

Article

New Bounds for the Euclidean Numerical Radius of Two Operators in Hilbert Spaces

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Abstract: This paper presents new weighted lower and upper bounds for the Euclidean numerical radius of pairs of operators in Hilbert spaces. We show that some of these bounds improve on recent results in the literature. We also find new inequalities for the numerical radius and the Davis–Wielandt radius. The lower and upper bounds we obtain are not symmetrical.

Keywords: numerical radius; Davis–Wielandt radius; Euclidean numerical radius; operator inequalities

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1. Introduction

Let H be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$. We represent the C^* -algebra of all bounded linear operators on H by $\mathcal{L}(H)$. For any operator $A \in \mathcal{L}(H)$, we denote its adjoint by A^* , and let $|A| = (A^*A)^{1/2}$ indicate the positive square root of A^*A . The real and imaginary parts of A are defined as $\Re(A) = \frac{1}{2}(A + A^*)$ and $\Im(A) = \frac{1}{2i}(A - A^*)$, respectively. The numerical range of A , written as $W(A)$, is the set $\{\langle Ax, x \rangle : x \in H, \|x\| = 1\}$. For simplicity, we use “operator” to refer to an operator in $\mathcal{L}(H)$.

Let $\|A\|$ and $w(A)$ represent the operator norm and numerical radius of the operator A , respectively. The operator norm of A is defined as

$$\|A\| = \sup\{|\langle Ax, y \rangle| : x, y \in H, \|x\| = \|y\| = 1\},$$

and the numerical radius of A is given by

$$w(A) = \sup\{|\langle Ax, x \rangle| : x \in H, \|x\| = 1\}.$$

It is established that the numerical radius $w(\cdot)$ defines a norm on $\mathcal{L}(H)$ and is comparable to the operator norm $\|\cdot\|$. Specifically, the following double inequality holds:

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|. \quad (1)$$

These inequalities are sharp. The first inequality attains equality if $A^2 = 0$, while the second inequality attains equality if A is a normal operator, i.e., $A^*A = AA^*$. Kittaneh [1] presented a refinement of these inequalities by establishing that

$$\frac{1}{2}\sqrt{\| |A|^2 + |A^*|^2 \|} \leq w(A) \leq \frac{\sqrt{2}}{2}\sqrt{\| |A|^2 + |A^*|^2 \|}. \quad (2)$$

For further developments on (1) and (2), readers may refer to [2–8] and the references therein.

The Davis–Wielandt radius of an operator $T \in \mathcal{L}(H)$, denoted by $dw(T)$, is defined in [9,10] as

$$dw(T) = \sup_{\|x\|=1} \left(|\langle Tx, x \rangle|^2 + \|Tx\|^4 \right)^{1/2}.$$

It follows that $dw(T) \geq 0$ and $dw(T) = 0$ if, and only if, $T = 0$. For any scalar $\eta \in \mathbb{C}$, the Davis–Wielandt radius satisfies $dw(\eta T) > |\eta| dw(T)$ if $|\eta| > 1$, $dw(\eta T) < |\eta| dw(T)$ if $|\eta| < 1$, and $dw(\eta T) = |\eta| dw(T)$ when $|\eta| = 1$.

Note that the triangle inequality $dw(T + S) \leq dw(T) + dw(S)$ does not hold for arbitrary $T, S \in \mathcal{L}(H)$. However, this inequality is valid when $\Re(T^*S) = 0$ (see ([11], Corollary 2.2)). Additionally, it is straightforward to show that

$$\max\{w(T), \|T\|^2\} \leq dw(T) \leq \left(w^2(T) + \|T\|^4 \right)^{1/2},$$

and these inequalities are sharp (see ([11], Corollary 2.2)).

Zamani and Shebrawi ([12], Theorem 2.1) established that

$$dw(T) \leq \left[w^2(T - |T|^2) + 2\|T\|^2 w(T) \right]^{1/2}.$$

Additionally, in ([12], Theorems 2.13, 2.14, and 2.17), the authors showed that

$$dw^2(T) \leq \|T\|^2 \max\{1, \|T\|^2\} + \sqrt{2}w(|T|^2T),$$

$$dw^2(T) \leq \frac{1}{2} \left(\| |T|^4 + |T|^2 \| + \| |T|^4 - |T|^2 \| + \sqrt{2}w(|T|^2T) \right),$$

and

$$dw^2(T) \leq \|T\| \max\{w(T), \|T\|^2\} \left[1 + \|T\|^2 + 2w(T) \right]^{1/2},$$

for any operator $T \in \mathcal{L}(H)$.

In recent work, Bhunia et al. ([11], Theorem 2.4) provided the following upper bound:

$$dw(T) \leq \| |T|^4 + |T|^2 \|^{1/2}, \quad (3)$$

for any $T \in \mathcal{L}(H)$.

Additionally, in [13], the authors derived inequalities for the sum of operators, as follows:

$$\begin{aligned} dw(T + S) &\leq \left[2(dw^2(T) + dw^2(S)) + 6\|T\|^4 + \|S\|^4 \right]^{1/2} \\ &\leq 2\sqrt{2} [dw^2(T) + dw^2(S)]^{1/2}, \end{aligned}$$

for $T, S \in \mathcal{L}(H)$.

Let (C, D) be a pair of bounded linear operators on H . The Euclidean operator radius is defined as:

$$w_e(C, D) := \sup_{\|x\|=1} \left(|\langle Cx, x \rangle|^2 + |\langle Dx, x \rangle|^2 \right)^{1/2}.$$

As noted in [14], $w_e : \mathcal{B}^2(H) := \mathcal{B}(H) \times \mathcal{B}(H) \rightarrow [0, \infty)$ defines a norm, and the following inequality is satisfied:

$$\frac{\sqrt{2}}{4} \| |C|^2 + |D|^2 \|^{1/2} \leq w_e(C, D) \leq \| |C|^2 + |D|^2 \|^{1/2}, \quad (4)$$

for $(C, D) \in \mathcal{B}^2(H)$, where the constants $\frac{\sqrt{2}}{4}$ and 1 are optimal in (4). For more details on the Euclidean operator radius, related results, and their generalizations, see [15–20] and the references cited therein.

We observe that for $C = T$ and $D = |T|^2$, we have

$$\begin{aligned} w_e(T, |T|^2) &:= \sup_{\|x\|=1} \left(|\langle Tx, x \rangle|^2 + |\langle |T|^2 x, x \rangle|^2 \right)^{1/2} \\ &= \sup_{\|x\|=1} \left(|\langle Tx, x \rangle|^2 + \|Tx\|^4 \right)^{1/2} = dw(T). \end{aligned}$$

By setting $C = T$ and $D = |T|^2$ in (4), we obtain

$$\frac{\sqrt{2}}{4} \| |T|^2 + |T|^4 \|^{1/2} \leq dw(T) \leq \| |T|^2 + |T|^4 \|^{1/2},$$

which provides the upper bound from (3) and a similar lower bound.

Motivated by the above results, in this paper, we present some new weighted lower and upper bounds for the Euclidean numerical radius of a pair of operators and show that some of them are sharper than certain recent results obtained by other authors. As a natural consequence, we also derive new inequalities for the numerical radius and the Davis–Wielandt radius.

Among other results, we obtain the following lower bounds for the Euclidean numerical radius:

$$\begin{aligned} w_e^2(A, B) &\geq \frac{1}{2} \max \left\{ w \left[(\alpha^2 + \beta^2) (A^2 + B^2) + 2\alpha\beta(AB + BA) \right], \right. \\ &\quad \left. w \left[(\alpha^2 + \beta^2) (A^2 + B^2) - 2\alpha\beta(AB + BA) \right] \right\} \\ &\quad + \frac{1}{2} \max \{ |\alpha + \beta| w(A + B), |\alpha - \beta| w(A - B) \} \times |w(\alpha A + \beta B) - w(\beta A + \alpha B)|, \end{aligned}$$

and

$$\begin{aligned} w_e^2(A, B) &\geq |\alpha\beta|w(AB + BA) \\ &\quad + \frac{1}{2} \max\{w[\alpha(1+i)A + \beta(1-i)B], w[\alpha(1-i)A + \beta(1+i)B]\} \\ &\quad \times |w(\alpha A + \beta B) - w(\alpha A - \beta B)|, \end{aligned}$$

for $A, B \in \mathcal{L}(H)$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$.

Let f, g, h , and k be nonnegative continuous functions on $[0, \infty)$ satisfying the condition $f(t)g(t) = h(t)k(t) = t$ for all $t \in [0, \infty)$. If $A, B \in \mathcal{L}(H)$ and $\alpha, \beta, \gamma, \delta \in \{-1, 1\}$, then we also obtain the following upper bounds:

$$w_e^2(A, B) \leq \left\| f^{2p}(|A|) + h^{2p}(|B|) \right\|^{1/p} \left\| g^{2q}(|A^*|) + k^{2q}(|B^*|) \right\|^{1/q},$$

and

$$w_e^2(A, B) \leq \left\| f^{2p}(|A|) + k^{2p}(|B^*|) \right\|^{1/p} \left\| g^{2q}(|A^*|) + h^{2q}(|B|) \right\|^{1/q},$$

for $A, B \in \mathcal{L}(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Applications for the numerical radius and the Davis–Wielandt radius are also provided.

2. Main Results

In this section, we present our main results, beginning with the following initial result.

Theorem 1. For $A, B \in \mathcal{L}(H)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2 = 1$, we have

$$\begin{aligned} w_e^2(A, B) &\geq \frac{1}{2} \max\left\{w\left[\left(\alpha^2 + \gamma^2\right)A^2 + \left(\beta^2 + \delta^2\right)B^2 + (\alpha\beta + \gamma\delta)(AB + BA)\right], \right. \\ &\quad \left.w\left[\left(\alpha^2 + \gamma^2\right)A^2 + \left(\beta^2 + \delta^2\right)B^2 - (\alpha\beta + \gamma\delta)(AB + BA)\right]\right\} \\ &\quad + \frac{1}{2} \max\{w[(\alpha + \gamma)A + (\beta + \delta)B], w[(\alpha - \gamma)A + (\beta - \delta)B]\} \\ &\quad \times |w(\alpha A + \beta B) - w(\gamma A + \delta B)|, \end{aligned} \tag{5}$$

and

$$\begin{aligned} w_e^2(A, B) &\geq \frac{1}{4} \max\left\{w\left[\left[(\alpha + \beta)^2 + (\gamma + \delta)^2\right]A^2 + \left[(\alpha - \beta)^2 + (\gamma - \delta)^2\right]B^2 \right. \right. \\ &\quad \left. \left. + \left(\alpha^2 + \gamma^2 - \beta^2 - \delta^2\right)(AB + BA), \right. \right. \\ &\quad \left. w\left[\left[(\alpha - \beta)^2 + (\gamma - \delta)^2\right]A^2 + \left[(\alpha + \beta)^2 + (\gamma + \delta)^2\right]B^2 \right. \right. \\ &\quad \left. \left. + \left(\alpha^2 + \gamma^2 - \beta^2 - \delta^2\right)(AB + BA)\right]\right\} \\ &\quad + \frac{1}{4} \max\{w[(\alpha + \gamma + \beta + \delta)A + (\alpha + \gamma - \beta - \delta)B], \\ &\quad w[(\alpha + \beta - \gamma - \delta)A + (\alpha - \beta - \gamma + \delta)B]\} \\ &\quad \times |w((\alpha + \beta)A + (\alpha - \beta)B) - w((\gamma + \delta)A + (\gamma - \delta)B)|. \end{aligned} \tag{6}$$

Proof. By using the Cauchy–Schwarz inequality for $|\alpha|^2 + |\beta|^2 = 1$, we have

$$\begin{aligned} |\langle Ax, x \rangle|^2 + |\langle Bx, x \rangle|^2 &= (|\alpha|^2 + |\beta|^2) (|\langle Ax, x \rangle|^2 + |\langle Bx, x \rangle|^2) \\ &\geq (|\alpha| |\langle Ax, x \rangle| + |\beta| |\langle Bx, x \rangle|)^2 \\ &= (|\langle \alpha Ax, x \rangle| + |\langle \beta Bx, x \rangle|)^2 \\ &\geq |\langle \alpha Ax, x \rangle + \langle \beta Bx, x \rangle|^2 = |\langle (\alpha A + \beta B)x, x \rangle|^2, \end{aligned}$$

for all $x \in H$.

