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A Robust-Fitted-Mesh-Based Finite Difference Approach for Solving a System of Singularly Perturbed Convection–Diffusion Delay Differential Equations with Two Parameters

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Abstract: This paper presents a robust fitted mesh finite difference method for solving a dynamical system of two parameter convection–reaction–diffusion delay differential equations defined on the interval $[0, 2]$. The method incorporates a piecewise uniform Shishkin mesh to accurately resolve the solution behavior caused by small perturbation parameters and delay terms. The proposed numerical scheme is proven to be parameter-robust and achieves almost first-order convergence. Numerical illustrations are provided to showcase the method's effectiveness, highlighting its capability to address boundary and interior layers with improved accuracy. The results, supported by symmetrical considerations in the figures, enhance the precision and serve as validation for the theoretical results.

Keywords: singularly perturbed delay differential equations; numerical methods; convection–diffusion; Shishkin meshes; boundary layer; uniform convergence

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1. Introduction

Singularly perturbed differential equations (SPDEs) emerge in numerous scientific and engineering fields, such as fluid dynamics, chemical reactor theory, population dynamics, particle fluid motion [1], and control theory [2]. A notable subset of these equations is the singularly perturbed delay differential equations (SPDDEs); these equations are characterized by the presence of two distinct small perturbation parameters, which introduce additional complexities due to the occurrence of boundary and interior layers. These layers arise because of the small perturbation parameters and delay terms, making the numerical approximation of SPDDEs particularly difficult. To tackle these challenges, numerous specialized numerical techniques,including fitted mesh methods [3] and fitted operator methods [4], are developed and well established. This approach offers robust and precise solutions to singularly perturbed differential equations (SPDEs). For instance, Cen [5] analyzed a class of first-order SPDDEs utilizing a hybrid difference scheme combined with a piecewise uniform Shishkin mesh, demonstrating that the approach achieves near second-order convergence. Gracia, O'Riordan, and Pickett [6] stated that a singularly perturbed ordinary differential equation is influenced by two small parameters that impact both the convection and diffusion terms . They proposed a second-order monotone numerical method to solve the problem. In [7], parabolic singularly perturbed parabolic differential equations involving two parameters on a rectangular domain were examined.

SPDDEs are typically defined a system of boundary value problems influenced by two small parameters, μ and ϵ_i. The interaction between these parameters produces intricate layers, determined by the ratio μ^2/ϵ_i. This work has focused on developing parameter-robust methods for solving these equations, especially as both ϵ_i and μ approach zero simultaneously. The most important papers among all are [6,8,9]. This paper focuses on the numerical solution of a boundary value problem for SPDDEs with two small perturbation parameters. The interaction between these parameters introduces complex, symmetry-influenced layer patterns in the solution, further compounded by the delay term. Our goal is to construct a parameter-robust numerical method that ensures the accurate solution of boundary and interior layers, regardless of the perturbation parameters. In doing so, we aim to establish the minimum principle and stability result of the solution and derive bounds for the solution's derivatives to support the convergence analysis of the proposed fitted mesh finite difference method, which will achieve nearly first-order convergence, regardless of the two perturbation parameters, and offer a robust layer-solving solution. Several studies have addressed singular perturbation problems, focusing on their asymptotic properties, parameter-uniform methods, and delay differential equations to ensure robust and accurate solutions across varying parameters [10–12].

The novelty of this paper lies in its approach to solving a system of two-parameter singularly perturbed delay differential equations (SPDDEs) of the convection–diffusion type. Unlike previous works, such as [9], which focuses on single delay differential equations, and [8], which addresses a system of convection–reaction–diffusion equations without delay terms, this paper considers interacting variables influenced by both delay and two distinct small perturbation parameters. This dual influence introduces significant complexities, particularly in the formation of boundary and interior layers. To tackle these challenges, a more robust numerical scheme integrating advanced mesh techniques, such as the use of a piecewise uniform Shishkin mesh, is proposed. Furthermore, this method employs more robust numerical schemes and advanced mesh techniques, potentially achieving almost first-order convergence, independent of the two perturbation parameters.

2. Formulation of the Problem

$$E\vec{\mathfrak{u}}''(\varkappa)+\mu\boldsymbol{\mathcal{A}}(\varkappa)\vec{\mathfrak{u}}'(\varkappa)-\boldsymbol{\mathcal{B}}(\varkappa)\vec{\mathfrak{u}}(\varkappa)+\boldsymbol{\mathcal{D}}(\varkappa)\vec{\mathfrak{u}}(\varkappa-1)=\vec{f}(\varkappa)\text{ for all }\varkappa\in\Omega=(0,2),\quad(1)$$

$$\vec{\mathfrak{u}}=\vec{\varphi}\quad\text{on }[-1,0],\quad\vec{\mathfrak{u}}(2)=\vec{\imath}.$$

Here, $\vec{\mathfrak{u}}=\begin{pmatrix}\mathfrak{u}_1\\ \mathfrak{u}_2\end{pmatrix}$, $\vec{f}=\begin{pmatrix}f_1\\ f_2\end{pmatrix}$, $E=\begin{pmatrix}\epsilon_1 & 0\\ 0 & \epsilon_2\end{pmatrix}$, $\boldsymbol{\mathcal{A}}(\varkappa)=\begin{pmatrix}\mathfrak{a}_1(\varkappa) & 0\\ 0 & \mathfrak{a}_2(\varkappa)\end{pmatrix}$, $\boldsymbol{\mathcal{B}}(\varkappa)=\begin{pmatrix}\mathfrak{b}_{11}(\varkappa) & -\mathfrak{b}_{12}(\varkappa)\\ -\mathfrak{b}_{21}(\varkappa) & \mathfrak{b}_{22}(\varkappa)\end{pmatrix}$, $\boldsymbol{\mathcal{D}}(\varkappa)=\begin{pmatrix}\mathfrak{d}_1(\varkappa) & 0\\ 0 & \mathfrak{d}_2(\varkappa)\end{pmatrix}$, where ϵ_i, for $i=1,2$ satisfy $0<\epsilon_1<\epsilon_2\leq 1$ and μ satisfy $0<\mu\leq 1$ are small parameters. The coefficient functions $\mathfrak{a}_i(\varkappa)$, $\mathfrak{b}_{ij}(\varkappa)$, $\mathfrak{d}_i(\varkappa)$, and $f_i(\varkappa)$ are all sufficiently smooth over the domain $\bar{\Omega}=[0,2]$ and $\mathfrak{a}_i(\varkappa)\geq\alpha>0, \mathfrak{b}_{ii}(\varkappa)+\mathfrak{b}_{ij}(\varkappa)\geq\beta>0, \mathfrak{d}_i(\varkappa)>0$, $\mathfrak{b}_{ii}(\varkappa)+\mathfrak{b}_{ij}(\varkappa)-\mathfrak{d}_i(\varkappa)\geq\kappa>0$, for $i,j=1,2$ and $i\neq j$. The $\vec{\varphi}(x)$ is sufficiently smooth over the interval $[-1,0]$. The value of γ is determined as $\gamma=\min\limits_{x\in\bar{\Omega}}\left(\dfrac{\sum_{j=1}^{2}\mathfrak{b}_{ij}(\varkappa)-\mathfrak{d}_i(\varkappa)}{\mathfrak{a}_i(\varkappa)}\right)$ for $i=1,2$ and $i\neq j$. When $\mu=0$, the above problem is considered in [13] without delay. The problem demonstrates boundary layers that are influenced by both ϵ_i and μ, in particular the layers are influenced by the ratio of $\frac{\mu^2}{\epsilon_i}$. If $\frac{\mu^2}{\epsilon_1}\leq\frac{\gamma}{\alpha}$, the reduced problem can be expressed as

$$-\boldsymbol{\mathcal{B}}(\varkappa)\vec{\mathfrak{u}}(\varkappa)+\boldsymbol{\mathcal{D}}(\varkappa)\vec{\mathfrak{u}}(\varkappa-1)=\vec{f}(\varkappa),\quad\varkappa\in(0,2],\vec{\mathfrak{u}}(\varkappa)=\vec{\varphi}(\varkappa)\quad\text{on}\quad[-1,0].\quad(2)$$

This predicts a boundary layer of width $O(\sqrt{\epsilon_i})$ near $\varkappa = 0$ and an interior layer at $\varkappa = 1$ (due to the unit delay component), assuming $\vec{\mathfrak{u}}(0) \neq \vec{\varphi}(0)$. Also, a boundary layer of width $O(\sqrt{\epsilon_i})$ is expected near $\varkappa = 2$ and an interior layer at $\varkappa = 1$ (again due to the unit delay component); if $\vec{\mathfrak{u}}(2) \neq \vec{\iota}$. If $\frac{\mu^2}{\epsilon_2} \geq \frac{\gamma}{\alpha}$, the reduced problem

$$\mu \boldsymbol{\mathcal{A}}(\varkappa)\vec{\mathfrak{u}}'(\varkappa) - \boldsymbol{\mathcal{B}}(\varkappa)\vec{\mathfrak{u}}(\varkappa) + \boldsymbol{\mathcal{D}}(\varkappa)\vec{\mathfrak{u}}(\varkappa - 1) = \vec{f}(\varkappa), \quad \varkappa \in (0,2), \tag{3}$$

$$\vec{\mathfrak{u}}(\varkappa) = \vec{\varphi}(\varkappa) \quad \text{on} \quad [-1,0], \quad \vec{\mathfrak{u}}(2) = \vec{\iota},$$

is once again singularly perturbed with the parameter μ. A boundary layer of width $O(\mu)$ is anticipated near $\varkappa = 2$ and an interior layer at $\varkappa = 1$ (due to the unit delay component), unless the solution at $\varkappa = 2$ differs from $\vec{\mathfrak{u}}(2)$; additionally, a boundary layer of width $O(\frac{\epsilon_i}{\mu})$ is anticipated near $\varkappa = 0$ and an interior layer at $\varkappa = 1$ (also due to the unit delay component), if $\vec{\mathfrak{u}}(0) \neq \vec{\varphi}(0)$. Numerical experiments suggest that the interior right layer weakens considerably when $\epsilon_i \ll \mu^2$. Consider the problem

$$\vec{L_1}\vec{z}(\varkappa) = E\vec{z}\,''(\varkappa) + \mu \boldsymbol{\mathcal{A}}(\varkappa)\vec{z}\,'(\varkappa) - \boldsymbol{\mathcal{B}}(\varkappa)\vec{z}(\varkappa) = \vec{g}(\varkappa), \quad \varkappa \in \Omega_1, \tag{4}$$

$$\vec{L_2}\vec{z}(\varkappa) \equiv E\vec{z}\,''(\varkappa) + \mu \boldsymbol{\mathcal{A}}(\varkappa)\vec{z}\,'(\varkappa) - \boldsymbol{\mathcal{B}}(\varkappa)\vec{z}(\varkappa) + \boldsymbol{\mathcal{D}}(\varkappa)\vec{z}(\varkappa - 1) = \vec{f}(\varkappa), \quad \varkappa \in \Omega_2, \tag{5}$$

with boundary conditions

$$\vec{z}(0) = \vec{\varphi}(0), \quad \vec{z}(1-) = \vec{z}(1+), \quad \vec{z}\,'(1-) = \vec{z}\,'(1+), \quad \vec{z}(2) = \vec{\iota},$$

where $\vec{g}(\varkappa) = \vec{f}(\varkappa) - \boldsymbol{\mathcal{D}}(\varkappa)\vec{\varphi}(\varkappa - 1)$, for $\varkappa \in \Omega_1$, $\Omega_1 = (0,1)$, $\Omega_2 = (1,2)$.

3. Analytical Results

This section presents a minimum principle and a stability result, and it provides estimates for the derivatives of the solution to the problem defined by Equation (1).

Lemma 1. *Let* $\vec{\boldsymbol{\psi}} \in C^2(\bar{\Omega})$ *be such that* $\vec{\boldsymbol{\psi}}(0) \geq \vec{0}$, $\vec{\boldsymbol{\psi}}(2) \geq \vec{0}$, $\vec{L}_1\vec{\boldsymbol{\psi}} \leq \vec{0}$ *on* $(0,1)$ *and* $\vec{L}_2\vec{\boldsymbol{\psi}} \leq \vec{0}$ *on* $(1,2)$, $\vec{\boldsymbol{\psi}}(1) = \vec{0}$, $\vec{\boldsymbol{\psi}}(1)' \leq \vec{0}$, *then* $\vec{\boldsymbol{\psi}} \geq \vec{0}$ *on* $[0,2]$.

Proof. Assume that $\varkappa^*$ and s^* be such that $\boldsymbol{\psi}_{s^*}(\varkappa^*) = \min\limits_{\varkappa \in \Omega, s=1,2} \boldsymbol{\psi}_s(\varkappa)$. Suppose $\boldsymbol{\psi}_{s^*}(\varkappa^*) < 0$. Then, $\varkappa^*$ cannot be at the boundaries 0 or 2. At $\varkappa^*$, the first derivative of $\boldsymbol{\psi}_{s^*}$, denoted as $\boldsymbol{\psi}'_{s^*}(\varkappa^*) = 0$ and the second derivative $\boldsymbol{\psi}''_{s^*}(\varkappa^*) \geq 0$. Claim (i): $\varkappa^* \notin (0,1)$. If $\varkappa^* \in (0,1)$, then

$$(\vec{L_1}\vec{\boldsymbol{\psi}})_{s^*}(\varkappa^*) = \epsilon_{s^*}\boldsymbol{\psi}''_{s^*}(\varkappa^*) + \mu \mathfrak{a}_{s^*}(\varkappa^*)\boldsymbol{\psi}'_{s^*}(\varkappa^*) - \sum_{j=1}^{2} \mathfrak{b}_{s^*j}(\varkappa^*)\boldsymbol{\psi}_j(\varkappa^*) > 0,$$

which contradicts the assumption that $\vec{L}_1\vec{\boldsymbol{\psi}} \leq \vec{0}$ on $(0,1)$. Thus, $\varkappa^* \notin (0,1)$. Claim (ii): $\varkappa^* \notin [1,2)$. If $\varkappa^* \in [1,2)$, then

$$(\vec{L_2}\vec{\boldsymbol{\psi}})_{s^*}(\varkappa^*) = \epsilon_{s^*}\boldsymbol{\psi}''_{s^*}(\varkappa^*) + \mu \mathfrak{a}_{s^*}(\varkappa^*)\boldsymbol{\psi}'_{s^*}(\varkappa^*) - \sum_{j=1}^{2} [\mathfrak{b}_{s^*j}(\varkappa^*)]\boldsymbol{\psi}_j(\varkappa^*)$$

$$-\mathfrak{d}_{s^*}(\varkappa^*)\boldsymbol{\psi}_{s^*}(\varkappa^* - 1) > 0, \tag{6}$$

which contradicts the assumption that $\vec{L}_2\vec{\boldsymbol{\psi}} \leq \vec{0}$ on $[1,2)$. Hence, $\varkappa^* \notin [1,2)$. When $\varkappa^* = 1$, the differentiability of $\boldsymbol{\psi}_{s^*}$ at $\varkappa = 1$ is investigated. If $\boldsymbol{\psi}'_{s^*}(1)$ does not exist, then $\boldsymbol{\psi}'_{s^*}(1) = \vec{\boldsymbol{\psi}}'_{s^*}(1+) - \vec{\boldsymbol{\psi}}'_{s^*}(1-) > 0$, which contradicts the condition $\boldsymbol{\psi}'_{s^*}(1) \leq 0$. If $\boldsymbol{\psi}_{s^*}$ is differentiable at $\varkappa = 1$, then $\mu \mathfrak{a}_{s^*}(1)\boldsymbol{\psi}'_{s^*}(1) - \sum_{j=1}^{2} \mathfrak{b}_{s^*j}(1)\boldsymbol{\psi}_j(1) > 0$. Since all

entries of $\mathcal{A}(\varkappa)$, $\mathcal{B}(\varkappa)$, and $\boldsymbol{\psi}_j(\varkappa)$ are continuous on $[0,2]$, there is an interval $[1-h,1)$ where $\mu \mathfrak{a}_{s^*}(\varkappa)\boldsymbol{\psi}'_{s^*}(\varkappa) - \sum_{j=1}^{2} \mathfrak{b}_{s^*j}(\varkappa)\boldsymbol{\psi}_j(\varkappa) > 0$. If $\boldsymbol{\psi}''_{s^*}(p) \geq 0$ for any $p \in [1-h,1)$, then $(\vec{L}_1\boldsymbol{\psi})_{s^*}(p) \geq 0$, which leads to a contradiction. Therefore, it is assumed that $\boldsymbol{\psi}''_{s^*}(\varkappa) < 0$ on the interval $[1-h,1)$. This indicates that $\boldsymbol{\psi}'_{s^*}(\varkappa)$ is strictly decreasing over $[1-h,1)$. Given that $\boldsymbol{\psi}'_{s^*}(1) = 0$ and $\boldsymbol{\psi}'_{s^*}$ is continuous on $(0,2)$, it follows that $\boldsymbol{\psi}'_{s^*}(\varkappa) > 0$ on $[1-h,1)$. As a result, the continuous function $\boldsymbol{\psi}_{s^*}(\varkappa)$ cannot attain a minimum at $\varkappa = 1$, which contradicts the assumption that $\varkappa^* = 1$. Thus, $\vec{\boldsymbol{\psi}} \geq \vec{0}$ on $[0,2]$. The proof of the lemma is complete. □

Lemma 2. *(Stability Result) Let* $\vec{\boldsymbol{\psi}} \in C^2(\bar{\Omega})$. *For* $\varkappa \in \bar{\Omega}$,

$$|\boldsymbol{\psi}_i(\varkappa)| \leq \max\left\{ |\vec{\boldsymbol{\psi}}(0)|, |\vec{\boldsymbol{\psi}}(2)|, \frac{1}{\kappa}\|\vec{L}_1\vec{\boldsymbol{\psi}}\|_{\bar{\Omega}_1}, \frac{1}{\kappa}\|\vec{L}_2\vec{\boldsymbol{\psi}}\|_{\bar{\Omega}_2} \right\}.$$

Proof. Define

$$M = \max\left\{ |\vec{\boldsymbol{\psi}}(0)|, |\vec{\boldsymbol{\psi}}(2)|, \frac{1}{\kappa}\|\vec{L}_1\vec{\boldsymbol{\psi}}\|_{\bar{\Omega}_1}, \frac{1}{\kappa}\|\vec{L}_2\vec{\boldsymbol{\psi}}\|_{\bar{\Omega}_2} \right\}.$$

Consider the functions $\vec{\theta}^{\pm}(\varkappa) = M\vec{e} \pm \vec{\boldsymbol{\psi}}(\varkappa)$, where $\vec{e} = (1,1)^T$. Clearly, $\vec{\theta}^{\pm} \in C^2(\Omega)$; at both 0 and 2, $\vec{\theta}^{\pm}$ is non-negative, i.e., $\vec{\theta}^{\pm}(0) \geq \vec{0}$ and $\vec{\theta}^{\pm}(2) \geq \vec{0}$. For $\varkappa \in (0,1)$,

$$\vec{L}_1\vec{\theta}^{\pm}(\varkappa) = -\boldsymbol{\mathcal{B}}(\varkappa)M\vec{e} \pm \vec{L}_1\vec{\boldsymbol{\psi}}(\varkappa) \leq \vec{0},$$

and for $\varkappa \in (1,2)$,

$$\vec{L}_2\vec{\theta}^{\pm}(\varkappa) = -\boldsymbol{\mathcal{B}}(\varkappa)M\vec{e} + \boldsymbol{\mathcal{D}}(\varkappa)M\vec{e} \pm \vec{L}_2\vec{\boldsymbol{\psi}}(\varkappa) \leq \vec{0}.$$

Also, $\vec{\theta}^{\pm}(1)$ is equal to $\pm\vec{\boldsymbol{\psi}}(1)$, both of which are zero, i.e., $\vec{\theta}^{\pm}(1) = \pm\vec{\boldsymbol{\psi}}(1) = \vec{0}$ and $\vec{\theta}^{\pm\prime}(1) = \pm\vec{\boldsymbol{\psi}}'(1) = \vec{0}$, then $\vec{\theta}^{\pm}(\varkappa) \geq \vec{0}$. By applying Lemma 1, it is obtained that $|\boldsymbol{\psi}_i(\varkappa)| \leq M$ for $\varkappa \in \bar{\Omega}$. The proof of the lemma is complete. □

Theorem 1. *Let* $\vec{u}$ *be the solution of* (1); *then, its derivatives satisfy the following bounds on* Ω,

$$|\mathbf{u}_i^{(k)}(\varkappa)| \leq \frac{C}{(\sqrt{\epsilon_i})^k}\left(1 + \left(\frac{\mu}{\sqrt{\epsilon_i}}\right)^k\right)\max\{\|\vec{\mathbf{u}}\|, \|\vec{f}\|\}, \tag{7}$$

$$\left|\mathbf{u}_i^{(3)}(\varkappa)\right| \leq \frac{C}{(\sqrt{\epsilon_i})^3}\left(1 + \left(\frac{\mu}{\sqrt{\epsilon_i}}\right)^3\right)\max\left\{\|\vec{\mathbf{u}}\|, \|\vec{f}\|, \|\vec{f}'\|\right\}, \tag{8}$$

$$|\mathbf{u}_i^{(4)}(\varkappa)| \leq \frac{C}{(\sqrt{\epsilon_i})^4}\left(1 + \left(\frac{\mu}{\sqrt{\epsilon_i}}\right)^4\right)\max\left\{\|\vec{\mathbf{u}}\|, \|\vec{f}\|, \|\vec{f}'\|, \|\vec{f}''\|\right\} \tag{9}$$

where the constant C *is independent of* ϵ_i *and* μ , $i, k = 1, 2$.

