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Galerkin-Type Solution of the Föppl–von Kármán Equations for Square Plates

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Abstract: The solution of the non-linear Föppl–von Kármán equations for square plates in the form of expansion over a system of eigenfunctions, generated by a linear self-adjoint operator, is obtained. The coefficients of the expansion are determined via the reduction method from the infinite-dimensional system of cubic equations. This allows the proposed solution to be considered as a non-linear generalization of the classical Galerkin approach. The novelty of the study is in the strict formulation of the auxiliary boundary problem, which makes it possible to take into account a rigid fixation against any displacements along the boundary. To verify the proposed solution, it is compared with experimental data. The latter is obtained by the holographic interferometry of small deflection increments superimposed on the large deflection caused by initial pressure. Experiment and theory show a good agreement.

Keywords: Föppl–von Kármán equations; finite deformations; clamped square plates; non-linear PDEs; expansion over a basis in space of square-integrable functions; Galerkin method; infinite-dimensional system of cubic equations; reduction; experimental verification



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1. Introduction

The Föppl–von Kármán equations, which describe the stress-strain state of flexible plates, have attracted the attention of physicists and mathematicians since August Föppl introduced their prototype in the treatise “Vorlesungen über technische Mechanik” [1]. A. Föppl assumed that in the case of finite deflections, both in-plane linear strains and the rotations are small with respect to unity, while the rotations and squares of in-plane strains are infinitesimals of the same order. Furthermore, he supposed that the bending strains can be thought of infinitesimally small with respect to membrane ones. With these assumptions he derived the following system of equations:

$$hL(w, \phi) + p = 0, \quad \nabla^2 \nabla^2 \phi + \frac{E}{2} L(w, w) = 0. \quad (1)$$

Here, h is plate thickness, E stands for Young’s modulus of the material, and p is the function of transversal load applied to the plate surface. The function $w(x, y)$ represents the deflections of the plate middle-plane, while $\phi(x, y)$ is the Airy stress function related to the components of the stress tensor within it:

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2}, \quad \sigma_{xy} = \sigma_{yx} = -\frac{\partial^2 \phi}{\partial x \partial y}. \quad (2)$$

Symbols $\nabla^2 \nabla^2$ and L stand for the biharmonic operator and the Monge–Ampère form [2], respectively. In Cartesian coordinates, they can be represented as follows:

$$\nabla^2 \nabla^2 f = \frac{\partial^4 f}{\partial x^4} + 2 \frac{\partial^4 f}{\partial x^2 \partial y^2} + \frac{\partial^4 f}{\partial y^4}, \quad L(f, g) = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 f}{\partial y^2} \frac{\partial^2 g}{\partial x^2} - 2 \frac{\partial^2 f}{\partial x \partial y} \frac{\partial^2 g}{\partial x \partial y}. \quad (3)$$

Theodor von Kármán improved the first equation in (1) by adding an item that accounts for the influence of in-plane stresses on bending and derived the system of equations in a well-known, almost symmetric form [3]:

$$D \nabla^2 \nabla^2 w - h L(w, \phi) = p, \quad \nabla^2 \nabla^2 \phi + \frac{E}{2} L(w, w) = 0. \quad (4)$$

Here, D denotes the bending stiffness of the plate, which is expressed through h , E , and the Poisson's ratio ν :

$$D = \frac{E h^3}{12(1 - \nu^2)}.$$

The Föppl–von Kármán equations were initially developed as a mathematical tool for buckling analysis of ship hull parts [4]. However, wide application of the structural elements that were able to bend sufficiently without undergoing plastic deformations calls for further development and improvement of the non-linear theories of plates, subjected to various loads, including transverse ones. The Föppl–von Kármán equations proved to be a good approach, taking into account non-linear effects but not having the intimidating bulky form of the non-linear elasticity equations. At present these equations are widely used to describe deformations of various elements of modern technical systems, particularly MEMS, for which the problem of accurate mathematical modeling of stress-strain state in flexible parts is extremely relevant [5,6].

First solutions of the non-linear bending problem were obtained for circular plates subjected to axially symmetric loading, which allowed one to significantly simplify the Föppl–von Kármán equations. Thus, the first attempt to solve them for a clamped circular plate under uniform pressure was undertaken by Nádai [7] in 1925. Another variant of the approximate solution was obtained by Timoshenko, who used the Rayleigh–Ritz method [8], and by Way through the power series expansions [9]. In the first part of their study [10], Friedrichs and Stoker presented numerical solutions for a simply supported circular plate under compressive forces at the boundary, obtained by three distinct algorithms. Furthermore, in [11], they introduced a change in variables, which reduced the order of the governing system of equations from 8th to 4th. In addition, in the second part of [10], they presented a proof of the existence theorem and demonstrated that, apart from a trivial solution, there is at most another one (accurate to a sign). In the present paper, only square plates will be discussed, so the part of the review focused on circular plates is exhausted by a few references. A detailed survey can be found in [12].

One of the first approximate solutions of the Föppl–von Kármán equations for a simply supported square plate under uniform pressure was derived by Kaiser [13] using the finite-difference method. An alternative approach to approximate solutions was proposed by Way [14]. He obtained a solution for clamped rectangular plates with various aspect ratios by the Ritz method. The Galerkin method was also successfully applied for this problem by Panov [15]. Levy first derived exact solutions in terms of double Fourier series for a simply supported rectangular plate [16] and a clamped square one [17]. Later, he and Greenman solved the problem for a clamped rectangular plate having an aspect ratio of 3/2 [18]. This method has also been applied in [19,20] for long clamped and simply supported plates under combined load. It is important to note that in works [16–19], it is assumed that clamped edges are movable, i.e., they are rigidly clamped against normal displacements

and rotations while allowing for movement in plane. A numerical solution for a clamped rectangular plate with immovable edges was obtained by Wang [21]. He also considered the important case of a riveted panel [22]. Lastly, it is worth noting the article by Green and Southwell on the application of the relaxation method to solve the Föppl–von Kármán system [23]. A detailed description of the majority of the aforementioned results, as well as many others, can be found in the remarkable books [4,24].

Briefly discuss the recent works on this topic. In the works [25,26], the solution for a simply supported plate was obtained by the “accelerated” iterative method. The perturbation method was successfully applied in the works [27–30]. In addition, the work [29] is notable because of the application of the spline function method for solving a set of linear equations. A number of authors successfully applied the finite-difference method [31–33]. A modification of this approach was proposed by Petit [34], who employed the finite-difference method in combination with the relaxation one. Furthermore, this method was also used in the work [35] for a plate under combined load. The finite element method was used extensively [36–39]. Also, a few attempts to apply the boundary element method were made [40–44]. The Ritz method was widely used as well. Boresi and Terner [45] applied it to obtain the solution for a simply supported rectangular plate with stress-free edges. In the work [46], the rectangular plates with various boundary conditions were considered using the automated Ritz method procedure. This procedure was also applied to calculate large deflections of orthotropic plates [47] and functionally graded material ones [48]. Enem [49] introduced the Ritz method solutions for rectangular plates with twelve different boundary conditions. Recently, the homotopy analysis method has begun to be successfully applied to solve the Föppl–von Kármán system [50–53]. A sufficiently detailed review of works managed before the 1970s is presented in the article [54]. Also, a comprehensive list of works managed before the 2000s is given in the book [55], while a detailed review containing more recent works can be found, for example, in the PhD thesis by Khoa [56].