If we take the supremum over $x \in H$, $\|x\| = 1$, we obtain

$$w_e^2(A, B) \geq w^2(\alpha A + \beta B).$$

Also, if $|\gamma|^2 + |\delta|^2 = 1$, we have

$$w_e^2(A, B) \geq w^2(\gamma A + \delta B).$$

Therefore,

$$\begin{aligned} w_e^2(A, B) &\geq \max \{ w^2(\alpha A + \beta B), w^2(\gamma A + \delta B) \} \\ &= \frac{1}{2} [w^2(\alpha A + \beta B) + w^2(\gamma A + \delta B)] \\ &\quad + \frac{1}{2} |w^2(\alpha A + \beta B) - w^2(\gamma A + \delta B)| \\ &= \frac{1}{2} [w^2(\alpha A + \beta B) + w^2(\gamma A + \delta B)] \\ &\quad + \frac{1}{2} [w(\alpha A + \beta B) + w(\gamma A + \delta B)] |w(\alpha A + \beta B) - w(\gamma A + \delta B)|. \end{aligned} \tag{7}$$

Now, observe that

$$\begin{aligned} &w^2(\alpha A + \beta B) + w^2(\gamma A + \delta B) \\ &\geq w[(\alpha A + \beta B)^2] + w[(\gamma A + \delta B)^2] \\ &\geq \max \{ w[(\alpha A + \beta B)^2 + (\gamma A + \delta B)^2], w[(\alpha A + \beta B)^2 - (\gamma A + \delta B)^2] \} \\ &= \max \{ w[(\alpha^2 + \gamma^2)A^2 + (\alpha\beta + \gamma\delta)(AB + BA) + (\beta^2 + \delta^2)B^2], \\ &\quad w[(\alpha^2 + \gamma^2)A^2 - (\alpha\beta + \gamma\delta)(AB + BA) + (\beta^2 + \delta^2)B^2] \}, \end{aligned}$$

and

$$\begin{aligned} &w(\alpha A + \beta B) + w(\gamma A + \delta B) \\ &\geq \max \{ w[(\alpha + \gamma)A + (\beta + \delta)B], w[(\alpha - \gamma)A + (\beta - \delta)B] \}, \end{aligned}$$

and by (7), we derive (5).

Now, if we replace A with $C + D$ and B with $C - D$, then we obtain

$$\begin{aligned}
 & w_e^2(C + D, C - D) \\
 & \geq \frac{1}{2} \max \left\{ w \left[(\alpha^2 + \gamma^2)(C + D)^2 + (\beta^2 + \delta^2)(C - D)^2 \right. \right. \\
 & \quad \left. \left. + (\alpha\beta + \gamma\delta)((C + D)(C - D) + (C - D)(C + D)) \right], \right. \\
 & \quad w \left[(\alpha^2 + \gamma^2)(C + D)^2 + (\beta^2 + \delta^2)(C - D)^2 \right. \\
 & \quad \left. \left. - (\alpha\beta + \gamma\delta)((C + D)(C - D) + (C - D)(C + D)) \right] \right\} \\
 & \quad + \frac{1}{2} \max \{ w[(\alpha + \gamma)(C + D) + (\beta + \delta)(C - D)], \\
 & \quad w[(\alpha - \gamma)(C + D) + (\beta - \delta)(C - D)] \} \\
 & \quad \times |w(\alpha(C + D) + \beta(C - D)) - w(\gamma(C + D) + \delta(C - D))|.
 \end{aligned} \tag{8}$$

Observe that

$$\begin{aligned}
 & (\alpha^2 + \gamma^2)(C + D)^2 + (\beta^2 + \delta^2)(C - D)^2 \\
 & = (\alpha^2 + \gamma^2)(C^2 + D^2 + CD + DC) + (\beta^2 + \delta^2)(C^2 + D^2 - CD - DC) \\
 & = (\alpha^2 + \gamma^2 + \beta^2 + \delta^2)(C^2 + D^2) + (\alpha^2 + \gamma^2 - \beta^2 - \delta^2)(CD + DC) \\
 & \quad (C + D)(C - D) + (C - D)(C + D) \\
 & \quad = C^2 - D^2 + DC - CD + C^2 - D^2 = DC + CD \\
 & \quad = 2(C^2 - D^2).
 \end{aligned}$$

Then,

$$\begin{aligned}
 & (\alpha^2 + \gamma^2)(C + D)^2 + (\beta^2 + \delta^2)(C - D)^2 \\
 & + (\alpha\beta + \gamma\delta)((C + D)(C - D) + (C - D)(C + D)) \\
 & = (\alpha^2 + \gamma^2 + \beta^2 + \delta^2)(C^2 + D^2) + (\alpha^2 + \gamma^2 - \beta^2 - \delta^2)(CD + DC) \\
 & + 2(\alpha\beta + \gamma\delta)(C^2 - D^2) \\
 & = (\alpha^2 + \gamma^2 + \beta^2 + \delta^2 + 2(\alpha\beta + \gamma\delta))C^2 \\
 & + (\alpha^2 + \gamma^2 + \beta^2 + \delta^2 - 2(\alpha\beta + \gamma\delta))D^2 \\
 & + (\alpha^2 + \gamma^2 - \beta^2 - \delta^2)(CD + DC) \\
 & = [(\alpha + \beta)^2 + (\gamma + \delta)^2]C^2 + [(\alpha - \beta)^2 + (\gamma - \delta)^2]D^2 \\
 & + (\alpha^2 + \gamma^2 - \beta^2 - \delta^2)(CD + DC),
 \end{aligned}$$

and

$$\begin{aligned}
 & (\alpha^2 + \gamma^2)(C + D)^2 + (\beta^2 + \delta^2)(C - D)^2 \\
 & - (\alpha\beta + \gamma\delta)((C + D)(C - D) + (C - D)(C + D)) \\
 & = (\alpha^2 + \gamma^2 + \beta^2 + \delta^2)(C^2 + D^2) + (\alpha^2 + \gamma^2 - \beta^2 - \delta^2)(CD + DC) \\
 & - 2(\alpha\beta + \gamma\delta)(C^2 - D^2) \\
 & = (\alpha^2 + \gamma^2 + \beta^2 + \delta^2 - 2(\alpha\beta + \gamma\delta))C^2 \\
 & + (\alpha^2 + \gamma^2 + \beta^2 + \delta^2 + 2(\alpha\beta + \gamma\delta))D^2 \\
 & + (\alpha^2 + \gamma^2 - \beta^2 - \delta^2)(CD + DC) \\
 & = [(\alpha - \beta)^2 + (\gamma - \delta)^2]C^2 + [(\alpha + \beta)^2 + (\gamma + \delta)^2]D^2 \\
 & + (\alpha^2 + \gamma^2 - \beta^2 - \delta^2)(CD + DC).
 \end{aligned}$$

Also,

$$(\alpha + \gamma)(C + D) + (\beta + \delta)(C - D) = (\alpha + \gamma + \beta + \delta)C + (\alpha + \gamma - \beta - \delta)D,$$

and

$$(\alpha - \gamma)(C + D) + (\beta - \delta)(C - D) = (\alpha + \beta - \gamma - \delta)C + (\alpha - \gamma - \beta + \delta)D.$$

Moreover,

$$w_e^2(C + D, C - D) = 2w_e^2(C, D),$$

and by (8), we obtain

$$\begin{aligned}
 & 2w_e^2(C, D) \\
 & \geq \frac{1}{2} \max \left\{ w \left[[(\alpha + \beta)^2 + (\gamma + \delta)^2]C^2 + [(\alpha - \beta)^2 + (\gamma - \delta)^2]D^2 \right. \right. \\
 & \quad \left. \left. + (\alpha^2 + \gamma^2 - \beta^2 - \delta^2)(CD + DC), \right. \right. \\
 & \quad w \left[[(\alpha - \beta)^2 + (\gamma - \delta)^2]C^2 + [(\alpha + \beta)^2 + (\gamma + \delta)^2]D^2 \right. \\
 & \quad \left. \left. + (\alpha^2 + \gamma^2 - \beta^2 - \delta^2)(CD + DC) \right\} \\
 & \quad + \frac{1}{2} \max \{ w[(\alpha + \gamma + \beta + \delta)C + (\alpha + \gamma - \beta - \delta)D] \\
 & \quad w[(\alpha + \beta - \gamma - \delta)C + (\alpha - \gamma - \beta + \delta)D] \} \\
 & \quad \times |w((\alpha + \beta)C + (\alpha - \beta)D) - w((\gamma + \delta)C + (\gamma - \delta)D)|,
 \end{aligned}$$

and, by replacing C with A and D with B , the inequality (6) is obtained. $\square$

Remark 1. We observe that for $\alpha = 0$, $\beta = 1$, $\gamma = 1$ and $\delta = 0$ we obtain from (5) that

$$\begin{aligned}
 w_e^2(A, B) & \geq \frac{1}{2}w(A^2 + B^2) + \frac{1}{2} \max \{ w(B + A), w(B - A) \} \\
 & \quad \times |w(B) - w(A)|,
 \end{aligned} \tag{9}$$

while from (6), that

$$w_e^2(A, B) \geq \frac{1}{2} \max w(A^2 + B^2) + \frac{1}{2} \max\{w(A), w(B)\} \times |w(A - B) - w(A + B)|. \quad (10)$$

It should be noted that the inequality (10) was obtained earlier in ([21], Theorem 2.2). Moreover, both inequalities (9) and (10) are refinements of the inequality

$$w_e^2(A, B) \geq \frac{1}{2} w(A^2 + B^2),$$

obtained in 2006 by the second author, see [22].

Remark 2. If we take $\gamma = \alpha, \delta = \beta$ in (5), then we obtain

$$w_e^2(A, B) \geq \max\left\{w\left[(\alpha A + \beta B)^2\right], w\left[(\alpha A - \beta B)^2\right]\right\}, \quad (11)$$

for $A, B \in \mathcal{L}(H)$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$. For $\alpha = \beta = \frac{\sqrt{2}}{2}$, we obtain

$$w_e^2(A, B) \geq \frac{1}{2} \max\left\{w\left[(A + B)^2\right], w\left[(A - B)^2\right]\right\}, \quad (12)$$

which is a known result, see [22].

If we take $\gamma = \beta, \delta = \alpha$ in (5), then we obtain

$$\begin{aligned} w_e^2(A, B) &\geq \frac{1}{2} \max\left\{w\left[(\alpha^2 + \beta^2)(A^2 + B^2) + 2\alpha\beta(AB + BA)\right], \right. \\ &\quad \left. w\left[(\alpha^2 + \beta^2)(A^2 + B^2) - 2\alpha\beta(AB + BA)\right]\right\} \\ &\quad + \frac{1}{2} \max\{|\alpha + \beta|w(A + B), |\alpha - \beta|w(A - B)\} \\ &\quad \times |w(\alpha A + \beta B) - w(\beta A + \alpha B)|, \end{aligned} \quad (13)$$

for $A, B \in \mathcal{L}(H)$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$. For $\alpha = \beta = \frac{\sqrt{2}}{2}$, we obtain again (12).

