all m and n that are an even number and an odd number, respectively, if there is a faulty vertex, a DBH in $T_{m,n} - f$ can be obtained. Therefore, this proof is completed. □

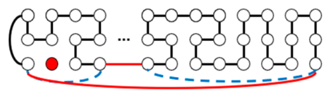

Figure 18. The DBH of $T_{4+8p,3} - f$.

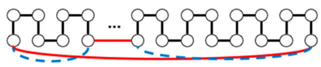

Figure 19. The DBH of $Y_{4+8p} - f$.

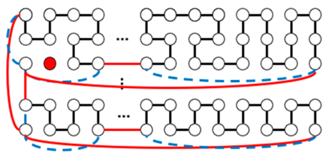

Figure 20. The DBH of $T_{4+8p,3+2q} - f$.

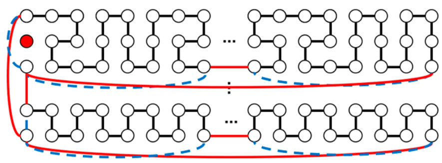

Figure 21. The DBH of $T_{8+8p,3+2q} - f$.

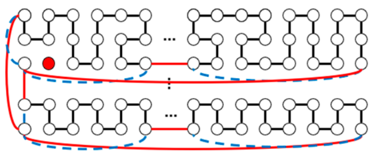

Figure 22. The DBH of $T_{6+8p,3+2q} - f$.

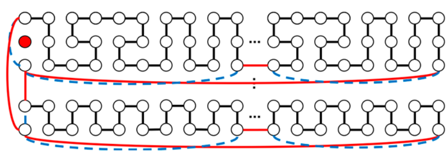

Figure 23. The DBH of $T_{10+8p,3+2q} - f$.

3. Conclusions

From the discussion in the previous section, it can be seen that in most cases of m and n, the dimension-balanced Hamiltonian cycle DBH can be found in $T_{m,n} - f$ for any faulty vertex f. The following corollary can be drawn.

Corollary 2. *Assuming that f is any vertex in the graph $T_{m,n}$, $T_{m,n} - f$ has a dimension-balanced Hamiltonian cycle DBH, except in the following cases:*

(1) When m and n are both even numbers;
(2) When one of m and n is 3, and the other satisfies mod 4 = 3 and is greater than 6.

Proof of Corollary 2. According to Theorem 1 and Corollary 1, we know there is no dimension-balanced Hamiltonian cycle on $T_{m,n} - f$ for any faulty vertex f when (1) both m and n are even or (2) one of m and n is 3 and the other satisfies mod 4 = 3 and is greater than 6. On the other hand, according to Theorems 2–4, $T_{m,n} - f$ has a dimension-balanced Hamiltonian cycle DBH when the above is not true for m and n. □

In addition, there are three conjectures in the literature [18]. If the following conjecture (that is, Conjecture 1) is true, the other two conjectures (that is, Conjectures 2 and 3) are also true, since it has already been proven in the remaining cases.

Conjecture 1 [18]**.** *When m and n* $\geq$ *5 are both odd numbers, for k =* $\lfloor (mn-1)/4 \rfloor$*, there is a DBC of length 4k in* $T_{m,n}$.

Conjecture 2 [18]**.** *For odd m and n* ≥ 5*,* $T_{m,n}$ *is (2max{m, n}* $-$ *1)-DBP.*

Conjecture 3 [18]**.** *For odd m and n* ≥ 5*,* $T_{m,n}$ *is (2max{m, n}* $-$ *1)-DBVP.*

Since it has been proven in [18] that when $k = \lfloor (mn-3)/4 \rfloor$, there is a DBC of length $4k$ in $T_{m,n}$, we only need to discuss when $k = (mn-1)/4$, that is, when $mn - 1 = 4k$. Theorem 3 (a), (c), (e), (g), (h), and (j) answer this question, so this paper also proves the correctness of the three conjectures in the literature [18]. And the following corollary can be concluded.

Corollary 3. *When m and n* $\geq$ *5 are both odd numbers, for k =* $\lfloor (mn-1)/4 \rfloor$*, there is a DBC of length 4k in* $T_{m,n}$*; therefore* $T_{m,n}$ *is (2max { m, n }* $-$ *1)-DBP and (2max { m, n }* $-$ *1)-DBVP.*

In addition, the following has been proven in [16]:

Theorem 5 [16]. *There are dimension-balanced Hamiltonian cycles for* $T_{m,n}$*, except mn* $\equiv$ *2 (mod 4), that is, one of m and n is odd and the other satisfies mod 4 = 2.*

Since a Hamiltonian cycle does not use all of its edges (in fact, there are two edges at any vertex that are not used), and because $T_{m,n}$ is a vertex-symmetric graph, if any edge is faulty, then through appropriate rotation, a dimension-balanced Hamilton cycle without using the faulty edge can be obtained. That is to say, the following corollary can be drawn.