Proof. The proof follows the methodology outlined in Lemma 2.2 of [14]. For any $\varkappa \in (0,1)$, a neighborhood $N_p = (p, p+\sqrt{\epsilon_i})$ can be constructed in such a way that $\varkappa \in N_p$ and $N_p \subset (0,1)$. According to the mean value theorem, there is a $y \in N_p$ satisfying

$$\mathbf{u}'_i(y) = \frac{\mathbf{u}_i(p+\sqrt{\epsilon_i}) - \mathbf{u}_i(p)}{\sqrt{\epsilon_i}} \implies |\mathbf{u}'_i(y)| \leq \frac{2\|\mathbf{u}_i\|}{\sqrt{\epsilon_i}}.$$

Now,

$$\mathbf{u}'_i(\varkappa) = \mathbf{u}'_i(y) + \int_y^{\varkappa} \mathbf{u}''_i(\eta)\, d\eta.$$

Thus,

$$|\mathbf{u}_i'(\varkappa)| \leq \frac{C}{\sqrt{\epsilon_i}}\left(1+\frac{\mu}{\sqrt{\epsilon_i}}\right)\max\{\|\vec{\mathbf{u}}\|,\|\vec{f}\|\}.$$

The bounds for $\mathbf{u}_i''$ are obtained from Equation (1) and the bounds from Equation (7) for $k=1$. Similarly, the bounds of $\mathbf{u}_i^{(3)}$ and $\mathbf{u}_i^{(4)}$ can be established for higher-order derivatives through analogous corresponding manipulations. The proof of the theorem is complete. □

4. Shishkin Decomposition of the Solution

For each of the cases $\alpha\mu^2 \leq \gamma\epsilon_1$ and $\alpha\mu^2 \geq \gamma\epsilon_2$, $\vec{\mathbf{u}}$ is expressed by

$$\vec{\mathbf{u}} = \vec{\mathfrak{v}} + \vec{w}^L + \vec{w}^R,$$

where

$$\vec{\mathfrak{v}}(\varkappa) = \begin{cases} \vec{r}(\varkappa), & \text{for } \varkappa \in [0,1) \\ \vec{s}(\varkappa), & \text{for } \varkappa \in [1,2] \end{cases}, \tag{10}$$

$$\vec{w}^L(\varkappa) = \begin{cases} \vec{w}^{L_1}(\varkappa) = \begin{cases} {w_1}^{L_1}(\varkappa) \\ {w_2}^{L_1}(\varkappa) \end{cases}, & \text{for } \varkappa \in [0,1) \\ \vec{w}^{L_2}(\varkappa) = \begin{cases} {w_1}^{L_2}(\varkappa) \\ {w_2}^{L_2}(\varkappa) \end{cases}, & \text{for } \varkappa \in [1,2] \end{cases}, \tag{11}$$

$$\vec{w}^R(\varkappa) = \begin{cases} \vec{w}^{R_1}(\varkappa) = \begin{cases} {w_1}^{R_1}(\varkappa) \\ {w_2}^{R_1}(\varkappa) \end{cases}, & \text{for } \varkappa \in [0,1) \\ \vec{w}^{R_2}(\varkappa) = \begin{cases} {w_1}^{R_2}(\varkappa) \\ {w_2}^{R_2}(\varkappa) \end{cases}, & \text{for } \varkappa \in [1,2] \end{cases}. \tag{12}$$

Case (i): $\alpha\mu^2 \leq \gamma\epsilon_1$
In this case,

$$\vec{L}_1\vec{r}(\varkappa) = \vec{g}(\varkappa), \quad \text{for } \varkappa \in (0,1), \quad \vec{r}(0) \text{ and } \vec{r}(1) \text{ are selected,} \tag{13}$$
$$\vec{L}_2\vec{s}(\varkappa) = \vec{f}(\varkappa), \quad \text{for } \varkappa \in (1,2), \quad \vec{s}(1) \text{ and } \vec{s}(2) \text{ are selected,} \tag{14}$$
$$\vec{s}(\varkappa) = \vec{r}(\varkappa) \text{ on } [0,1) \tag{15}$$

$$\vec{L}_1\vec{w}^{L_1} = \vec{0}, \quad \text{for } \varkappa \in (0,1), \quad \vec{w}^{L_1}(0) = \vec{u}(0) - \vec{v}(0) - c_1(\epsilon_i,\mu), \quad \vec{w}^{L_1}(1) = \vec{0}, \tag{16}$$
$$\vec{L}_2\vec{w}^{L_2} = \vec{0}, \quad \text{for } \varkappa \in (1,2), \quad \vec{w}^{L_2}(1) = k_1(\epsilon_i,\mu) - c_2(\epsilon_i,\mu), \quad \vec{w}^{L_2}(2) = \vec{0}, \tag{17}$$
$$\vec{w}^{L_2}(\varkappa) = \vec{w}^{L_1}(\varkappa), \quad \text{for } \varkappa \in [0,1) \tag{18}$$

$$\vec{L}_1\vec{w}^{R_1} = \vec{0}, \quad \text{for } \varkappa \in (0,1), \quad \vec{w}^{R_1}(0) = c_1(\epsilon_i,\mu), \quad \vec{w}^{R_1}(1) = k_2(\epsilon_i,\mu), \tag{19}$$
$$\vec{L}_2\vec{w}^{R_2} = \vec{0}, \quad \text{for } \varkappa \in (1,2), \quad \vec{w}^{R_2}(1) = c_2(\epsilon_i,\mu), \quad \vec{w}^{R_2}(2) = \vec{u}(2) - \vec{v}(2), \tag{20}$$
$$\vec{w}^{R_2}(\varkappa) = \vec{w}^{R_1}(\varkappa), \quad \text{for } \varkappa \in [0,1) \tag{21}$$

Case (ii): $\alpha\mu^2 \geq \gamma\epsilon_2$
In this case,

$$\vec{L}_1\vec{r}(\varkappa) = \vec{g}(\varkappa), \quad \text{for } \varkappa \in (0,1), \quad \vec{r}(0) \text{ and } \vec{r}(1) \text{ are selected,} \tag{22}$$
$$\vec{L}_2\vec{s}(\varkappa) = \vec{f}(\varkappa), \quad \text{for } \varkappa \in (1,2), \quad \vec{s}(1) \text{ and } \vec{s}(2) \text{ are selected,} \tag{23}$$
$$\vec{s}(\varkappa) = \vec{r}(\varkappa) \text{ on } [0,1) \tag{24}$$

$$\vec{L}_1\vec{w}^{L_1} = \vec{0}, \quad \text{for } \varkappa \in (0,1), \quad \vec{w}^{L_1}(0) = \vec{u}(0) - \vec{v}(0) - c_1(\epsilon_i,\mu), \quad \vec{w}^{L_1}(1) = \vec{0}, \tag{25}$$

$$\vec{L}_2\vec{w}^{L_2} = \vec{0}, \quad \text{for } \varkappa \in (1,2), \quad \vec{w}^{L_2}(1) = k_1(\epsilon_i,\mu) - c_2(\epsilon_i,\mu), \quad \vec{w}_2^{L}(2) = \vec{0}, \tag{26}$$

$$\vec{w}^{L_2}(\varkappa) = \vec{w}^{L_1}(\varkappa), \quad \text{for } \varkappa \in [0,1) \tag{27}$$

$$\vec{L}_1\vec{w}^{R_1} = \vec{0}, \quad \text{for } \varkappa \in (0,1), \quad \vec{w}^{R_1}(0) = c_1(\epsilon_i,\mu), \quad \vec{w}^{R_1}(1) = k_2(\epsilon_i,\mu), \tag{28}$$

$$\vec{L}_2\vec{w}^{R_2} = \vec{0}, \quad \text{for } \varkappa \in (1,2), \quad \vec{w}^{R_2}(1) = c_2(\epsilon_i,\mu), \quad \vec{w}^{R_2}(2) = \vec{u}(2) - \vec{v}(2), \tag{29}$$

$$\vec{w}^{R_2}(\varkappa) = \vec{w}^{R_1}(\varkappa), \quad \text{for } \varkappa \in [0,1) \tag{30}$$

To ensure the jump conditions at $\varkappa = 1$ in Equations (13) and (22) are satisfied, the constants $k_1(\epsilon_i,\mu)$ and $k_2(\epsilon_i,\mu)$ must be selected appropriately. Additionally, the constants $c_1(\epsilon_i,\mu)$ and $c_2(\epsilon_i,\mu)$ should be determined independently for the cases $\alpha\mu^2 \le \gamma\epsilon_1$ and $\alpha\mu^2 \ge \gamma\epsilon_2$, ensuring they meet the bounds required for the singular component. Given that $\vec{\mathbf{u}}(0)$ and $\vec{\mathbf{u}}(1)$ are bounded by constants that do not depend on ϵ_i and μ, even though c_1, c_2, k_1, and k_2 are functions of ϵ_i and μ, the magnitudes $|c_1|$, $|c_2|$, $|k_1|$, and $|k_2|$ are constants independent of ϵ_i and μ.

5. Bounds on the Regular Component and Its Derivatives

Case (i): $\alpha\mu^2 \le \gamma\epsilon_1$

Establishing the bounds of the regular components $\vec{r}$ and $\vec{s}$, they are broken down as in [9]

$$\vec{r} = \vec{y}_0 + \sqrt{\epsilon_2}\vec{y}_1 + (\sqrt{\epsilon_2})^2\vec{y}_2 + (\sqrt{\epsilon_2})^3\vec{y}_3, \bar{\Omega}_1 = [0,1] \tag{31}$$

$$\vec{s} = \vec{z}_0 + \sqrt{\epsilon_2}\vec{z}_1 + (\sqrt{\epsilon_2})^2\vec{z}_2 + (\sqrt{\epsilon_2})^3\vec{z}_3, \bar{\Omega}_2 = [1,2] \tag{32}$$

where $\vec{y}_i = (y_{i1}, y_{i2})^T, \vec{z}_i = (z_{i1}, z_{i2})^T, i = 0,1,2,3$, and their respective equations are on the intervals $[0,1]$ and $[1,2]$, where $\vec{y}_i(\varkappa)$ and $\vec{z}_i(\varkappa)$, $i = 0,1,2,3$ are defined by

$$-\boldsymbol{\mathcal{B}}(\varkappa)\vec{y}_0(\varkappa) + \boldsymbol{\mathcal{D}}(\varkappa)\vec{\varphi}(\varkappa - 1) = \vec{f}(\varkappa), \quad \text{for } \varkappa \in [0,1], \tag{33}$$

$$-\boldsymbol{\mathcal{B}}(\varkappa)\vec{z}_0(\varkappa) + \boldsymbol{\mathcal{D}}(\varkappa)\vec{y}_0(\varkappa - 1) = \vec{f}(\varkappa), \quad \text{for } \varkappa \in [1,2], \tag{34}$$

$$\boldsymbol{\mathcal{B}}(\varkappa)\vec{y}_1(\varkappa) = \vec{g_1}, \quad \text{for } \varkappa \in [0,1], \tag{35}$$

$$\boldsymbol{\mathcal{B}}(\varkappa)\vec{z}_1(\varkappa) - \boldsymbol{\mathcal{D}}(\varkappa)\vec{y}_1(\varkappa - 1) = \vec{q_1}, \quad \text{for } \varkappa \in [1,2], \tag{36}$$

$$\boldsymbol{\mathcal{B}}(\varkappa)\vec{y}_2(\varkappa) = \vec{g_2}, \quad \text{for } \varkappa \in [0,1], \tag{37}$$

$$\boldsymbol{\mathcal{B}}(\varkappa)\vec{z}_2(\varkappa) - \boldsymbol{\mathcal{D}}(\varkappa)\vec{y}_2(\varkappa - 1) = \vec{q_2}, \quad \text{for } \varkappa \in [1,2], \tag{38}$$

$$\vec{L}_1\vec{y}_3(\varkappa) = -\vec{g_3}, \quad \text{for } \varkappa \in (0,1), \tag{39}$$

$$\vec{y}_3(0) = 0, \quad \vec{y}_3(1) = 0, \tag{40}$$

$$\vec{L}_2\vec{z}_3(\varkappa) = -\vec{q_3}, \quad \text{for } \varkappa \in (1,2), \tag{41}$$

$$\vec{z}_3(1) = 0, \quad \vec{z}_3(2) = 0, \quad \vec{z}_3(\varkappa) = \vec{y}_3(\varkappa), \text{ on } [0,1). \tag{42}$$

where $\vec{g}_i = (g_{i1}, g_{i2})^T$, $g_{ij} = \frac{\epsilon_j}{\sqrt{\epsilon_2}}y''_{i-1j} + \frac{\mu}{\sqrt{\epsilon_2}}\mathfrak{a}_j y'_{i-1j}$, and $\vec{q}_i = (q_{i1}, q_{i2})^T$, $q_{ij} = \frac{\epsilon_j}{\sqrt{\epsilon_2}}z''_{i-1j} + \frac{\mu}{\sqrt{\epsilon_2}}\mathfrak{a}_j z'_{i-1j}$, $i = 1,2,3$, $j = 1,2$, Equations (33)–(37) give

$$\|\vec{y}_0^{(k)}\|_{\bar{\Omega}_1} \le C, \quad \|\vec{z}_0^{(k)}\|_{\bar{\Omega}_2} \le C, \quad \text{for } 0 \le k \le 7, \tag{43}$$

$$\|\vec{y}_1^{(k)}\|_{\bar{\Omega}_1} \le C, \quad \|\vec{z}_1^{(k)}\|_{\bar{\Omega}_2} \le C, \quad \text{for } 0 \le k \le 5, \tag{44}$$

$$\|\vec{y}_2^{(k)}\|_{\bar{\Omega}_1} \le C, \quad \|\vec{z}_2^{(k)}\|_{\bar{\Omega}_2} \le C, \quad \text{for } 0 \le k \le 3. \tag{45}$$

Now, from (39)–(42) and using Lemma 2, $\|\vec{y}_3\| \leq C$. Using the estimates $\|y_{32}^{(k)}\| \leq C\epsilon_2^{-k/2}, k = 1, 2$. From the defining equation of y_{31} in (39),

$$\epsilon_1 y_{31}'' + \mu \mathfrak{a}_1 y_{31}' - \mathfrak{b}_{11} y_{31} = -\frac{\epsilon_1}{\sqrt{\epsilon_2}} y_{21}'' - \frac{\mu}{\sqrt{\epsilon_2}} \mathfrak{a}_1 y_{21}' - \mathfrak{b}_{12} y_{32}.$$

Decomposing y_{31} as $y_{31} = p_0 + \sqrt{\epsilon_1} p_1 + (\sqrt{\epsilon_1})^2 p_2 + (\sqrt{\epsilon_1})^3 p_3$ where

$$\mathfrak{b}_{11} p_0 = \frac{\epsilon_1}{\sqrt{\epsilon_2}} y_{21}'' + \frac{\mu}{\sqrt{\epsilon_2}} \mathfrak{a}_1 y_{21}' + \mathfrak{b}_{12} y_{32} \tag{46}$$

$$\mathfrak{b}_{11} p_1 = \sqrt{\epsilon_1} p_0'' + \frac{\mu}{\sqrt{\epsilon_1}} \mathfrak{a}_1 p_0' \tag{47}$$

$$\mathfrak{b}_{11} p_2 = \sqrt{\epsilon_1} p_1'' + \frac{\mu}{\sqrt{\epsilon_1}} \mathfrak{a}_1 p_1' \tag{48}$$

$$\epsilon_1 p_3'' + \mu \mathfrak{a}_1 p_3' - \mathfrak{b}_{11} p_3 = -\sqrt{\epsilon_1} p_2'' - \frac{\mu}{\sqrt{\epsilon_1}} \mathfrak{a}_1 p_2' \tag{49}$$

$$p_3(0) = p_3(1) = 0 \tag{50}$$

using the above, p_0, p_1, p_2 , and p_3 are found and then, from (46), $|y_{31}^{(k)}(x)| \leq C, k = 0, 1$, $|y_{31}''(x)| \leq C\sqrt{\epsilon_2}^{-1}$, $|y_{31}'''(\varkappa)| \leq C\sqrt{\epsilon_1}^{-1}\sqrt{\epsilon_2}^{-2}$. By differentiating the defining equation of y_{32} once and using the bounds of $\vec{y}_0, \vec{y}_1, \vec{y}_2$, and y_{31}, the following bound $\|y_{32}'''(x)\| \leq C\sqrt{\epsilon_2}^{-2}$ is obtained. Now, by choosing

$$\vec{r}(0) = \vec{y}_0(0) + \sqrt{\epsilon_2}\vec{y}_1(0) + (\sqrt{\epsilon_2})^2\vec{y}_2(0) + (\sqrt{\epsilon_2})^3\vec{y}_3(0) \tag{51}$$

$$\vec{r}(1) = \vec{y}_0(1) + \sqrt{\epsilon_2}\vec{y}_1(1) + (\sqrt{\epsilon_2})^2\vec{y}_2(1) + (\sqrt{\epsilon_2})^3\vec{y}_3(1) \tag{52}$$

from the above, it is derived that

$$|r_1^{(k)}(\varkappa)| \leq C, where\ k = 0, 1, 2, \quad |r_1'''(\varkappa)| \leq \frac{C}{\sqrt{\epsilon_1}}, \quad |r_2^{(k)}(\varkappa)| \leq C, where\ k = 0, 1, 2, 3. \tag{53}$$

Similarly,

$$|s_1^{(k)}(\varkappa)| \leq C, where\ k = 0, 1, 2, \quad |s_1'''(\varkappa)| \leq \frac{C}{\sqrt{\epsilon_1}}, \quad |s_2^{(k)}(\varkappa)| \leq C, where\ k = 0, 1, 2, 3. \tag{54}$$

Case (ii): $\alpha\mu^2 \geq \gamma\epsilon_2$

Establishing the bounds of the regular components $\vec{r}$ and $\vec{s}$, they are broken down as

$$\vec{r} = \vec{y}_0 + \epsilon_2\vec{y}_1 + \epsilon_2^2\vec{y}_2 + \epsilon_2^3\vec{y}_3, \bar{\Omega}_1 = [0, 1] \tag{55}$$

$$\vec{s} = \vec{z}_0 + \epsilon_2\vec{z}_1 + \epsilon_2^2\vec{z}_2 + \epsilon_2^3\vec{z}_3, \bar{\Omega}_2 = [1, 2] \tag{56}$$

where $\vec{y}_i = \begin{pmatrix} y_{i1} \\ y_{i2} \end{pmatrix}, \vec{z}_i = \begin{pmatrix} z_{i1} \\ z_{i2} \end{pmatrix}, i = 0, 1, 2, 3.$

To estimate the smooth components $\vec{r}\,(\varkappa)$ and $\vec{s}\,(\varkappa)$ for $\alpha\mu^2 \geq \gamma\epsilon_2$, the decomposition of $\vec{r}(\varkappa)$ and $\vec{s}(\varkappa)$ is analyzed on two levels. Furthermore, the maximum principle for a linear first-order operator in the context of a terminal value problem is established. Define the operators

$$\vec{L}_3 \equiv \mu\boldsymbol{\mathcal{A}}(\varkappa)\vec{\mathfrak{u}}\,'(\varkappa) - \boldsymbol{\mathcal{B}}(\varkappa)\vec{\mathfrak{u}}(\varkappa), \tag{57}$$

$$\vec{L}_4 \equiv \mu\boldsymbol{\mathcal{A}}(\varkappa)\vec{\mathfrak{u}}\,'(\varkappa) - \boldsymbol{\mathcal{B}}(\varkappa)\vec{\mathfrak{u}}(\varkappa) + \boldsymbol{\mathcal{D}}(\varkappa)\vec{\mathfrak{u}}(\varkappa - 1). \tag{58}$$

Consider the following decompositions:

$$\vec{r}\,(\varkappa;\epsilon,\mu)=\sum_{i=0}^{3}\epsilon_2^i\vec{y}_i(\varkappa;\mu),\quad \varkappa\in\bar{\Omega}_1,\vec{s}\,(\varkappa;\epsilon,\mu)=\sum_{i=0}^{3}\epsilon_2^i\vec{z}_i(\varkappa;\mu),\quad \varkappa\in\bar{\Omega}_2,$$

where $\vec{y}_i=\begin{pmatrix}y_{i1}\\ y_{i2}\end{pmatrix}$, $\vec{z}_i=\begin{pmatrix}z_{i1}\\ z_{i2}\end{pmatrix}$, $i=0,1,2,3$ are defined as follows:

$$\vec{L}_3\vec{y}_0(\varkappa)=\vec{g}(\varkappa),\quad \varkappa\in[0,1),\quad \vec{y}_0(1)\quad \text{is taken,} \tag{59}$$

$$\vec{L}_4\vec{z}_0(\varkappa)=\vec{f}(\varkappa),\quad \varkappa\in[1,2),\quad \vec{z}_0(2)\quad \text{is taken,} \tag{60}$$

$$\vec{z}_0(\varkappa)=\vec{y}_0(\varkappa),\quad \varkappa\in[0,1),$$

$$\vec{L}_3\vec{y}_1(\varkappa)=-\vec{m}_1(\varkappa),\quad \varkappa\in[0,1), \tag{61}$$

$$\vec{L}_4\vec{z}_1(\varkappa)=-\vec{n}_1(\varkappa),\quad \varkappa\in[1,2), \tag{62}$$

$$\vec{z}_1(\varkappa)=\vec{y}_1(\varkappa),\quad \varkappa\in[0,1),$$

$$\vec{L}_3\vec{y}_2(\varkappa)=-\vec{m}_2(\varkappa),\quad \varkappa\in[0,1),\quad \vec{y}_2(1)=\vec{0}, \tag{63}$$

$$\vec{L}_4\vec{z}_2(\varkappa)=-\vec{n}_2(\varkappa),\quad \varkappa\in[1,2),\quad \vec{z}_2(2)=\vec{0}, \tag{64}$$

$$\vec{z}_2(\varkappa)=\vec{y}_2(\varkappa),\quad \varkappa\in[0,1),$$

$$\vec{L}_1\vec{y}_3(\varkappa)=-\vec{m}_3(\varkappa),\quad \varkappa\in(0,1),\quad \vec{y}_3(0)=\vec{0},\quad \vec{y}_3(1)=\vec{0}, \tag{65}$$

$$\vec{L}_2\vec{z}_3(\varkappa)=-\vec{n}_3(\varkappa),\quad \varkappa\in(1,2),\quad \vec{z}_3(1)=\vec{0},\quad \vec{z}_3(2)=\vec{0}, \tag{66}$$

$$\vec{z}_3(\varkappa)=\vec{y}_3(\varkappa),\quad \varkappa\in[0,1).$$

where $\vec{m}_i=(m_{i1},m_{i2})^T$, $m_{ij}=\frac{\epsilon_j}{\sqrt{\epsilon_2}}y''_{i-1j}$, and $\vec{n}_i=(n_{i1},n_{i2})^T$, $n_{ij}=\frac{\epsilon_j}{\sqrt{\epsilon_2}}z''_{i-1j}$. Here, $\vec{g}(\varkappa)$ and $\vec{f}(\varkappa)$ are given functions, and the functions $\vec{y}_i(\varkappa)$ and $\vec{z}_i(\varkappa)$ are chosen to satisfy the boundary and continuity conditions.

Lemma 3. *Let* $\vec{\Psi}\in C^1([0,2])$ *with* $\vec{\Psi}(1)\geq\vec{0}$ *and* $\vec{\Psi}(2)\geq\vec{0}$ *. If* $\vec{L}_3\vec{\Psi}\leq\vec{0}$ *on* $[0,1)$ *and* $\vec{L}_4\vec{\Psi}\leq\vec{0}$ *on* $[1,2)$ *, then* $\vec{\Psi}\geq\vec{0}$ *on* $[0,2]$.