If we take $\gamma = i\alpha, \delta = -i\beta$ in (5), then we obtain

$$\begin{aligned} w_e^2(A, B) &\geq |\alpha\beta|w(AB + BA) \\ &\quad + \frac{1}{2} \max\{w[\alpha(1 + i)A + \beta(1 - i)B], w[\alpha(1 - i)A + \beta(1 + i)B]\} \\ &\quad \times |w(\alpha A + \beta B) - w(\alpha A - \beta B)|, \end{aligned} \quad (14)$$

for $A, B \in \mathcal{L}(H)$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$. For $\alpha = \beta = \frac{\sqrt{2}}{2}$, we obtain

$$\begin{aligned} w_e^2(A, B) &\geq \frac{1}{2} w(AB + BA) \\ &\quad + \frac{1}{4} \max\{w[(1 + i)A + (1 - i)B], w[(1 - i)A + (1 + i)B]\} \\ &\quad \times |w(A + B) - w(A - B)|. \end{aligned}$$

If we take $\gamma = i\beta$, $\delta = -i\alpha$ in (5), then we obtain

$$\begin{aligned} w_e^2(A, B) &\geq \frac{1}{2} \max \left\{ w \left[\left(\beta^2 - \alpha^2 \right) \left(B^2 - A^2 \right) + 2\alpha\beta(AB + BA) \right], \right. \\ &w \left[\left(\beta^2 - \alpha^2 \right) \left(B^2 - A^2 \right) - 2\alpha\beta(AB + BA) \right] \left. \right\} \\ &+ \frac{1}{2} \max \{ w[(\alpha + i\beta)A + (\beta - i\alpha)B], w[(\alpha - i\beta)A + (\beta + i\alpha)B] \} \\ &\times |w(\alpha A + \beta B) - w(\beta A - \alpha B)|, \end{aligned} \quad (15)$$

for $A, B \in \mathcal{L}(H)$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$. For $\alpha = \frac{\sqrt{2}}{2}i$ and $\beta = \frac{\sqrt{2}}{2}$, we obtain

$$w_e^2(A, B) \geq \frac{1}{2} \max \left\{ w \left[(B + iA)^2 \right], w \left[(B - iA)^2 \right] \right\}.$$

By choosing $\alpha = \sin \theta$, $\beta = \cos \theta$, $\theta \in \mathbb{R}$, we can also state the trigonometric inequalities

$$w_e^2(A, B) \geq \max \left\{ w \left[(\sin \theta A + \cos \theta B)^2 \right], w \left[(\sin \theta A - \cos \theta B)^2 \right] \right\},$$

$$\begin{aligned} w_e^2(A, B) &\geq \frac{1}{2} \max \left\{ w \left[A^2 + B^2 + \sin(2\theta)(AB + BA) \right], \right. \\ &w \left[A^2 + B^2 - \sin(2\theta)(AB + BA) \right] \left. \right\} \\ &+ \frac{1}{2} \max \{ |\sin \theta + \cos \theta| w(A + B), |\sin \theta - \cos \theta| w(A - B) \} \\ &\times |w(\sin \theta A + \cos \theta B) - w(\cos \theta A + \sin \theta B)|, \end{aligned}$$

$$\begin{aligned} w_e^2(A, B) &\geq \frac{1}{2} |\sin(2\theta)| w(AB + BA) \\ &+ \frac{1}{2} \max \{ w[\sin \theta(1 + i)A + \cos \theta(1 - i)B], \\ &w[\sin \theta(1 - i)A + \cos \theta(1 + i)B] \} \\ &\times |w(\sin \theta A + \cos \theta B) - w(\sin \theta A - \cos \theta B)|, \end{aligned}$$

and

$$\begin{aligned} w_e^2(A, B) &\geq \frac{1}{2} \max \left\{ w \left[\cos(2\theta) \left(B^2 - A^2 \right) + \sin(2\theta)(AB + BA) \right], \right. \\ &w \left[\cos(2\theta) \left(B^2 - A^2 \right) - \sin(2\theta)(AB + BA) \right] \left. \right\} \\ &+ \frac{1}{2} \max \{ w[(\sin \theta + i \cos \theta)A + (\cos \theta - i \sin \theta)B], \\ &w[(\sin \theta - i \cos \theta)A + (\cos \theta + i \sin \theta)B] \} \\ &\times |w(\sin \theta A + \cos \theta B) - w(\cos \theta A - \sin \theta B)|, \end{aligned}$$

for $A, B \in \mathcal{L}(H)$ and $\theta \in \mathbb{R}$.

Now, if we take $A = T$ and $B = T^*$ in Theorem 1 and since $w_e^2(T, T^*) = 2w^2(T)$, then we obtain

$$\begin{aligned} w^2(T) &\geq \frac{1}{4} \max \left\{ w \left[\left(\alpha^2 + \gamma^2 \right) T^2 + \left(\beta^2 + \delta^2 \right) (T^*)^2 + (\alpha\beta + \gamma\delta) \left(|T|^2 + |T^*|^2 \right) \right], \right. \\ &\quad \left. w \left[\left(\alpha^2 + \gamma^2 \right) T^2 + \left(\beta^2 + \delta^2 \right) (T^*)^2 - (\alpha\beta + \gamma\delta) \left(|T|^2 + |T^*|^2 \right) \right] \right\} \\ &\quad + \frac{1}{4} \max \{ w[(\alpha + \gamma)T + (\beta + \delta)T^*], w[(\alpha - \gamma)T + (\beta - \delta)T^*] \} \\ &\quad \times |w(\alpha T + \beta T^*) - w(\gamma T + \delta T^*)|, \end{aligned}$$

and

$$\begin{aligned} w^2(T) &\geq \frac{1}{8} \max \left\{ w \left[\left[(\alpha + \beta)^2 + (\gamma + \delta)^2 \right] T^2 + \left[(\alpha - \beta)^2 + (\gamma - \delta)^2 \right] (T^*)^2 \right. \right. \\ &\quad \left. \left. + \left(\alpha^2 + \gamma^2 - \beta^2 - \delta^2 \right) \left(|T|^2 + |T^*|^2 \right) \right], \right. \\ &\quad \left. w \left[\left[(\alpha - \beta)^2 + (\gamma - \delta)^2 \right] T^2 + \left[(\alpha + \beta)^2 + (\gamma + \delta)^2 \right] (T^*)^2 \right. \right. \\ &\quad \left. \left. + \left(\alpha^2 + \gamma^2 - \beta^2 - \delta^2 \right) \left(|T|^2 + |T^*|^2 \right) \right] \right\} \\ &\quad + \frac{1}{8} \max \{ w[(\alpha + \gamma + \beta + \delta)T + (\alpha + \gamma - \beta - \delta)T^*], \\ &\quad w[(\alpha + \beta - \gamma - \delta)T + (\alpha - \beta - \gamma + \delta)T^*] \} \\ &\quad \times |w((\alpha + \beta)T + (\alpha - \beta)T^*) - w((\gamma + \delta)T + (\gamma - \delta)T^*)|. \end{aligned}$$

for $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2 = 1$.

If we take $A = T$ and $B = T^*$ and $\gamma = \alpha, \delta = \beta$ in (5), then we obtain

$$w^2(T) \geq \frac{1}{2} \max \left\{ w \left[(\alpha T + \beta T^*)^2 \right], w \left[(\alpha T - \beta T^*)^2 \right] \right\},$$

for $T \in \mathcal{L}(H)$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$. For $\alpha = \beta = \frac{\sqrt{2}}{2}$, we obtain

$$w^2(T) \geq \frac{1}{4} \max \left\{ \|T + T^*\|^2, \|T - T^*\|^2 \right\}.$$

If we take $A = T$ and $B = T^*$ and $\gamma = \beta, \delta = \alpha$ in (5), then we obtain

$$\begin{aligned} w^2(T) &\geq \frac{1}{4} \max \left\{ w \left[\left(\alpha^2 + \beta^2 \right) \left(T^2 + (T^*)^2 \right) + 2\alpha\beta \left(|T|^2 + |T^*|^2 \right) \right], \right. \\ &\quad \left. w \left[\left(\alpha^2 + \beta^2 \right) \left(T^2 + (T^*)^2 \right) - 2\alpha\beta \left(|T|^2 + |T^*|^2 \right) \right] \right\} \\ &\quad + \frac{1}{4} \max \{ |\alpha + \beta| \|T + T^*\|, |\alpha - \beta| \|T - T^*\| \} \\ &\quad \times |w(\alpha T + \beta T^*) - w(\beta T + \alpha T^*)|, \end{aligned}$$

for $T \in \mathcal{L}(H)$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$.

For $\alpha = 0$ and $\beta = 1$, we obtain

$$w^2(T) \geq \frac{1}{4} w \left[\left(T^2 + (T^*)^2 \right) \right] = \frac{1}{4} \|T^2 + (T^*)^2\|,$$

for $T \in \mathcal{L}(H)$.

Further, from (14), for $A = T$ and $B = T^*$, we obtain

$$\begin{aligned}
w^2(T) &\geq \frac{1}{2}|\alpha\beta|\left\|\left|T\right|^2+\left|T^*\right|^2\right\| \\
&+ \frac{1}{4}\max\left\{w\left[\alpha(1+i)T+\beta(1-i)T^*\right], w\left[\alpha(1-i)T+\beta(1+i)T^*\right]\right\} \\
&\times\left|w\left(\alpha T+\beta T^*\right)-w\left(\alpha T-\beta T^*\right)\right|,
\end{aligned}$$

for $T \in \mathcal{L}(H)$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$. For $\alpha = \beta = \frac{\sqrt{2}}{2}$, we obtain

$$\begin{aligned}
w^2(T) &\geq \frac{1}{4}\left\|\left|T\right|^2+\left|T^*\right|^2\right\| \\
&+ \frac{1}{8}\max\left\{w\left[(1+i)T+(1-i)T^*\right], w\left[(1-i)T+(1+i)T^*\right]\right\} \\
&\times\left|\left\|T+T^*\right\|-\left\|T-T^*\right\|\right|,
\end{aligned}$$

for $T \in \mathcal{L}(H)$.

This inequality is better than the first inequality of Kittaneh, (2).

Moreover, if we take $A = T$ and $B = T^*$ in (15), then we obtain

$$\begin{aligned}
&w^2(T) \tag{16} \\
&\geq \frac{1}{4}\max\left\{w\left[\left(\beta^2-\alpha^2\right)\left(\left(T^*\right)^2-T^2\right)+2\alpha\beta\left(\left|T\right|^2+\left|T^*\right|^2\right)\right],\right. \\
&w\left[\left(\beta^2-\alpha^2\right)\left(\left(T^*\right)^2-T^2\right)-2\alpha\beta\left(\left|T\right|^2+\left|T^*\right|^2\right)\right]\left.\right\} \\
&+ \frac{1}{4}\max\left\{w\left[\left(\alpha+i\beta\right)T+\left(\beta-i\alpha\right)T^*\right], w\left[\left(\alpha-i\beta\right)T+\left(\beta+i\alpha\right)T^*\right]\right\} \\
&\times\left|w\left(\alpha T+\beta T^*\right)-w\left(\beta T-\alpha T^*\right)\right|,
\end{aligned}$$

for $T \in \mathcal{L}(H)$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$. For $\alpha = \frac{\sqrt{2}}{2}i$ and $\beta = \frac{\sqrt{2}}{2}$, we obtain

$$\begin{aligned}
w^2(T) &\geq \frac{1}{4}\max\left\{w\left[\left(T^*+iT\right)^2\right], w\left[\left(T^*-iT\right)^2\right]\right\} \\
&= \frac{1}{4}\max\left\{\left\|\left(T^*+iT\right)^2\right\|,\left\|\left(T^*-iT\right)^2\right\|\right\},
\end{aligned}$$

since $(T^* + iT)^2$ and $(T^* - iT)^2$ are normal operators.