An original approach to solve the Föppl–von Kármán system was proposed by H. Berger in his PhD thesis [57]. This approach was based on the hypothesis that the strain tensor second invariant makes a small contribution to the full potential energy in comparison with the contribution made by the first invariant. Although this hypothesis lacked a physical basis, it allowed one to decouple the Föppl–von Kármán system. Berger obtained solutions for circular and rectangular plates with various boundary conditions using this method. The further development of Berger’s technique was continued by a number of other scientists [58–61]. A similar method for decoupling the Föppl–von Kármán equations was employed in the recent works [62–64]. An alternative approach to decoupling of this system was proposed by Banerjee and Datta [65].

A large amount of results were obtained by the Galerkin method. Yamaki [66] used a modification of the one-term Galerkin method to solve the dynamic analogue of the Föppl–von Kármán equations. He also gave a classification of flexible plate boundary conditions. In their work [67], Iyengar and Naqvi applied biorthogonal expansions of the displacements and the Airy function to obtain solutions for a simply supported plate and a clamped one. For each boundary condition type, two options were considered: all edges are stress-free, and all edges are immovable. In the papers [68,69], the authors obtained the Galerkin method-based system of cubic equations for Kármán plates in an explicit form. In the work [70], this result was supplemented with the explicit expression of the inverted Galerkin matrix. This procedure allowed us to obtain the solution more effectively due to avoiding a laborious calculation of the inverted Galerkin matrix. Recently, the wavelet-Galerkin method has been widely used [71,72]. This method leads to simplification of calculations due to the choice of wavelet functions as the basis.

It is advisable to consider some works devoted to the analysis of the Föppl–von Kármán equations in terms of pure math. Initially, they did not have a rigorous mathematical justification and were derived through an engineering approach significantly based on a few auxiliary hypotheses [2]. The justification of this system has been provided by Ciarlet, who has demonstrated [2,73] that this system can be obtained using the asymptotic expansion of the equations of non-linear elasticity theory by a small parameter, assuming certain hypotheses about the asymptotic orders of displacement vector components. This approach was also applied in the recent work [74] to construct the family of Kármán-like plate models.

The existence theorem for the Föppl–von Kármán boundary-value problem with sufficiently smooth boundary and homogeneous boundary conditions was proved by Morozov [75] using the Leray–Schauder method [76]. The conditions for the existence of a non-trivial equilibrium state of a Kármán plate under compression forces were formulated by M. Berger and Fife [77]. Knightly [78] demonstrated the existence of a solution for a plate subjected to combined loading, and moreover, he proven that if the magnitude of the loading is sufficiently small, then this problem has a unique solution. Further, Knightly and Sather [79] have modified this uniqueness result. Hlaváček and Naumann have obtained similar results for the various types of inhomogeneous boundary conditions [80,81]. It is important to highlight that results of Knightly, Hlaváček, and Naumann are applicable to plates with the boundary that consists of curves from C^3 and may have corners at junctions of the arcs, i.e., they are valid for the majority of plates used in engineering.

From this brief analysis, it is observed that, to date, there are no long-standing methods to solve Föppl–von Kármán equations for square and rectangular plates fixed at the boundary with respect to all displacement components. Moreover, direct 3D FEM modeling of thin plates results in an ill-conditioned algebraic system due to the small parameter, which is the plate thickness. However, it is the full fixation that is usually implemented in experiment and practice. In this context, the development of Föppl–von Kármán equations solution methods for square and rectangular plates, fully fixed at the boundary, is an urgent problem in computational mechanics. This is the subject of the present paper. A new solution is proposed in it via the form of decomposition over a basis in functional space of square-integrable functions. The coefficients of expansion are determined from the infinite-dimensional cubic system via the reduction method. This approach could provide a non-linear counterpart to the Galerkin method. The principal difference from similar studies [67,70] consists in a strict formulation of the auxiliary boundary problem, allowing for accounting natural ways of plate fixing.

2. Problem Statement

Consider a clamped square plate of side $2a$ and thickness h , made from a linear-elastic material with moduli E and ν and subjected to an arbitrary transversal load $p(x, y)$ (see Figure 1). For a closed formulation of the initial boundary problem, we should complement Equation (4) with suitable boundary conditions, which we will now deal with. The boundary value problem will be formulated in the region:

$$\Omega = (-a, a) \times (-a, a) \subset \mathbb{R}^2,$$

with the boundary

$$\partial\Omega = \mathcal{E}_1 \cup \mathcal{E}_2,$$

where the parts of the boundary $\mathcal{E}_1, \mathcal{E}_2$ correspond to pairs of opposite edges, i.e.,

$$\mathcal{E}_1 = (\{-a\} \times [-a, a]) \cup (\{a\} \times [-a, a]), \quad \mathcal{E}_2 = ([-a, a] \times \{-a\}) \cup ([-a, a] \times \{a\}).$$

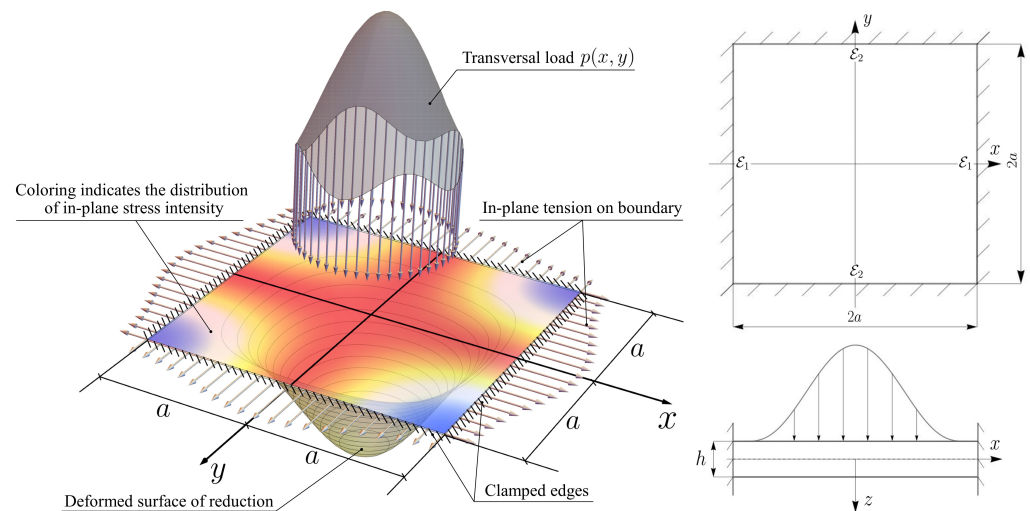


Figure 1. Problem statement.