Proof. Let $\varkappa^*$ be a point , where $\vec{\Psi}(\varkappa^*)=\min\limits_{\varkappa\in[0,1]}\vec{\Psi}(\varkappa)$. Assume for contradiction that $\vec{\Psi}(\varkappa^*)<\vec{0}$; this implies that $\varkappa^*\neq 1$ and $\vec{\Psi}'(\varkappa^*)\geq\vec{0}$. If $\varkappa^*\in[0,1)$. Then, this can analyzed as

$$\vec{L}_3\vec{\Psi}(\varkappa^*)=\mu\mathfrak{a}(\varkappa^*)\vec{\Psi}'(\varkappa^*)-\mathfrak{b}(\varkappa^*)\vec{\Psi}(\varkappa^*)>\vec{0},$$

which contradicts the assumption that $L_3\vec{\Psi}\leq\vec{0}$ on $[0,1)$. Furthermore, if $\varkappa^*\in[1,2)$

$$\vec{L}_4\vec{\Psi}(\varkappa^*)=\mu\mathfrak{a}(\varkappa^*)\vec{\Psi}'(\varkappa^*)-\mathfrak{b}(\varkappa^*)\vec{\Psi}(\varkappa^*)+\mathfrak{d}(\varkappa^*)\vec{\Psi}(\varkappa^*-1)>\vec{0},$$

which also leads to a contradiction. Thus, $\vec{\Psi}(\varkappa)\geq\vec{0}$ on $[0,2]$.
The decomposition of $\vec{y}_0(\varkappa;\mu)$ and $\vec{z}_0(\varkappa;\mu)$ is given by

$$\vec{y}_0(\varkappa;\mu)=\vec{q}_0(\varkappa)+\mu\vec{q}_1(\varkappa)+\mu^2\vec{q}_2(\varkappa)+\mu^3\vec{q}_3(\varkappa),\quad \text{for } \varkappa\in\Omega_1,$$

$$\vec{z}_0(\varkappa;\mu)=\vec{\rho}_0(\varkappa)+\mu\vec{\rho}_1(\varkappa)+\mu^2\vec{\rho}_2(\varkappa)+\mu^3\vec{\rho}_3(\varkappa),\quad \text{for } \varkappa\in\Omega_2,$$

where

$$-\boldsymbol{\mathcal{B}}(\varkappa)\vec{q}_0(\varkappa)=\vec{g}(\varkappa),\quad \text{for } \varkappa\in[0,1], \tag{67}$$

$$-\boldsymbol{\mathcal{B}}(\varkappa)\vec{\rho}_0(\varkappa)+\boldsymbol{\mathcal{D}}(\varkappa)\vec{q}_0(\varkappa-1)=\vec{f}(\varkappa),\quad \text{for } \varkappa\in[1,2], \tag{68}$$

$$\boldsymbol{\mathcal{B}}(\varkappa)\vec{q}_1(\varkappa) = \boldsymbol{\mathcal{A}}(\varkappa)\vec{q}\,'_0(\varkappa), \quad \text{for } \varkappa \in [0,1], \tag{69}$$

$$\boldsymbol{\mathcal{B}}(\varkappa)\vec{\rho}_1(\varkappa) - \boldsymbol{\mathcal{D}}(\varkappa)\vec{q}_1(\varkappa-1) = \mathcal{A}(\varkappa)\vec{\rho}\,'_0(\varkappa), \quad \text{for } \varkappa \in [1,2], \tag{70}$$

$$\boldsymbol{\mathcal{B}}(\varkappa)\vec{q}_2(\varkappa) = \boldsymbol{\mathcal{A}}(\varkappa)\vec{q}\,'_1(\varkappa), \quad \text{for } \varkappa \in [0,1], \tag{71}$$

$$\boldsymbol{\mathcal{B}}(\varkappa)\vec{\rho}_2(\varkappa) - \boldsymbol{\mathcal{D}}(\varkappa)\vec{q}_2(\varkappa-1) = \boldsymbol{\mathcal{A}}(\varkappa)\vec{\rho}\,'_1(\varkappa), \quad \text{for } \varkappa \in [1,2], \tag{72}$$

$$\vec{L}_3\vec{q}_3(\varkappa) = -\boldsymbol{\mathcal{A}}(\varkappa)\vec{q}\,'_2(\varkappa), \quad \text{for } \varkappa \in [0,1), \quad \vec{q}_3(1) = \vec{0}, \tag{73}$$

$$\vec{L}_4\vec{\rho}_3(\varkappa) = -\boldsymbol{\mathcal{A}}(\varkappa)\vec{\rho}\,'_2(\varkappa), \quad \text{for } \varkappa \in [1,2), \quad \vec{\rho}_3(2) = \vec{0}, \tag{74}$$

$$\vec{\rho}_3(\varkappa) = \vec{q}_3(\varkappa) \quad \text{on } [0,1). \tag{75}$$

Since $\boldsymbol{\mathcal{A}}(\varkappa)$, $\boldsymbol{\mathcal{B}}(\varkappa)$, $\boldsymbol{\mathcal{D}}(\varkappa)$, and $\vec{f}(\varkappa)$ are sufficiently smooth, it follows that

$$||\,\vec{q}_{0,0}^{(k)}\,||_{\Omega_1} \le C, \quad ||\,\vec{q}_{0,1}^{(k)}\,||_{\Omega_1} \le C, \quad ||\,\vec{q}_{0,2}^{(k)}\,||_{\Omega_1} \le C, \quad \text{for } 0 \le k \le 7,$$

$$||\,\vec{\rho}_{0,0}^{(k)}\,||_{\Omega_2} \le C, \quad ||\,\vec{\rho}_{0,1}^{(k)}\,||_{\Omega_2} \le C, \quad ||\,\vec{\rho}_{0,2}^{(k)}\,||_{\Omega_2} \le C, \quad \text{for } 0 \le k \le 7.$$

Using Lemma 3, it can be proved that $||\,\vec{q}_3(1)\,|| \le C$, $||\,\vec{q}_3(2)\,|| \le C$, $||\,\vec{\rho}_3(1)\,|| \le C$, $||\,\vec{\rho}_3(2)\,|| \le C$. Also, from (73),

$$||\vec{q}_{31}|| \le C, \quad ||\vec{q}_{32}|| \le C, \quad ||\vec{\rho}_{31}|| \le C \quad \text{and} \quad ||\vec{\rho}_{32}|| \le C.$$

Hence, choosing

$$\vec{y}_0(1) = \vec{q}_0(1) + \mu\vec{q}_1(1) + \mu^2\vec{q}_2(1), \vec{z}_0(2) = \vec{\rho}_0(2) + \mu\vec{\rho}_1(2) + \mu^2\vec{\rho}_2(2),$$

yields $||\vec{y}_0^{(k)}|| \le C$ for $0 \le k \le 7$ and $||\vec{\rho}_0^{(k)}|| \le C$ for $0 \le k \le 3$. Similarly, finding $\vec{y}_1, \vec{y}_2$, and $\vec{y}_3$ and using Lemma 3, $||\vec{y}_1^{(k)}|| \le C(1+\mu^{-(k-1)})$, $0 \le k \le 3$ and $||\vec{y}_2^{(k)}|| \le C\mu^{-(k+1)}$, $0 \le k \le 3$. Utilizing the bounds from Theorem 1, the resulting estimates are as follows:

$$||y_{32}|| \le C\mu^{-3}, \quad ||y'_{32}|| \le C\epsilon_2^{-1}\mu^{-2}, \quad ||y''_{32}|| \le C\epsilon_2^{-2}\mu^{-1}.$$

Also , from the equation of $\vec{y}_{31}$ in (65),

$$\epsilon_1 y''_{31} + \mu\mathfrak{a}_1 y'_{31} - \mathfrak{b}_{11}y_{31} = -\frac{\epsilon_1}{\epsilon_2}y''_{21} - \mathfrak{b}_{12}y_{32},$$

and these bounds are obtained

$$||y_{31}|| \le C\mu^{-3}, \quad ||y'_{31}|| \le C\mu^{-4}, \quad ||y''_{31}|| \le C\epsilon_2^{-1}\mu^{-3}, \text{and} \quad ||y'''_{31}|| \le C(\epsilon_1^{-1} + \epsilon_2^{-2}\mu^{-3}).$$

Differentiating the equation of $\vec{y}_{32}$ and using the bounds of $\vec{y}_{31}$ and its derivatives,

$$||y'''_{32}|| \le C\epsilon_2^{-3} + C\epsilon_2^{-1}\mu^{-4} + C\epsilon_2^{-2}\mu^{-2}.$$

similarly,

$$||z_{31}|| \le C\mu^{-3}, \quad ||z'_{31}|| \le C\mu^{-4}, \quad ||z''_{31}|| \le C\epsilon_2^{-1}\mu^{-3}, \quad ||z'''_{31}|| \le C(\epsilon_1^{-1} + \epsilon_2^{-2}\mu^{-3})$$

$$\text{and} \qquad ||z'''_{32}|| \le C\epsilon_2^{-3} + C\epsilon_2^{-1}\mu^{-4} + C\epsilon_2^{-2}\mu^{-2}.$$

Hence,

$$\vec{r}(0) = \vec{y}_0(0) + \epsilon_2\vec{y}_1(0) + \epsilon_2^2\vec{y}_2(0) + \epsilon_3^2\vec{y}_3(0), \quad \vec{r}(1) = \vec{y}_0(1) + \epsilon_2\vec{y}_1(1), \tag{76}$$

$$\vec{s}(0) = \vec{y}_0(1) + \epsilon_2\vec{y}_1(1) + \epsilon_2^2\vec{y}_2(1) + \epsilon_3^2\vec{y}_3(1), \quad \vec{s}(2) = \vec{y}_0(2) + \epsilon_2\vec{y}_1(2) \tag{77}$$

and by utilizing all the aforementioned bounds, the bounds of the regular component are obtained by

$$||\vec{r}^{(k)}|| \leq C \quad \text{for } k = 0, 1, 2, \quad ||r_1'''|| \leq C\epsilon_1^{-1}, \quad ||r_2'''|| \leq C. \tag{78}$$

Similarly, it can be determined that

$$||\vec{s}^{(k)}|| \leq C \quad \text{for } k = 0, 1, 2, \quad ||s_1'''|| \leq C\epsilon_1^{-1}, \quad ||s_2'''|| \leq C. \tag{79}$$

The proof of the lemma is complete. □

6. Layer Functions

The functions for the layers are denoted by $\mathfrak{B}_i^{l_1}(\varkappa)$ and $\mathfrak{B}_i^{r_1}(\varkappa), i = 1, 2$ and are specified over the interval $[0, 1]$,

$$\mathfrak{B}_1^{l_1}(\varkappa) = \begin{cases} e^{-\theta_1\varkappa}, & \alpha\mu^2 \leq \gamma\epsilon_1 \\ e^{-\lambda_1\varkappa}, & \alpha\mu^2 \geq \gamma\epsilon_2 \end{cases}, \mathfrak{B}_1^{r_1}(\varkappa) = \begin{cases} e^{-\theta_1(1-\varkappa)}, & \alpha\mu^2 \leq \gamma\epsilon_1 \\ e^{-\kappa(1-\varkappa)}, & \alpha\mu^2 \geq \gamma\epsilon_2 \end{cases}, \tag{80}$$

$$\mathfrak{B}_2^{l_1}(\varkappa) = \begin{cases} e^{-\theta_2\varkappa}, & \alpha\mu^2 \leq \gamma\epsilon_1 \\ e^{-\lambda_2\varkappa}, & \alpha\mu^2 \geq \gamma\epsilon_2 \end{cases}, \mathfrak{B}_2^{r_1}(\varkappa) = e^{-\theta_2(1-\varkappa)}, \quad \alpha\mu^2 \leq \gamma\epsilon_1 \tag{81}$$

The layer functions $\mathfrak{B}_i^{l_2}(\varkappa), \mathfrak{B}_i^{r_2}(\varkappa), i = 1, 2$ are specified over $[1, 2]$,

$$\mathfrak{B}_1^{l_2}(\varkappa) = \begin{cases} e^{-\theta_1(\varkappa-1)}, & \alpha\mu^2 \leq \gamma\epsilon_1 \\ e^{-\lambda_1(\varkappa-1)}, & \alpha\mu^2 \geq \gamma\epsilon_2 \end{cases}, \mathfrak{B}_1^{r_2}(\varkappa) = \begin{cases} e^{-\theta_1(2-\varkappa)}, & \alpha\mu^2 \leq \gamma\epsilon_1 \\ e^{-\kappa(2-\varkappa)}, & \alpha\mu^2 \geq \gamma\epsilon_2 \end{cases}, \tag{82}$$

$$\mathfrak{B}_2^{l_2}(\varkappa) = \begin{cases} e^{-\theta_2(\varkappa-1)}, & \alpha\mu^2 \leq \gamma\epsilon_1 \\ e^{-\lambda_2(\varkappa-1)}, & \alpha\mu^2 \geq \gamma\epsilon_2 \end{cases}, \mathfrak{B}_2^{r_2}(\varkappa) = e^{-\theta_2(2-\varkappa)}, \quad \alpha\mu^2 \leq \gamma\epsilon_1 \tag{83}$$

where $\theta_i = \sqrt{\frac{\gamma\alpha}{\epsilon_i}}$; $\lambda_i = \frac{\alpha\mu}{\epsilon_i}$; $\kappa = \frac{\gamma}{2\mu}$, for $i = 1, 2$. Following the Lemma 5 presented in [13], the points $\varkappa_s \in \left(0, \frac{1}{2}\right)$, which satisfy the conditions for the case $\alpha\mu^2 \leq \gamma\epsilon_1$, can be proved.

$$\frac{\mathfrak{B}_1^{li}(\varkappa_s)}{\epsilon_1^s} = \frac{\mathfrak{B}_2^{li}(\varkappa_s)}{\epsilon_2^s}, \frac{\mathfrak{B}_1^{ri}(1-\varkappa_s)}{\epsilon_1^s} = \frac{\mathfrak{B}_2^{ri}(1-\varkappa_s)}{\epsilon_2^s},$$

$$\frac{\mathfrak{B}_1^{li}(\varkappa_s-1)}{\epsilon_1^s} = \frac{\mathfrak{B}_2^{li}(\varkappa_s-1)}{\epsilon_2^s}, \quad \text{and} \quad \frac{\mathfrak{B}_1^{ri}(2-\varkappa_s)}{\epsilon_1^s} = \frac{\mathfrak{B}_2^{ri}(2-\varkappa_s)}{\epsilon_2^s}, \quad 0 \leq s \leq \frac{3}{2}.$$

Similarly, for the case $\alpha\mu^2 \geq \gamma\epsilon_2$, it can be shown that there are points $\varkappa_k, k = 1, 2, 3$ in $\left(0, \frac{1}{2}\right)$ such that

$$\frac{\mathfrak{B}_1^{li}(\varkappa_k)}{\epsilon_1^k} = \frac{\mathfrak{B}_2^{li}(\varkappa_k)}{\epsilon_2^k}.$$

7. Bounds on the Singular Component and Its Derivatives

Theorem 2. *Let $\vec{w}^L, \vec{w}^R$ satisfy problems (16), (19) and (25), (28) for the cases $\alpha\mu^2 \le \gamma\epsilon_1$ and $\alpha\mu^2 \ge \gamma\epsilon_2$, respectively. Consequently, the components of $\vec{w}^{L_1}$ and $\vec{w}^{R_1}$ satisfy the following bounds on $(0,1)$.*

For the case $\alpha\mu^2 \le \gamma\epsilon_1$,

$$\begin{aligned}
|w_1^{L_1}(\varkappa)| \le C\mathfrak{B}_2^{l_1}(\varkappa); \quad |w_2^{L_1}(\varkappa)| \le C\mathfrak{B}_2^{l_1}(\varkappa);\\
|w_1^{L_1,(k)}(\varkappa)| \le C\Big(\epsilon_1^{-k/2}\mathfrak{B}_1^{l_1}(\varkappa) + \epsilon_2^{-k/2}\mathfrak{B}_2^{l_1}(\varkappa)\Big), \quad k = 1,2,3;\\
|w_2^{L_1,(k)}(\varkappa)| \le C\epsilon_2^{-k/2}\mathfrak{B}_2^{l_1}(\varkappa), \quad k = 1,2;\\
|w_2^{L_1,(3)}(\varkappa)| \le C\epsilon_2^{-1}\Big(\epsilon_1^{-1/2}\mathfrak{B}_1^{l_1}(\varkappa) + \epsilon_2^{-1/2}\mathfrak{B}_2^{l_1}(\varkappa)\Big).
\end{aligned}$$

For the case $\alpha\mu^2 \ge \gamma\epsilon_2$,

$$\begin{aligned}
|w_1^{L_1}(\varkappa)| \le C\mathfrak{B}_2^{l_1}(\varkappa); \quad |w_2^{L_1}(\varkappa)| \le C\mathfrak{B}_2^{l_1}(\varkappa);\\
|w_1^{L_1,(k)}(\varkappa)| \le C\mu^k\Big(\epsilon_1^{-k}\mathfrak{B}_1^{l_1}(\varkappa) + \epsilon_2^{-k}\mathfrak{B}_2^{l_1}(\varkappa)\Big), \quad k = 1,2,3;\\
|w_2^{L_1,(k)}(\varkappa)| \le C\mu^k\epsilon_2^{-k}\mathfrak{B}_2^{l_1}(\varkappa), \quad k = 1,2;\\
|w_2^{L_1,(3)}(\varkappa)| \le C\left(\frac{\mu^3}{\epsilon_1\epsilon_2^2}\mathfrak{B}_1^{l_1}(\varkappa) + \frac{\mu^3}{\epsilon_2^3}\mathfrak{B}_2^{l_1}(\varkappa)\right).
\end{aligned}$$

Moreover, the components satisfy the following bounds of $\vec{w}^{R_1}$. For the case $\alpha\mu^2 \le \gamma\epsilon_1$,

$$\begin{aligned}
|w_1^{R_1}(\varkappa)| \le C\mathfrak{B}_2^{r_1}(\varkappa); \quad |w_2^{R_1}(\varkappa)| \le C\mathfrak{B}_2^{r_1}(\varkappa);\\
|w_1^{R_1,(k)}(\varkappa)| \le C\Big(\epsilon_1^{-k/2}\mathfrak{B}_1^{r_1}(\varkappa) + \epsilon_2^{-k/2}\mathfrak{B}_2^{r_1}(\varkappa)\Big), \quad k = 1,2,3;\\
|w_2^{R_1,(k)}(\varkappa)| \le C\epsilon_2^{-k/2}\mathfrak{B}_2^{r_1}(\varkappa), \quad k = 1,2;\\
|w_2^{R_1,(3)}(\varkappa)| \le C\epsilon_2^{-1}\Big(\epsilon_1^{-1/2}\mathfrak{B}_1^{r_1}(\varkappa) + \epsilon_2^{-1/2}\mathfrak{B}_2^{r_1}(\varkappa)\Big).
\end{aligned}$$

For the case $\alpha\mu^2 \ge \gamma\epsilon_2$,

$$\begin{aligned}
|w_1^{R_1}(\varkappa)| \le C; \quad |w_2^{R_1}(\varkappa)| \le C;\\
|w_1^{R_1,(k)}(\varkappa)| \le C\mu^{-k}, \quad k = 1,2;\\
|w_1^{R_1,(3)}(\varkappa)| \le C\mu^{-3} + C\epsilon_1^{-1};\\
|w_2^{R_1,(k)}(\varkappa)| \le C\mu^{-k}, \quad k = 1,2,3.
\end{aligned}$$

Components of $\vec{w}^{L_2}$ and $\vec{w}^{R_2}$on $(1,2)$, for the case $\alpha\mu^2 \le \gamma\epsilon_1$,

$$\begin{aligned}
|w_1^{L_2}(\varkappa)| \le C\mathfrak{B}_2^{l_2}(\varkappa); \quad |w_2^{L_2}(\varkappa)| \le C\mathfrak{B}_2^{l_2}(\varkappa);\\
|w_1^{L_2,(k)}(\varkappa)| \le C\Big(\epsilon_1^{-k/2}\mathfrak{B}_1^{l_2}(\varkappa) + \epsilon_2^{-k/2}\mathfrak{B}_2^{l_2}(\varkappa)\Big), \quad k = 1,2,3;\\
|w_2^{L_2,(k)}(\varkappa)| \le C\epsilon_2^{-k/2}\mathfrak{B}_2^{l_2}(\varkappa), \quad k = 1,2;\\
|w_2^{L_2,(3)}(\varkappa)| \le C\epsilon_2^{-1}\Big(\epsilon_1^{-1/2}\mathfrak{B}_1^{l_2}(\varkappa) + \epsilon_2^{-1/2}\mathfrak{B}_2^{l_2}(\varkappa)\Big).
\end{aligned}$$

For the case $\alpha\mu^2 \geq \gamma\epsilon_2$,

$$\begin{aligned}
|w_1^{L_2}(\varkappa)| \leq C\mathfrak{B}_2^{l_2}(\varkappa); \quad |w_2^{L_2}(\varkappa)| \leq C\mathfrak{B}_2^{l_2}(\varkappa);\\
|w_1^{L_2,(k)}(\varkappa)| \leq C\mu^k\Big(\epsilon_1^{-k}\mathfrak{B}_1^{l_2}(\varkappa) + \epsilon_2^{-k}\mathfrak{B}_2^{l_2}(\varkappa)\Big), \quad k = 1,2,3;\\
|w_2^{L_2,(k)}(\varkappa)| \leq C\mu^k\epsilon_2^{-k}\mathfrak{B}_2^{l_2}(\varkappa), \quad k = 1,2;\\
|w_2^{L_2,(3)}(\varkappa)| \leq C\left(\frac{\mu^3}{\epsilon_1\epsilon_2^2}\mathfrak{B}_1^{l_2}(\varkappa) + \frac{\mu^3}{\epsilon_2^3}\mathfrak{B}_2^{l_2}(\varkappa)\right).
\end{aligned}$$

Moreover, the components satisfy the following bounds of $\vec{w}^{R_2}$. *For the case* $\alpha\mu^2 \leq \gamma\epsilon_1$,

$$\begin{aligned}
|w_1^{R_2}(\varkappa)| \leq C\mathfrak{B}_2^{r_2}(\varkappa); \quad |w_2^{R_2}(\varkappa)| \leq C\mathfrak{B}_2^{r_2}(\varkappa);\\
|w_1^{R_2,(k)}(\varkappa)| \leq C\Big(\epsilon_1^{-k/2}\mathfrak{B}_1^{r_2}(\varkappa) + \epsilon_2^{-k/2}\mathfrak{B}_2^{r_2}(\varkappa)\Big), \quad k = 1,2,3;\\
|w_2^{R_2,(k)}(\varkappa)| \leq C\epsilon_2^{-k/2}\mathfrak{B}_2^{r_2}(\varkappa), \quad k = 1,2;\\
|w_2^{R_2,(3)}(\varkappa)| \leq C\epsilon_2^{-1}\Big(\epsilon_1^{-1/2}\mathfrak{B}_1^{r_2}(\varkappa) + \epsilon_2^{-1/2}\mathfrak{B}_2^{r_2}(\varkappa)\Big).
\end{aligned}$$

For the case $\alpha\mu^2 \geq \gamma\epsilon_2$,

$$\begin{aligned}
|w_1^{R_2}(\varkappa)| \leq C; \quad |w_2^{R_2}(\varkappa)| \leq C;\\
|w_1^{R_2,(k)}(\varkappa)| \leq C\mu^{-k}, \quad k = 1,2;\\
|w_1^{R_2,(3)}(\varkappa)| \leq C\mu^{-3} + C\epsilon_1^{-1};\\
|w_2^{R_2,(k)}(\varkappa)| \leq C\mu^{-k}, \quad k = 1,2,3.
\end{aligned}$$

Proof. The bounds on $\vec{w}^{R_1}$ and its derivatives are established for the case $\alpha\mu^2 \geq \gamma\epsilon_2$. In this scenario, $\vec{w}^{R_1}$ is decomposed over the interval $(0,1)$.

$$\vec{w}^{R_1} = \vec{q}_0^{R_1} + \epsilon_2\vec{q}_1^{R_1} + \epsilon_2^2\vec{q}_2^{R_1} + \epsilon_2^3\vec{q}_3^{R_1}, \tag{84}$$

where $\vec{q}_i^{R_1} = \begin{pmatrix} q_{i1}^{R_1} \\ q_{i2}^{R_1} \end{pmatrix}$, $i = 0,1,2,3$, and their equations are given as follows:

$$\mu\boldsymbol{\mathcal{A}}(\varkappa)\vec{q}_0^{R_1,\prime} - \boldsymbol{\mathcal{B}}(\varkappa)\vec{q}_0^{R_1} = \vec{0}; \quad q_{01}^{R_1}(1) = u_1(1) - v_1(1); \quad q_{02}^{R_1}(1) = u_2(1) - v_2(1), \tag{85}$$

$$\mu\boldsymbol{\mathcal{A}}(\varkappa)\vec{q}_1^{R_1,\prime} - \boldsymbol{\mathcal{B}}(\varkappa)\vec{q}_1^{R_1} = \left(-\frac{\epsilon_1}{\epsilon_2}q_{01}^{R_1,\prime\prime}, -q_{02}^{R_1,\prime\prime}\right)^T, \quad q_{11}^{R_1}(1) = q_{12}^{R_1}(1) = 0, \tag{86}$$

$$\mu\boldsymbol{\mathcal{A}}(\varkappa)\vec{q}_2^{R_1,\prime} - \boldsymbol{\mathcal{B}}(\varkappa)\vec{q}_2^{R_1} = \left(-\frac{\epsilon_1}{\epsilon_2}q_{11}^{R_1,\prime\prime}, -q_{12}^{R_1,\prime\prime}\right)^T, \quad q_{21}^{R_1}(1) = q_{22}^{R_1}(1) = 0, \tag{87}$$

$$\epsilon_1\vec{q}_3^{R_1,\prime\prime} + \mu\boldsymbol{\mathcal{A}}(\varkappa)\vec{q}_3^{R_1,\prime} - \boldsymbol{\mathcal{B}}(\varkappa)\vec{q}_3^{R_1} = \left(-\frac{\epsilon_1}{\epsilon_2}q_{21}^{R_1,\prime\prime}, -q_{22}^{R_1,\prime\prime}\right)^T, \tag{88}$$

$$q_{31}^{R_1}(0) = q_{32}^{R_1}(0) = 0; \quad q_{31}^{R_1}(1) = q_{32}^{R_1}(1) = 0.$$

It can be easily derived from Equations (85)–(87) that,

$$|\vec{q}_0^{R_1,(k)}| \leq \frac{C}{\mu^k}, \quad 0 \leq k \leq 5, \quad |\vec{q}_1^{R_1,(k)}| \leq \frac{C}{\mu^{(k+2)}}, \quad 0 \leq k \leq 4,$$

$$|\vec{q}_2^{R_1,(k)}| \leq \frac{C}{\mu^{(k+4)}}, \quad 0 \leq k \leq 3.$$

Also, from the equation of $q_{32}^{R_1}$ in (88) using Theorem 1 and the fact that $\alpha\mu^2 \geq \gamma\epsilon_2$, the following can be derived:

$$|q_{32}^{R_1}| \leq C\mu^{-6}, \quad |q_{32}^{R_1,\prime}| \leq C\epsilon_2^{-1}\mu^{-5}, \quad |q_{32}^{R_1,\prime\prime}| \leq C\epsilon_2^{-2}\mu^{-4}.$$