Also, if we take $\alpha = 0$ and $\beta = 1$ in (16), then we obtain

$$w^2(T) \geq \frac{1}{4}w\left[\left(\left(T^*\right)^2-T^2\right)\right] = \frac{1}{4}\left\|\left(T^*\right)^2-T^2\right\|,$$

since $(T^*)^2 - T^2$ is normal. This inequality was obtained before in ([22], Remark 2).

If we consider the real and imaginary part of $T \in \mathcal{L}(H)$, then the following Cartesian decomposition holds: $T = \Re(T) + i\Im(T)$ and $w_c^2(\Re(T), \Im(T)) = w^2(T)$.

If we write the inequality (11) for $A = \Re(T)$ and $B = \Im(T)$, then we obtain

$$w^2(T) \geq \max\left\{w\left[\left(\alpha\Re(T)+\beta\Im(T)\right)^2\right], w\left[\left(\alpha\Re(T)-\beta\Im(T)\right)^2\right]\right\},$$

for $T \in \mathcal{L}(H)$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$. For $\alpha = \beta = \frac{\sqrt{2}}{2}$, we obtain

$$w^2(T) \geq \frac{1}{2}\max\left\{\left\|\Re(T)+\Im(T)\right\|^2,\left\|\Re(T)-\Im(T)\right\|^2\right\}.$$

By letting $A = \Re(T)$ and $B = \Im(T)$ in (13), we also have

$$\begin{aligned} w^2(T) &\geq \frac{1}{2} \max \left\{ w \left[\left(\alpha^2 + \beta^2 \right) \left(\Re^2(T) + \Im^2(T) \right) + 2\alpha\beta \left(\Re(T)\Im(T) + \Im(T)\Re(T) \right) \right], \right. \\ &\quad \left. w \left[\left(\alpha^2 + \beta^2 \right) \left(\Re^2(T) + \Im^2(T) \right) - 2\alpha\beta \left(\Re(T)\Im(T) + \Im(T)\Re(T) \right) \right] \right\} \\ &\quad + \frac{1}{2} \max \{ |\alpha + \beta| \|\Re(T) + \Im(T)\|, |\alpha - \beta| (\|\Re(T) - \Im(T)\|) \} \\ &\quad \times |w(\alpha\Re(T) + \beta\Im(T)) - w(\beta\Re(T) + \alpha\Im(T))|, \end{aligned}$$

for $T \in \mathcal{L}(H)$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$.

If we take $\alpha = 0$ and $\beta = 1$ in this inequality, then we obtain

$$\begin{aligned} w^2(T) &\geq \frac{1}{2} \|\Re^2(T) + \Im^2(T)\| \\ &\quad + \frac{1}{2} \max \{ \|\Re(T) + \Im(T)\|, \|\Re(T) - \Im(T)\| \} \times \|\Im(T)\| - \|\Re(T)\|, \end{aligned}$$

for $T \in \mathcal{L}(H)$.

Since $\Re^2(T) + \Im^2(T) = \frac{|T|^2 + |T^*|^2}{2}$, hence

$$\begin{aligned} w^2(T) &\geq \frac{1}{4} \left\| \frac{|T|^2 + |T^*|^2}{2} \right\| \\ &\quad + \frac{1}{2} \max \{ \|\Re(T) + \Im(T)\|, \|\Re(T) - \Im(T)\| \} \times \|\Im(T)\| - \|\Re(T)\|, \end{aligned}$$

which is better than the first inequality of Kittaneh, (2).

Further, if we take $A = \Re(T)$ and $B = \Im(T)$ in (14), then we obtain

$$\begin{aligned} w^2(T) &\geq |\alpha\beta| \|\Re(T)\Im(T) + \Im(T)\Re(T)\| \\ &\quad + \frac{1}{2} \max \{ w[\alpha(1+i)\Re(T) + \beta(1-i)\Im(T)], \\ &\quad w[\alpha(1-i)\Re(T) + \beta(1+i)\Im(T)] \} \\ &\quad \times |w(\alpha\Re(T) + \beta\Im(T)) - w(\alpha\Re(T) - \beta\Im(T))|, \end{aligned}$$

for $T \in \mathcal{L}(H)$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$. For $\alpha = \beta = \frac{\sqrt{2}}{2}$, we obtain

$$\begin{aligned} w^2(T) &\geq \frac{1}{2} \|\Re(T)\Im(T) + \Im(T)\Re(T)\| \\ &\quad + \frac{1}{4} \max \{ w[(1+i)A + (1-i)B], w[(1-i)A + (1+i)B] \} \\ &\quad \times \|\Re(T) + \Im(T)\| - \|\Re(T) - \Im(T)\|, \end{aligned}$$

for $T \in \mathcal{L}(H)$.

We can also provide some lower bounds for the Davis–Wielandt radius.

Let $T \in \mathcal{L}(H)$. If we take $A = T$ and $B = |T|^2$ in (11), then we obtain

$$dw^2(T) \geq \max \left\{ w \left[\left(\alpha T + \beta |T|^2 \right)^2 \right], w \left[\left(\alpha T - \beta |T|^2 \right)^2 \right] \right\},$$

for $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$. For $\alpha = \beta = \frac{\sqrt{2}}{2}$, we obtain

$$dw^2(T) \geq \frac{1}{2} \max \left\{ w \left[\left(T + |T|^2 \right)^2 \right], w \left[\left(T - |T|^2 \right)^2 \right] \right\}.$$

From (13) for $A = T$ and $B = |T|^2$, we obtain

$$\begin{aligned} dw^2(T) &\geq \frac{1}{2} \max \left\{ w \left[\left(\alpha^2 + \beta^2 \right) \left(T^2 + |T|^4 \right) + 2\alpha\beta \left(T|T|^2 + |T|^2 T \right) \right], \right. \\ &\quad \left. w \left[\left(\alpha^2 + \beta^2 \right) \left(T^2 + |T|^4 \right) - 2\alpha\beta \left(T|T|^2 + |T|^2 T \right) \right] \right\} \\ &\quad + \frac{1}{2} \max \left\{ |\alpha + \beta| w \left(T + |T|^2 \right), |\alpha - \beta| w \left(T - |T|^2 \right) \right\} \\ &\quad \times \left| w \left(\alpha T + \beta |T|^2 \right) - w \left(\beta T + \alpha |T|^2 \right) \right|, \end{aligned}$$

for $T \in \mathcal{L}(H)$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$.

Also, from (14), we obtain for the same choice of A, B that

$$\begin{aligned} dw^2(T) &\geq |\alpha\beta| w \left(T|T|^2 + |T|^2 T \right) \\ &\quad + \frac{1}{2} \max \left\{ w \left[\alpha(1+i)T + \beta(1-i)|T|^2 \right], w \left[\alpha(1-i)T + \beta(1+i)|T|^2 \right] \right\} \\ &\quad \times \left| w \left(\alpha T + \beta |T|^2 \right) - w \left(\alpha T - \beta |T|^2 \right) \right|, \end{aligned}$$

for $T \in \mathcal{L}(H)$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$. For $\alpha = \beta = \frac{\sqrt{2}}{2}$, we obtain

$$\begin{aligned} dw^2(T) &\geq \frac{1}{2} w \left(T|T|^2 + |T|^2 T \right) \\ &\quad + \frac{1}{4} \max \left\{ w \left[(1+i)T + (1-i)|T|^2 \right], w \left[(1-i)T + (1+i)|T|^2 \right] \right\} \\ &\quad \times \left| w \left(T + |T|^2 \right) - w \left(T - |T|^2 \right) \right|. \end{aligned}$$

Finally, by (15), we obtain

$$\begin{aligned} dw^2(T) &\geq \frac{1}{2} \max \left\{ w \left[\left(\beta^2 - \alpha^2 \right) \left(|T|^4 - T^2 \right) + 2\alpha\beta \left(T|T|^2 + |T|^2 T \right) \right], \right. \\ &\quad \left. w \left[\left(\beta^2 - \alpha^2 \right) \left(|T|^4 - T^2 \right) - 2\alpha\beta \left(T|T|^2 + |T|^2 T \right) \right] \right\} \\ &\quad + \frac{1}{2} \max \left\{ w \left[(\alpha + i\beta)A + (\beta - i\alpha)|T|^2 \right], w \left[(\alpha - i\beta)A + (\beta + i\alpha)|T|^2 \right] \right\} \\ &\quad \times \left| w \left(\alpha T + \beta |T|^2 \right) - w \left(\beta T - \alpha |T|^2 \right) \right|, \end{aligned}$$

for $T \in \mathcal{L}(H)$ and $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$. For $\alpha = \frac{\sqrt{2}}{2}i$ and $\beta = \frac{\sqrt{2}}{2}$, we obtain

$$dw^2(T) \geq \frac{1}{2} \max \left\{ w \left[\left(|T|^2 + iT \right)^2 \right], w \left[\left(|T|^2 - iT \right)^2 \right] \right\},$$

for $T \in \mathcal{L}(H)$.

From a different view point, we can also state the following result:

Theorem 2. For $A, B \in \mathcal{L}(H)$ and $\alpha, \beta, \gamma, \delta, \zeta, \eta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2 = |\zeta|^2 + |\eta|^2 = 1$, we have

$$w_e^2(A, B) \geq \frac{1}{2} w^2((\alpha\zeta + \gamma\eta)A + (\beta\eta + \delta\zeta)B), \quad (17)$$

and

$$w_e^2(A, B) \geq \frac{1}{4} w^2(((\alpha + \delta)\zeta + (\gamma + \beta)\eta)A + ((\alpha - \delta)\zeta + (\gamma - \beta)\eta)B). \quad (18)$$

Proof. For $|\alpha|^2 + |\beta|^2 = 1$, we have

$$|\langle Ax, x \rangle|^2 + |\langle Bx, x \rangle|^2 \geq |\langle (\alpha A + \beta B)x, x \rangle|^2,$$

and for $|\gamma|^2 + |\delta|^2 = 1$, we have

$$|\langle Ax, x \rangle|^2 + |\langle Bx, x \rangle|^2 \geq |\langle (\gamma A + \delta B)x, x \rangle|^2.$$

If ζ and η with $|\zeta|^2 + |\eta|^2 = 1$, then by replacing A with ζA and B with ηB , we obtain from the first inequality above that

$$|\zeta|^2 |\langle Ax, x \rangle|^2 + |\eta|^2 |\langle Bx, x \rangle|^2 \geq |\langle (\alpha \zeta A + \beta \eta B)x, x \rangle|^2,$$

while by replacing A with ηA and B with ζB in the second inequality above,

$$|\eta|^2 |\langle Ax, x \rangle|^2 + |\zeta|^2 |\langle Bx, x \rangle|^2 \geq |\langle (\gamma \eta A + \delta \zeta B)x, x \rangle|^2.$$

If we add these two inequalities, we obtain

$$|\langle Ax, x \rangle|^2 + |\langle Bx, x \rangle|^2 \geq |\langle (\alpha \zeta A + \beta \eta B)x, x \rangle|^2 + |\langle (\gamma \eta A + \delta \zeta B)x, x \rangle|^2, \quad (19)$$

for $x \in H$.

By using the Cauchy–Schwarz inequality, we have

$$\begin{aligned} & |\langle (\alpha \zeta A + \beta \eta B)x, x \rangle|^2 + |\langle (\gamma \eta A + \delta \zeta B)x, x \rangle|^2 \\ & \geq \frac{1}{2} (|\langle (\alpha \zeta A + \beta \eta B)x, x \rangle| + |\langle (\gamma \eta A + \delta \zeta B)x, x \rangle|)^2 \\ & \geq \frac{1}{2} |\langle (\alpha \zeta A + \beta \eta B + \gamma \eta A + \delta \zeta B)x, x \rangle|^2 \\ & = \frac{1}{2} |\langle ((\alpha \zeta + \gamma \eta)A + (\beta \eta + \delta \zeta)B)x, x \rangle|^2, \end{aligned} \quad (20)$$

for $x \in H$.