The boundary conditions for the flexible plates usually are gathered into two sequences associated with plate bending and with plane stress-strain state, respectively. Consider the first one. In the case of clamped plates, it can be written as

$$\left. \frac{\partial w}{\partial x} \right|_{\varepsilon_1} = 0, \quad \left. \frac{\partial w}{\partial y} \right|_{\varepsilon_2} = 0, \quad w|_{\partial\Omega} = 0. \quad (5)$$

Other variants of the boundary conditions related to bending are well known [24], and we will not comment on them, as (5) are the ones that will be stated hereinafter. The boundary conditions associated with the in-plane stress-strain state are not so widely covered in literature, and we will discuss their formulation in more detail. Following [17,21], we consider two cases:

- movable edges:

$$u|_{\varepsilon_1} = 0, \quad v|_{\varepsilon_2} = 0, \quad \left. \frac{\partial^2 \phi}{\partial x \partial y} \right|_{\partial\Omega} = 0; \quad (6)$$

- immovable edges:

$$u|_{\partial\Omega} = 0, \quad v|_{\partial\Omega} = 0. \quad (7)$$

Note that despite the designation, moving edges are not free from fixing at all. The points on them can only be moved along the edges, while their displacement in a perpendicular direction is constrained. In Equations (6) and (7), $u(x, y)$ and $v(x, y)$ denote the in-plane plate displacements (the in-plane displacements of the points on the plate middle plane). These functions are not explicitly included in the governing system (4), so in order to solve the boundary-value problem, it is necessary to express them through the desired functions w, ϕ . The sought relations can be obtained using the Cesàro formula, which expresses the displacement vector $\mathbf{u}$ and the rotation vector $\boldsymbol{\omega}$ via the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ [82]:

$$\mathbf{u}_M = \mathbf{u}_0 + \boldsymbol{\omega}_0 \times (\mathbf{x} - \mathbf{x}_0) + \int_{M_0}^M [\boldsymbol{\varepsilon} + (\mathbf{x} - \mathbf{x}_M) \times (\nabla \times \boldsymbol{\varepsilon})] \cdot d\mathbf{x}. \quad (8)$$

Note that the displacement vector $\mathbf{u}$ is defined with account of the kinematic hypothesis of the Kirchhoff-like plate theory:

$$\mathbf{u} = \left(u - z \frac{\partial w}{\partial x}\right) \mathbf{i} + \left(v - z \frac{\partial w}{\partial y}\right) \mathbf{j} + w \mathbf{k}.$$

Therefore, to relate the displacements $\mathbf{u}$ to the Airy stress function ϕ , it is necessary to express the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ through the stress tensor $\boldsymbol{\sigma}$. The essence of the Kármán theory is based on a number of simplifications that allow only the use of the Saint-Venant–Kirchhoff constitutive equation [73,83]:

$$\boldsymbol{\sigma} = \frac{E}{1+\nu} \mathbf{E}^* + \frac{\nu E}{1-\nu^2} (\text{tr} \mathbf{E}^*) \mathbf{I}, \quad (9)$$

where $\mathbf{E}^*$ is the reduced Green–Saint-Venant strain tensor (its components are also known as the Kármán strains):

$$\mathbf{E}^* = \boldsymbol{\varepsilon} + \frac{1}{2} (\nabla w) \otimes (\nabla w) = \frac{1}{2} \left[\nabla \mathbf{u}^T + \nabla \mathbf{u} + (\nabla w) \otimes (\nabla w) \right]. \quad (10)$$

Finally, substituting the relation (10) into the Cesàro formula (8), one can obtain:

$$\begin{aligned} \mathbf{u}_M &= \mathbf{u}_0 + \boldsymbol{\omega}_0 \times (\mathbf{x} - \mathbf{x}_0) \\ &+ \int_{M_0}^M \left\{ \mathbf{E}^* - \frac{1}{2} (\nabla w) \otimes (\nabla w) + (\mathbf{x} - \mathbf{x}_M) \times \left[\nabla \times \left(\mathbf{E}^* - \frac{1}{2} (\nabla w) \otimes (\nabla w) \right) \right] \right\} \cdot d\mathbf{x}. \end{aligned} \quad (11)$$

Remark 1. This representation of displacements does not reproduce the accurate non-linear analogues of the Cesàro formula [84–87]. However, since the displacements are calculated only in the plane of reduction and the reduced finite strain measure (10) takes into account quadratic terms only for the transversal component of displacements, such an inaccurate formulation is usually acceptable for engineering calculations [4,21,66].

Formula (11) can be significantly simplified by choosing an appropriate path of integration. Taking into account that displacements and rotations of the whole plate are defined up to an additive constants any point of it can be fixed. Apart from that, if the transversal load function is odd or even by both coordinates, then the center of a plate will be displaced vertically without changing its in-plane position. For the sake of simplicity, assume that the load function p and, as a consequence, sought functions w , ϕ are even by both coordinates (three other cases of symmetry are derived from reasoning completely similar to the following):

$$\{w, \phi, p\} \subset \mathcal{F}_s = \left\{ f \in \mathcal{L}^2(\Omega) : f(x, y) = f(-x, y), f(x, y) = f(y, x) \right\}. \quad (12)$$

Here, we also restrict the class of considered functions p , w , ϕ only to square-integrable ones. This allows us to decompose them over basis and reduce the boundary-value problem to a system of cubic equations.

With account of the transversal loading function symmetry (12), the displacement on $\mathcal{E}_1$ can be calculated as follows:

$$u|_{x=\pm a} = \int_0^{\pm a} \left(E_{xx}^* - \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right) dx, \quad v|_{x=\pm a} = \int_0^y \left(E_{yy}^* - \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right) dy \Big|_{x=\pm a}.$$

Substitution of relations (9) and (2) into the above formula results in

$$\begin{aligned} u|_{x=\pm a} &= \int_0^{\pm a} \left[\frac{\partial^2 \phi}{\partial y^2} - \nu \frac{\partial^2 \phi}{\partial x^2} - \frac{E}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] dx = 0, \\ v|_{x=\pm a} &= \int_0^y \left[\frac{\partial^2 \phi}{\partial x^2} - \nu \frac{\partial^2 \phi}{\partial y^2} - \frac{E}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right] dy \Big|_{x=\pm a} = 0. \end{aligned} \quad (13)$$

However, taking into account that a function on a simply connected region is identically zero iff it takes value zero at least at one point and its first derivative is zero also, the latter integral condition can be reduced to a differential relation:

$$\left[\frac{\partial^2 \phi}{\partial x^2} - \nu \frac{\partial^2 \phi}{\partial y^2} - \frac{E}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right] \Big|_{x=\pm a} = 0.$$

The last terms in the left-hand side of this relation can be omitted due to the fact that the deflection function is identically zero on the edges (5):

$$\left(\frac{\partial^2 \phi}{\partial x^2} - \nu \frac{\partial^2 \phi}{\partial y^2} \right) \Big|_{x=\pm a} = 0.$$

Thus, the above relation, together with the first integral equality (13), fully describes the boundary conditions for the in-plane displacements of the plate on $\mathcal{E}_1$. The formula for the boundary conditions on $\mathcal{E}_2$ can be written in a similar way up to the replacement of the variable x by y and function u by v , respectively.