Corollary 4. *There are dimension-balanced Hamiltonian cycles for* $T_{m,n} - e$*, except mn* $\equiv$ *2 (mod 4), where e is any edge of* $T_{m,n}$.

According to Corollaries 2 and 4 and the definition of one-fault dimension-balanced Hamiltonian, we draw the following conclusion.

Corollary 5. $T_{m,n}$ *is one-fault dimension-balanced Hamiltonian, except in the following cases:*
(1) When m and n are both even numbers;
(2) When one of m and n is 3, and the other satisfies mod 4 = 3 and is greater than 6;
(3) When one of m and n is odd, and the other satisfies mod 4 = 2.

Proof of Corollary 5. Corollary 2 shows that $T_{m,n} - f$ has a DBH for any node f, except (1) or (2). So, $T_{m,n}$ is not one-fault dimension-balanced Hamiltonian when (1) or (2) holds. Corollary 4 shows that $T_{m,n} - e$ has a DBH for any edge e, except (3). So, $T_{m,n}$ is not one-fault dimension-balanced Hamiltonian when (3) holds. In addition, since it is true that $T_{m,n} - f$ and $T_{m,n} - e$ both have a DBH, except (1), (2), and (3), $T_{m,n}$ is one-fault dimension-balanced Hamiltonian, except (1), (2), and (3). □

Table 3 shows a comparison of this study with previous works. Where two-fault H means two-fault Hamiltonian. Note that the DBP and WDBH problem under other conditions (both m and n are even, or one of them is even and the other is odd) has also been studied already. The DBP and WDBP problem is completed by [18] and [20], respectively (except for the DBP problem, three conjectures remain unsolved). From Table 3, it can be observed that, compared to previous works, our study is the first to investigate the one-fault DBH problem on $T_{m,n}$. Moreover, we resolved a conjecture previously left open in the DBP problem on $T_{m,n}$, thereby completing the solution of the DBP problem.

Table 3. A brief summary of previously obtained results and our results.

Reference	Multiprocessor Systems	Problems
[5]	Toroidal mesh graph ($T_{m,n}$)	two-fault H
[16]	Toroidal mesh graph ($T_{m,n}$)	DBH
[18]	Toroidal mesh graph ($T_{m,n}$)	DBP
[19]	Toroidal mesh graph ($T_{m,n}$)	WDBH
[20]	Toroidal mesh graph ($T_{m,n}$)	WDBP
[21]	3-dimensional toroidal mesh graph ($T_{m,n,r}$)	DBH
This paper	Toroidal mesh graph ($T_{m,n}$)	one-fault DBH and DBP

It is not difficult to see, based on Theorems 2–5, that an algorithm for identifying a dimension-balanced Hamiltonian cycle on a toroidal mesh network ($T_{m,n}$) can be obtained when given the input size m, n and the locations of faulty nodes. Since the automorphism adjustment based on the faulty node or edge is performed in linear time, the time complexity of the algorithm is proportional to the length of the Hamiltonian cycle, which is $O(mn)$.

In 2000, [5] proved that if $m \geq 3$, $n \geq 3$, and n is odd, $T_{m,n} - F$ has a Hamiltonian cycle for any F with $|F| \leq 2$. And whether $T_{m,n} - F$ has a DBH for any F with $|F| = 1$ has been solved in this paper. Therefore, building on the current results, a potential future research direction is to explore the conditions under which a toroidal mesh graph $T_{m,n}$ in this partition is two-fault dimension-balanced Hamiltonian based on the values of m and n. Additionally, another avenue for future study could involve investigating the dimension-balanced Hamiltonian cycle problem on a toroidal mesh graph under different partitioning schemes, expanding the applicability of the proposed approach. Please note that the DBHs constructed in this paper are all discussed based on the edge partitions E_1 and E_2 defined at the beginning of this paper, and there is no guarantee that there is a DBH on $T_{m,n} - f$ or $T_{m,n} - e$ for any partition of the edge set. Therefore, such studies could provide deeper insights into the fault tolerance and adaptability of Hamiltonian cycles in various configurations of toroidal mesh graphs.

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