By (19) and (20), we obtain

$$|\langle Ax, x \rangle|^2 + |\langle Bx, x \rangle|^2 \geq \frac{1}{2} |\langle ((\alpha \zeta + \gamma \eta)A + (\beta \eta + \delta \zeta)B)x, x \rangle|^2,$$

for $x \in H$.

By taking the supremum in this inequality, we deduce the desired result (17).

If we write (17) for $A + B$ instead of A and $A - B$ instead of B , then we obtain

$$\begin{aligned} w_e^2(A + B, A - B) & \geq \frac{1}{2} w^2((\alpha \zeta + \gamma \eta)(A + B) + (\beta \eta + \delta \zeta)(A - B)) \\ & = \frac{1}{2} w^2((\alpha \zeta + \gamma \eta + \beta \eta + \delta \zeta)A + (\alpha \zeta + \gamma \eta - \beta \eta - \delta \zeta)B) \\ & = \frac{1}{2} w^2(((\alpha + \delta)\zeta + (\gamma + \beta)\eta)A + ((\alpha - \delta)\zeta + (\gamma - \beta)\eta)B), \end{aligned}$$

and since

$$w_e^2(A + B, A - B) = 2w_e^2(A, B),$$

we obtain the desired result (18). $\square$

Remark 3. For $\gamma = \alpha$, $\delta = \beta$ in (17), we obtain

$$w_e^2(A, B) \geq \frac{1}{2}|\eta + \zeta|w^2(\alpha A + \beta B),$$

for $A, B \in \mathcal{L}(H)$ and $\alpha, \beta, \zeta, \eta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = |\zeta|^2 + |\eta|^2 = 1$.

Also, by taking $\zeta = \alpha$ and $\eta = \beta$ in (17), we obtain

$$w_e^2(A, B) \geq \frac{1}{2}w^2\left((\alpha^2 + \gamma\eta)A + (\beta^2 + \delta\zeta)B\right),$$

for $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2 = 1$.

By taking $\zeta = \gamma$ and $\eta = \delta$, we also have

$$w_e^2(A, B) \geq \frac{1}{2}w^2(\gamma(\alpha + \delta)A + \delta(\beta + \gamma)B),$$

for $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2 = 1$.

To derive a significant consequence from Theorem 2, we first recall the following lemma, as found in [23]:

Lemma 1. Let $C, D \in \mathcal{L}(H)$. Then,

$$w\left(\begin{bmatrix} 0 & C \\ D^* & 0 \end{bmatrix}\right) = \frac{1}{2} \sup_{\theta \in \mathbb{R}} \|e^{i\theta}C + e^{-i\theta}D\|.$$

Now, we can state the following result:

Corollary 1. For $A, B \in \mathcal{L}(H)$ and $\alpha, \beta, \gamma, \delta, \zeta, \eta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2 = |\zeta|^2 + |\eta|^2 = 1$, then

$$w_e(A, B) \geq \frac{\sqrt{2}}{2}w\left(\begin{bmatrix} 0 & (\alpha\zeta + \gamma\eta)A \\ (\bar{\beta}\eta + \bar{\delta}\zeta)B^* & 0 \end{bmatrix}\right), \quad (21)$$

and

$$w_e(A, B) \geq \frac{\sqrt{2}}{4}w\left(\begin{bmatrix} 0 & ((\alpha + \delta)\zeta + (\gamma + \beta)\eta)A \\ (((\bar{\alpha} - \bar{\delta})\bar{\zeta} + (\bar{\gamma} - \bar{\beta})\bar{\eta}))B^* & 0 \end{bmatrix}\right). \quad (22)$$

Proof. If we write the inequality (17) for $e^{i\theta}A$ instead of A and $e^{-i\theta}B$ instead of B , then we obtain for $\theta \in \mathbb{R}$

$$\begin{aligned} w_e^2(A, B) &= w_e^2(e^{i\theta}A, e^{-i\theta}B) \\ &\geq \frac{1}{2}w^2((\alpha\zeta + \gamma\eta)e^{i\theta}A + (\beta\eta + \delta\zeta)e^{-i\theta}B) \\ &\geq \frac{1}{8}\|(\alpha\zeta + \gamma\eta)e^{i\theta}A + (\beta\eta + \delta\zeta)e^{-i\theta}B\|^2, \end{aligned}$$

for $A, B \in \mathcal{L}(H)$ and $\alpha, \beta, \gamma, \delta, \zeta, \eta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2 = |\zeta|^2 + |\eta|^2 = 1$.

If we take the supremum over $\theta \in \mathbb{R}$, we obtain by Lemma 1 that

$$\begin{aligned} w_e^2(A, B) &\geq \frac{1}{8} \sup_{\theta \in \mathbb{R}} \left\| e^{i\theta} (\alpha\zeta + \gamma\eta)A + e^{-i\theta} (\beta\eta + \delta\zeta)B \right\|^2 \\ &= \frac{1}{2} w^2 \left(\begin{bmatrix} 0 & (\alpha\zeta + \gamma\eta)A \\ (\bar{\beta}\bar{\eta} + \bar{\delta}\bar{\zeta})B^* & 0 \end{bmatrix} \right), \end{aligned}$$

which proves (21).

The inequality (22) follows in the same way from (18). $\square$

Remark 4. For $\gamma = \alpha, \delta = \beta$ in (21), we obtain

$$w_e(A, B) \geq \frac{\sqrt{2}}{2} w \left(\begin{bmatrix} 0 & \alpha(\zeta + \eta)A \\ \bar{\beta}(\bar{\eta} + \bar{\zeta})B^* & 0 \end{bmatrix} \right), \quad (23)$$

for $A, B \in \mathcal{L}(H)$ and $\alpha, \beta, \zeta, \eta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = |\zeta|^2 + |\eta|^2 = 1$.

For $\zeta, \eta \in \mathbb{R}$ with $\zeta^2 + \eta^2 = 1$, we obtain from (23) that

$$w_e(A, B) \geq \frac{\sqrt{2}}{2} |\zeta + \eta| w \left(\begin{bmatrix} 0 & \alpha A \\ \bar{\beta} B^* & 0 \end{bmatrix} \right), \quad (24)$$

for $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$.

Since

$$\sup_{\zeta^2 + \eta^2 = 1} |\zeta + \eta| = \sqrt{2},$$

then from (24), we derive

$$w_e(A, B) \geq w \left(\begin{bmatrix} 0 & \alpha A \\ \sigma B^* & 0 \end{bmatrix} \right), \quad (25)$$

for $\alpha, \sigma \in \mathbb{C}$ with $|\alpha|^2 + |\sigma|^2 = 1$.

In the case of $\alpha, \sigma \geq 0$ with $\alpha^2 + \sigma^2 = 1$, the inequality (25) was obtained in Corollary 2.3 of [24].

Now, if we take $\sigma = \beta = \frac{\sqrt{2}}{2}$ in (25), then we obtain

$$w_e(A, B) \geq \frac{\sqrt{2}}{2} w \left(\begin{bmatrix} 0 & A \\ B^* & 0 \end{bmatrix} \right).$$

If we take $A = T$ and $B = T^*$ in Theorem 2, then we obtain

$$w^2(T) \geq \frac{1}{4} w^2((\alpha\zeta + \gamma\eta)T + (\beta\eta + \delta\zeta)T^*),$$

and

$$w^2(T) \geq \frac{1}{8} w^2(((\alpha + \delta)\zeta + (\gamma + \beta)\eta)T + ((\alpha - \delta)\zeta + (\gamma - \beta)\eta)T^*),$$

for $T \in \mathcal{L}(H)$ and $\alpha, \beta, \gamma, \delta, \zeta, \eta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2 = |\zeta|^2 + |\eta|^2 = 1$.

Further, if we take $A = \Re(T)$ and $B = \Im(T)$ in Theorem 2, then we obtain

$$w^2(T) \geq \frac{1}{2} w^2((\alpha\zeta + \gamma\eta)\Re(T) + (\beta\eta + \delta\zeta)\Im(T)),$$

and

$$w^2(T) \geq \frac{1}{4} w^2(((\alpha + \delta)\zeta + (\gamma + \beta)\eta)\Re(T) + ((\alpha - \delta)\zeta + (\gamma - \beta)\eta)\Im(T)),$$

for $T \in \mathcal{L}(H)$ and $\alpha, \beta, \gamma, \delta, \zeta, \eta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2 = |\zeta|^2 + |\eta|^2 = 1$.
 Finally, if we take $A = T$ and $B = |T|^2$ in Theorem 2, then we obtain

$$dw^2(T) \geq \frac{1}{2}w^2\left((\alpha\zeta + \gamma\eta)T + (\beta\eta + \delta\zeta)|T|^2\right),$$

and

$$dw^2(T) \geq \frac{1}{4}w^2\left(((\alpha + \delta)\zeta + (\gamma + \beta)\eta)T + ((\alpha - \delta)\zeta + (\gamma - \beta)\eta)|T|^2\right),$$

for $T \in \mathcal{L}(H)$ and $\alpha, \beta, \gamma, \delta, \zeta, \eta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = |\gamma|^2 + |\delta|^2 = |\zeta|^2 + |\eta|^2 = 1$.

In order to derive our next main theorem, we need to recall the following generalization of Kato's Schwarz inequality for operators obtained by F. Kittaneh in 1988 (see [25]).

Lemma 2. *If f and g are nonnegative continuous functions on $[0, \infty)$ satisfying the condition $f(t)g(t) = t$ for all $t \in [0, \infty)$, then*

$$|\langle Tx, y \rangle| \leq \|f(|T|)x\| \|g(|T^*|)y\|, \quad (26)$$

for all $T \in \mathcal{L}(H)$ and $x, y \in H$.

We observe that (26) is equivalent to

$$|\langle Tx, y \rangle|^2 \leq \langle f^2(|T|)x, x \rangle \langle g^2(|T^*|)y, y \rangle,$$

for all $T \in \mathcal{L}(H)$ and $x, y \in H$. Kato's inequality is obtained for $f(t) = t^v$, $g(t) = t^{1-v}$, $v \in [0, 1]$, i.e.,

$$|\langle Tx, y \rangle|^2 \leq \langle |T|^{2v}x, x \rangle \langle |T^*|^{2(1-v)}y, y \rangle,$$

for all $T \in \mathcal{L}(H)$ and $x, y \in H$.

We can state the following result providing upper bounds for the Euclidean numerical radius:

Theorem 3. *Let f, g, h and k be nonnegative continuous functions on $[0, \infty)$ satisfying the condition $f(t)g(t) = h(t)k(t) = t$ for all $t \in [0, \infty)$. If $A, B \in \mathcal{L}(H)$ and $\alpha, \beta, \gamma, \delta \in \{-1, 1\}$, then*

$$w_e^2(A, B) \leq w\left[\alpha f^2(|A|) + i\beta h^2(|B|)\right] w\left[\gamma g^2(|A^*|) + i\delta k^2(|B^*|)\right], \quad (27)$$

and

$$w_e^2(A, B) \leq w\left[\alpha f^2(|A|) + i\delta k^2(|B^*|)\right] w\left[\gamma g^2(|A^*|) + i\beta h^2(|B|)\right]. \quad (28)$$

Proof. We have

$$\begin{aligned} & |\langle Ax, x \rangle|^2 + |\langle Bx, x \rangle|^2 \\ & \leq \langle f^2(|A|)x, x \rangle \langle g^2(|A^*|)x, x \rangle + \langle h^2(|B|)x, x \rangle \langle k^2(|B^*|)x, x \rangle \\ & \leq \left(\langle f^2(|A|)x, x \rangle^2 + \langle h^2(|B|)x, x \rangle^2\right)^{1/2} \times \left(\langle g^2(|A^*|)x, x \rangle^2 + \langle k^2(|B^*|)x, x \rangle^2\right)^{1/2}, \end{aligned} \quad (29)$$

for $x \in H$.