Now we can move on to the formulation of a boundary value problem. To this end, introduce the following dimensionless variables:

$$\tilde{x} := \frac{x}{a}, \quad \tilde{y} := \frac{y}{a}, \quad \tilde{z} := \frac{z}{h}, \quad \tilde{w} := \frac{w}{h}, \quad \tilde{\phi} := \frac{\phi h}{D}, \quad \tilde{p} := \frac{pa^4}{Dh}.$$

In the new variables the Föppl–von Kármán Equation (4) and corresponding boundary conditions can be written as

$$\tilde{\nabla}^2 \tilde{\nabla}^2 \tilde{w} - \tilde{L}(\tilde{w}, \tilde{\phi}) = \tilde{p}, \quad \tilde{\nabla}^2 \tilde{\nabla}^2 \tilde{\phi} + \kappa \tilde{L}(\tilde{w}, \tilde{w}) = 0, \quad (14)$$

$$\frac{\partial \tilde{w}}{\partial \tilde{x}} \Big|_{\tilde{x}=\pm 1} = 0, \quad \frac{\partial \tilde{w}}{\partial \tilde{y}} \Big|_{\tilde{y}=\pm 1} = 0, \quad \tilde{w} \Big|_{\tilde{x}=\pm 1} = 0, \quad \tilde{w} \Big|_{\tilde{y}=\pm 1} = 0, \quad (15)$$

$$\int_0^{\pm 1} \left[\frac{\partial^2 \tilde{\phi}}{\partial \tilde{y}^2} - \nu \frac{\partial^2 \tilde{\phi}}{\partial \tilde{x}^2} - \kappa \left(\frac{\partial \tilde{w}}{\partial \tilde{x}} \right)^2 \right] d\tilde{x} = 0, \quad \int_0^{\pm 1} \left[\frac{\partial^2 \tilde{\phi}}{\partial \tilde{x}^2} - \nu \frac{\partial^2 \tilde{\phi}}{\partial \tilde{y}^2} - \kappa \left(\frac{\partial \tilde{w}}{\partial \tilde{y}} \right)^2 \right] d\tilde{y} = 0; \quad (16)$$

– movable edges:

$$\frac{\partial^2 \tilde{\phi}}{\partial \tilde{x} \partial \tilde{y}} \Big|_{\tilde{x}=\pm 1} = 0, \quad \frac{\partial^2 \tilde{\phi}}{\partial \tilde{x} \partial \tilde{y}} \Big|_{\tilde{y}=\pm 1} = 0; \quad (17)$$

– immovable edges:

$$\left(\frac{\partial^2 \tilde{\phi}}{\partial \tilde{y}^2} - \nu \frac{\partial^2 \tilde{\phi}}{\partial \tilde{x}^2} \right) \Big|_{\tilde{y}=\pm 1} = 0, \quad \left(\frac{\partial^2 \tilde{\phi}}{\partial \tilde{x}^2} - \nu \frac{\partial^2 \tilde{\phi}}{\partial \tilde{y}^2} \right) \Big|_{\tilde{x}=\pm 1} = 0, \quad (18)$$

where $\kappa := 6(1 - \nu^2)$, $\tilde{\nabla}^2 := a^2 \nabla^2$, $\tilde{L}(\cdot, \cdot) := a^4 L(\cdot, \cdot)$. In what follows, we will omit the tildes above these operators for brevity. The Equations (14)–(18) constitute two different boundary-value problems for a clamped square plate.

To obtain an approximate solution of the problems (14)–(17) or (18), a recursive approach is often used. Briefly describe its essence for comparison with the proposed solution below. The basis of the recursion $\tilde{w}_0$ can be taken as a solution of the Sophie Germain–Lagrange equation:

$$\tilde{w}_0 := \mathcal{M}_w^{-1} \tilde{p},$$

while the step of the recursion is defined as follows (here subscript indicates recursion step number):

$$\tilde{w}_k = \mathcal{M}_w^{-1} \left(\tilde{p} - \kappa L \left(\tilde{w}_{k-1}, \mathcal{M}_\phi^{-1} L(\tilde{w}_{k-1}, \tilde{w}_{k-1}) \right) \right).$$

Stress function $\tilde{\phi}_k$ can be obtained on each step from the relation

$$\tilde{\phi}_k := -\kappa \mathcal{M}_\phi^{-1} L(\tilde{w}_k, \tilde{w}_k).$$

Here, $\mathcal{M}_w, \mathcal{M}_\phi$ stand for differential operators, defined by the same biharmonic differential expression (3), but by different boundary conditions, corresponding to deflection (15) and stress function (16)–(18). Computational efficiency of recursive processes depends on the way to represent inverse linear operators $\mathcal{M}_w$ and $\mathcal{M}_\phi$. It can be performed differently, but for square plates, the usage of the complex potentials method (Goursat–Muskhelishvili–Kalandiya) [88] seems to be most appropriate. Within this method one can define the solution of linear problem $\mathcal{M}_w \tilde{w} = \tilde{p}$ as the sum of some particular solution $\tilde{w}_*$ that satisfies the non-homogeneous biharmonic differential equation but does not meet boundary conditions and the boundary solution of the homogeneous equation $\tilde{w}_0$, such that $\tilde{w} = \tilde{w}_* + \tilde{w}_0$ satisfies the conditions on the boundary.

Particular solution $\tilde{w}_*$ can be obtained as

$$\tilde{w}_*(z) = \frac{1}{16} \int_0^z \int_0^{\bar{z}} (z - \beta)(\bar{z} - \alpha) \tilde{p} \left(\frac{\alpha + \beta}{2}, \frac{\beta - \alpha}{2i} \right) d\alpha d\beta,$$

or

$$\begin{aligned} \tilde{w}_*(x, y) = & \frac{1}{16} \left[\int_0^x \int_0^x (x - iy - \alpha)(x + iy - \beta) \tilde{p} \left(\frac{\alpha + \beta}{2}, \frac{\beta - \alpha}{2i} \right) d\alpha d\beta \right. \\ & + \int_0^x \int_0^y (y - \alpha)(x + iy - \beta) \tilde{p} \left(\frac{x - i\alpha + \beta}{2}, \frac{-x + i\alpha + \beta}{2i} \right) d\alpha d\beta \\ & - \int_0^x \int_0^y (y - \alpha)(x - iy - \beta) \tilde{p} \left(\frac{x + i\alpha + \beta}{2}, \frac{x + i\alpha - \beta}{2i} \right) d\alpha d\beta \\ & \left. + \int_0^y \int_0^y (y - \alpha)(y - \beta) \tilde{p} \left(\frac{2x - i\alpha + i\beta}{2}, \frac{\alpha + \beta}{2} \right) d\alpha d\beta \right]. \end{aligned}$$

while the boundary solution w_0 can be expressed in terms of complex-valued series with respect to $z = x + iy$ as follows:

$$\tilde{w}_0 = 2\text{Re}(\gamma \bar{\psi} + \chi).$$

Here,

$$\chi = \int \psi \frac{\partial \gamma}{\partial z} dz, \quad \phi = \sum a_k z^k, \quad \psi = \sum b_k z^k,$$

where a_k, b_k can be obtained from the solution of the algebraic counterpart for the boundary problem (see [88,89] for details). In the above formula, $\bar{z}$ indicates the complex conjugate of z , while γ stands for the conformal map between the unit disk and rectangular domain.

This mapping can be exactly expressed via hypergeometric function, but for computations it is more convenient to use polynomial approximation of order n :

$$\gamma = e^{-\frac{i\pi}{4}} \frac{\sqrt{2}\Gamma(\frac{3}{4})}{\sqrt{\pi}\Gamma(\frac{5}{4})} \sum_{k=0}^n \frac{z^{4k+1}}{(4k+1)k!} \prod_{s=0}^{k-1} \left(s + \frac{1}{2}\right).$$

The inverse operator M_ϕ^{-1} can be obtained in a similar way. Note, that the advantage of complex potential method is that the integral boundary conditions can be expressed directly via potentials with Love relations:

$$u = \frac{1}{2\mu} \operatorname{Re} \left(\frac{3-\nu}{1+\nu} \varphi - \gamma \frac{\partial \bar{\phi}}{\partial \gamma} - \bar{\psi} \right), \quad v = \frac{1}{2\mu} \operatorname{Im} \left(\frac{3-\nu}{1+\nu} \varphi - \gamma \frac{\partial \bar{\phi}}{\partial \gamma} - \bar{\psi} \right).$$

Before embarking on the general procedure of problems (14)–(18), let us try to approach it with the recursive method mentioned above. In doing so, keep in mind that in order to identify the characteristic features of non-linear deformation of the plate, it is sufficient to perform at least one step of recursion. The calculation results for the model problem ($a = 1$, $D = 1$, $p = 1$, $\nu = 1/3$, $h = 1/150$) are shown in Figure 2. In it, the subfigure (a) shows the distribution of deflections $\tilde{w}_0$ obtained without accounting for non-linear effects (it is the base of recursion, i.e., the solution of the Sophie Germain–Lagrange problem). Subfigure (b) shows the distribution of plain deformation incompatibility, induced by bending, namely $L(\tilde{w}_0, \tilde{w}_0)$. In subfigure (c), one can see stress function $\tilde{\phi}$, corresponding to such incompatibility. In the second line of figures, the distributions for normal and tangential in-plane stresses are shown, while the last subfigure (f) demonstrates dummy transverse loading $\tilde{q} := \kappa L(\tilde{w}_0, \mathcal{M}_\phi^{-1} L(\tilde{w}_0, \tilde{w}_0))$ that has to be subtracted from actual loading on the first recursive step.