Consider the equation of $q_{31}^{R_1}$ in (88), which can be written as,

$$\epsilon_1 q_{31}^{R_1,\prime\prime} + \mu\mathfrak{a}_1 q_{31}^{R_1,\prime} - \mathfrak{b}_{11} q_{31}^{R_1} = -\frac{\epsilon_1}{\epsilon_2} q_{21}^{R_1,\prime\prime} - \mathfrak{b}_{12} q_{32}^{R_1}.$$

The decomposition of $q_{31}^{R_1}$ is as follows:

$$q_{31}^{R_1} = \wp_0 + \epsilon_1\wp_1 + \epsilon_1^2\wp_2 + \epsilon_1^3\wp_3,$$

where $\wp_i, i = 0, 1, 2, 3$ are characterized by,

$$\mu\mathfrak{a}_1\wp_0' - \mathfrak{b}_{11}\wp_0 = -\frac{\epsilon_1}{\epsilon_2} q_{21}^{R_1,\prime\prime} - \mathfrak{b}_{12} q_{32}^{R_1}, \quad \wp_0(1) = 0,$$

$$\mu\mathfrak{a}_1\wp_1' - \mathfrak{b}_{11}\wp_1 = -\wp_0'', \quad \wp_1(1) = 0,$$

$$\mu\mathfrak{a}_1\wp_2' - \mathfrak{b}_{11}\wp_2 = -\wp_1'', \quad \wp_2(1) = 0,$$

$$\epsilon_1\wp_3'' + \mu\mathfrak{a}_1\wp_3' - \mathfrak{b}_{11}\wp_3 = -\wp_2'', \quad \wp_3(0) = \wp_3(1) = 0.$$

From the above equations,

$$|\wp_0^{(k)}| \leq C\mu^{-(6+k)}, \quad k = 0, 1, \quad |\wp_0^{(k)}| \leq C\mu^{-(6+k)} + C\epsilon_2^{-(k-1)}\mu^{-(8-k)}, \quad k = 2, 3;$$

$$|\wp_1^{(k)}| \leq C\mu^{-(8+k)} + C\epsilon_2^{-1}\mu^{-(6+k)}, |\wp_2^{(k)}| \leq C\mu^{-(10+k)} + C\epsilon_2^{-2}\mu^{-(6+k)}, \quad k = 0, 1;$$

$$|\wp_1^{(k)}| \leq C\mu^{-(8+k)} + C\epsilon_2^{-k}\mu^{-(8-k)} + C\epsilon_1^{(2-k)}\epsilon_2^{(2-k)}\mu^{-(k-1)}, \quad k = 2, 3;$$

$$|\wp_2^{(k)}| \leq C\mu^{-(10+k)} + C\epsilon_2^{-(k+1)}\mu^{-(8-k)} + C\epsilon_1^{(1-k)}\epsilon_2^{-1}\mu^{-(k+1)}, \quad k = 2, 3.$$

By applying Theorem 3 of [6], the following bounds are established by

$$|\wp_3| \leq C\mu^{-12} + C\epsilon_2^{-3}\mu^{-6} + C\epsilon_1^{-1}\epsilon_2^{-1}\mu^{-3};$$

$$|\wp_3^{(k)}| \leq C\epsilon_1^{-k}(\mu^{-(12-k)} + \epsilon_2^{-3}\mu^{-(6-k)} + \epsilon_1^{-1}\epsilon_2^{-1}\mu^{-(3-k)}), \quad k = 1, 2, 3.$$

Using $\wp_0$, $\wp_1$, $\wp_2$, and $\wp_3$, the following bounds of $q_{31}^{R_1}$ are obtained by

$$|q_{31}^{R_1,(k)}| \leq C\mu^{-(6+k)} + C\epsilon_2^{-k}\mu^{-(6-k)}, \quad k = 0, 1, 2;$$

$$|q_{31}^{R_1,(3)}| \leq C\mu^{-9} + C\epsilon_2^{-2}\mu^{-5} + C\epsilon_2^{-3}\mu^{-3} + C\epsilon_1^{-1}\epsilon_2^{-1}.$$

Now, using the bounds of $q_{01}^{R_1}, q_{11}^{R_1}, q_{21}^{R_1}$, and $q_{31}^{R_1}$and

$$|q_{32}^{R_1,(3)}| \leq C\mu^{-7}\epsilon_2^{-1} + C\epsilon_2^{-2}\mu^{-5} + C\epsilon_2^{-3}\mu^{-3},$$

using the above bounds, the following bounds of $\vec{w}^{R_1}$ and its derivatives are found for $k = 0, 1, 2, 3$

$$|w_1^{R_1,(k)}| \leq C\mu^{-k}, \quad k = 0, 1, 2; \quad |w_1^{R_1,(3)}| \leq C\epsilon_1^{-1} + C\mu^{-3}; \quad |w_2^{R_1,(k)}| \leq C\mu^{-k}. \tag{89}$$

For the case $\alpha\mu^2 \geq \gamma\epsilon_2$, we define the barrier functions $\vec{\psi}^{\pm} = \begin{pmatrix} \psi_1^{\pm} \\ \psi_2^{\pm} \end{pmatrix}$, where $\psi_i^{\pm} = C\mathfrak{B}_2^{l_1} \pm w_i^{L_1}$ for $i = 1, 2$. It is evident that $\vec{\psi}_i^{\pm}(0) \geq \vec{0}$ and $\vec{\psi}_i^{\pm}(1) \geq \vec{0}$. Additionally, $\vec{L}_1\psi^{\pm}(x) \leq 0$ for all x in the interval $(0, 1)$. Therefore, it can demonstrate that $|w_1^{L_1}(\varkappa)| \leq C\mathfrak{B}_2^{l_1}(\varkappa)$ and $|w_2^{L_1}(\varkappa)| \leq C\mathfrak{B}_2^{l_1}(\varkappa)$.
Considering the equation of $w_2^{L_1}$ from (25),

$$\epsilon_2 w_2^{L_1,''}(\varkappa) + \mu\mathfrak{a}_2(\varkappa)w_2^{L_1,'}(\varkappa) + \mathfrak{b}_{21}(\varkappa)w_1^{L_1}(\varkappa) - \mathfrak{b}_{22}(\varkappa)w_2^{L_1}(\varkappa) = 0. \tag{90}$$

This can also be written as,

$$w_2^{L_1,''}(\varkappa) + \frac{\mu}{\epsilon_2}\mathfrak{a}_2(\varkappa)w_2^{L_1,'}(\varkappa) = \frac{1}{\epsilon_2}\mathfrak{b}_{22}(\varkappa)w_2^{L_1}(\varkappa) - \frac{1}{\epsilon_2}\mathfrak{b}_{21}(\varkappa)w_1^{L_1}(\varkappa)) \equiv \frac{1}{\epsilon_2}h_2(\varkappa),$$

where

$$h_2(\varkappa) = \mathfrak{b}_{22}(\varkappa)w_2^{L_1}(\varkappa) - \mathfrak{b}_{21}(\varkappa)w_1^{L_1}(\varkappa).$$

Now, taking $A_2(0) = a_0$,

$$w_2^{L_1,'}(\varkappa) = w_2^{L_1,'}(0)e^{-\frac{\mu}{\epsilon_2}(A_2(\varkappa)-a_0)} + \epsilon_2^{-1}\int_0^{\varkappa} h_2(s)e^{\frac{\mu}{\epsilon_2}(A_2(s)-A_2(\varkappa))}\,ds,$$

where $A_2(\varkappa)$ is the indefinite integral of $a_2(\varkappa)$. Using the bounds on $\vec{\mathfrak{u}}$, it is established that $|w_2^{L_1,'}(0)| \leq C\epsilon_2^{-1}$. Using the inequality $e^{-\frac{\mu}{\epsilon_2}(A_2(s)-A_2(\varkappa))} \leq e^{-\frac{\mu}{\epsilon_2}\beta(\varkappa-s)}$ and applying integration by parts, it is derived from the above that

$$|w_2^{L_1,'}(\varkappa)| \leq C\epsilon_2^{-1}\mathfrak{B}_2^{l_1}(\varkappa).$$

Using a similar argument, it can be

$$|w_1^{L_1,'}(\varkappa)| \leq C\epsilon_1^{-1}\mathfrak{B}_1^{l_1}(\varkappa) + C\epsilon_2^{-1}\mathfrak{B}_2^{l_1}(\varkappa).$$

Differentiating the equation and using a similar procedure as above, it can be shown that

$$|w_1^{L_1,''}(\varkappa)| \leq C\epsilon_1^{-2}\mathfrak{B}_1^{l_1}(\varkappa) + C\epsilon_2^{-2}\mathfrak{B}_2^{l_1}(\varkappa).$$

Similarly,

$$|w_1^{L_1,'''}(\varkappa)| \leq C\epsilon_1^{-3}\mathfrak{B}_1^{l_1}(\varkappa) + C\epsilon_2^{-3}\mathfrak{B}_2^{l_1}(\varkappa).$$

The bounds on $w_2^{L_1,'''}(\varkappa)$ can be obtained from the equation of $w_2^{L_1}$ in (25) and the bounds on $w_2^{L_1}(x)$, $w_2^{L_1,'}(x)$, and $w_1^{L_1}(x)$. The bound on $w_1^{L_1,'''}$ is obtained by differentiating the equation of $w_1^{L_1}$ in (25) twice and using a similar argument as above. Now, by differentiating the defining equation of $w_2^{L_1}$ in (25) once and using the bounds on $w_2^{L_1,''}(\varkappa)$, $w_1^{L_1,'}(\varkappa)$ and $w_2^{L_1,'}(\varkappa)$, the following can be derived below

$$|w_2^{L_1,'''}(\varkappa)| \leq C\epsilon_2^{-1}\Big(\epsilon_1^{-1}\mathfrak{B}_1^{l_1}(\varkappa) + \epsilon_2^{-2}\mathfrak{B}_2^{l_2}(\varkappa)\Big).$$

It has been established that

$$|w_1^{L_1,'}(\varkappa)| \leq C\Big(\epsilon_1^{-1}\mathfrak{B}_1^{l_1}(\varkappa) + \epsilon_2^{-1}\mathfrak{B}_2^{l_1}(\varkappa)\Big).$$

Consequently,

$$|\vec{L}_1 w_1^{L_1}(\varkappa)| \leq C\left(\frac{\mu}{\epsilon_1}\mathfrak{B}_1^{l_1}(\varkappa) + \frac{\mu}{\epsilon_2}\mathfrak{B}_2^{l_1}(\varkappa)\right),$$

where

$$\begin{aligned}\vec{L}_1 w_1^{L_1,'}(\varkappa) &\equiv \epsilon_1 (w_1^{L_1,'})''(\varkappa) + \mu \mathfrak{a}_1(\varkappa)(w_1^{L_1,'})'(\varkappa) - \mathfrak{b}_{11}(\varkappa) w_1^{L_1,'}(\varkappa) \\ &= -\mu \mathfrak{a}_1(\varkappa) w_1^{L_1,'}(\varkappa) - \mathfrak{b}_{12}(\varkappa) w_2^{L_1,'}(\varkappa) - \mathfrak{b}_{12}'(\varkappa) w_2^{L_1}(\varkappa) + \mathfrak{b}_{11}(\varkappa) w_1^{L_1}(\varkappa).\end{aligned}$$

By introducing the barrier function

$$\psi^{\pm}(\varkappa) = C\left(\frac{\mu}{\epsilon_1}\mathfrak{B}_1^{l_1}(\varkappa) + \frac{\mu}{\epsilon_2}\mathfrak{B}_2^{l_1}(\varkappa)\right) \pm w_1^{L_1,'}(\varkappa),$$

it can be demonstrated that $\vec{\psi}^{\pm} \geq \vec{0}$ on $[0,1]$ and $\vec{L}_1 \vec{\psi}^{\pm}(\varkappa) \leq \vec{0}$ on $[0,1]$, which implies

$$|w_1^{L_1,'}(\varkappa)| \leq C\left(\frac{\mu}{\epsilon_1}\mathfrak{B}_1^{l_1}(\varkappa) + \frac{\mu}{\epsilon_2}\mathfrak{B}_2^{l_1}(\varkappa)\right),$$

By introducing another barrier function

$$\phi^{\pm}(\varkappa) = C\left(\frac{\mu}{\epsilon_1^2}\mathfrak{B}_1^{l_1}(\varkappa) + \frac{\mu}{\epsilon_2^2}\mathfrak{B}_2^{l_1}(\varkappa)\right) \pm w_1^{L_1,''}(\varkappa),$$

it is derived that

$$|w_1^{L_1,''}(\varkappa)| \leq C\mu\left(\epsilon_1^{-2}\mathfrak{B}_1^{l_1}(\varkappa) + \epsilon_2^{-2}\mathfrak{B}_2^{l_1}(\varkappa)\right)$$

$$|w_2^{L_1,''}(\varkappa)| \leq C\frac{\mu^2}{\epsilon_2^2}\mathfrak{B}_2^{l_1}(\varkappa).$$

By differentiating the equations of $w_1^{L_1}$ and $w_2^{L_1}$ once and applying the bounds of $w_1^{L_1,'}$, $w_1^{L_1,''}$, $w_2^{L_1,'}$, and $w_2^{L_1,''}$, it is found that

$$|w_1^{L_1,'''}(\varkappa)| \leq C\left(\frac{\mu^3}{\epsilon_1^3}\mathfrak{B}_1^{l_1}(\varkappa) + \frac{\mu^3}{\epsilon_1\epsilon_2^2}\mathfrak{B}_2^{l_1}(\varkappa)\right),$$

$$|w_2^{L_1,'''}(\varkappa)| \leq C\left(\frac{\mu^3}{\epsilon_1\epsilon_2^2}\mathfrak{B}_1^{l_1}(\varkappa) + \frac{\mu^3}{\epsilon_2^3}\mathfrak{B}_2^{l_1}(\varkappa)\right).$$

Similarly, analogous results can be derived for $\vec{w}^{L_2}$ on $[1,2]$.
Next, the bounds on $\vec{w}^{L_1}$ for the case $\alpha\mu^2 \leq \gamma\epsilon_1$ are derived. The bounds on $w_1^{L_1}(\varkappa)$ and $w_2^{L_1}(\varkappa)$ can be derived by defining the barrier functions

$$\psi_i^{\pm}(\varkappa) = C\mathfrak{B}_2^{l_1}(\varkappa) \pm w_i^{L_1}(\varkappa), \quad i = 1,2.$$

To bound $w_1^{L_1,'}(\varkappa)$ and $w_2^{L_1,'}(\varkappa)$, the argument proceeds as follows, and from the equation of $w_2^{L_1}$ in Equation (16) and Theorem 1, it is obtained that $|w_2^{L_1,'}(\varkappa)| \leq C\epsilon_2^{-1/2}\mathfrak{B}_2^{l_1}(\varkappa)$. From the equation of $w_1^{L_1}$ in (16) and Theorem 1, $|w_1^{L_1,'}(\varkappa)| \leq C\epsilon_1^{-1/2}\mathfrak{B}_2^{l_1}(\varkappa)$. To improve the above bound on $|w_1^{L_1}(\varkappa)|$, proceed with the following and differentiate $w_1^{L_1}$ in (16) once, obtaining

$$|\vec{L}_1 w_1^{L_1,'}(\varkappa)| \leq C\epsilon_2^{-1/2}\mathfrak{B}_2^{l_1}(x),$$

To establish the necessary bounds, define the barrier functions as follows:

$$\phi_i^{\pm}(x) = C\left(\epsilon_1^{-1/2}\mathfrak{B}_1^{l_1}(\varkappa) + \epsilon_2^{-1/2}\mathfrak{B}_2^{l_1}(\varkappa)\right) \pm w_i^{L_1,'}(\varkappa), i = 1,2$$

and apply the minimum principle for the operator $\vec{L}_1$. The bound on $w_2^{L_1,''}(\varkappa)$ is obtained from the equation of $w_2^{L_1}$ in (16). To bound $w_1^{L_1,''}(\varkappa)$ and $w_1^{L_1,'''}(\varkappa)$, the defining equation of $w_1^{L_1}$ in (16) is differentiated twice and thrice, respectively, and an argument analogous to the one used to bound $w_1^{L_1,'}(\varkappa)$ leads to the required bounds. The bound on $w_2^{L_1,'''}(\varkappa)$ is obtained by differentiating the defining equation of $w_2^{L_1}$ in (16) once and using the bounds of $w_1^{L_1,''}(\varkappa)$, $w_2^{L_1,''}(\varkappa)$, $w_1^{L_1,'}(\varkappa)$, and $w_2^{L_1,'}(\varkappa)$; it can be seen that

$$|w_1^{L_1,'''}(\varkappa)| \leq C\Big(\epsilon_1^{-3/2}\mathfrak{B}_1^{l_1}(\varkappa) + \epsilon_2^{-3/2}\mathfrak{B}_2^{l_1}(\varkappa)\Big),$$

$$|w_2^{L_1,'''}(\varkappa)| \leq C\epsilon_2^{-1}\Big(\epsilon_1^{-1/2}\mathfrak{B}_1^{l_1}(\varkappa) + \epsilon_2^{-1/2}\mathfrak{B}_2^{l_1}(\varkappa)\Big).$$

Similarly, using an analogous way, it can be derived for $\vec{w}^{L_2}$ on [1,2].

For the case $\alpha\mu^2 \leq \gamma\epsilon_1$, the bounds on $\vec{w}^{R_2}$ are obtained by the above analogous arguments to those applied in bounding $\vec{w}^{L}$. The proof of the theorem is complete. □

8. Sharper Bounds for $w_1^{L_1}, w_2^{L_1}, w_1^{L_2}$, and $w_2^{L_2}$

To achieve sharper bounds on the derivatives of the singular components $w_1^{L_1}, w_2^{L_1}, w_1^{L_2}$, and $w_2^{L_2}$, decompose these components further for [0,1] and [1,2]. This refinement will help to demonstrate the nearly first-order convergence of the proposed method. Now, focusing on the case $\alpha\mu^2 \geq \gamma\epsilon_2$. For $\vec{w}^{L_1}$ and $\vec{w}^{L_2}$, decompose it as follows: $\vec{w}^{L_1}(\varkappa) = w_1^{L_1}(\varkappa) + w_2^{L_1}(\varkappa)$ on [0,1] and $\vec{w}^{L_2}(\varkappa) = w_1^{L_2}(\varkappa) + w_2^{L_2}(\varkappa)$ on [1,2]. Decompose of $w_1^{L_1}(\varkappa)$

$$w_1^{L_1}(\varkappa) = w_{11}^{L_1}(\varkappa) + w_{12}^{L_1}(\varkappa)$$

where

$$w_{11}^{L_1}(\varkappa) = \begin{cases} \sum_{k=0}^{3}\left(\frac{(\varkappa-\varkappa_2)^k}{k!}\right)w_1^{L_1,(k)}(\varkappa_2) & \text{for } \varkappa \in [0,\varkappa_2) \\ w_1^{L_1}(\varkappa) & \text{for } \varkappa \in [\varkappa_2,1] \end{cases} \tag{91}$$

$$w_{12}^{L_1}(\varkappa) = w_1^{L_1}(\varkappa) - w_{11}^{L_1}(\varkappa).$$

Decompose of $w_2^{L_1}(\varkappa)$

$$w_2^{L_1}(\varkappa) = w_{21}^{L_1}(\varkappa) + w_{22}^{L_1}(\varkappa)$$

where

$$w_{21}^{L_1}(\varkappa) = \begin{cases} \sum_{k=0}^{3}\left(\frac{(\varkappa-\varkappa_1)^k}{k!}\right)w_2^{L_1,(k)}(\varkappa_1) & \text{for } \varkappa \in [0,\varkappa_1) \\ w_2^{L_1}(\varkappa) & \text{for } \varkappa \in [\varkappa_1,1] \end{cases} \tag{92}$$

$$w_{22}^{L_1}(\varkappa) = w_2^{L_1}(\varkappa) - w_{21}^{L_1}(\varkappa).$$

Decompose of $w_1^{L_2}(\varkappa)$

$$w_1^{L_2}(\varkappa) = w_{11}^{L_2}(\varkappa) + w_{12}^{L_2}(\varkappa)$$

where

$$w_{11}^{L_2}(\varkappa) = \begin{cases} \sum_{k=0}^{3}\left(\frac{(\varkappa-\varkappa_2)^k}{k!}\right)w_1^{L_2,(k)}(\varkappa_2) & \text{for } \varkappa \in [1,\varkappa_2) \\ w_1^{L_2}(\varkappa) & \text{for } \varkappa \in [x_2,2] \end{cases} \tag{93}$$

$$w_{12}^{L_2}(\varkappa) = w_1^{L_2}(\varkappa) - w_{11}^{L_2}(\varkappa).$$

Decompose of $w_2^{L_2}(\varkappa)$

$$w_2^{L_2}(\varkappa) = w_{21}^{L_2}(\varkappa) + w_{22}^{L_2}(\varkappa)$$

where

$$w_{21}^{L_2}(\varkappa) = \begin{cases} \sum_{k=0}^{3}(\frac{(x-x_1)^k}{k!})w_2^{L_2,(k)}(\varkappa_1) & \text{for } \varkappa \in [1,\varkappa_1) \\ w_2^{L_2}(\varkappa) & \text{for } \varkappa \in [\varkappa_1,2] \end{cases} \tag{94}$$

$$w_{22}^{L_2}(\varkappa) = w_2^{L_2}(\varkappa) - w_{21}^{L_2}(\varkappa).$$

Lemma 4. *Given the decompositions of $w_{11}^{L_1}, w_{12}^{L_1}, w_{21}^{L_1}$, and $w_{22}^{L_1}$, the following bounds hold on [0,1]:*

$$\begin{aligned} |w_{11}^{L_1,'''}(\varkappa)| &\le C\mu^3\epsilon_1^{-1}\epsilon_2^{-2}\mathfrak{B}_2^{l_1}(\varkappa), \\ |w_{12}^{L_1,''}(\varkappa)| &\le C\mu^2\epsilon_1^{-2}\mathfrak{B}_1^{l_1}(\varkappa), \\ |w_{21}^{L_1,'''}(\varkappa)| &\le C\mu^3\epsilon_2^{-3}\mathfrak{B}_2^{l_1}(\varkappa), \\ |w_{22}^{L_1,''}(\varkappa)| &\le C\mu^2\epsilon_2^{-2}\mathfrak{B}_2^{l_1}(\varkappa). \end{aligned} \tag{95}$$

Similarly, the decompositions of $w_{11}^{L_2}, w_{12}^{L_2}, w_{21}^{L_2}$ and $w_{22}^{L_2}$, the following bounds hold on [1,2]:

$$\begin{aligned} |w_{11}^{L_2,'''}(\varkappa)| &\le C\mu^3\epsilon_1^{-1}\epsilon_2^{-2}\mathfrak{B}_2^{l_2}(\varkappa), \\ |w_{12}^{L_2,''}(\varkappa)| &\le C\mu^2\epsilon_1^{-2}\mathfrak{B}_1^{l_2}(\varkappa), \\ |w_{21}^{L_2,'''}(\varkappa)| &\le C\mu^3\epsilon_2^{-3}\mathfrak{B}_2^{l_2}(\varkappa), \\ |w_{22}^{L_2,''}(\varkappa)| &\le C\mu^2\epsilon_2^{-2}\mathfrak{B}_2^{l_2}(\varkappa). \end{aligned} \tag{96}$$

Proof. For $w_{11}^{L_1,'''}(\varkappa)$,
On $[0, x_2)$

$$\begin{aligned} |w_{11}^{L_1,'''}(\varkappa)| = |w_1^{L_1,'''}(\varkappa_2)| &\le C\mu^3(\epsilon_1^{-3}\mathfrak{B}_1^{l_1}(\varkappa_2) + \epsilon_1^{-1}\epsilon_2^{-2}\mathfrak{B}_2^{l_1}(\varkappa_2)) \\ &\le C\mu^3\epsilon_1^{-1}\epsilon_2^{-2}\mathfrak{B}_2^{l_1}(\varkappa). \end{aligned}$$

On $[\varkappa_2, 1]$
$|w_{11}^{L_1,'''}(\varkappa)| = |w_1^{L_1,'''}(\varkappa)| \le C\mu^3\epsilon_1^{-1}\epsilon_2^{-2}\mathfrak{B}_2^{l_1}(\varkappa)$.
For $w_{21}^{L_1,'''}(\varkappa)$, On $[0, \varkappa_1)$

$$\begin{aligned} |w_{21}^{L_1,'''}(\varkappa)| = |w_2^{L_1,'''}(\varkappa_1)| &\le C\mu^3\epsilon_2^{-2}(\epsilon_1^{-1}\mathfrak{B}_1^{l_1}(\varkappa_1) + \epsilon_2^{-1}\mathfrak{B}_2^{l_1}(\varkappa_1)) \\ &\le C\mu^3\epsilon_2^{-3}\mathfrak{B}_2^{l_1}(\varkappa) \end{aligned}.$$

On $[\varkappa_1, 1]$ $|w_{21}^{L_1,'''}(\varkappa)| = |w_2^{L_1,'''}(\varkappa)| \le C\mu^3\epsilon_2^{-3}\mathfrak{B}_2^{l_1}(\varkappa)$.
For $w_{12}^{L_1,''}(\varkappa)$ and $w_{12}^{L_2,''}(\varkappa)$, similar bounds are obtained. For [1,2] , $w_{11}^{L_2,'''}(\varkappa)$ is considered and on $[1, \varkappa_2)$

$$\begin{aligned} |w_{11}^{L_2,'''}(\varkappa)| = |w_1^{L_2,'''}(\varkappa_2)| &\le C(\mu^3\epsilon_1^{-3}\mathfrak{B}_1^{l_2}(\varkappa_2) + \epsilon_1^{-1}\epsilon_2^{-2}\mathfrak{B}_2^{l_2}(\varkappa_2)) \\ &\le C\mu^3\epsilon_1^{-1}\epsilon_2^{-2}\mathfrak{B}_2^{l_2}(\varkappa) \end{aligned}.$$

On $[\varkappa_2, 2]$
$|w_{11}^{L_2,'''}(\varkappa)| = |w_1^{L_2,'''}(\varkappa)| \le C\mu^3\epsilon_1^{-1}\epsilon_2^{-2}\mathfrak{B}_2^{l_1}(\varkappa)$.
For $w_{21}^{L_2,'''}(\varkappa)$, on $[1, \varkappa_1)$

$$\begin{aligned} |w_{21}^{L_2,'''}(\varkappa)| = |w_2^{L_2,'''}(\varkappa_1)| &\le C\mu^3\epsilon_2^{-2}(\epsilon_1^{-1}\mathfrak{B}_1^{l_2}(\varkappa_1) + \epsilon_2^{-1}\mathfrak{B}_2^{l_2}(\varkappa_1)) \\ &\le C\mu^3\epsilon_2^{-3}\mathfrak{B}_2^{l_2}(\varkappa) \end{aligned}.$$

On $[\varkappa_1, 2]$
$|w_{21}^{L_2,'''}(\varkappa)| = |w_2^{L_2,'''}(\varkappa)| \le C\mu^3\epsilon_2^{-3}\mathfrak{B}_2^{l_2}(\varkappa)$.