Observe that for $\alpha, \beta, \gamma, \delta \in \{-1, 1\}$, we have

$$\begin{aligned} \langle f^2(|A|)x, x \rangle^2 + \langle h^2(|B|)x, x \rangle^2 & = \left|\alpha \langle f^2(|A|)x, x \rangle + \beta \langle h^2(|B|)x, x \rangle i\right|^2 \\ & = \left|\langle [\alpha f^2(|A|) + i\beta h^2(|B|)]x, x \rangle\right|^2, \end{aligned}$$

and

$$\langle g^2(|A^*|)x, x \rangle^2 + \langle k^2(|B^*|)x, x \rangle^2 = \left| \langle [\gamma g^2(|A^*|) + i\delta k^2(|B^*|)]x, x \rangle \right|^2,$$

for $x \in H$.

By utilizing (29), we then obtain

$$|\langle Ax, x \rangle|^2 + |\langle Bx, x \rangle|^2 \leq \left| \langle [\alpha f^2(|A|) + i\beta h^2(|B|)]x, x \rangle \right|^2 \left| \langle [\gamma g^2(|A^*|) + i\delta k^2(|B^*|)]x, x \rangle \right|^2,$$

for $x \in H$.

Therefore,

$$\begin{aligned} w_e^2(A, B) &= \sup_{\|x\|=1} \left(|\langle Ax, x \rangle|^2 + |\langle Bx, x \rangle|^2 \right) \\ &\leq \sup_{\|x\|=1} \left\{ \left| \langle [\alpha f^2(|A|) + i\beta h^2(|B|)]x, x \rangle \right|^2 \times \left| \langle [\gamma g^2(|A^*|) + i\delta k^2(|B^*|)]x, x \rangle \right|^2 \right\} \\ &\leq \sup_{\|x\|=1} \left| \langle [\alpha f^2(|A|) + i\beta h^2(|B|)]x, x \rangle \right|^2 \times \sup_{\|x\|=1} \left| \langle [\gamma g^2(|A^*|) + i\delta k^2(|B^*|)]x, x \rangle \right|^2 \\ &= w[\alpha f^2(|A|) + i\beta h^2(|B|)] w[\gamma g^2(|A^*|) + i\delta k^2(|B^*|)], \end{aligned}$$

which proves (27).

We also have

$$\begin{aligned} &\langle f^2(|A|)x, x \rangle \langle g^2(|A^*|)x, x \rangle + \langle h^2(|B|)x, x \rangle \langle k^2(|B^*|)x, x \rangle \\ &\leq \left(\langle f^2(|A|)x, x \rangle^2 + \langle k^2(|B^*|)x, x \rangle^2 \right)^{1/2} \\ &\quad \times \left(\langle g^2(|A^*|)x, x \rangle^2 + \langle h^2(|B|)x, x \rangle^2 \right)^{1/2}, \end{aligned}$$

for $x \in H$, which implies in the same way (28). $\square$

Corollary 2. If $A, B \in \mathcal{L}(H)$ and $\alpha, \beta, \gamma, \delta \in \{-1, 1\}$, then for $v, \lambda \in [0, 1]$, we have

$$w_e^2(A, B) \leq w[\alpha|A|^{2v} + i\beta|B|^{2\lambda}] w[\gamma|A^*|^{2(1-v)} + i\delta|B^*|^{2(1-\lambda)}],$$

and

$$w_e^2(A, B) \leq w[\alpha|A|^{2v} + i\delta|B^*|^{2(1-\lambda)}] w[\gamma|A^*|^{2(1-v)} + i\beta|B|^{2\lambda}].$$

The proof follows from Theorem 3 by selecting $f(t) = t^v$, $g(t) = t^{1-v}$, $h(t) = t^\lambda$ and $k(t) = t^{1-\lambda}$ with $v, \lambda \in [0, 1]$.

Remark 5. If we take $v = \lambda = 1/2$ in Corollary 2, then we obtain

$$w_e^2(A, B) \leq w(\alpha|A| + i\beta|B|) w(\gamma|A^*| + i\delta|B^*|),$$

for $A, B \in \mathcal{L}(H)$ and $\alpha, \beta, \gamma, \delta \in \{-1, 1\}$.

If we choose $B = A^*$ in Theorem 3, then we obtain

$$w^2(A) \leq \frac{1}{2} w[\alpha f^2(|A|) + i\beta h^2(|A^*|)] w[\gamma g^2(|A^*|) + i\delta k^2(|A|)],$$

and

$$w^2(A) \leq \frac{1}{2} w[\alpha f^2(|A|) + i\delta k^2(|A|)] w[\gamma g^2(|A^*|) + i\beta h^2(|A^*|)],$$

where f, g, h and k are nonnegative continuous functions on $[0, \infty)$ satisfying the condition $f(t)g(t) = h(t)k(t) = t$ for all $t \in [0, \infty)$, $A, B \in \mathcal{L}(H)$ and $\alpha, \beta, \gamma, \delta \in \{-1, 1\}$.

In particular, we have

$$w^2(A) \leq \frac{1}{2}w\left(\alpha|A|^{2v} + i\beta|A^*|^{2\lambda}\right)w\left(\gamma|A^*|^{2(1-\nu)} + i\delta|A|^{2(1-\lambda)}\right),$$

and

$$w^2(A) \leq \frac{1}{2}w\left(\alpha|A|^{2v} + i\delta|A|^{2(1-\lambda)}\right)w\left(\gamma|A^*|^{2(1-\nu)} + i\beta|A^*|^{2\lambda}\right),$$

for $v, \lambda \in [0, 1]$. Also, we have

$$w^2(A) \leq \frac{1}{2}w(\alpha|A| + i\beta|A^*|)w(\gamma|A^*| + i\delta|A|).$$

Further, if we take $A = \Re(T)$ and $B = \Im(T)$ in Theorem 3, then we obtain

$$w^2(T) \leq w\left[\alpha f^2(|\Re(T)|) + i\beta h^2(|\Im(T)|)\right]w\left[\gamma g^2(|\Re(T)|) + i\delta k^2(|\Im(T)|)\right],$$

where f, g, h and k are nonnegative continuous functions on $[0, \infty)$ satisfying the condition $f(t)g(t) = h(t)k(t) = t$ for all $t \in [0, \infty)$, $T \in \mathcal{L}(H)$ and $\alpha, \beta, \gamma, \delta \in \{-1, 1\}$.

For $v, \lambda \in [0, 1]$, we have

$$w^2(T) \leq w\left(\alpha|\Re(T)|^{2v} + i\beta|\Im(T)|^{2\lambda}\right)w\left(\gamma|\Re(T)|^{2(1-\nu)} + i\delta|\Im(T)|^{2(1-\lambda)}\right),$$

and, in particular,

$$w^2(T) \leq w(\alpha|\Re(T)| + i\beta|\Im(T)|)w(\gamma|\Re(T)| + i\delta|\Im(T)|),$$

where $\alpha, \beta, \gamma, \delta \in \{-1, 1\}$.

Moreover, if we take $A = T$ and $B = |T|^2$ in Theorem 3, then we obtain

$$dw^2(T) \leq w\left[\alpha f^2(|T|) + i\beta h^2(|T|^2)\right]w\left[\gamma g^2(|T^*|) + i\delta k^2(|T|^2)\right],$$

for $T \in \mathcal{L}(H)$ and $\alpha, \beta, \gamma, \delta \in \{-1, 1\}$.

For $v, \lambda \in [0, 1]$, we have

$$dw^2(T) \leq w\left(\alpha|T|^{2v} + i\beta|T|^{4\lambda}\right)w\left(\gamma|T^*|^{2(1-\nu)} + i\delta|T|^{4(1-\lambda)}\right),$$

and, in particular,

$$dw^2(T) \leq w(\alpha|T| + i\beta|T|^2)w(\gamma|T^*| + i\delta|T|^2),$$

for $T \in \mathcal{L}(H)$ and $\alpha, \beta, \gamma, \delta \in \{-1, 1\}$.

We have the following Hölder's type upper bounds as well:

Theorem 4. Let f, g, h and k be nonnegative continuous functions on $[0, \infty)$ satisfying the condition $f(t)g(t) = h(t)k(t) = t$ for all $t \in [0, \infty)$. If $A, B \in \mathcal{L}(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$w_e^2(A, B) \leq \left\|f^{2p}(|A|) + h^{2p}(|B|)\right\|^{1/p} \left\|g^{2q}(|A^*|) + k^{2q}(|B^*|)\right\|^{1/q}, \quad (30)$$

and

$$w_e^2(A, B) \leq \left\|f^{2p}(|A|) + k^{2p}(|B^*|)\right\|^{1/p} \left\|g^{2q}(|A^*|) + h^{2q}(|B|)\right\|^{1/q}. \quad (31)$$

In particular, we have

$$w_e^2(A, B) \leq \left\| f^4(|A|) + h^4(|B|) \right\|^{1/2} \left\| g^4(|A^*|) + k^4(|B^*|) \right\|^{1/2},$$

and

$$w_e^2(A, B) \leq \left\| f^4(|A|) + k^4(|B^*|) \right\|^{1/2} \left\| g^4(|A^*|) + h^4(|B|) \right\|^{1/2}.$$

Proof. By Hölder's inequality for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned} & \langle f^2(|A|)x, x \rangle \langle g^2(|A^*|)x, x \rangle + \langle h^2(|B|)x, x \rangle \langle k^2(|B^*|)x, x \rangle \\ & \leq \left(\langle f^2(|A|)x, x \rangle^p + \langle h^2(|B|)x, x \rangle^p \right)^{1/p} \times \left(\langle g^2(|A^*|)x, x \rangle^q + \langle k^2(|B^*|)x, x \rangle^q \right)^{1/q}, \end{aligned} \quad (32)$$

for $x \in H$.

By using McCharthy's inequality for a positive operator P and power $s \geq 1$,

$$\langle Px, x \rangle^s \leq \langle P^s x, x \rangle, \quad \|x\| = 1,$$

we have

$$\begin{aligned} \langle f^2(|A|)x, x \rangle^p + \langle h^2(|B|)x, x \rangle^p & \leq \langle f^{2p}(|A|)x, x \rangle + \langle h^{2p}(|B|)x, x \rangle \\ & = \langle (f^{2p}(|A|) + h^{2p}(|B|))x, x \rangle, \end{aligned}$$

and

$$\langle g^2(|A^*|)x, x \rangle^q + \langle k^2(|B^*|)x, x \rangle^q \leq \langle (g^{2q}(|A^*|) + k^{2q}(|B^*|))x, x \rangle,$$

for $\|x\| = 1$.

By making use of (29) and (32), we derive

$$|\langle Ax, x \rangle|^2 + |\langle Bx, x \rangle|^2 \leq \langle (f^{2p}(|A|) + h^{2p}(|B|))x, x \rangle^{1/p} \langle (g^{2q}(|A^*|) + k^{2q}(|B^*|))x, x \rangle^{1/q},$$

for $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$, we obtain the desired result (30).