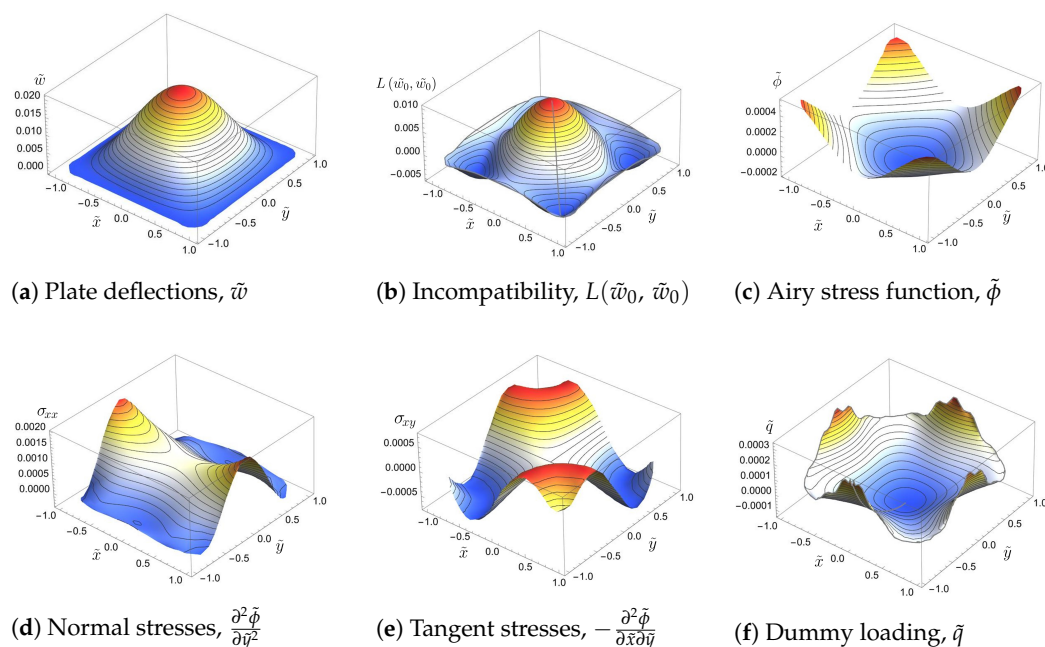


Figure 2. Results for the model problem.

Already from the analysis of the results obtained in the first step of recursion, it is possible to give a qualitative estimate of the effect of the dummy load and, accordingly, of the non-linear effects depending on the expected value of the maximal deflections. In particular, with evenly distributed load, the maximal value of dummy load (according to the distribution shown in Figure 2f for some load $\alpha \tilde{p}$ with dimensionless intensity α)

can be estimated as $4 \times 10^{-4} \kappa \alpha^3$. Therefore, the given and dummy loads will be of the same order when $\alpha \approx 20$, which, with account of distribution for deflections (Figure 2a), is approximately equivalent to maximal deflections, obtained in a linear approach, of 20% from the edge length of the plate. This value is a very rough estimate from above, because in fact the displacements will be much lower if we take into account non-linear effects, calculating the following recursion steps. Also, the following can be noticed:

- The influence of the dummy load increases proportionally to the cube of maximum deflection;
- The geometric shape of the bending surface at the final deformation significantly differs from the corresponding shape for small deflections because the dummy load changes the distribution of RHS of the bending equation on successive recurrent steps. We will see this again later when we consider the general procedure of solution.

The mentioned strategy for the solution of the non-linear bending problem has computational stability for moderate bending only. When the amount of deflection increases, the dummy load is no longer a small adjustment and may even exceed the given transverse load. In this case, the sequence of recursively constructed approximations tends to diverge. Although special regularization procedures [5] can be used, the diverging cannot be avoided. For this reason, below a different strategy is proposed, implementing the full algebraization of the problem and transforming it to the infinite-dimensional system of cubic equations. Reducing this system to a finite-dimensional one and using any variation of the successive approximation method to find the solution of the reduced system, it is possible to avoid computational instability even for deflections that are comparable with the size of the plate in the plane. The rest of the article is devoted to the construction and analysis of such algebraization.

3. Solution of the Linear Problem

Since the solution of the non-linear problem will be obtained in the form of decomposition over some basis in $\mathcal{L}^2(\Omega)$, generated by a linear self-adjoint operator, we will start with the linear problem. It is advisable to consider the linear bending of a square plate as such a prototype for this problem because, first, it generates a basis in terms of well-known Krylov–Duncan functions, and second, the problem itself can be used for testing a non-linear solution at zero approach.

The boundary-value problem for clamped Sophie Germain's plate can be stated as follows:

$$\begin{aligned} \nabla^2 \nabla^2 \tilde{w} &= \tilde{p}, \\ \frac{\partial \tilde{w}}{\partial \tilde{x}} \Big|_{\tilde{x}=\pm 1} &= 0, \quad \frac{\partial \tilde{w}}{\partial \tilde{y}} \Big|_{\tilde{y}=\pm 1} = 0, \quad \tilde{w} \Big|_{\tilde{x}=\pm 1} = 0, \quad \tilde{w} \Big|_{\tilde{y}=\pm 1} = 0. \end{aligned} \quad (19)$$

In order to expand the deflection function $\tilde{w}$ into a Fourier series, it is necessary to choose a basis in $\mathcal{L}^2(\Omega)$. According to the Hilbert–Schmidt theorem [90], a system of eigenfunctions of any compact self-adjoint operator constitutes a basis of $\mathcal{L}^2(\Omega)$. However, the biharmonic operator has a sufficiently complicated system of eigenfunctions even with homogeneous boundary conditions, so it is more preferable to use another self-adjoint operator. It is appropriate to use the following operator:

$$\mathcal{L} = \frac{\partial^4}{\partial \tilde{x}^4} + \frac{\partial^4}{\partial \tilde{y}^4}.$$

The sequence of eigenfunctions for this operator can be obtained as a combination of solutions of an auxiliary Sturm–Liouville problem.