For $w_{12}^{L_2,''}(\varkappa)$ and $w_{12}^{L_2,''}(\varkappa)$, similar bounds are obtained. Similarly, find the sharper bounds for the case $\alpha\mu^2 \leq \gamma\epsilon_1$. In this scenario, the decomposition of $w_1^{L_1}$ and $w_1^{L_2}$ remains as before. The proof of the lemma is complete. □

Lemma 5. *Given the decompositions of $w_{11}^{L_1}, w_{12}^{L_1}, w_{21}^{L_1}$, and $w_{22}^{L_1}$, the following bounds hold on [0,1]:*

$$\begin{aligned}
|w_{11}^{L_1,'''}(\varkappa)| &\leq C\epsilon_2^{-\frac{3}{2}}\mathfrak{B}_2^{l_1}(\varkappa),\\
|w_{12}^{L_1,''}(\varkappa)| &\leq C\Big(\epsilon_1^{-1}\mathfrak{B}_1^{l_1}(\varkappa) + \epsilon_2^{-1}\mathfrak{B}_2^{l_1}(\varkappa)\Big),\\
|w_{21}^{L_1,'''}(\varkappa)| &\leq C\epsilon_2^{-\frac{3}{2}}\mathfrak{B}_2^{l_1}(\varkappa),\\
|w_{22}^{L_1,''}(\varkappa)| &\leq C\epsilon_2^{-1}\mathfrak{B}_2^{l_1}(\varkappa),
\end{aligned} \tag{97}$$

and given the decompositions of $w_{11}^{L_2}, w_{12}^{L_2}, w_{21}^{L_2}$, and $w_{22}^{L_2}$, the following bounds hold for [1,2]:

$$\begin{aligned}
|w_{11}^{L_2,'''}(\varkappa)| &\leq C\epsilon_2^{-\frac{3}{2}}\mathfrak{B}_2^{l_2}(\varkappa),\\
|w_{12}^{L_2,''}(\varkappa)| &\leq C\Big(\epsilon_1^{-1}\mathfrak{B}_1^{l_2}(\varkappa) + \epsilon_2^{-1}\mathfrak{B}_2^{l_1}(\varkappa)\Big),\\
|w_{21}^{L_2,'''}(\varkappa)| &\leq C\epsilon_2^{-\frac{3}{2}}\mathfrak{B}_2^{l_2}(\varkappa),\\
|w_{22}^{L_2,''}(\varkappa)| &\leq C\epsilon_2^{-1}\mathfrak{B}_2^{l_2}(\varkappa).
\end{aligned} \tag{98}$$

Proof. The proof follows the same logic as Lemma 4. □

Analogously, the decompositions can be made for $\vec{w}^{R_1}$ and $\vec{w}^{R_2}$ in both cases. Corresponding bounds for these components can be similarly demonstrated in a similar manner.

9. Numerical Method

This section presents the proposed numerical method for solving (1) and utilizing a Shishkin mesh for the discretization.

For the cases $\alpha\mu^2 \leq \gamma\epsilon_1$ and $\alpha\mu^2 \geq \gamma\epsilon_2$, appropriate Shishkin meshes are developed over interval $[0, 2]$.

Case (i): $\alpha\mu^2 \leq \gamma\epsilon_1$

In this case, the Shishkin mesh for $\bar{\Omega}^N$ over the interval $[0, 2]$ is determined by the transition parameters $\daleth_1$ and $\daleth_2$. The interval $[0, 2]$ is partitioned into the subintervals $[0, \daleth_1]$, $[\daleth_1, \daleth_2]$, $[\daleth_2, 1-\daleth_2]$, $[1-\daleth_2, 1-\daleth_1]$, $[1-\daleth_1, 1]$,$[1, 1+\daleth_1]$,$[1+\daleth_1, 1+\daleth_2]$, $[1+\daleth_2, 2-\daleth_2]$, $[2-\daleth_2, 2-\daleth_1]$, and $[2-\daleth_1, 2]$, where $\daleth_1$ and $\daleth_2$ are defined as,

$$\daleth_2 = \min\left(\frac{1}{4}, \frac{2\sqrt{\epsilon_2}}{\sqrt{\gamma\alpha}}\ln N\right), \daleth_1 = \min\left(\frac{\tau_2}{2}, \frac{2\sqrt{\epsilon_1}}{\sqrt{\gamma\alpha}}\ln N\right).$$

Each of the intervals $[0, \daleth_1]$, $[\daleth_1, \daleth_2]$, $[1-\daleth_2, 1-\daleth_1]$, $[1-\daleth_1, 1]$, $[1, 1+\daleth_1]$, $[1+\daleth_1, 1+\daleth_2]$, $[2-\daleth_2, 2-\daleth_1]$, and $[2-\daleth_1, 2]$ are divided into $\frac{N}{16}$ subintervals, while the intervals $[\daleth_2, 1-\daleth_2]$ and $[1+\daleth_2, 2-\daleth_2]$ are divided into $\frac{N}{4}$ subintervals. When $\daleth_2 = \frac{1}{4}$ and $\daleth_1 = \frac{1}{8}$, the mesh is uniform. Let H_1, H_2, and H_3 denote the step sizes on the intervals $[0, \daleth_1]$, $[\daleth_1, \daleth_2]$, and $[\daleth_2, 1-\daleth_2]$, respectively. Thus, $H_1 = \frac{16\daleth_1}{N}, H_2 = \frac{16(\daleth_2-\daleth_1)}{N}, H_3 = \frac{4(1-2\daleth_2)}{N}$. The step sizes in the intervals $[1-\daleth_2, 1-\daleth_1]$ and $[1-\daleth_1, 1]$ are H_2 and H_1, respectively. Based on the selection for $\daleth_1$ and $\daleth_2$, there are four possible meshes in this case.

Case (ii): $\alpha\mu^2 \geq \gamma\epsilon_2$

In this case, the Shishkin mesh for $\bar{\Omega}^N$on $[0, 2]$ is defined by the transition parameters $\daleth_1$, $\daleth_2$, and σ_1. The interval $[0, 2]$ is divided into the subintervals $[0, \daleth_1]$, $[\daleth_1, \daleth_2]$, $[\daleth_2, 1-\sigma_1]$,

$[1-\sigma_1, 1]$, $[1, 1+\urcorner_1]$,$[1+\urcorner_1, 1+\urcorner_2]$, $[1+\urcorner_2, 2-\sigma_1]$, and $[2-\sigma_1, 2]$, where $\urcorner_1$, $\urcorner_2$, and σ_1 are defined as

$$\urcorner_2 = \min\left(\frac{1}{4}, \frac{\epsilon_2}{\mu\alpha}\ln N\right), \urcorner_1 = \min\left(\frac{\tau_2}{2}, \frac{\epsilon_1}{\mu\alpha}\ln N\right), \sigma_1 = \min\left(\frac{1}{4}, \frac{\mu}{\gamma}\ln N\right).$$

Each of the intervals $[0, \urcorner_1]$, $[\urcorner_1, \urcorner_2]$, $[1, 1+\urcorner_1]$, and $[1+\urcorner_1, 1+\urcorner_2]$ is divided into $\frac{N}{16}$ subintervals, the intervals $[1-\sigma_1, 1]$ and $[2-\sigma_1, 2]$ are divided into $\frac{N}{8}$ subintervals, and the intervals $[\urcorner_2, 1-\sigma_1]$ and $[1+\urcorner_2, 2-\sigma_1]$ are divided into $\frac{N}{4}$ subintervals. Let the step length for each of the intervals be $[0, \urcorner_1]$, $[\urcorner_1, \urcorner_2]$, $[\urcorner_2, 1-\sigma_1]$, and let $[1-\sigma_1, 1]$ be denoted by H_1, H_2, H_3, and H_4, respectively. Thus, $H_1 = \frac{16\urcorner_1}{N}$, $H_2 = \frac{16(\urcorner_2-\urcorner_1)}{N}$, $H_3 = \frac{4(1-\sigma_1-\urcorner_2)}{N}$, $H_4 = \frac{8\sigma_1}{N}$. Although there are eight possible meshes $\bar{\Omega}^N$ resulting from the various options for the transition parameters, $\tau_2 = \frac{1}{4}$, σ_1 should take the value $\frac{1}{4}$. Otherwise, the case of inequality $\alpha\mu^2 \leq \gamma\epsilon_2$ would be true, leading to a contradiction. Therefore, there are only six possible meshes in this case.

10. The Discrete Problem

The discrete problem is defined as follows:

$$\begin{aligned}\vec{L}^N\vec{\mathfrak{U}}(\varkappa_j) \equiv E\delta^2\vec{\mathfrak{U}}(\varkappa_j) + \mu\boldsymbol{\mathcal{A}}(\varkappa_j)\mathfrak{D}^+\vec{\mathfrak{U}}(\varkappa_j) - \boldsymbol{\mathcal{B}}(\varkappa_j)\vec{\mathfrak{U}}(\varkappa_j) \\ +\boldsymbol{\mathcal{D}}(\varkappa_j)\vec{\mathfrak{U}}(\varkappa_j - 1) = \vec{f}(\varkappa_j) \text{ on } \Omega^N, \quad 0 \leq j \leq N-1,\end{aligned} \tag{99}$$

with boundary conditions specified as follows:

$$\vec{\mathfrak{U}}(\varkappa_j - 1) = \vec{\varphi}(\varkappa_j - 1) \quad \text{,for } 0 \leq j \leq \frac{N}{2}, \quad \vec{\mathfrak{U}}(\varkappa_N) = \vec{\mathfrak{u}}(\varkappa_N).$$

Let

$$\vec{L}_1^N\vec{\mathfrak{U}}(\varkappa_j) = E\delta^2\vec{\mathfrak{U}}(\varkappa_j) + \mu\boldsymbol{\mathcal{A}}(\varkappa_j)\mathfrak{D}^+\vec{\mathfrak{U}}(\varkappa_j) - \boldsymbol{\mathcal{B}}(\varkappa_j)\vec{\mathfrak{U}}(\varkappa_j) = \vec{g}(\varkappa_j) \text{ on } \Omega_1^N, \tag{100}$$

$$\begin{aligned}\vec{L}_2^N\vec{\mathfrak{U}}(\varkappa_j) = E\delta^2\vec{\mathfrak{U}}(\varkappa_j) + \mu\boldsymbol{\mathcal{A}}(\varkappa_j)\mathfrak{D}^+\vec{\mathfrak{U}}((\varkappa_j) - \boldsymbol{\mathcal{B}}(\varkappa_j)\vec{\mathfrak{U}}(\varkappa_j) + \mathfrak{D}(\varkappa_j)\vec{\mathfrak{U}}(\varkappa_j - 1) \\ = \vec{f}(\varkappa_j) \text{ on } \Omega_2^N.\end{aligned} \tag{101}$$

The discrete derivatives are defined as

$$\mathfrak{D}^+\vec{\mathfrak{U}}(\varkappa_j) = \frac{\vec{\mathfrak{U}}(\varkappa_{j+1}) - \vec{\mathfrak{U}}(\varkappa_j)}{h_{j+1}}, \mathfrak{D}^-\vec{\mathfrak{U}}(\varkappa_j) = \frac{\vec{\mathfrak{U}}(\varkappa_j) - \vec{\mathfrak{U}}(\varkappa_{j-1})}{h_j},$$

$$\delta^2\vec{\mathfrak{U}}(\varkappa_j) = \frac{1}{\bar{h}_j}(\mathfrak{D}^+\vec{\mathfrak{U}}(\varkappa_j) - \mathfrak{D}^-\vec{\mathfrak{U}}(\varkappa_j)),$$

with $h_j = \varkappa_j - \varkappa_{j-1}$, $\bar{h}_j = \frac{h_j+h_{j+1}}{2}$, $\varkappa_j \in \bar{\Omega}^N$.

11. Numerical Results

This section establishes a discrete minimum principle, a discrete stability result, and the first-order convergence of the proposed numerical method.

Lemma 6. *(Discrete Minimum Principle) Assume that the mesh function $\vec{\Psi}(\varkappa_j)$ satisfies $\vec{\Psi}(\varkappa_0) \geq \vec{0}$ and $\vec{\Psi}(\varkappa_N) \geq \vec{0}$. Then, if $\vec{L}_1^N\vec{\Psi}(\varkappa_j) \leq \vec{0}$ for $1 \leq j \leq \frac{N}{2} - 1$, $\vec{L}_2^N\vec{\Psi}(\varkappa_j) \leq \vec{0}$ for $\frac{N}{2} \leq j \leq N-1$, and $\mathfrak{D}^+\vec{\Psi}(\varkappa_{\frac{N}{2}}) - \mathfrak{D}^-\vec{\Psi}(\varkappa_{\frac{N}{2}}) \leq \vec{0}$, it implies that $\vec{\Psi}(\varkappa_j) \geq \vec{0}$ for all $0 \leq j \leq N$.*

Proof. Let i^* and j^* be such that $\Psi_{i^*}(\varkappa_{j^*}) = \min_{i,j} \Psi_i(\varkappa_j)$ and suppose $\Psi_{i^*}(\varkappa_{j^*}) < 0$. Then, $j^* \notin \{0, N\}$, $\Psi_{i^*}(\varkappa_{j^*}) \leq \Psi_{i^*}(\varkappa_{j^*+1})$, and $\Psi_{i^*}(\varkappa_{j^*}) \leq \Psi_{i^*}(\varkappa_{j^*-1})$.Therefore, $\delta^2\Psi_{i^*}(\varkappa_{j^*}) \geq 0$. Conside the two cases, for $1 \leq j^* \leq \frac{N}{2} - 1$,if $\varkappa_{j^*} \in \Omega_1^N$, then

$$(\vec{L}_1^N\vec{\Psi})_{i^*}(\varkappa_{j^*}) = \epsilon_{i^*}\delta^2\Psi_{i^*}(\varkappa_{j^*}) + \mu\mathfrak{a}_{i^*}(\varkappa_{j^*})\mathfrak{D}^+\Psi_{i^*}(\varkappa_{j^*}) - \sum_{j=1}^{2}\mathfrak{b}_{i^*j}(\varkappa_{j^*})\Psi_j(\varkappa_{j^*}) > \vec{0}$$

which is a contradiction, which gives $(\vec{L}_1^N\vec{\Psi})_{i^*}(\varkappa_{j^*}) \leq \vec{0}$. For $\frac{N}{2} \leq j^* \leq N-1$,if $\varkappa_{j^*} \in \Omega_2^N$, then

$$\begin{aligned}(\vec{L}_2^N\vec{\Psi})_{i^*}(\varkappa_{j^*}) = \epsilon_{i^*}\delta^2\Psi_{i^*}(\varkappa_{j^*}) + \mu\mathfrak{a}_{i^*}(\varkappa_{j^*})\mathfrak{D}^+\Psi_{i^*}(\varkappa_{j^*}) - \sum_{j=1}^{2}\mathfrak{b}_{i^*j}(\varkappa_{j^*})\Psi_j(\varkappa_{j^*})\\ -\mathfrak{d}_{i^*}(x_{j^*})\Psi_{i^*}(\varkappa_{j^*}-1) > \vec{0}\end{aligned}$$

which is a contradiction, which gives $(\vec{L}_2^N\vec{\Psi})_{i^*}(\varkappa_{j^*}) \leq \vec{0}$. The only remaining possibility is that $\varkappa_{j^*} = \varkappa_{\frac{N}{2}}$. Thus, by hypothesis, it follows that

$$\mathfrak{D}^-\Psi_{i^*}(\varkappa_{\frac{N}{2}}) \leq 0 \quad \text{and} \quad \mathfrak{D}^+\Psi_{i^*}(\varkappa_{\frac{N}{2}}) \geq 0.$$

This implies

$$\mathfrak{D}^+\Psi_{i^*}(\varkappa_{\frac{N}{2}}) - \mathfrak{D}^-\Psi_{i^*}(\varkappa_{\frac{N}{2}}) \leq 0,$$

and since $\mathfrak{D}^-\Psi_{i^*}(\varkappa_{\frac{N}{2}}) \leq 0 \leq \mathfrak{D}^+\Psi_{i^*}(\varkappa_{\frac{N}{2}})$, it follows that

$$\mathfrak{D}^+\Psi_{i^*}(\varkappa_{\frac{N}{2}}) \leq \mathfrak{D}^-\Psi_{i^*}(\varkappa_{\frac{N}{2}}).$$

Consequently, it follows that

$$\Psi_{i^*}(\varkappa_{\frac{N}{2}-1}) = \Psi_{i^*}(\varkappa_{\frac{N}{2}}) = \Psi_{i^*}(\varkappa_{\frac{N}{2}+1}) < 0.$$

Now, consider the operator acting on the solution at $\varkappa_{N/2-1}$

$$\begin{aligned}(\vec{L}_1^N\vec{\Psi})_{i^*}(\varkappa_{\frac{N}{2}-1}) &= \epsilon_{i^*}\delta^2\Psi_{i^*}(\varkappa_{\frac{N}{2}-1}) + \mathfrak{a}_{i^*}(\varkappa_{\frac{N}{2}-1})\Psi'_{i^*}(\varkappa_{\frac{N}{2}-1}) - \sum_{j=1}^{2}\mathfrak{b}_{i^*j}(\varkappa_{\frac{N}{2}-1})\Psi_j(\varkappa_{\frac{N}{2}-1})\\ &= \epsilon_{i^*}\delta^2\Psi_{i^*}(\varkappa_{\frac{N}{2}-1}) - \sum_{j=1, j\neq i^*}^{2}\mathfrak{b}_{i^*j}(\varkappa_{\frac{N}{2}-1})\Psi_{i^*}(\varkappa_{N/2-1}) > \vec{0},\end{aligned}$$

leading to a contradiction, implying that $\vec{\Psi}(\varkappa_j) \geq \vec{0}$ for all $0 \leq j \leq N$. □

Lemma 7. *(Discrete Stability Result) If $\vec{\Psi}(\varkappa_j)$ is any mesh function, then*

$$|\Psi_i(\varkappa_j)| \leq \max\left(|\vec{\Psi}(\varkappa_0)|, |\vec{\Psi}(\varkappa_N)|, \max_{1\leq j\leq \frac{N}{2}-1}|L_1^N\vec{\Psi}(\varkappa_j)|, \max_{\frac{N}{2}\leq j\leq N-1}|L_2^N\vec{\Psi}(\varkappa_j)|\right).$$

Error Estimate

Analogous to the continuous case, the discrete solution $\vec{\mathfrak{U}}$ can be decomposed into $\vec{V}$ and $\vec{W}$ as defined below.

$$\vec{L}_1^N\vec{V}_1(\varkappa_j) = \vec{g}(\varkappa_j), \quad \text{for} \quad 0 < j < \frac{N}{2} - 1, \quad \vec{V}_1(\varkappa_0) = \vec{r}(0), \quad \vec{V}_1\left(\varkappa_{\frac{N}{2}}\right) = \vec{r}(1), \tag{102}$$

$$\vec{L}_2^N \vec{V}_2(\varkappa_j) = \vec{f}(\varkappa_j), \quad \text{for} \quad \frac{N}{2} < j < N-1, \quad \vec{V}_2\left(\varkappa_{\frac{N}{2}}\right) = \vec{s}(1), \quad \vec{V}_2(\varkappa_N) = \vec{s}(2), \tag{103}$$

$$\vec{V}_2(\varkappa_j - 1) = \vec{V}_1\left(\varkappa_{j-\frac{N}{2}}\right), \quad \text{for} \quad \frac{N}{2} \le j < N,$$

$$\vec{L}_1^N \vec{W}^{L_1}(\varkappa_j) = 0, \quad \text{for} \quad 0 < j < \frac{N}{2} - 1, \quad \vec{W}^{L_1}(\varkappa_0) = \vec{w}^{L_1}(0), \quad \vec{W}^{L_1}\left(\varkappa_{\frac{N}{2}}\right) = \vec{w}^{L_1}(1), \tag{104}$$

$$\vec{L}_2^N \vec{W}^{L_2}(\varkappa_j) = 0, \quad \text{for} \quad \frac{N}{2} < j < N-1, \quad \vec{W}^{L_2}\left(\varkappa_{\frac{N}{2}}\right) = \vec{w}^{L_2}(1), \quad \vec{W}^{L_2}(\varkappa_N) = \vec{w}^{L_2}(2), \tag{105}$$

$$\vec{W}^{L_1}(\varkappa_j - 1) = \vec{W}^{L_1}\left(\varkappa_{j-\frac{N}{2}}\right), \quad \text{for} \quad \frac{N}{2} \le j < N.$$

It is clear that

$$\vec{V}(\varkappa_j) = \begin{cases} \vec{V}_1(\varkappa_j), & \text{for} \quad 0 \le j \le \frac{N}{2} - 1 \\ \vec{V}_2(\varkappa_j), & \text{for} \quad \frac{N}{2} \le j \le N \end{cases},$$

$$\vec{W}^{L}(\varkappa_j) = \begin{cases} \vec{W}^{L_1}(\varkappa_j), & \text{for} \quad 0 \le j \le \frac{N}{2} - 1 \\ \vec{W}^{L_2}(\varkappa_j), & \text{for} \quad \frac{N}{2} \le j \le N \end{cases}.$$

Lemma 8. *If $\vec{v}$ is the solution of* (10), (13), (22) *and $\vec{V}$ is the solution of* (102) *and* (103) *, then*

$$|(\vec{V} - \vec{\mathfrak{v}})(\varkappa_j)| \le CN^{-1}, \quad \textit{for} \quad 0 \le j \le N.$$

Proof.

$$\vec{\mathfrak{v}}(\varkappa) = \begin{cases} \vec{r}(\varkappa), & \text{for} \quad \varkappa \in [0,1) \\ \vec{s}(\varkappa), & \text{for} \quad \varkappa \in [1,2] \end{cases}$$