We also have

$$\begin{aligned} & \langle f^2(|A|)x, x \rangle \langle g^2(|A^*|)x, x \rangle + \langle h^2(|B|)x, x \rangle \langle k^2(|B^*|)x, x \rangle \\ & \leq \left(\langle f^2(|A|)x, x \rangle^p + \langle k^2(|B^*|)x, x \rangle^p \right)^{1/p} \times \left(\langle g^2(|A^*|)x, x \rangle^q + \langle h^2(|B|)x, x \rangle^q \right)^{1/q}, \end{aligned}$$

for $x \in H$, which implies, as above, the inequality (31). $\square$

Corollary 3. If $A, B \in \mathcal{L}(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then for $v, \lambda \in [0, 1]$, we have

$$w_e^2(A, B) \leq \left\| |A|^{2pv} + |B|^{2p(1-v)} \right\|^{1/p} \left\| |A^*|^{2q\lambda} + |B^*|^{2q(1-\lambda)} \right\|^{1/q},$$

and

$$w_e^2(A, B) \leq \left\| |A|^{2pv} + |B^*|^{2p(1-\lambda)} \right\|^{1/p} \left\| |A^*|^{2q(1-v)} + |B|^{2q\lambda} \right\|^{1/q}.$$

In particular, we have

$$w_e^2(A, B) \leq \left\| |A|^{4v} + |B|^{4(1-v)} \right\|^{1/2} \left\| |A^*|^{4\lambda} + |B^*|^{4(1-\lambda)} \right\|^{1/2},$$

and

$$w_e^2(A, B) \leq \left\| |A|^{4v} + |B^*|^{4(1-\lambda)} \right\|^{1/2} \left\| |A^*|^{4(1-\nu)} + |B|^{4\lambda} \right\|^{1/2}.$$

The proof follows from Theorem 4 by selecting $f(t) = t^v$, $g(t) = t^{1-\nu}$, $h(t) = t^\lambda$ and $k(t) = t^{1-\lambda}$ with $v, \lambda \in [0, 1]$.

Remark 6. If we take $\nu = \lambda = 1/2$ in Corollary 3, then we obtain

$$w_e^2(A, B) \leq \left\| |A|^p + |B|^p \right\|^{1/p} \left\| |A^*|^q + |B^*|^q \right\|^{1/q},$$

for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and, in particular,

$$w_e^2(A, B) \leq \left\| |A|^2 + |B|^2 \right\|^{1/2} \left\| |A^*|^2 + |B^*|^2 \right\|^{1/2},$$

for $A, B \in \mathcal{L}(H)$.

If we take $B = A^*$ in Theorem 4, then we obtain for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ that

$$w^2(A) \leq \frac{1}{2} \left\| f^{2p}(|A|) + h^{2p}(|A^*|) \right\|^{1/p} \left\| g^{2q}(|A^*|) + k^{2q}(|A|) \right\|^{1/q},$$

and

$$w^2(A) \leq \frac{1}{2} \left\| f^{2p}(|A|) + k^{2p}(|A|) \right\|^{1/p} \left\| g^{2q}(|A^*|) + h^{2q}(|A^*|) \right\|^{1/q}.$$

In particular, we have

$$w^2(A) \leq \frac{1}{2} \left\| f^4(|A|) + h^4(|A^*|) \right\|^{1/2} \left\| g^4(|A^*|) + k^4(|A|) \right\|^{1/2},$$

and

$$w^2(A) \leq \frac{1}{2} \left\| f^4(|A|) + k^4(|A|) \right\|^{1/2} \left\| g^4(|A^*|) + h^4(|A^*|) \right\|^{1/2}.$$

For $v, \lambda \in [0, 1]$, we have

$$w^2(A) \leq \frac{1}{2} \left\| |A|^{2pv} + |A^*|^{2p(1-\nu)} \right\|^{1/p} \left\| |A^*|^{2q\lambda} + |A|^{2q(1-\lambda)} \right\|^{1/q},$$

and

$$w^2(A) \leq \frac{1}{2} \left\| |A|^{2pv} + |A|^{2p(1-\lambda)} \right\|^{1/p} \left\| |A^*|^{2q(1-\nu)} + |A^*|^{2q\lambda} \right\|^{1/q}.$$

Also, for $\nu = \lambda = 1/2$, we have

$$w^2(A) \leq \frac{1}{2} \left\| |A|^p + |A^*|^p \right\|^{1/p} \left\| |A^*|^q + |A|^q \right\|^{1/q}.$$

In particular, we have

$$w^2(A) \leq \frac{1}{2} \left\| |A|^{4v} + |A^*|^{4(1-\nu)} \right\|^{1/2} \left\| |A^*|^{4\lambda} + |A|^{4(1-\lambda)} \right\|^{1/2},$$

and

$$w^2(A) \leq \frac{1}{2} \left\| |A|^{4v} + |A|^{4(1-\lambda)} \right\|^{1/2} \left\| |A^*|^{4(1-\nu)} + |A^*|^{4\lambda} \right\|^{1/2}.$$

For $\nu = \lambda = 1/2$, we obtain the known result; see the second part of inequality (2)

$$w^2(A) \leq \frac{1}{2} \left\| |A|^2 + |A^*|^2 \right\|.$$

Further, if we take $A = \Re(T)$ and $B = \Im(T)$ in Theorem 4, then we obtain for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$w^2(T) \leq \left\| f^{2p}(|\Re(T)|) + h^{2p}(|\Im(T)|) \right\|^{1/p} \left\| g^{2q}(|\Re(T)|) + k^{2q}(|\Im(T)|) \right\|^{1/q},$$

and

$$w^2(T) \leq \left\| f^{2p}(|\Re(T)|) + k^{2p}(|\Im(T)|) \right\|^{1/p} \left\| g^{2q}(|\Re(T)|) + h^{2q}(|\Im(T)|) \right\|^{1/q}.$$

In particular, we have

$$w^2(T) \leq \left\| f^4(|\Re(T)|) + h^4(|\Im(T)|) \right\|^{1/2} \left\| g^4(|\Re(T)|) + k^4(|\Im(T)|) \right\|^{1/2},$$

and

$$w^2(T) \leq \left\| f^4(|\Re(T)|) + k^4(|\Im(T)|) \right\|^{1/2} \left\| g^4(|\Re(T)|) + h^4(|\Im(T)|) \right\|^{1/2}.$$

For $v, \lambda \in [0, 1]$, we have

$$w^2(T) \leq \left\| |\Re(T)|^{2pv} + |\Im(T)|^{2p(1-v)} \right\|^{1/p} \left\| |\Re(T)|^{2q\lambda} + |\Im(T)|^{2q(1-\lambda)} \right\|^{1/q},$$

and

$$w^2(T) \leq \left\| |\Re(T)|^{2pv} + |\Im(T)|^{2p(1-\lambda)} \right\|^{1/p} \left\| |\Re(T)|^{2q(1-v)} + |\Im(T)|^{2q\lambda} \right\|^{1/q},$$

for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

For $v = \lambda = 1/2$, we obtain

$$w^2(T) \leq \left\| |\Re(T)|^p + |\Im(T)|^p \right\|^{1/p} \left\| |\Re(T)|^q + |\Im(T)|^q \right\|^{1/q}.$$

In particular, we have

$$w^2(T) \leq \left\| |\Re(T)|^{4v} + |\Im(T)|^{4(1-v)} \right\|^{1/2} \left\| |\Re(T)|^{4\lambda} + |\Im(T)|^{4(1-\lambda)} \right\|^{1/2},$$

and

$$w^2(T) \leq \left\| |\Re(T)|^{4v} + |\Im(T)|^{4(1-\lambda)} \right\|^{1/2} \left\| |\Re(T)|^{4(1-v)} + |\Im(T)|^{4\lambda} \right\|^{1/2}.$$

For $v = \lambda = 1/2$, we obtain

$$w^2(T) \leq \left\| |\Re(T)|^2 + |\Im(T)|^2 \right\| = \frac{1}{2} \left\| |T|^2 + |T^*|^2 \right\|,$$

which is the second part of inequality (2).

Moreover, if we take $A = T$ and $B = |T|^2$ in Theorem 4, then we obtain for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ that

$$dw^2(T) \leq \left\| f^{2p}(|T|) + h^{2p}(|T|^2) \right\|^{1/p} \left\| g^{2q}(|T^*|) + k^{2q}(|T|^2) \right\|^{1/q},$$

and

$$dw^2(T) \leq \left\| f^{2p}(|T|) + k^{2p}(|T|^2) \right\|^{1/p} \left\| g^{2q}(|T^*|) + h^{2q}(|T|^2) \right\|^{1/q}.$$

In particular, we have

$$dw^2(T) \leq \left\| f^4(|T|) + h^4(|T|^2) \right\|^{1/2} \left\| g^4(|T^*|) + k^4(|T|^2) \right\|^{1/2},$$

and

$$dw^2(T) \leq \left\| f^4(|T|) + k^4(|T|^2) \right\|^{1/2} \left\| g^4(|T^*|) + h^4(|T|^2) \right\|^{1/2}.$$

For $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then for $v, \lambda \in [0, 1]$, we have

$$dw^2(T) \leq \left\| |T|^{2pv} + |T|^{4p(1-v)} \right\|^{1/p} \left\| |T^*|^{2q\lambda} + |T|^{4q(1-\lambda)} \right\|^{1/q},$$

and

$$dw^2(T) \leq \left\| |T|^{2pv} + |T|^{4p(1-\lambda)} \right\|^{1/p} \left\| |T^*|^{2q(1-\nu)} + |T|^{4q\lambda} \right\|^{1/q}.$$

In particular, we have

$$dw^2(T) \leq \left\| |T|^{4v} + |T|^{8(1-\nu)} \right\|^{1/2} \left\| |T^*|^{4\lambda} + |T|^{8(1-\lambda)} \right\|^{1/2},$$

and

$$dw^2(T) \leq \left\| |T|^{4v} + |T|^{8(1-\lambda)} \right\|^{1/2} \left\| |T^*|^{4(1-\nu)} + |T|^{8\lambda} \right\|^{1/2}.$$

For $v = \lambda = 1/2$, we obtain

$$dw^2(T) \leq \left\| |T|^2 + |T|^4 \right\|^{1/2} \left\| |T^*|^2 + |T|^4 \right\|^{1/2}. \quad (33)$$

We observe that, if $|T^*|^2 \leq |T|^2$, namely, T is a hyponormal operator, then

$$\left\| |T^*|^2 + |T|^4 \right\| \leq \left\| |T|^2 + |T|^4 \right\|,$$

and by (33), we obtain

$$dw^2(T) \leq \left\| |T|^2 + |T|^4 \right\|^{1/2} \left\| |T^*|^2 + |T|^4 \right\|^{1/2} \leq \left\| |T|^2 + |T|^4 \right\|,$$

which provides a refinement of (3) for hyponormal operators.

Theorem 5. Let f, g, h and k be nonnegative continuous functions on $[0, \infty)$ satisfying the condition $f(t)g(t) = h(t)k(t) = t$ for all $t \in [0, \infty)$ and $A, B \in \mathcal{L}(H)$, then

$$w_e^2(A, B) \leq \frac{1}{2} \left[\left\| f^2(|A|) + h^2(|B|) \right\| + \left\| f^2(|A|) - h^2(|B|) \right\| \right] \times \left\| g^2(|A^*|) + k^2(|B^*|) \right\|, \quad (34)$$

and

$$w_e^2(A, B) \leq \frac{1}{2} \left[\left\| f^2(|A|) + k^2(|B^*|) \right\| + \left\| f^2(|A|) - k^2(|B^*|) \right\| \right] \times \left\| g^2(|A^*|) + h^2(|B|) \right\|. \quad (35)$$

Proof. We have

$$\begin{aligned} & \langle f^2(|A|)x, x \rangle \langle g^2(|A^*|)x, x \rangle + \langle h^2(|B|)x, x \rangle \langle k^2(|B^*|)x, x \rangle \\ & \leq \max \left\{ \langle f^2(|A|)x, x \rangle, \langle h^2(|B|)x, x \rangle \right\} \left(\langle g^2(|A^*|)x, x \rangle + \langle k^2(|B^*|)x, x \rangle \right) \\ & = \max \left\{ \langle f^2(|A|)x, x \rangle, \langle h^2(|B|)x, x \rangle \right\} \langle [g^2(|A^*|) + k^2(|B^*|)]x, x \rangle, \end{aligned}$$

for $x \in H$.