3.1. Auxiliary Sturm–Liouville Problem

Let us consider a Sturm–Liouville problem:

$$\frac{\partial^4 f}{\partial v^4} - \eta^4 f = 0, \quad f|_{v=\pm 1} = 0, \quad \frac{\partial f}{\partial v}|_{v=\pm 1} = 0.$$

A fundamental system of solutions for this equation is well known:

$$\Psi = \{\sin(\eta v), \cos(\eta v), \sinh(\eta v), \cosh(\eta v)\}.$$

By substituting a linear combination of these functions into the boundary conditions, we obtain the spectral equation with respect to eigenvalues η :

$$[\sinh(\eta) \cos(\eta) - \sin(\eta) \cosh(\eta)][\sinh(\eta) \cos(\eta) + \sin(\eta) \cosh(\eta)] = 0.$$

Thus, a spectrum of the operator consists of two sequences of eigenvalues (in the strict sense, λ_m, μ_m are not eigenvalues; they are fourth power roots of eigenvalues; however, for brevity, we will not specify this further):

$$\{\eta_m\}_{m=1}^{\infty} = \{\lambda_m\}_{m=1}^{\infty} \cup \{\mu_m\}_{m=1}^{\infty},$$

which are defined as roots of transcendental equations:

$$\tanh(\lambda_m) + \tan(\lambda_m) = 0, \quad \tanh(\mu_m) - \tan(\mu_m) = 0.$$

These roots can be found numerically with arbitrary precision, but for sufficiently large values of m , very precise approximations can be obtained from asymptotic formula:

$$\lambda_m = \pi m - \frac{\pi}{4}, \quad \mu_m = \pi m + \frac{\pi}{4}. \quad (20)$$

The question arises: what does it mean to be large enough? The following computation (Tables 1 and 2) shows that for standard machine precision, it is enough to refine only the five first values, while all the rest can be calculated directly by Formula (20). Of course, for calculations with enhanced accuracy, the number of refined roots should be increased.

Table 1. Eigenvalues λ .

Eigenvalue Number	Numerical Solution	Approximation Formula	Relative Error
1	2.36502	2.35619	0.00372
2	5.49780	5.49779	$3.05126 \cdot 10^{-6}$
3	8.63938	8.63938	$3.62616 \cdot 10^{-6}$
4	11.7810	11.7810	$4.96600 \cdot 10^{-12}$
5	14.9226	14.9226	$7.38037 \cdot 10^{-15}$

Table 2. Eigenvalues μ .

Eigenvalue Number	Numerical Solution	Approximation Formula	Relative Error
1	3.92660	3.92699	0.00010
2	7.06858	7.06858	$1.02559 \cdot 10^{-7}$
3	10.2102	10.2102	$1.32593 \cdot 10^{-10}$
4	13.3518	13.3518	$1.89320 \cdot 10^{-13}$
5	16.4934	16.4934	$2.15403 \cdot 10^{-16}$

Each sequence of the eigenvalues is associated with a sequence of the auxiliary eigenfunctions:

$$\begin{aligned} u_m(v) &= \cos(\lambda_m v) \cosh(\lambda_m) - \cos(\lambda_m) \cosh(\lambda_m v), \\ v_m(v) &= \sin(\mu_m v) \cosh(\mu_m) - \cos(\mu_m) \sinh(\mu_m v). \end{aligned} \quad (21)$$

Such sequences of functions are convenient for derivations due to their compact view; however, for calculations, it is more convenient to use normalized functions:

$$\tilde{u}_m(v) = \frac{u_m}{\sqrt{\cosh(\lambda_m)^2 + \cos(\lambda_m)^2}}, \quad \tilde{v}_m(v) = \frac{v_m}{\sqrt{\cosh(\mu_m)^2 - \cos(\mu_m)^2}}.$$

Curves of the first several normalized eigenfunctions are shown in Figure 3.

It is evident that each pairwise combination of functions (21) presents an eigenfunction of two-dimensional operator $\mathcal{L}$. Thus, we arrive at four families of eigenfunctions for $\mathcal{L}$:

$$\{u_i(\tilde{x})u_j(\tilde{y}), u_i(\tilde{x})v_j(\tilde{y}), v_i(\tilde{x})u_j(\tilde{y}), v_i(\tilde{x})v_j(\tilde{y})\}. \quad (22)$$

Although each term of these families is marked by two indexes, it is preferable to use a one-indexed numeration system. Such a system can be provided by any bijection $\mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$. The optimal numeration system is well known from classical courses of calculus:

$$k = \frac{1}{2}(i+j)(i+j-1) - j + 1. \quad (23)$$

This bijection is named as the Cantor pairing function for the positive integers and was proposed by him in the article [91]. The correspondence between two systems of numeration can be illustrated by a diagram as shown in Figure 3c. This way of numeration is the most preferable for square regions because it allows one to construct the reduced sums symmetrically with respect to both indices. In order to achieve this, it is sufficient to include complete diagonals in the sum.

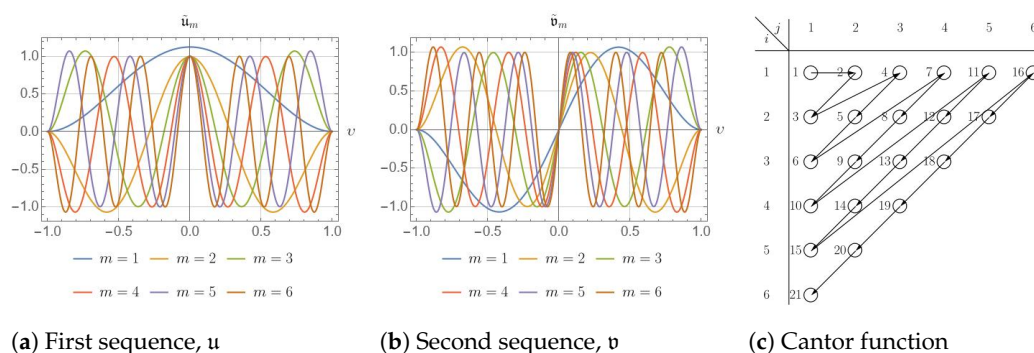


Figure 3. Normalized auxiliary eigenfunctions and Cantor pairing function diagram.

Thus the two-indexed system of basis functions (22) can be replaced by the following one:

$$\{f_{1k}(\tilde{x}, \tilde{y}), f_{2k}(\tilde{x}, \tilde{y}), f_{3k}(\tilde{x}, \tilde{y}), f_{4k}(\tilde{x}, \tilde{y})\}. \quad (24)$$

One can go back to the functions (22) with the reverse function to (23). In order to construct this function, it is necessary to relate a number of the current term k with the number of diagonal N on which this term is located:

$$N = \left\lfloor \frac{1 + \sqrt{1 + 8(k-1)}}{2} \right\rfloor,$$

where the symbols $\lfloor \dots \rfloor$ denote the floor. From here, it is simple to obtain the inverse function:

$$i = k - \frac{N(N-1)}{2} =: S(k), \quad j = \frac{N(N+1)}{2} - k + 1 =: C(k).$$

Hereinafter, S and C are brief notations for components of the inverse function.