Hence,

$$|(\vec{V} - \vec{\mathfrak{v}})(\varkappa_j)| = \begin{cases} |(\vec{V}_1 - \vec{r})(\varkappa_j)|, & \text{for} \quad 0 \le j \le \frac{N}{2} - 1 \\ |(\vec{V}_2 - \vec{s})(\varkappa_j)|, & \text{for} \quad \frac{N}{2} \le j \le N \end{cases}.$$

$$\vec{L}_1^N(\vec{V} - \vec{r})(x_j) = \vec{g}(\varkappa_j) - \vec{L}_1^N \vec{r}(\varkappa_j) = (\vec{L}_1 - \vec{L}_1^N)\vec{r}(\varkappa_j) \tag{106}$$

$$= E\left(\frac{d^2}{d\varkappa^2} - \delta^2\right)\vec{r}(\varkappa_j) + \mu\left(\frac{d}{d\varkappa} - \mathfrak{D}^+\right)\boldsymbol{\mathcal{A}}(\varkappa_j)\vec{r}(x_j) \tag{107}$$

$$= \begin{cases} \epsilon_1\left(\frac{d^2}{d\varkappa^2} - \delta^2\right)r_1(\varkappa_j) + \mu\mathfrak{a}_1(\varkappa_j)\left(\frac{d}{d\varkappa} - \mathfrak{D}^+\right)r_1(x_j) \\ \epsilon_2\left(\frac{d^2}{d\varkappa^2} - \delta^2\right)r_2(\varkappa_j) + \mu\mathfrak{a}_2(\varkappa_j)\left(\frac{d}{d\varkappa} - \mathfrak{D}^+\right)r_2(\varkappa_j) \end{cases} \tag{108}$$

Determining the local truncation error

$$\left|\epsilon_i\left(\frac{d^2}{d\varkappa^2} - \delta^2\right)r_i(\varkappa_j) + \mu\mathfrak{a}_i(\varkappa_j)\left(\frac{d}{d\varkappa} - \mathfrak{D}^+\right)r_i((\varkappa_j)\right| \le C(\varkappa_{j+1} - \varkappa_{j-1})(\epsilon_i\|r_i^{'''}\| + \mu\|r_i^{''}\|),$$

for $i = 1, 2$. It is established that $(\varkappa_{j+1} - \varkappa_{j-1}) \le CN^{-1}$. In this case $\alpha\mu^2 \le \gamma\epsilon_1$, from (53) and (54),

$$|\vec{L}_1^N(\vec{V}_1 - \vec{r})(\varkappa_j)| \le CN^{-1}\begin{pmatrix} \epsilon_1\|r_1^{'''}\| + \mu\|r_1^{''}\| \\ \epsilon_2\|r_2^{'''}\| + \mu\|r_2^{''}\| \end{pmatrix}$$

$$\le CN^{-1}\begin{pmatrix} \epsilon_1 C{\epsilon_1}^{-1/2} + \mu\, C \\ \epsilon_2 C + \mu C \end{pmatrix}$$

$$\leq CN^{-1}\begin{pmatrix} C{\epsilon_1}^{1/2} + \mu\, C \\ \epsilon_2 C + \mu C \end{pmatrix} \leq CN^{-1}$$

$$|\vec{L}_2^N(\vec{V}_1 - \vec{s})(\varkappa_j)| \leq CN^{-1}\begin{pmatrix} \epsilon_1 \|s_1^{'''}\| + \mu\|s_1^{''}\| \\ \epsilon_2 \|s_2^{'''}\| + \mu\|s_2^{''}\| \end{pmatrix}$$

$$\leq CN^{-1}\begin{pmatrix} \epsilon_1 C{\epsilon_1}^{-1/2} + \mu\, C \\ \epsilon_2 C + \mu C \end{pmatrix} \leq CN^{-1}.$$

Using Lemma 7, consider the mesh function,

$$\vec{\Psi}^{\pm}(\varkappa_j) = CN^{-1}(\vec{e}^{\,T}) \pm (\vec{V}_1 - \vec{r})(\varkappa_j), \quad 0 \leq j \leq \frac{N}{2}.$$

Provided that the value of C is sufficiently large, it follows that $\vec{L}_1^N\vec{\Psi}^{\pm}(\varkappa_j) \leq \vec{0}$, for $1 \leq j \leq \frac{N}{2} - 1$, $\vec{\Psi}^{\pm}(\varkappa_0) \geq \vec{0}$, and $\vec{\Psi}^{\pm}(\varkappa_{\frac{N}{2}}) \geq \vec{0}$.

$$|(\vec{V}_1 - \vec{r})(\varkappa_j)| \leq CN^{-1}, \quad \text{for } 0 \leq j \leq \frac{N}{2}.$$

Similarly,

$$|(\vec{V}_1 - \vec{s})(\varkappa_j)| \leq CN^{-1}, \quad \text{for } \frac{N}{2} \leq j \leq N.$$

For the case $\alpha\mu^2 \geq \gamma\epsilon_2$

$$|(\vec{V}_2 - \vec{r})(\varkappa_j)| \leq CN^{-1}, \quad \text{for } 0 \leq j \leq \frac{N}{2}.$$

Similarly,

$$|(\vec{V}_2 - \vec{s})(\varkappa_j)| \leq CN^{-1}, \quad \text{for } \frac{N}{2} \leq j \leq N.$$

It can be stated that for both cases,

$$|(\vec{V} - \vec{\mathfrak{v}})(\varkappa_j)| = \begin{cases} |(\vec{V}_1 - \vec{r})(\varkappa_j)| \leq CN^{-1}, & \text{for } 0 \leq j \leq \frac{N}{2} \\ |(\vec{V}_2 - \vec{r})(\varkappa_j)| \leq CN^{-1}, & \text{for } 0 \leq j \leq \frac{N}{2} \\ |(\vec{V}_1 - \vec{s})(\varkappa_j)| \leq CN^{-1}, & \text{for } \frac{N}{2} \leq j \leq N \\ |(\vec{V}_2 - \vec{s})(\varkappa_j)| \leq CN^{-1}, & \text{for } \frac{N}{2} \leq j \leq N. \end{cases}$$

Thus,

$$|(\vec{V} - \vec{\mathfrak{v}})(\varkappa_j)| \leq CN^{-1}, \quad \text{for } 0 \leq j \leq N. \tag{109}$$

The proof of the lemma is complete. □

The bounds on the error in the singular components $\vec{w}^{L_1}$ and $\vec{w}^{L_2}$ are estimated for the case $\alpha\mu^2 \geq \gamma\epsilon_2$, utilizing the mesh functions defined on $\bar{\Omega}^N$,

$$\boldsymbol{B}_1^{(l_p,N)}(\varkappa_j) = \prod_{i=1}^{j}\left(1 + \frac{\alpha\mu h_i}{2\epsilon_1}\right)^{-1}, \quad \boldsymbol{B}_2^{(l_p,N)}(\varkappa_j) = \prod_{i=1}^{j}\left(1 + \frac{\alpha\mu h_i}{2\epsilon_2}\right)^{-1},$$

with $\boldsymbol{B}_1^{(l_p,N)}(\varkappa_0) = \boldsymbol{B}_2^{(l_p,N)}(\varkappa_0) = 1$ for p=1,2.

Lemma 9. *For the case $\alpha\mu^2 \geq \gamma\epsilon_2$, the layer components $\vec{W}^{L_1}$ and $\vec{W}^{L_2}$ satisfy the following bounds on $\bar{\Omega}^N$:*

$$|W_1^{L_1}(\varkappa_j)| \leq C\boldsymbol{B}_2^{(l_1,N)}(\varkappa_j), \quad |W_1^{L_2}(\varkappa_j)| \leq C\boldsymbol{B}_2^{(l_2,N)}(\varkappa_j)$$

$$|W_2^{L_1}(\varkappa_j)| \leq C\boldsymbol{B}_2^{(l_1,N)}(\varkappa_j), \quad |W_2^{L_2}(\varkappa_j)| \leq C\boldsymbol{B}_2^{(l_2,N)}(\varkappa_j).$$

Proof. This result can be demonstrated by defining the mesh functions $\psi_i^{\pm}(\varkappa_j) = C\boldsymbol{B}_2^{(l_p,N)}(\varkappa_j) \pm W_i^{L_1}(\varkappa_j)$; $\psi_i^{\pm}(\varkappa_j) = C\boldsymbol{B}_2^{(l_p,N)}(\varkappa_j) \pm W_i^{L_2}(\varkappa_j)$; i = 1, 2; and p =1,2 and noticing that $\psi_i^{\pm}(\varkappa_0) \geq 0$ and $\psi_i^{\pm}(\varkappa_N) \geq 0$. Furthermore, $(\vec{L}_1^N\vec{\psi}^{\pm})_i(\varkappa_j) \leq \vec{0}$, and $(\vec{L}_2^N\vec{\psi}^{\pm})_i(\varkappa_j) \leq \vec{0}$, $j = 1, 2, \cdots, N-1$. Therefore, the discrete minimum principle provides the desired outcome. The proof of the lemma is complete. □

Lemma 10. *Let $\vec{w}^{L_1}$ and $\vec{w}^{L_2}$ satisfy* (25) *, and let $\vec{W}^{L_1}$ and $\vec{W}^{L_2}$ satisfy* (104) *and* (105)*. Then,*

$$\|\vec{W}^{L_1} - \vec{w}^{L_1}\| \leq CN^{-1}\ln N.$$

$$\|\vec{W}^{L_2} - \vec{w}^{L_2}\| \leq CN^{-1}\ln N.$$

Proof. The local truncation error is given by

$$|\vec{L}_1^N(\vec{W}^{L_1} - \vec{w}^{L_1})| \leq C(\varkappa_{j+1} - \varkappa_{j-1}) \begin{pmatrix} \epsilon_1\|w_1^{L_1,'''}\|_D + \mu\|w_1^{L_1,''}\|_D \\ \epsilon_2\|w_2^{L_1,'''}\|_D + \mu\|w_2^{L_1,''}\|_D \end{pmatrix} \tag{110}$$

$$|\vec{L}_2^N(\vec{W}^{L_2} - \vec{w}^{L_2})| \leq C(\varkappa_{j+1} - \varkappa_{j-1}) \begin{pmatrix} \epsilon_1\|w_1^{L_2,'''}\|_D + \mu\|w_1^{L_2,''}\|_D \\ \epsilon_2\|w_2^{L_2,'''}\|_D + \mu\|w_2^{L_2,''}\|_D \end{pmatrix} \tag{111}$$

where $D = [\varkappa_{j-1}, \varkappa_{j+1}]$. When $\urcorner_2 = \frac{1}{4}$, $\urcorner_1 = \frac{1}{8}$ and $\sigma_1 = \frac{1}{4}$, the mesh $\bar{\Omega}^N$ is uniform; then, the value of $h = N^{-1}$. In this instance, $\mu\epsilon_1^{-1} \leq C\ln N$, $\mu\epsilon_2^{-1} \leq C\ln N$, and $\mu^{-1} \leq C\ln N$.

$$|\vec{L}_1^N(\vec{W}^{L_1} - \vec{w}^{L_1})| \leq CN^{-1}\mu^3 \begin{pmatrix} \left(\frac{1}{\epsilon_1^2}\mathfrak{B}_1^{l_1}(\varkappa_{j-1}) + \frac{1}{\epsilon_2^2}\mathfrak{B}_2^{l_1}(\varkappa_{j-1})\right) \\ \left(\frac{1}{\epsilon_1}\frac{1}{\epsilon_2}\mathfrak{B}_1^{l_1}(\varkappa_{j-1}) + \frac{1}{\epsilon_2^2}\mathfrak{B}_2^{l_1}(\varkappa_{j-1})\right) \end{pmatrix} \tag{112}$$

$$|\vec{L}_2^N(\vec{W}^{L_2} - \vec{w}^{L_2})| \leq CN^{-1}\mu^3 \begin{pmatrix} \left(\frac{1}{\epsilon_1^2}\mathfrak{B}_1^{l_2}(\varkappa_{j-1}) + \frac{1}{\epsilon_2^2}\mathfrak{B}_2^{l_2}(x_{j-1})\right) \\ \left(\frac{1}{\epsilon_1}\frac{1}{\epsilon_2}\mathfrak{B}_1^{l_2}(\varkappa_{j-1}) + \frac{1}{\epsilon_2^2}\mathfrak{B}_2^{l_2}(\varkappa_{j-1})\right) \end{pmatrix} \tag{113}$$

Develop the mesh functions $\vec{\phi}$ that were specified in $\bar{\Omega}^N$ by

$$\phi_1(\varkappa_j) = \frac{CN^{-1}}{\nu(\alpha-\nu)}\left(\exp\left(\frac{2\nu h\mu}{\epsilon_1}\right)\epsilon_1^{-1}\mu Y_j + \exp\left(\frac{2\nu h\mu}{\epsilon_2}\right)\epsilon_2^{-1}\mu Z_j\right), \tag{114}$$

$$\phi_2(\varkappa_j) = \frac{CN^{-1}}{\nu(\alpha-\nu)}\exp\left(\frac{2\nu h\mu}{\epsilon_2}\right)\epsilon_1^{-1}\mu Z_j, \tag{115}$$

where ν is a constant and it satisfies $0 < \nu < \alpha$,

$$Y_j = \frac{\lambda^{N-j} - 1}{\lambda^N - 1}, \quad \lambda = 1 + \frac{\nu\mu h}{\epsilon_1}, \quad Z_j = \frac{\zeta^{N-j} - 1}{\zeta^N - 1}, \quad \zeta = 1 + \frac{\nu\mu h}{\epsilon_2}.$$

The mesh functions described above are inspired by those constructed in [15]. Now, that $0 \leq Y_j, Z_j \leq 1$, $(\epsilon_1\delta^2 + \mu\nu\mathfrak{D}^+)Y_j = 0$, $(\epsilon_2\delta^2 + \mu\nu\mathfrak{D}^+)Z_j = 0$, $\mathfrak{D}^+Y_j \leq -\nu\mu\epsilon_1^{-1}\exp\left(-\nu\mu\varkappa_{j+1}\epsilon_1^{-1}\right)$ and $\mathfrak{D}^+Z_j \leq -\nu\mu\epsilon_2^{-1}\exp\left(-\nu\mu\varkappa_{j+1}\epsilon_2^{-1}\right)$. Then, define $\vec{\psi}^{\pm}(\varkappa_j) = \vec{\phi}(\varkappa_j) \pm (\vec{W}^L - \vec{w}^L)(\varkappa_j)$. It is easy to observe that $\vec{\psi}(\varkappa_j) \geq \vec{0},\ j = 0, ..., N$ and $\vec{L}_1^N\vec{\psi}(\varkappa_j) \leq \vec{0}, \quad \vec{L}_2^N\vec{\psi}(\varkappa_j) \leq \vec{0},\ 1 \leq j \leq N-1$. Hence, by applying minimum principle,

$$|(\vec{W}^{L_1} - \vec{w}^{L_1})(\varkappa_j)| \leq CN^{-1}\begin{pmatrix}\left(\epsilon_1^{-1}\mu + \epsilon_2^{-1}\mu\right)\\ \epsilon_2^{-1}\mu\end{pmatrix} \leq CN^{-1}\ln N.$$

Simillarly,

$$|(\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)| \leq CN^{-1}\begin{pmatrix}\epsilon_1^{-1}\mu + \epsilon_2^{-1}\mu\\ \epsilon_2^{-1}\mu\end{pmatrix} \leq CN^{-1}\ln N.$$

Now, when $\urcorner_2 = \frac{\epsilon_2}{\mu\alpha}\ln N$, for the mesh elements $\varkappa_j$, for $\frac{N}{8} \leq j \leq \frac{N}{2}$

$$\begin{aligned}|({W_1}^{L_1} - {w_1}^{L_1})(\varkappa_j)| &\leq |{W_1}^{L_1}(\varkappa_j)| + |{w_1}^{L_1}(\varkappa_j)|\\ &\leq C\boldsymbol{B}_2^{(l_1,N)}(\varkappa_j) + C\mathfrak{B}_2^{l_1}(\varkappa_j)\\ &\leq C\boldsymbol{B}_2^{(l_1,N)}(\urcorner_2) + C\mathfrak{B}_2^{l_1}(\urcorner_2) \leq CN^{-1}.\end{aligned}$$

Similarly,

$$|({W_2}^{L_1} - {w_2}^{L_1})(\varkappa_j)| \leq CN^{-1}.$$

For $\frac{5N}{8} \leq j \leq N$

$$\begin{aligned}|({W_1}^{L_2} - {w_1}^{L_2})(\varkappa_j)| &\leq |{W_1}^{L_2}(\varkappa_j)| + |{w_1}^{L_2}(\varkappa_j)|\\ &\leq C\boldsymbol{B}_2^{(l_2,N)}(\varkappa_j) + C\mathfrak{B}_2^{l_2}(\varkappa_j)\\ &\leq C\boldsymbol{B}_2^{(l_2,N)}(\urcorner_2) + C\mathfrak{B}_2^{l_2}(\urcorner_2) \leq CN^{-1}.\end{aligned}$$

Similarly,

$$|({W_2}^{L_2} - {w_2}^{L_2})(\varkappa_j)| \leq CN^{-1}.$$

When $\urcorner_2 = \frac{\epsilon_2}{\mu\alpha}\ln N$, there are two distinct cases that are considered: $\epsilon_1 \geq \frac{\epsilon_2}{2}$ and $\epsilon_1 < \frac{\epsilon_2}{2}$. In the case $\frac{\epsilon_2}{2} \leq \epsilon_1 \leq \epsilon_2$, for $\frac{N}{16} \leq j \leq \frac{N}{8}$, $\urcorner_2 \leq \frac{2\epsilon_1}{\mu\alpha}\ln N$. Hence,

$$|\vec{L}_1^N(\vec{W}^{L_1} - \vec{w}^{L_1})(\varkappa_j)| \leq CN^{-1}\ln N\begin{pmatrix}\epsilon_1^{-1}\mu^2\mathfrak{B}_1^{l_1}(\varkappa_{j-1}) + \epsilon_2^{-1}\mu^2\mathfrak{B}_2^{l_1}(\varkappa_{j-1})\\ \epsilon_2^{-1}\mu^2\mathfrak{B}_2^{l_1}(\varkappa_{j-1})\end{pmatrix}.$$

For $\frac{9N}{16} \leq j \leq \frac{5N}{8}$

$$|\vec{L}_2^N(\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)| \leq CN^{-1}\ln N\begin{pmatrix}\epsilon_1^{-1}\mu^2\mathfrak{B}_1^{l_2}(\varkappa_{j-1}) + \epsilon_2^{-1}\mu^2\mathfrak{B}_2^{l_2}(\varkappa_{j-1})\\ \epsilon_2^{-1}\mu^2\mathfrak{B}_2^{l_2}(\varkappa_{j-1})\end{pmatrix}.$$

For this case, $\epsilon_2 > 2\epsilon_1$ for $\frac{N}{16} \leq j \leq \frac{N}{8}$; using the decomposition of $\vec{w}^{L_1}$ and $\vec{w}^{L_2}$, the following results are derived based on (95)–(98). Let $\bar{h} = \varkappa_{j+1} - \varkappa_{j-1}$; then,

$$|\vec{L}_1^N(\vec{W}^{L_1} - \vec{w}^{L_1})(\varkappa_j)| \leq \begin{pmatrix}\bar{h}\left(\epsilon_1||{w_{11}^{L_1}}'''|| + C\mu||{w_{11}^{L_1}}''||\right) + \epsilon_1||{w_{12}^{L_1}}'''|| + C\mu||{w_{12}^{L_1}}'||\\ \epsilon_2\bar{h}||{w_{21}^{L_1}}'''|| + \epsilon_1||{w_{22}^{L_1}}''|| + C\mu\bar{h}||{w_2^{L_1}}''||\end{pmatrix}$$

$$|\vec{L}_1^N(\vec{W}^{L_1} - \vec{w}^{L_1})(\varkappa_j)| \leq \begin{pmatrix} CN^{-1}\ln N\mu^2\epsilon_2^{-1}\mathfrak{B}_2^{l_1}(\varkappa_{j-1}) + C\epsilon_1^{-1}\mu^2\mathfrak{B}_1^{l_1}(\varkappa_{j-1}) \\ CN^{-1}\ln N\mu^2\epsilon_2^{-1}\mathfrak{B}_2^{l_1}(\varkappa_{j-1}) + C\epsilon_2^{-1}\mu^2\mathfrak{B}_1^{l_1}(\varkappa_{j-1}) \end{pmatrix}.$$

For $\frac{9N}{16} \leq j \leq \frac{5N}{8}$

$$|\vec{L}_2^N(\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)| \leq \begin{pmatrix} \bar{h}\Big(\epsilon_1||w_{11}^{L_2,'''}|| + C\mu||w_{11}^{L_2,''}||\Big) + \epsilon_1||w_{12}^{L_2,'''}|| + C\mu||w_{12}^{L_2,'}|| \\ \epsilon_2\bar{h}||w_{21}^{L_2,'''}|| + \epsilon_1||w_{22}^{L_2,''}|| + C\mu\bar{h}||w_2^{L_2,''}|| \end{pmatrix}$$

$$|\vec{L}_2^N(\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)| \leq \begin{pmatrix} CN^{-1}\ln N\mu^2\epsilon_2^{-1}\mathfrak{B}_2^{l_2}(\varkappa_{j-1}) + C\epsilon_1^{-1}\mu^2\mathfrak{B}_1^{l_2}(\varkappa_{j-1}) \\ CN^{-1}\ln N\mu^2\epsilon_2^{-1}\mathfrak{B}_2^{l_2}(\varkappa_{j-1}) + C\epsilon_2^{-1}\mu^2\mathfrak{B}_1^{l_2}(\varkappa_{j-1}) \end{pmatrix}.$$

For $0 \leq j \leq \frac{N}{16}$, $\daleth_1 \leq \frac{\epsilon_1}{\mu\alpha}\ln N$,

$$|\vec{L}_1^N(\vec{W}^{l_1} - \vec{w}^{l_1})(\varkappa_j)| \leq CN^{-1}\mu^2\ln N\begin{pmatrix} \epsilon_1^{-1}\mathfrak{B}_1^{l_1}(\varkappa_{j-1}) + \epsilon_2^{-1}\mathfrak{B}_2^{l_1}(\varkappa_{j-1}) \\ \epsilon_2^{-1}\mathfrak{B}_2^{l_1}(\varkappa_{j-1}) \end{pmatrix}.$$

For $\frac{N}{2} \leq j \leq \frac{9N}{16}$,

$$|\vec{L}_2^N(\vec{W}^{L_2} - \vec{w}^{l_2})(\varkappa_j)| \leq CN^{-1}\mu^2\ln N\begin{pmatrix} \epsilon_1^{-1}\mathfrak{B}_1^{l_2}(\varkappa_{j-1}) + \epsilon_2^{-1}\mathfrak{B}_2^{l_2}(\varkappa_{j-1}) \\ \epsilon_2^{-1}\mathfrak{B}_2^{l_2}(\varkappa_{j-1}) \end{pmatrix}.$$

Let us define the mesh functions $\vec{\Phi}$ on the domain $\bar{\Omega}^N$ for the value of j such that $0 \leq j \leq \frac{N}{16}$, $p = 1$ and for $\frac{N}{2} \leq j \leq \frac{9N}{16}$, $p = 2$ as follows:

$$\Phi_1(\varkappa_j) = CN^{-1}\ln N\left[\exp\left(\frac{2\alpha\mu H_1}{\epsilon_1}\right)\boldsymbol{B}_1^{(l_p,N)}(\varkappa_j) + \exp\left(\frac{2\alpha\mu H_1}{\epsilon_2}\right)\boldsymbol{B}_2^{(l_p,N)}(\varkappa_j)\right],$$

$$\Phi_2(\varkappa_j) = CN^{-1}\ln N\exp\left(\frac{2\alpha\mu H_1}{\epsilon_2}\right)\boldsymbol{B}_2^{(l_p,N)}(\varkappa_j),$$

and for $\frac{N}{16} \leq j \leq \frac{N}{8}$, $p = 1$ and $\frac{9N}{16} \leq j \leq \frac{5N}{8}$, $p = 2$ as

$$\Phi_1(\varkappa_j) = CN^{-1}\ln N\left[\exp\left(\frac{2\alpha\mu H_2}{\epsilon_2}\right)\boldsymbol{B}_2^{(l_p,N)}(\varkappa_j) + C\boldsymbol{B}_1^{(l_p,N)}(\varkappa_j)\right],$$

$$\Phi_2(\varkappa_j) = CN^{-1}\ln N\left[\exp\left(\frac{2\alpha\mu H_2}{\epsilon_2}\right)\boldsymbol{B}_2^{(l_p,N)}(\varkappa_j) + CN^{-1}\Big((\tau_2 - \varkappa_j)\epsilon_2^{-1} + 1\Big)\right].$$