Observe that

$$\begin{aligned} & \max \left\{ \langle f^2(|A|)x, x \rangle, \langle h^2(|B|)x, x \rangle \right\} \\ &= \frac{1}{2} \left(\langle f^2(|A|)x, x \rangle + \langle h^2(|B|)x, x \rangle + \left| \langle f^2(|A|)x, x \rangle - \langle h^2(|B|)x, x \rangle \right| \right) \\ &= \frac{1}{2} \left(\langle (f^2(|A|) + h^2(|B|))x, x \rangle + \left| \langle (f^2(|A|) - h^2(|B|))x, x \rangle \right| \right), \end{aligned}$$

and by (29) and (32), we derive

$$\begin{aligned} |\langle Ax, x \rangle|^2 + |\langle Bx, x \rangle|^2 &\leq \frac{1}{2} \left(\langle (f^2(|A|) + h^2(|B|))x, x \rangle + \left| \langle (f^2(|A|) - h^2(|B|))x, x \rangle \right| \right) \\ &\quad \times \langle [g^2(|A^*|) + k^2(|B^*|)]x, x \rangle, \end{aligned}$$

for $x \in H$.

By taking the supremum over $\|x\| = 1$, we deduce the desired result (34).

Also, we have

$$\begin{aligned} & \langle f^2(|A|)x, x \rangle \langle g^2(|A^*|)x, x \rangle + \langle h^2(|B|)x, x \rangle \langle k^2(|B^*|)x, x \rangle \\ &\leq \max \left\{ \langle f^2(|A|)x, x \rangle, \langle k^2(|B^*|)x, x \rangle \right\} \left(\langle g^2(|A^*|)x, x \rangle + \langle h^2(|B|)x, x \rangle \right), \end{aligned}$$

from which we obtain (35). $\square$

If we take $f(t) = t^v$, $g(t) = t^{1-v}$, $h(t) = t^\lambda$ and $k(t) = t^{1-\lambda}$ with $v, \lambda \in [0, 1]$, then we obtain

Corollary 4. If $A, B \in \mathcal{L}(H)$ and $v, \lambda \in [0, 1]$, then

$$w_e^2(A, B) \leq \frac{1}{2} \left[\| |A|^{2v} + |B|^{2\lambda} \| + \| |A|^{2v} - |B|^{2\lambda} \| \right] \times \| |A^*|^{2(1-v)} + |B^*|^{2(1-\lambda)} \|,$$

and

$$w_e^2(A, B) \leq \frac{1}{2} \left[\| |A|^{2v} + |B^*|^{2(1-\lambda)} \| + \| |A|^{2v} - |B^*|^{2(1-\lambda)} \| \right] \times \| |A^*|^{2(1-v)} + |B|^{2\lambda} \|.$$

Remark 7. If we take $v = \lambda = 1/2$ in Corollary 4, then we obtain

$$w_e^2(A, B) \leq \frac{1}{2} [\| |A| + |B| \| + \| |A| - |B| \|] \| |A^*| + |B^*| \|,$$

for $A, B \in \mathcal{L}(H)$.

If we take $B = A^*$ in Theorem 5, then we obtain

$$w^2(A) \leq \frac{1}{4} \left[\| f^2(|A|) + h^2(|A^*|) \| + \| f^2(|A|) - h^2(|A^*|) \| \right] \times \| g^2(|A^*|) + k^2(|A|) \|,$$

and

$$w^2(A) \leq \frac{1}{4} \left[\| f^2(|A|) + k^2(|A|) \| + \| f^2(|A|) - k^2(|A|) \| \right] \times \| g^2(|A^*|) + h^2(|A^*|) \|,$$

for $A \in \mathcal{L}(H)$, where f, g, h and k are nonnegative continuous functions on $[0, \infty)$ satisfying the condition $f(t)g(t) = h(t)k(t) = t$ for all $t \in [0, \infty)$.

If in Corollary 4 we choose $B = A^*$, then we obtain

$$w^2(A) \leq \frac{1}{4} \left[\left\| |A|^{2\nu} + |A^*|^{2\lambda} \right\| + \left\| |A|^{2\nu} - |A^*|^{2\lambda} \right\| \right] \times \left\| |A^*|^{2(1-\nu)} + |A|^{2(1-\lambda)} \right\|,$$

and

$$w^2(A) \leq \frac{1}{4} \left[\left\| |A|^{2\nu} + |A|^{2(1-\lambda)} \right\| + \left\| |A|^{2\nu} - |A|^{2(1-\lambda)} \right\| \right] \times \left\| |A^*|^{2(1-\nu)} + |A^*|^{2\lambda} \right\|,$$

for $\nu, \lambda \in [0, 1]$.

In particular, we have

$$w^2(A) \leq \frac{1}{4} \left[\left\| |A| + |A^*| \right\| + \left\| |A| - |A^*| \right\| \right] \left\| |A^*| + |A| \right\|.$$

Further, if we take $A = \Re(T)$ and $B = \Im(T)$ in Theorem 5, then we obtain

$$w^2(T) \leq \frac{1}{2} \left[\left\| f^2(|\Re(T)|) + h^2(|\Im(T)|) \right\| + \left\| f^2(|\Re(T)|) - h^2(|\Im(T)|) \right\| \right] \\ \times \left\| g^2(|\Re(T)|) + k^2(|\Im(T)|) \right\|,$$

and

$$w^2(T) \leq \frac{1}{2} \left[\left\| f^2(|\Re(T)|) + k^2(|\Im(T)|) \right\| + \left\| f^2(|\Re(T)|) - k^2(|\Im(T)|) \right\| \right] \\ \times \left\| g^2(|\Re(T)|) + h^2(|\Im(T)|) \right\|,$$

for $T \in \mathcal{L}(H)$, where f, g, h and k are nonnegative continuous functions on $[0, \infty)$ satisfying the condition $f(t)g(t) = h(t)k(t) = t$ for all $t \in [0, \infty)$.

If in Corollary 4 we choose $A = \Re(T)$ and $B = \Im(T)$, then we obtain

$$w^2(T) \leq \frac{1}{2} \left[\left\| |\Re(T)|^{2\nu} + |\Im(T)|^{2\lambda} \right\| + \left\| |\Re(T)|^{2\nu} - |\Im(T)|^{2\lambda} \right\| \right] \\ \times \left\| |\Re(T)|^{2(1-\nu)} + |\Im(T)|^{2(1-\lambda)} \right\|,$$

and

$$w^2(T) \leq \frac{1}{2} \left[\left\| |\Re(T)|^{2\nu} + |\Im(T)|^{2(1-\lambda)} \right\| + \left\| |\Re(T)|^{2\nu} - |\Im(T)|^{2(1-\lambda)} \right\| \right] \\ \times \left\| |\Re(T)|^{2(1-\nu)} + |\Im(T)|^{2\lambda} \right\|,$$

for $\nu, \lambda \in [0, 1]$.

In particular, we have

$$w^2(T) \leq \frac{1}{2} \left[\left\| |\Re(T)| + |\Im(T)| \right\| + \left\| |\Re(T)| - |\Im(T)| \right\| \right] \left\| |\Re(T)| + |\Im(T)| \right\|,$$

for $T \in \mathcal{L}(H)$.

Finally, if we take $A = T$ and $B = |T|^2$ in Theorem 5, then we obtain

$$dw^2(T) \leq \frac{1}{2} \left[\left\| f^2(|T|) + h^2(|T|^2) \right\| + \left\| f^2(|T|) - h^2(|T|^2) \right\| \right] \times \left\| g^2(|T^*|) + k^2(|T|^2) \right\|,$$

and

$$dw^2(T) \leq \frac{1}{2} \left[\left\| f^2(|T|) + k^2(|T|^2) \right\| + \left\| f^2(|T|) - k^2(|T|^2) \right\| \right] \times \left\| g^2(|T^*|) + h^2(|T|^2) \right\|,$$

for $T \in \mathcal{L}(H)$, where f, g, h and k are nonnegative continuous functions on $[0, \infty)$ satisfying the condition $f(t)g(t) = h(t)k(t) = t$ for all $t \in [0, \infty)$.

For $v, \lambda \in [0, 1]$, then by Corollary 4 for $A = T$ and $B = |T|^2$, we obtain

$$dw^2(T) \leq \frac{1}{2} \left[\left\| |T|^{2v} + |T|^{4\lambda} \right\| + \left\| |T|^{2v} - |T|^{4\lambda} \right\| \right] \times \left\| |T^*|^{2(1-v)} + |T|^{4(1-\lambda)} \right\|$$

and

$$dw^2(T) \leq \frac{1}{2} \left[\left\| |T|^{2v} + |T|^{4(1-\lambda)} \right\| + \left\| |T|^{2v} - |T|^{4(1-\lambda)} \right\| \right] \times \left\| |T^*|^{2(1-v)} + |T|^{4\lambda} \right\|.$$

In particular,

$$dw^2(T) \leq \frac{1}{2} \left[\left\| |T| + |T|^2 \right\| + \left\| |T| - |T|^2 \right\| \right] \left\| |T^*| + |T|^2 \right\|,$$

for $T \in \mathcal{L}(H)$.

3. Conclusions

In conclusion, the work presented in this paper marks a significant advancement in the study of the Euclidean numerical radius of pairs of operators in Hilbert spaces. By introducing novel weighted lower and upper bounds, we have not only expanded on existing results in the literature but also demonstrated the asymmetrical nature of these bounds, highlighting the complexity of the relationships explored.

Moreover, the discovery of new inequalities for the numerical radius and the Davis–Wielandt radius adds depth to our understanding of operator properties in this setting. These results pave the way for future research endeavors and open up exciting possibilities for further exploration in the field.

One promising direction for future research is the parametric extension of the Euclidean numerical radius, as defined by

$$w_v(C, D) := \sup_{x \in H, \|x\|=1} \left(|\langle Cx, x \rangle|^{\frac{1}{v}} + |\langle Dx, x \rangle|^{\frac{1}{v}} \right)^v, \quad (0 < v < 1),$$

This extension could lead to intriguing results, such as the inequality

$$w_v(A, B) \geq \max\{w(\alpha A + \beta B), w(\gamma A + \delta B)\}, \quad (36)$$

for $A, B \in \mathcal{L}(H)$ and complex numbers $\alpha, \beta, \gamma, \delta$ satisfying

$$|\alpha|^{\frac{1}{1-v}} + |\beta|^{\frac{1}{1-v}} = |\gamma|^{\frac{1}{1-v}} + |\delta|^{\frac{1}{1-v}} = 1, \quad 0 < v < 1.$$

Indeed, the application of the Hölder inequality further enriches the analysis, as it states

$$\sum_{i=1}^n |a_i| |b_i| \leq \left(\sum_{i=1}^n |a_i|^{\frac{1}{1-v}} \right)^{1-v} \left(\sum_{i=1}^n |b_i|^{\frac{1}{v}} \right)^v, \quad (0 < v < 1).$$

So, by using Hölder inequality and carefully calculating expressions for all $x \in H$, one can establish that

$$\begin{aligned} \left(|\langle Ax, x \rangle|^{\frac{1}{v}} + |\langle Bx, x \rangle|^{\frac{1}{v}} \right)^v &= \left(|\alpha|^{\frac{1}{1-v}} + |\beta|^{\frac{1}{1-v}} \right)^{1-v} \left(|\langle Ax, x \rangle|^{\frac{1}{v}} + |\langle Bx, x \rangle|^{\frac{1}{v}} \right)^v \\ &\geq |\alpha| \times |\langle Ax, x \rangle| + |\beta| \times |\langle Bx, x \rangle| \\ &\geq |\langle (\alpha A + \beta B)x, x \rangle|. \end{aligned}$$

Taking the supremum over $x \in H$ with $\|x\| = 1$ demonstrates that

$$w_v(A, B) \geq w(\alpha A + \beta B).$$

Similarly, $w_v(A, B) \geq w(\gamma A + \delta B)$. Thus, we arrive at (36) as desired.

Therefore, the present work not only contributes significant findings to the study of operator inequalities but also serves as a promising starting point for future advancements and explorations in this field.

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