3.2. Solution of the Linear Boundary-Value Problem

Now it is time to recover the boundary-value problem (19). We will seek the solution of this problem in the form of expansion in terms of a series of normalized basis functions (24):

$$\tilde{w} = \sum_{r=1}^4 \sum_{k=1}^{\infty} g_{rk} \frac{f_{rk}}{\|f_{rk}\|} = \sum_{r=1}^4 \sum_{k=1}^{\infty} g_{rk} \tilde{f}_{rk}. \quad (25)$$

This solution is guaranteed to satisfy the boundary conditions (15) due to the appropriate choice of the basis. Consequently, in order to solve the boundary-value problem, it is sufficient to find the Fourier coefficients g_{rk} , with which (25) satisfies the inhomogeneous Sophie Germain–Lagrange Equation (19):

$$\sum_{r=1}^4 \sum_{k=1}^{\infty} g_{rk} \nabla^2 \nabla^2 \tilde{f}_{rk} = \sum_{r=1}^4 \sum_{k=1}^{\infty} g_{rk} \left(\xi_{rk} \tilde{f}_{rk} + 2 \frac{\partial^4 \tilde{f}_{rk}}{\partial \tilde{x}^2 \partial \tilde{y}^2} \right) = \tilde{p}. \quad (26)$$

Here with ξ_{rk} , we denote the eigenvalues of the operator $\mathcal{L}$:

$$\begin{aligned} \xi_{1k} &= \lambda_{S(k)}^4 + \lambda_{C(k)}^4, & \xi_{2k} &= \lambda_{S(k)}^4 + \mu_{C(k)}^4, \\ \xi_{3k} &= \mu_{S(k)}^4 + \lambda_{C(k)}^4, & \xi_{4k} &= \mu_{S(k)}^4 + \mu_{C(k)}^4. \end{aligned}$$

One can find g_{rk} by acting the projection operators:

$$\mathcal{P}_{sq}(F(\tilde{x}, \tilde{y})) = \int_{-1}^1 \int_{-1}^1 F \tilde{f}_{sq} d\tilde{x} d\tilde{y}, \quad q = 1, \dots, \infty, \quad (27)$$

on both sides of Equation (26):

$$g_{sq} \xi_{sq} + \sum_{r=1}^4 \sum_{k=1}^{\infty} g_{rk} \int_{-1}^1 \int_{-1}^1 2 \frac{\partial^4 \tilde{f}_{rk}}{\partial \tilde{x}^2 \partial \tilde{y}^2} \tilde{f}_{sq} d\tilde{x} d\tilde{y} = \int_{-1}^1 \int_{-1}^1 \tilde{p} \tilde{f}_{sq} d\tilde{x} d\tilde{y}.$$

It is essential to note that the basis functions lying in different sequences have a different parity. Consequently, due to the fact that the second derivative of any function has the same parity as the function itself, the double sum in the previous relation can be reduced to the sum:

$$g_{sq} \xi_{sq} + \sum_{k=1}^{\infty} g_{sk} \int_{-1}^1 \int_{-1}^1 2 \frac{\partial^4 \tilde{f}_{sk}}{\partial \tilde{x}^2 \partial \tilde{y}^2} \tilde{f}_{sq} d\tilde{x} d\tilde{y} = \int_{-1}^1 \int_{-1}^1 \tilde{p} \tilde{f}_{sq} d\tilde{x} d\tilde{y}.$$

Thus, the differential equation is reduced to an infinite-dimensional system of algebraic equations. By applying the reduction method [92,93], we arrive at the sequence of finite-dimensional systems of linear algebraic equations, which can be written in matrix form:

$$\mathbf{G}^s \mathbf{g}^s = \mathbf{b}^s, \quad s = \{1, \dots, 4\}, \quad (28)$$

where $\mathbf{G}^s$ are matrices $n \times n$ known as Galerkin matrices. Their components, G_{qk}^s , can be obtained via formula (here, δ_{qk} is the Kronecker delta):

$$[G^s]_{qk} = G_{qk}^s = \zeta_{sq} \delta_{qk} + \int_{-1}^1 \int_{-1}^1 2 \frac{\partial^4 \tilde{f}_{sk}}{\partial \tilde{x}^2 \partial \tilde{y}^2} \tilde{f}_{sq} d\tilde{x} d\tilde{y}.$$

Vector $\mathbf{b}^s$ contains Fourier coefficients of transverse load function b_q^s (for brevity, we will refer to it as the load vector):

$$[\mathbf{b}^s]_q = b_q^s = \int_{-1}^1 \int_{-1}^1 \tilde{p} \tilde{f}_{sq} d\tilde{x} d\tilde{y}. \quad (29)$$

Recall now that we consider functions only from $\mathcal{F}$ (12), so it is sufficient to calculate only the first Galerkin matrix $\mathbf{G}^1$ and the first load vector $\mathbf{b}^1$, which correspond to even by both variables functions. In this regard, hereinafter we will omit the first index and adopt the following notations:

$$\begin{aligned} f_k &:= f_{1k}, & \tilde{f}_k &:= \tilde{f}_{1k}, & \mathcal{P}_q &:= \mathcal{P}_{1q}, & \zeta_k &:= \zeta_{1k}, & \mathbf{G} &:= \mathbf{G}^1, & G_{qk} &:= G_{qk}^1, \\ \mathbf{b} &:= \mathbf{b}^1, & b_q &:= b_q^1, & \mathbf{g} &:= \mathbf{g}^1, & g_k &:= g_k^1. \end{aligned} \quad (30)$$

Thus, the problem has been reduced to one matrix equation. In the case of an arbitrary non-symmetric load function, further steps would be similar; however, they will require finding of all matrices $\mathbf{G}^s$ and vectors $\mathbf{b}^s, \mathbf{g}^s$.

Proceed now to calculating the Galerkin matrix and load vector components. It is convenient to represent an expression for them using an auxiliary integral I_G (the auxiliary integrals are provided in the Appendix A):

$$G_{qk} = \zeta_q \delta_{qk} + \frac{2\lambda_{S(k)}^2 \lambda_{C(k)}^2}{\|f_k\| \|f_q\|} I_G(\lambda_{S(q)}, \lambda_{S(k)}) I_G(\lambda_{C(q)}, \lambda_{C(k)}) \quad (31)$$

where $\|f_k\|$ denote norms of the basis functions:

$$\|f_k\| = \sqrt{\cosh^2(\lambda_{S(k)}) + \cos^2(\lambda_{S(k)})} \sqrt{\cosh^2(\lambda_{C(k)}) + \cos^2(\lambda_{C(k)})}.$$

The components of the load vector are calculated by Formula (29) if we assume $s = 1$.

Now, the Fourier coefficients g_k can be calculated from the first matrix equation system (28) using, for example, Cramer's rule:

$$g_k = \frac{\det \mathbf{G}_k}{\det \mathbf{G}}, \quad \mathbf{G}_k = \mathbf{G} \cdot (\mathbf{E} - \mathbf{E}_{kk}) + \sum_{i=1}^n b_i \mathbf{E}_{ik}. \quad (32)$$

Here, $\mathbf{E}$ is the identity matrix, and $\mathbf{E}_{ij}$ are so-called matrix units which have only one unit component with indexes ij , while other components are zero.

Finally, the solution can be obtained by summing the basis functions with the calculated Fourier coefficients g_k :

$$\tilde{w}_n = \sum_{k=1}^n \frac{\det \mathbf{G}_k}{\det \mathbf{G}} \tilde{f}_k(\tilde{x}, \tilde{y}), \quad w_n = \frac{pa^4}{D} \sum_{k=1}^n \frac{\det \mathbf{G}_k}{\det \mathbf{G}} \tilde{f}_k\left(\frac{x}{a}, \frac{x}{a}\right), \quad w = \lim_{n \rightarrow \infty} w_n. \quad (33)$$

The above limit exists because the original infinite system is fully regular (see [93]). It should be noted that the Formula (33), based on Cramer's rule, are merely of academic interest, while in numerical realization it is convenient to use more efficient algorithms of matrix inverting.