By defining the barrier functions $\vec{\Psi}^{\pm}(\varkappa_j) = \vec{\Phi}(\varkappa_j) \pm (\vec{W}^{L_1} - \vec{w}^{L_1})(\varkappa_j)$ for $0 \leq j \leq \frac{N}{8}$ and utilizing the minimum principle, it can be established that $\vec{\Psi}^{\pm}(\varkappa_j) \geq \vec{0}$ for all $0 \leq j \leq \frac{N}{8}$. Similarly, by defining the barrier functions $\vec{\Psi}^{\pm}(\varkappa_j) = \vec{\Phi}(x_j) \pm (\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)$ for $\frac{N}{2} \leq j \leq \frac{5N}{8}$ and utilizing the minimum principle, it can be established that $\vec{\Psi}^{\pm}(\varkappa_j) \geq \vec{0}$ for all $\frac{N}{2} \leq j \leq \frac{5N}{8}$. Consequently, it follows that

$$|(\vec{W}^{L_1} - \vec{w}^{L_1})(\varkappa_j)| \leq CN^{-1}\ln N, \quad \text{for } 0 \leq j \leq \frac{N}{8},$$

$$|(\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)| \leq CN^{-1}\ln N, \quad \text{for } \frac{N}{2} \leq j \leq \frac{5N}{8}.$$

Now, consider the case $\daleth_2 = \frac{1}{4}$, $\sigma_1 = \frac{1}{4}$, and $\daleth_1 = \frac{\epsilon_1}{\mu\alpha} \ln N$. In this scenario, $\epsilon_2^{-1}\mu \leq C \ln N$. For $x_j \in [0, \daleth_1)$, it follows that $(\varkappa_{j+1} - \varkappa_{j-1}) \leq C\epsilon_1\mu^{-1}N^{-1} \ln N$. Hence,

$$|\vec{L}_1^N(\vec{W}^{L_1} - \vec{w}^{L_1})(\varkappa_j)| \leq C(\varkappa_{j+1} - \varkappa_{j-1}) \begin{pmatrix} \epsilon_1 \|w_1^{L_1,'''}\|_D + \mu \|w_1^{L_1,''}\|_D \\ \epsilon_2 \|w_2^{L_1,'''}\|_D + \mu \|w_2^{L_1,''}\|_D \end{pmatrix}$$

$$\leq CN^{-1} \ln N \mu^2 \begin{pmatrix} \epsilon_1^{-1}\mathfrak{B}_1^{l_1}(\varkappa_{j-1}) + \epsilon_2^{-1}\mathfrak{B}_2^{l_1}(\varkappa_{j-1}) \\ \epsilon_2^{-1}\mathfrak{B}_2^{l_1}(\varkappa_{j-1}) \end{pmatrix}$$

For $\varkappa_j \in [1, 1 + \daleth_1)$, it follows that $(\varkappa_{j+1} - \varkappa_{j-1}) \leq C\epsilon_1\mu^{-1}N^{-1} \ln N$. Hence,

$$|\vec{L}_2^N(\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)| \leq C(\varkappa_{j+1} - \varkappa_{j-1}) \begin{pmatrix} \epsilon_1 \|w_1^{L_2,'''}\|_D + \mu \|w_1^{L_2,''}\|_D \\ \epsilon_2 \|w_2^{L_2,'''}\|_D + \mu \|w_2^{L_2,''}\|_D \end{pmatrix}$$

$$\leq CN^{-1} \ln N \mu^2 \begin{pmatrix} \epsilon_1^{-1}\mathfrak{B}_1^{l_2}(\varkappa_{j-1}) + \epsilon_2^{-1}\mathfrak{B}_2^{l_2}(\varkappa_{j-1}) \\ \epsilon_2^{-1}\mathfrak{B}_2^{l_2}(\varkappa_{j-1}) \end{pmatrix}.$$

For $\varkappa_j \in [\daleth_1, \daleth_2]$, by utilizing the decomposition in the previous case, the following is derived

$$|\vec{L}_1^N(\vec{W}^{L_1} - \vec{w}^{L_1})(\varkappa_j)| \leq \begin{pmatrix} \bar{h}\Big(\epsilon_1 ||w_{11}^{L_1,'''}|| + C\mu ||w_{11}^{L_1,''}||\Big) + \epsilon_1 ||w_{12}^{L_1,'''}|| + C\mu ||w_{12}^{L_1,'}|| \\ \epsilon_2\bar{h}||w_{21}^{L_1,'''}|| + \epsilon_1 ||w_{22}^{L_1,''}|| + C\mu\bar{h}||w_2^{L_1,''}|| \end{pmatrix}$$

$$|\vec{L}_1^N(\vec{W}^{L_1} - \vec{w}^{L_1})(\varkappa_j)| \leq \begin{pmatrix} CN^{-1} \ln N\mu^2\epsilon_2^{-1}\mathfrak{B}_2^{l_1}(\varkappa_{j-1}) + C\epsilon_1^{-1}\mu^2\mathfrak{B}_1^{l_1}(\varkappa_{j-1}) \\ CN^{-1} \ln N\mu^2\epsilon_2^{-1}\mathfrak{B}_2^{l_1}(\varkappa_{j-1}) + C\epsilon_2^{-1}\mu^2\mathfrak{B}_1^{l_1}(\varkappa_{j-1}) \end{pmatrix}.$$

For $\varkappa_j \in [1 + \daleth_1, 1 + \daleth_2]$,

$$|\vec{L}_2^N(\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)| \leq \begin{pmatrix} \bar{h}\Big(\epsilon_1 ||w_{11}^{L_2,'''}|| + C\mu ||w_{11}^{L_2,''}||\Big) + \epsilon_1 ||w_{12}^{L_2,'''}|| + C\mu ||w_{12}^{L_2,'}|| \\ \epsilon_2\bar{h}||w_{21}^{L_2,'''}|| + \epsilon_1 ||w_{22}^{L_2,''}|| + C\mu\bar{h}||w_2^{L_2,''}|| \end{pmatrix}$$

$$|\vec{L}_2^N(\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)| \leq \begin{pmatrix} CN^{-1} \ln N\mu^2\epsilon_2^{-1}\mathfrak{B}_2^{l_2}(\varkappa_{j-1}) + C\epsilon_1^{-1}\mu^2\mathfrak{B}_1^{l_2}(\varkappa_{j-1}) \\ CN^{-1} \ln N\mu^2\epsilon_2^{-1}\mathfrak{B}_2^{l_2}(\varkappa_{j-1}) + C\epsilon_2^{-1}\mu^2\mathfrak{B}_1^{l_2}(\varkappa_{j-1}) \end{pmatrix}.$$

Similarly, for $\varkappa_j \in [\daleth_2, 1)$, it follows that $(\varkappa_{j+1} - \varkappa_{j-1}) \leq CN^{-1} \ln N$. Hence,

$$|\vec{L}_1^N(\vec{W}^{L_1} - \vec{w}^{L_1})(\varkappa_j)| \leq C(\varkappa_{j+1} - \varkappa_{j-1}) \begin{pmatrix} \epsilon_1 \|w_1^{L_1,'''}\|_D + \mu \|w_1^{L_1,''}\|_D \\ \epsilon_2 \|w_2^{L_1,'''}\|_D + \mu \|w_2^{L_1,''}\|_D \end{pmatrix}$$

$$\leq \begin{pmatrix} CN^{-1} \ln N\mu^2\epsilon_2^{-1}\mathfrak{B}_2^{l_1}(\varkappa_{j-1}) + C\epsilon_1^{-1}\mu^2\mathfrak{B}_1^{l_1}(\varkappa_{j-1}) \\ CN^{-1} \ln N\mu^2\epsilon_2^{-1}\mathfrak{B}_2^{l_1}(\varkappa_{j-1}) + C\epsilon_2^{-1}\mu^2\mathfrak{B}_2^{l_1}(\varkappa_{j-1}) \end{pmatrix}.$$

For $\varkappa_j \in [1 + \daleth_2, 2)$, it follows that $(\varkappa_{j+1} - \varkappa_{j-1}) \leq CN^{-1} \ln N$. Hence,

$$|\vec{L}_2^N(\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)| \leq C(\varkappa_{j+1} - \varkappa_{j-1}) \begin{pmatrix} \epsilon_1 \|w_1^{L_2,'''}\|_D + \mu \|w_1^{L_2,''}\|_D \\ \epsilon_2 \|w_2^{L_2,'''}\|_D + \mu \|w_2^{L_2,''}\|_D \end{pmatrix}$$

$$|\vec{L}_2^N(\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)| \leq \begin{pmatrix} CN^{-1} \ln N\mu^2\epsilon_2^{-1}\mathfrak{B}_2^{l_2}(\varkappa_{j-1}) + C\epsilon_1^{-1}\mu^2\mathfrak{B}_1^{l_2}(\varkappa_{j-1}) \\ CN^{-1} \ln N\mu^2\epsilon_2^{-1}\mathfrak{B}_2^{l_2}(\varkappa_{j-1}) + C\epsilon_2^{-1}\mu^2\mathfrak{B}_2^{l_2}(\varkappa_{j-1}) \end{pmatrix}.$$

Finally, by utilizing the minimum principle along with the specified barrier functions, the required bounds can be obtained. Let $\vec{\psi}^{\pm}(\varkappa_j) = \vec{\phi}(\varkappa_j) \pm (\vec{W}^{L_1} - \vec{w}^{L_1})(\varkappa_j)$, where $\vec{\phi} = (\phi_1, \phi_2)^T$ is defined for $0 \le j < \frac{N}{16}, p = 1$ and $\vec{\psi}^{\pm}(\varkappa_j) = \vec{\phi}(\varkappa_j) \pm (\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)$ for $\frac{N}{2} \le j \le \frac{9N}{16}, p = 2$ as follows:

$$\phi_1(\varkappa_j) = CN^{-1} \ln N \left[\exp\left(\frac{2\alpha\mu H_1}{\epsilon_1} \right) \boldsymbol{B}_1^{(l_p,N)}(\varkappa_j) + \exp\left(\frac{2\alpha\mu H_1}{\epsilon_2} \right) \boldsymbol{B}_2^{(l_p,N)}(\varkappa_j) \right],$$

$$\phi_2(\varkappa_j) = CN^{-1} \ln N \exp\left(\frac{2\alpha\mu H_1}{\epsilon_2} \right) \boldsymbol{B}_2^{(l_p,N)}(\varkappa_j),$$

and for $\frac{N}{16} \le j \le \frac{N}{8}, p = 1$ and $\frac{9N}{16} \le j \le \frac{5N}{8}, p = 2$ as

$$\phi_1(\varkappa_j) = CN^{-1} \ln N \left[\exp\left(\frac{2\alpha\mu H_2}{\epsilon_2} \right) \boldsymbol{B}_2^{(l_p,N)}(\varkappa_j) + C\boldsymbol{B}_1^{(l_p,N)}(\varkappa_j) \right],$$

$$\phi_2(\varkappa_j) = CN^{-1} \ln N \left[\exp\left(\frac{2\alpha\mu H_2}{\epsilon_2} \right) \boldsymbol{B}_2^{(l_p,N)}(\varkappa_j) + CN^{-1} \left((\urcorner_2 - \varkappa_j)\epsilon_2^{-1} + 1 \right) \right].$$

and for $\frac{N}{8} \le j \le \frac{N}{2}, p = 1$ and for $\frac{5N}{8} \le j \le N, p = 2$ as

$$\phi_1(\varkappa_j) = CN^{-1} \ln N \left[\exp\left(\frac{2\alpha\mu H_3}{\epsilon_2} \right) \boldsymbol{B}_2^{(l_p,N)}(\varkappa_j) + C\boldsymbol{B}_1^{(l_p,N)}(\varkappa_j) \right],$$

$$\phi_2(\varkappa_j) = CN^{-1} \ln N \left[\exp\left(\frac{2\alpha\mu H_3}{\epsilon_2} \right) \boldsymbol{B}_2^{(l_p,N)}(\varkappa_j) + C\boldsymbol{B}_1^{(l_p,N)}(\varkappa_j) \right].$$

Thus, for each of the cases, the barrier function is constructed, and, using maximum principle, it has been derived that

$$|(\vec{W}^{L_1} - \vec{w}^{L_1})(\varkappa_j)| \le CN^{-1} \ln N,$$

$$|(\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)| \le CN^{-1} \ln N.$$

Therefore,

$$|(\vec{W}^{L} - \vec{w}^{L})(\varkappa_j)| \le CN^{-1} \ln N.$$

The proof of the lemma is complete. □

For the case $\alpha\mu^2 \le \gamma\epsilon_1$, the error in the component $\vec{w}^L$ is bounded as follows:

Lemma 11. *Let $\vec{w}^{L_1}$ and $\vec{w}^{L_2}$ satisfy (16) , and $\vec{W}^{L_1}$ and $\vec{W}^{L_2}$ satisfy (104) and (105). Then,*

$$\|\vec{W}^{L_1} - \vec{w}^{L_1}\| \le CN^{-1} \ln N.$$

$$\|\vec{W}^{L_2} - \vec{w}^{L_2}\| \le CN^{-1} \ln N.$$

Proof. The local truncation error is given by

$$|\vec{L}_1^N(\vec{W}^{L_1} - \vec{w}^{L_1})(\varkappa_j)| \le C(\varkappa_{j+1} - \varkappa_{j-1}) \begin{pmatrix} \epsilon_1 \|w_1^{L_1,'''}\|_D + \mu \|w_1^{L_1,''}\|_D \\ \epsilon_2 \|w_2^{L_1,'''}\|_D + \mu \|w_2^{L_1,''}\|_D \end{pmatrix} \tag{116}$$

$$|\vec{L}_2^N(\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)| \le C(\varkappa_{j+1} - \varkappa_{j-1}) \begin{pmatrix} \epsilon_1 \|w_1^{L_2,'''}\|_D + \mu \|w_1^{L_2,''}\|_D \\ \epsilon_2 \|w_2^{L_2,'''}\|_D + \mu \|w_2^{L_2,''}\|_D \end{pmatrix}. \tag{117}$$

For $\urcorner_1 = \frac{1}{8}$ and $\urcorner_2 = \frac{1}{4}$, with a uniform mesh size $h = N^{-1}$, it follows that $\epsilon_1^{-1/2} \leq C \ln N$ and $\epsilon_2^{-1/2} \leq C \ln N$,

$$|\vec{L}_1^N(\vec{W}^{L_1} - \vec{w}^{L_1})(\varkappa_j)| \leq CN^{-1} \begin{pmatrix} \frac{1}{\sqrt{\epsilon_1}}\mathfrak{B}_1^{l_1}(\varkappa_{j-1}) + \frac{1}{\sqrt{\epsilon_2}}\mathfrak{B}_2^{l_1}(\varkappa_{j-1}) \\ \frac{1}{\sqrt{\epsilon_1}}\mathfrak{B}_1^{l_1}(\varkappa_{j-1}) + \frac{1}{\sqrt{\epsilon_2}}\mathfrak{B}_2^{l_1}(\varkappa_{j-1}) \end{pmatrix} \leq CN^{-1} \ln N \quad (118)$$

$$|\vec{L}_2^N(\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)| \leq CN^{-1} \begin{pmatrix} \frac{1}{\sqrt{\epsilon_1}}\mathfrak{B}_1^{l_2}(\varkappa_{j-1}) + \frac{1}{\sqrt{\epsilon_2}}\mathfrak{B}_2^{l_2}(\varkappa_{j-1}) \\ \frac{1}{\sqrt{\epsilon_1}}\mathfrak{B}_1^{l_2}(\varkappa_{j-1}) + \frac{1}{\sqrt{\epsilon_2}}\mathfrak{B}_2^{l_2}(\varkappa_{j-1}) \end{pmatrix} \leq CN^{-1} \ln N. \quad (119)$$

Consider the intervals separately for the case when $\urcorner_1 = \sqrt{\frac{\epsilon_1}{\gamma\alpha}} \ln N$ and $\urcorner_2 = \frac{1}{4}$, for $x_j \in (\urcorner_2, 1 - \urcorner_2), (1 + \urcorner_2, 2 - \urcorner_2)$

$$|\vec{L}_1^N(\vec{W}^{L_1} - \vec{w}^{L_1})(\varkappa_j)| \leq C \begin{pmatrix} N^{-1}\frac{1}{\sqrt{\epsilon_2}}\mathfrak{B}_2^{l_1}(\varkappa_{j-1}) \\ N^{-1}\frac{1}{\sqrt{\epsilon_2}}\mathfrak{B}_2^{l_1}(\varkappa_{j-1}) \end{pmatrix} \leq CN^{-1} \ln N$$

$$|\vec{L}_2^N(\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)| \leq C \begin{pmatrix} N^{-1}\frac{1}{\sqrt{\epsilon_2}}\mathfrak{B}_2^{l_2}(\varkappa_{j-1}) \\ N^{-1}\frac{1}{\sqrt{\epsilon_2}}\mathfrak{B}_2^{l_2}(\varkappa_{j-1}) \end{pmatrix} \leq CN^{-1} \ln N.$$

For $\varkappa_j \in (\urcorner_1, \urcorner_2), (1 - \urcorner_2, 1 - \urcorner_1)$,using the decomposition of $\vec{w}^{L_1}$ and $\vec{w}^{L_2}$, it follows that

$$|\vec{L}_1^N(\vec{W}^{L_1} - \vec{w}^{L_1})(\varkappa_j)| \leq CN^{-1} \begin{pmatrix} \frac{1}{\sqrt{\epsilon_1}}\mathfrak{B}_1^{l_1}(\varkappa_{j-1}) + \frac{1}{\sqrt{\epsilon_2}}\mathfrak{B}_2^{l_1}(\varkappa_{j-1}) \\ \frac{1}{\sqrt{\epsilon_1}}\mathfrak{B}_1^{l_1}(\varkappa_{j-1}) + \frac{1}{\sqrt{\epsilon_2}}\mathfrak{B}_2^{l_1}(\varkappa_{j-1}) \end{pmatrix} \leq CN^{-1} \ln N. \quad (120)$$

For $\varkappa_j \in (1 + \urcorner_1, 1 + \urcorner_2), (2 - \urcorner_2, 2 - \urcorner_1)$

$$|\vec{L}_2^N(\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)| \leq CN^{-1} \begin{pmatrix} \frac{1}{\sqrt{\epsilon_1}}\mathfrak{B}_1^{l_2}(\varkappa_{j-1}) + \frac{1}{\sqrt{\epsilon_2}}\mathfrak{B}_2^{l_2}(\varkappa_{j-1}) \\ \frac{1}{\sqrt{\epsilon_1}}\mathfrak{B}_1^{l_2}(\varkappa_{j-1}) + \frac{1}{\sqrt{\epsilon_2}}\mathfrak{B}_2^{l_2}(\varkappa_{j-1}) \end{pmatrix} \leq CN^{-1} \ln N. \quad (121)$$

For $\varkappa_j \in (0, \urcorner_1), (1 - \urcorner_1, 1)$, it follows that $(\varkappa_{j+1} - \varkappa_{j-1}) \leq CN^{-1}\sqrt{\epsilon_1} \ln N$; then,

$$|\vec{L}_1^N(\vec{W}^{L_1} - \vec{w}^{L_1})(\varkappa_j)| \leq CN^{-1}\sqrt{\epsilon_1} \ln N \begin{pmatrix} \frac{1}{\sqrt{\epsilon_1}} + \frac{1}{\sqrt{\epsilon_2}} \\ \frac{1}{\sqrt{\epsilon_1}} + \frac{1}{\sqrt{\epsilon_2}} \end{pmatrix} \leq CN^{-1} \ln N. \quad (122)$$

For $\varkappa_j \in (1, 1 + \urcorner_1), (2 - \urcorner_1, 2)$

$$|\vec{L}_2^N(\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)| \leq CN^{-1}\sqrt{\epsilon_1} \ln N \begin{pmatrix} \frac{1}{\sqrt{\epsilon_1}} + \frac{1}{\sqrt{\epsilon_2}} \\ \frac{1}{\sqrt{\epsilon_1}} + \frac{1}{\sqrt{\epsilon_2}} \end{pmatrix} \leq CN^{-1} \ln N. \quad (123)$$

To estimate this bound, the following mesh functions are defined on $\bar{\Omega}^N$

$$\boldsymbol{B}_1^{(l_p,N)}(\varkappa_j) = \prod_{i=1}^{j} \left(1 + \sqrt{\frac{\gamma\alpha}{\epsilon_1}} h_i\right)^{-1}, \quad (124)$$

$$\boldsymbol{B}_2^{(l_p,N)}(\varkappa_j) = \prod_{i=1}^{j} \left(1 + \sqrt{\frac{\gamma\alpha}{\epsilon_2}} h_i\right)^{-1}, \quad (125)$$

with $\boldsymbol{B}_1^{(l_p,N)}(\varkappa_0) = \boldsymbol{B}_2^{(l_p,N)}(\varkappa_0) = 1$, for $\Omega_1, p = 1$, for $\Omega_2, p = 2$. When $\daleth_2 = \sqrt{\frac{\epsilon_2}{\gamma\alpha}} \ln N$ and $\daleth_1 = \frac{\daleth_2}{2}$, it follows that $\frac{\sqrt{\epsilon_2}}{2} \leq \sqrt{\epsilon_1} < \sqrt{\epsilon_2}$; hence, $\daleth_2 \leq C\sqrt{\epsilon_1} \ln N$. For $\varkappa_j \in [\daleth_2, 1 - \daleth_2]$, it follows that

$$\begin{aligned}|(W_1{}^{L_1} - w_1{}^{L_1})(\varkappa_j)| &\leq |W_1{}^{L_1}(\varkappa_j)| + |w_1{}^{L_1}(\varkappa_j)| \\ &\leq C\boldsymbol{B}_2^{(l_1,N)}(\varkappa_j) + C\mathfrak{B}_2^{l_1}(\varkappa_j) \leq \prod_{i=1}^{j}\left(1 + \sqrt{\frac{\gamma\alpha}{\epsilon_1}}h_i\right)^{-1} + C\mathfrak{B}_2^{l_1}(\daleth_2) \\ &\leq (1 + 16N^{-1}\ln N)^{\frac{-N}{16}} + CN^{-1} \leq CN^{-1}.\end{aligned}$$

Similarly,

$$|(W_2{}^{L_1} - w_2{}^{L_1})(\varkappa_j)| \leq CN^{-1}.$$

For $\varkappa_j \in [1 + \daleth_2, 2 - \daleth_2]$, it follows that

$$\begin{aligned}|(W_1{}^{L_2} - w_1{}^{L_2})(\varkappa_j)| &\leq |W_1{}^{L_2}(\varkappa_j)| + |w_1{}^{L_2}(\varkappa_j)| \\ &\leq C\boldsymbol{B}_2^{(l_2,N)}(\varkappa_j) + C\mathfrak{B}_2^{l_2}(\varkappa_j) \leq \prod_{i=1}^{j}\left(1 + \sqrt{\frac{\gamma\alpha}{\epsilon_1}}h_i\right)^{-1} + C\mathfrak{B}_2^{l_1}(\daleth_2) \\ &\leq (1 + 16N^{-1}\ln N)^{\frac{-N}{16}} + CN^{-1} \leq CN^{-1}.\end{aligned}$$

Simillarly,

$$|(W_2{}^{L_2} - w_2{}^{L_2})(\varkappa_j)| \leq CN^{-1}.$$

When $\varkappa_j \in (\daleth_1, \daleth_2)$ or $(1 - \daleth_2, 1 - \daleth_1)$, $\varkappa_{j+1} - \varkappa_{j-1} \leq CN^{-1}\sqrt{\epsilon_1}\ln N$. Therefore,

$$|\vec{L}_1^N(\vec{W}^{L_1} - \vec{w}^{L_1})(\varkappa_j)| \leq CN^{-1}\sqrt{\epsilon_1}\ln N\begin{pmatrix}\frac{1}{\sqrt{\epsilon_1}} + \frac{1}{\sqrt{\epsilon_2}} \\ \frac{1}{\sqrt{\epsilon_1}} + \frac{1}{\sqrt{\epsilon_2}}\end{pmatrix} \leq CN^{-1}\ln N. \tag{126}$$

For $\varkappa_j \in (1 + \daleth_1, 1 + \daleth_2), (2 - \daleth_2, 2 - \daleth_1)$

$$|\vec{L}_2^N(\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)| \leq CN^{-1}\sqrt{\epsilon_1}\ln N\begin{pmatrix}\frac{1}{\sqrt{\epsilon_1}} + \frac{1}{\sqrt{\epsilon_2}} \\ \frac{1}{\sqrt{\epsilon_1}} + \frac{1}{\sqrt{\epsilon_2}}\end{pmatrix} \leq CN^{-1}\ln N. \tag{127}$$

When $\varkappa_j \in (0, \daleth_1)$ or $(1 - \daleth_1, 1)$, $(\varkappa_{j+1} - \varkappa_{j-1}) \leq C\sqrt{\epsilon_1}N^{-1}\ln N$, and the required bounds are derived as

$$\begin{aligned}|\vec{L}_1^N(\vec{W}^{L_1} - \vec{w}^{L_1})(\varkappa_j)| &\leq CN^{-1}\ln\ N \\ |\vec{L}_2^N(\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)| &\leq CN^{-1}\ln N.\end{aligned}$$

When $\daleth_2 = \sqrt{\frac{\epsilon_2}{\gamma\alpha}}\ln N$ and $\daleth_1 = \sqrt{\frac{\epsilon_1}{\gamma\alpha}}\ln N$ and when $\varkappa_j \in [\daleth_2, 1 - \daleth_2]$ or $[1 + \daleth_2, 2 - \daleth_2]$,

$$\begin{aligned}|\vec{L}_1^N(\vec{W}^{L_1} - \vec{w}^{L_1})(\varkappa_j)| &\leq CN^{-1}\ln\ N \\ |\vec{L}_2^N(\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)| &\leq CN^{-1}\ln N.\end{aligned}$$

For $\varkappa_j \in (0, \daleth_1]$ or $(1 - \daleth_2, 1)$, analogous arguments to those applied in the corresponding intervals of earlier cases yield the required bounds,$(\varkappa_{j+1} - \varkappa_{j-1}) \leq C\sqrt{\epsilon_1}N^{-1}\ln N$,

$$|\vec{L}_1^N(\vec{W}^{L_1} - \vec{w}^{L_1})(\varkappa_j)| \leq CN^{-1}\ln N.$$

For $\varkappa_j \in (1, 1+\urcorner_1]$ or $(2-\urcorner_1, 2)$,

$$|\vec{L}_2^N(\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)| \leq CN^{-1}\ln N.$$

When $\varkappa_j \in (\urcorner_1, \urcorner_2)$ or $(1-\urcorner_2, 1-\urcorner_1)$, $\varkappa_{j+1} - \varkappa_{j-1} \leq CN^{-1}\urcorner_2 \leq CN^{-1}\sqrt{\epsilon_2}\ln N$. Hence,

$$|\vec{L}_1^N(\vec{W}^{L_1} - \vec{w}^{L_1})(\varkappa_j)| \leq CN^{-1}\ln N.$$

For $\varkappa_j \in (1+\urcorner_1, 1+\urcorner_2)$ or $(2-\urcorner_2, 2-\urcorner_1)$,

$$|\vec{L}_2^N(\vec{W}^{L_2} - \vec{w}^{L_2})(\varkappa_j)| \leq CN^{-1}\ln N.$$

Therefore,

$$|(\vec{W}^{L} - \vec{w}^{L})(\varkappa_j)| \leq CN^{-1}\ln N.$$

The proof of the lemma is complete. □

For the case $\alpha\mu^2 \geq \gamma\epsilon_2$, the error in the component $\vec{w}^{R_1}$ is bounded as follows:

Lemma 12. *When $\vec{w}^{R_1}(\varkappa)$ is decomposed as in (84), the following bounds hold $|\vec{w}^{R_1}(0)| \leq Ce^{-\frac{\gamma}{\mu}}$.*

Proof. Decomposition of (84),

$$\vec{w}^{R_1}(0) = \vec{w}_0^{R_1}(0) + \epsilon_2\vec{w}_1^{R_1}(0) + \epsilon_2^2\vec{w}_2^{R_1}(0).$$

Let $\psi_i^{\pm}(x) = Ce^{-\frac{\gamma}{\mu}(1-x)} \pm w_{0i}^{R_1}(x)$, for $i = 1, 2$, where C is chosen to be sufficiently large and it is evident that $\psi_i^{\pm}(0) \geq 0$. Additionally, it is known that $\vec{L_1}\vec{\psi}_1 \leq 0$ as defined in Lemma 3. By utilizing the minimum principle, it follows that $\psi_i^{\pm}(x) \geq 0$, leading to the inequality

$$|w_{0i}^{R_1}(\varkappa)| \leq Ce^{-\frac{\gamma}{\mu}(1-\varkappa)} \quad \text{for } i = 1, 2.$$

Using this result and Equation (85), the bound is derived as follows:

$$|w_{0i}^{R_1,\prime}(\varkappa)| \leq \frac{C}{\mu}e^{-\frac{\gamma}{\mu}(1-\varkappa)}, \quad \text{for } i = 1, 2.$$

By differentiating Equation (85) and applying the previous bounds, it is obtained that

$$|w_{0i}^{R_1,\prime\prime}(\varkappa)| \leq \frac{C}{\mu^2}e^{-\frac{\gamma}{\mu}(1-\varkappa)}, \quad \text{for } i = 1, 2.$$

Thus,

$$|w_{0i}^{R_1,(k)}(\varkappa)| \leq \frac{C}{\mu^k}e^{-\frac{\gamma}{\mu}(1-\varkappa)}, \quad \text{for } i = 1, 2.$$

Next, considering $w_1^{R_1}(\varkappa)$ and using the established bounds on $w_{0i}^{R_1}(\varkappa)$ and their derivatives, it is concluded that

$$|w_{1i}^{R_1,(k)}(\varkappa)| \leq \frac{C}{\mu^{(k+2)}}e^{-\frac{\gamma}{\mu}(1-\varkappa)}, \quad \text{where } k = 0, 1, 2 \text{ and } i = 1, 2.$$

Similarly, it is found that

$$|w_{2i}^{R_1,(k)}(\varkappa)| \leq \frac{C}{\mu^{(k+4)}}e^{-\frac{\gamma}{\mu}(1-\varkappa)}, \quad \text{where } k = 0, 1, 2 \text{ and } i = 1, 2.$$

By combining all these results, the bound is obtained as follows: $|\vec{w}^{R_1}(\varkappa)| \leq Ce^{-\frac{\gamma}{\mu}(1-\varkappa)}$ for [0,1]. Analogously, it is easy to find $|\vec{w}^{R_2}(\varkappa)| \leq Ce^{-\frac{\gamma}{\mu}(2-\varkappa)}$ for [1,2]. The proof of the lemma is complete. □

Lemma 13. *The singular component $\vec{w}^{R_1}(\varkappa)$ and $\vec{w}^{R_2}(\varkappa)$ for the case $\alpha\mu^2 \geq \gamma\epsilon_2$ satisfies the following bounds:*

$$|\vec{w}^{R_1}(\varkappa)| \leq Ce^{-\frac{\gamma}{\mu}(1-\varkappa)},$$
$$|\vec{w}^{R_2}(\varkappa)| \leq Ce^{-\frac{\gamma}{\mu}(2-\varkappa)}.$$

Proof. Utilizing the previous lemma, deriving the above bound is not hard. To establish the bounds on the error $|(W^{R_1} - w^{R_1})(\varkappa_j)|$, the mesh function is defined over $\bar{\Omega}^N$

$$\boldsymbol{B}^{(r_p,N)}(\varkappa_j) = \prod_{i=j+1}^{N}\left(1+\frac{\gamma h_i}{2\mu}\right)^{-1}, \quad \boldsymbol{B}^{(r_p,N)}(\varkappa_N) = 1, \text{where } \boldsymbol{B}^{(r_p,N)}, i = 1,2.$$

Now, consider $\vec{\Psi}^{R_1}(\varkappa_j) = C\boldsymbol{B}^{(r_1,N)}(\varkappa_j) \pm \vec{W}^{R_1}$ and $\vec{\Psi}^{R_2}(\varkappa_j) = C\boldsymbol{B}^{(r_2,N)}(\varkappa_j) \pm \vec{W}^{R_2}$. Also, since $\vec{W}^{R_1}(0) \leq e^{-\frac{\gamma}{\mu}}$, $\vec{W}^{R_1}(0) \leq \boldsymbol{B}^{(r_1,N)}(\varkappa_0)$. Hence, $\vec{\Psi}^{R_1}(0) \geq 0$. Also, for an appropriate choice of C, it follows that $\vec{\Psi}^{R_1}(\varkappa_N) \geq 0$. Further, $\vec{L}_1^N\vec{\Psi}^{R_1}(\varkappa_j) \leq 0$ and $\vec{L}_2^N\vec{\Psi}^{R_2}(\varkappa_j) \leq 0$. Hence, by the minimum principle , $\vec{\Psi}^{R_1}(\varkappa_j) \geq 0$ and $\vec{\Psi}^{R_2}(\varkappa_j) \geq 0$, for $0 \leq j \leq N$. Hence, similarly, it can be stated for $|W^{R_2}(\varkappa_j)|$,

$$|W^{R_1}(\varkappa_j)| \leq C\boldsymbol{B}^{(r_1,N)}(\varkappa_j) \text{ and } \quad |W^{R_2}(\varkappa_j)| \leq C\boldsymbol{B}^{(r_2,N)}(\varkappa_j) \text{ on } \bar{\Omega}^N.$$

The proof of the lemma is complete. □

Lemma 14. *At each point $\varkappa_j \in \bar{\Omega}^N$, $|(\vec{W}^R - \vec{w}^R)(\varkappa_j)| \leq CN^{-1}\ln N$, for the case $\alpha\mu^2 \geq \gamma\epsilon_2$.*

Proof. The local truncation error is given by

$$|\vec{L}_1^N(\vec{W}^{R_1} - \vec{w}^{R_1})| \leq C(\varkappa_{j+1} - \varkappa_{j-1})\begin{pmatrix}\epsilon_1\|w_1^{R_1,'''}\|_D + \mu\|w_1^{R_1,''}\|_D \\ \epsilon_2\|w_2^{R_1,'''}\|_D + \mu\|w_2^{R_1,''}\|_D\end{pmatrix} \tag{128}$$

$$|\vec{L}_2^N(\vec{W}^{R_2} - \vec{w}^{R_2})| \leq C(\varkappa_{j+1} - \varkappa_{j-1})\begin{pmatrix}\epsilon_1\|w_1^{R_2,'''}\|_D + \mu\|w_1^{R_2,''}\|_D \\ \epsilon_2\|w_2^{R_2,'''}\|_D + \mu\|w_2^{R_2,''}\|_D\end{pmatrix} \tag{129}$$

where $D = [\varkappa_{j-1}, \varkappa_{j+1}]$. Consider the case $\sigma_1 = \frac{\mu}{\gamma}\ln N$, $x_j \in (0, 1-\sigma_1]$. Hence,

$$\begin{aligned}|(\vec{W}^{R_1} - \vec{w}^{R_1})(\varkappa_j)| &\leq |\vec{W}^{R_1}(\varkappa_j)| + |\vec{w}^{R_1}(\varkappa_j)| \\ &\leq C\boldsymbol{B}^{(r_1,N)}(\varkappa_j) + C\mathfrak{B}_2^{r_1}(\varkappa_j) \\ &\leq C\boldsymbol{B}^{(r_1,N)}(\sigma_1) + C\mathfrak{B}_2^{r_1}(\sigma_1) \leq CN^{-1},\end{aligned}$$

for $\varkappa_j \in (1, 2-\sigma_1]$

$$\begin{aligned}|(\vec{W}^{R_2} - \vec{w}^{R_2})(\varkappa_j)| &\leq |\vec{W}^{R_2}(\varkappa_j)| + |\vec{w}^{R_2}(\varkappa_j)| \\ &\leq C\boldsymbol{B}^{(r_2,N)}(\varkappa_j) + C\mathfrak{B}_2^{r_2}(\varkappa_j) \\ &\leq C\boldsymbol{B}^{(r_2,N)}(\sigma_1) + C\mathfrak{B}_2^{r_2}(\sigma_1) \leq CN^{-1}.\end{aligned}$$

Examine the mesh region $(1-\sigma_1, 1]$; it is known that $\bar{h} = \varkappa_{j+1} - \varkappa_{j-1}$, then, $\varkappa_{j+1} - \varkappa_{j-1} \leq C\mu N^{-1} \ln N$

$$|\vec{L}_1^N(\vec{W}^{R_1} - \vec{w}^{R_1})(\varkappa_j)| \leq C\mu N^{-1} \ln N \begin{pmatrix} \epsilon_1(\frac{C}{\mu^3} + \frac{C}{\epsilon_1}) + \mu(\frac{C}{\mu^2}) \\ \epsilon_2(\frac{C}{\mu^3}) + \mu(\frac{C}{\mu^2}) \end{pmatrix}.$$

$$\leq CN^{-1} \ln N.$$

For $(2-\sigma_1, 2]$

$$|\vec{L}_2^N(\vec{W}^{R_2} - \vec{w}^{R_2})(\varkappa_j)| \leq C\mu N^{-1} \ln N \begin{pmatrix} \epsilon_1(\frac{C}{\mu^3} + \frac{C}{\epsilon_1}) + \mu(\frac{C}{\mu^2}) \\ \epsilon_2(\frac{C}{\mu^3}) + \mu(\frac{C}{\mu^2}) \end{pmatrix}.$$

$$\leq CN^{-1} \ln N.$$

When $\sigma_1 = \frac{1}{4}$, $\mu^{-1} \leq C \ln N$. Hence,

$$|(\vec{W}^{R_1} - \vec{w}^{R_1})(\varkappa_j)| \leq CN^{-1}\mu^{-1} \leq CN^{-1} \ln N,$$

$$|(\vec{W}^{R_2} - \vec{w}^{R_2})(\varkappa_j)| \leq CN^{-1}\mu^{-1} \leq CN^{-1} \ln N.$$

Thus,

$$\|\vec{W}^R - \vec{w}^R\|_{\bar{\Omega}^N} \leq CN^{-1} \ln N.$$

The proof of the lemma is complete. □

From the previous Lemmas 10–14 , for $\alpha\mu^2 \leq \gamma\epsilon_1$, analogously $\|\vec{W}^R - \vec{w}^R\|_{\bar{\Omega}^N} \leq CN^{-1} \ln N$ is obtained.

Theorem 3. *Let* $\vec{\mathfrak{u}}$ *be the solution of* (1) *and* $\vec{\mathfrak{U}}$ *be the solution of* (99)–(101)*. Then, for each mesh point* $\varkappa_j \in \bar{\Omega}^N$,

$$\|\vec{\mathfrak{U}} - \vec{\mathfrak{u}}\|_{\bar{\Omega}^N} \leq CN^{-1} \ln N,$$

for both of the cases $\alpha\mu^2 \leq \gamma\epsilon_1$ *and* $\alpha\mu^2 \geq \gamma\epsilon_2$.

Proof. The proof follows Lemmas 8, 10, 11, and 14. □

12. Numerical Illustration

Example

The numerical approximated the solution to the following system on the interval $(0, 2)$, applying the proposed method to both cases where $\alpha\mu^2 \leq \gamma\varepsilon_1$ and $\alpha\mu^2 \geq \gamma\varepsilon_2$

$$\epsilon_1 \mathfrak{u}_1''(\varkappa) + 0.5\mu\mathfrak{u}_1'(\varkappa) - 6\mathfrak{u}_1(\varkappa) + 2\mathfrak{u}_2(\varkappa) + 0.8\mathfrak{u}_1(\varkappa - 1) = 1.0$$

$$\epsilon_2 \mathfrak{u}_2''(\varkappa) + 0.5\mu\mathfrak{u}_2'(\varkappa) + 2\mathfrak{u}_1(\varkappa) - 6\mathfrak{u}_2(\varkappa) + 0.8\mathfrak{u}_2(\varkappa - 1) = 2.0$$

with boundary conditions specified as $\mathfrak{u}_1(0) = \mathfrak{u}_1(2) = 1$ and $\mathfrak{u}_2(0) = \mathfrak{u}_2(2) = 1$.

To evaluate the order of convergence, maximum pointwise errors, and error constants, a modified two-mesh algorithm was utilized . The findings are summarized in Tables 1 and 2. Notably, as η decreases, the error stabilizes for each N, while the maximum

error D_N reduces with increasing N. Furthermore, the order of convergence p_N rises as N increases, confirming the theoretical results.

Figures 1 and 2 illustrate the graphs of the solutions obtained and demonstrate how symmetry in the layers contributes to the accuracy and validation of the theoretical results. In Figure 1, where the ratio $\frac{\mu^2}{\varepsilon_1} \leq \frac{\gamma}{\alpha}$, boundary layers for $\mathbf{u}_1$ and $\mathbf{u}_2$ are observed near $\varkappa = 0$, $\varkappa = 1$, and $\varkappa = 2$, which is consistent with expectations. In contrast, Figure 2 shows that when $\frac{\mu^2}{\varepsilon_2} \geq \frac{\gamma}{\alpha}$, the initial layers for $\mathbf{u}_1$ and $\mathbf{u}_2$ are present near $\varkappa = 0$, while boundary layers appear for both $\mathbf{u}_1$ and $\mathbf{u}_2$ near $\varkappa = 2$ and a delay appears near $\varkappa = 1$.

Table 1. Values of D_ε^N, D^N, p^N, p^* , and $C_{p^*}^N$ when $\epsilon_1 = \frac{\eta}{64}, \epsilon_2 = \frac{\eta}{32}, \mu = \frac{\eta}{4}$ for $\alpha\mu^2 \leq \gamma\epsilon_1$.

η	Number of Mesh Points N			
	64	128	256	512
0.250	0.326×10^{-2}	0.242×10^{-2}	0.137×10^{-2}	0.691×10^{-3}
0.625×10^{-1}	0.326×10^{-2}	0.216×10^{-2}	0.131×10^{-2}	0.756×10^{-3}
0.156×10^{-1}	0.326×10^{-2}	0.216×10^{-2}	0.131×10^{-2}	0.756×10^{-3}
0.391×10^{-2}	0.326×10^{-2}	0.216×10^{-2}	0.131×10^{-2}	0.756×10^{-3}
0.977×10^{-3}	0.325×10^{-2}	0.215×10^{-2}	0.131×10^{-2}	0.755×10^{-3}
D^N	0.326×10^{-2}	0.242×10^{-2}	0.137×10^{-2}	0.756×10^{-3}
p^N	0.429×10^{0}	0.823×10^{0}	0.857×10^{0}	
C_p^N	0.754×10^{-1}	0.754×10^{-1}	0.574×10^{-1}	0.427×10^{-1}
The order of convergence $p^* = 0.429 \times 10^{0}$				
Computed error constant, $C_{p^*}^N = 0.754 \times 10^{-1}$				

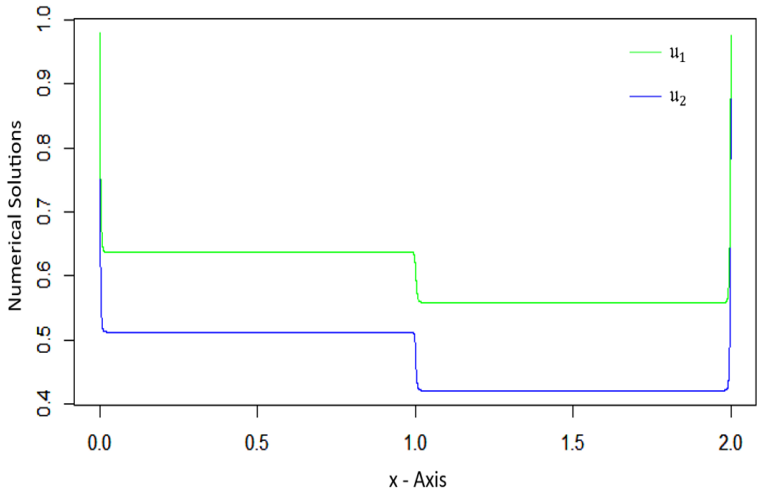


Figure 1. Graphical representation of numerical solutions for the case $\alpha\mu^2 \leq \gamma\epsilon_1$.

Table 2. Values of D_{ε}^{N}, D^{N}, p^{N}, p^{*}, and $C_{p^{*}}^{N}$ when $\epsilon_1 = \frac{\eta}{64}, \epsilon_2 = \frac{\eta}{32}, \mu = \frac{\eta}{4}$ for $\alpha\mu^2 \geq \gamma\epsilon_2$.

η	Number of Mesh Points N			
	64	128	256	512
0.625×10^{-1}	0.159×10^{-1}	0.111×10^{-1}	0.367×10^{-2}	0.958×10^{-3}
0.156×10^{-1}	0.123×10^{-1}	0.158×10^{-1}	0.109×10^{-1}	0.357×10^{-2}
0.391×10^{-2}	0.169×10^{-1}	0.123×10^{-1}	0.154×10^{-1}	0.106×10^{-1}
0.977×10^{-3}	0.499×10^{-1}	0.337×10^{-1}	0.201×10^{-1}	0.147×10^{-1}
0.244×10^{-3}	0.889×10^{-1}	0.761×10^{-1}	0.576×10^{-1}	0.372×10^{-1}
D^N	0.889×10^{-1}	0.761×10^{-1}	0.575×10^{-1}	0.371×10^{-1}
p^N	0.224×10^{0}	0.403×10^{0}	0.631×10^{0}	
C_p^N	0.157×10^{1}	0.157×10^{1}	0.138×10^{1}	0.104×10^{1}
The order of convergence $p^* = 0.224 \times 10^{0}$				
Computed error constant, $C_{p^*}^N = 0.157 \times 10^{1}$				

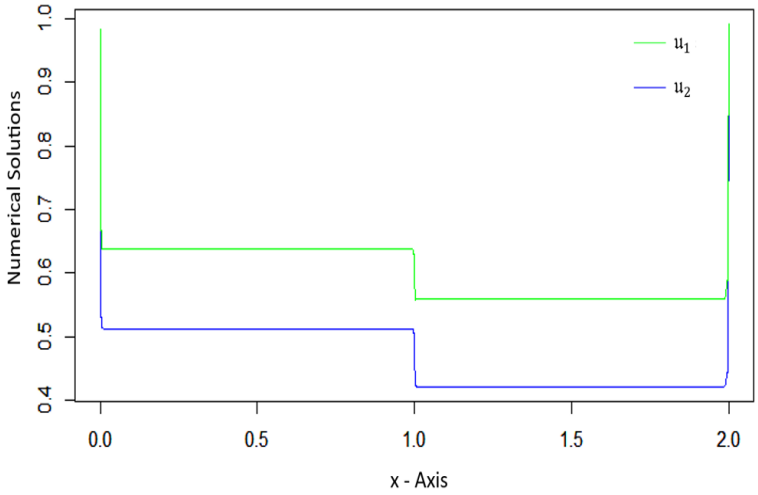


Figure 2. Graphical representation of Numerical solutions for the case: $\alpha\mu^2 \geq \gamma\epsilon_2$.

13. Conclusions

In this paper, a robust fitted mesh finite difference method is introduced for solving a system of two-parameter singularly perturbed delay differential equation of the convection–reaction–diffusion type. By employing a piecewise uniform Shishkin mesh, our robust fitted mesh method effectively addresses the complexities arising from two distinct small perturbation parameters, delay terms, and resolved boundary and interior layers in SPDDEs. The theoretical analysis confirms that the proposed scheme achieves almost first-order convergence in the maximum norm, independent of these two distinct perturbation parameters. Numerical illustrations validate the method's effectiveness, showcasing its ability to resolve intricate boundary and interior layers. This work highlights the importance in numerical techniques for SPDDEs and lays the groundwork for further advancements. Future research could focus on extending this methodology to multidimensional and time-dependent problems, exploring adaptive mesh strategies and applying the approach to real-world applications in engineering and science.

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