To illustrate the obtained solution model, calculations were performed for two types of load function: uniform load, $\tilde{p} \equiv 1$, and bump-function, which is defined within the region Ω as polynomial function $\tilde{p} = (1 - \tilde{x}^2)^2(1 - \tilde{y}^2)^2$ and is zero outside of it. The trial load function and its approximations by reduced Fourier sums are shown in Figure 4. The purpose of these calculations is to show how the convergence rate of the sequence of partial sums $\tilde{w}_n$ depends on the smoothness of the load function and its belonging to the differential operator's domain (obviously the first function does not belong to the domain, whereas the second one belongs to it).

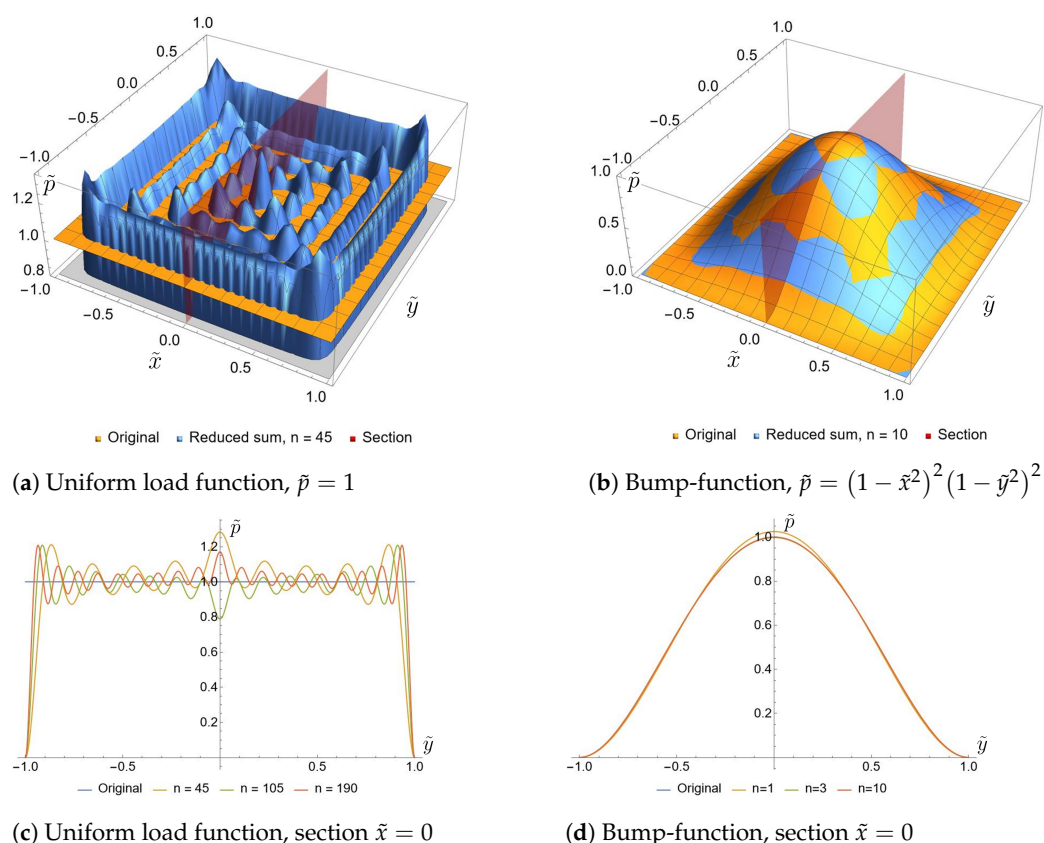


Figure 4. Trial load functions and their reduced Fourier sums.

Results of these calculations are presented in Figures 5 and 6. Subfigures (a) contain deflected shapes of plate. In subfigures (b), the inaccuracy functions for the corresponding load functions are shown, which are defined as distributions of unbalanced transversal forces absolute values:

$$\delta_n = \left| \tilde{p} - \sum_{k=1}^n g_k \nabla^2 \nabla^2 \tilde{f}_k \right|.$$

Subfigures (c) give integral estimations of the inaccuracy functions versus the number of accounted terms in reduced sums:

$$\varepsilon_n = \left| 1 - \frac{\sum_{k=1}^n g_k \int_{-1}^1 \int_{-1}^1 \nabla^2 \nabla^2 \tilde{f}_k d\tilde{x} d\tilde{y}}{\int_{-1}^1 \int_{-1}^1 \tilde{p} d\tilde{x} d\tilde{y}} \right|.$$

In these subfigures, the orange points correspond to sums that are symmetric with respect to the coordinates, i.e., with any term $g_k \tilde{f}_k = g_k \tilde{u}_i(\tilde{x}) \tilde{u}_j(\tilde{y})$, they contain a symmetric term $g_q \tilde{u}_j(\tilde{x}) \tilde{u}_i(\tilde{y})$.

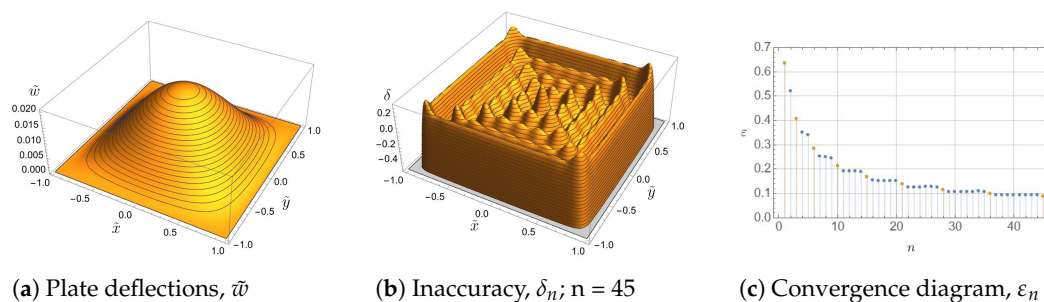


Figure 5. Test solutions for plate under uniform load.

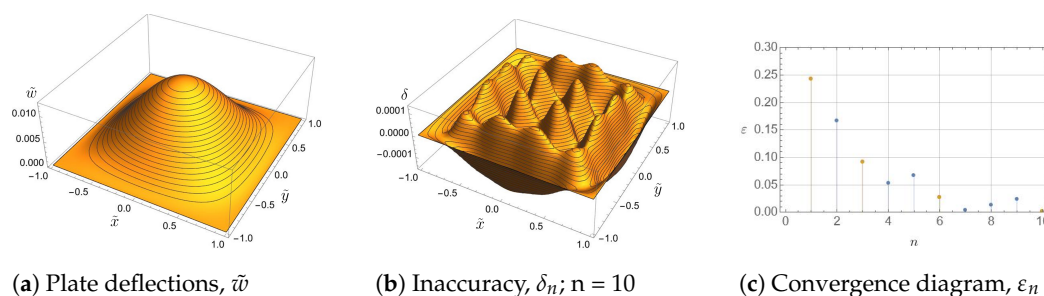


Figure 6. Test solutions for plate under bump-function load.

4. Solution of the Föppl–von Kármán System

Proceed to solving the Föppl–von Kármán boundary-value problems (14)–(18). As above, we will represent the unknown functions $\tilde{w}$ and $\tilde{\phi}$ as decompositions over basis functions (24). However, in this case such expansions do not allow the boundary conditions to be satisfied automatically. In order to overcome this difficulty, we augment the decompositions by the solution of the specific boundary problem.

4.1. Auxiliary Boundary Problem

Consider a homogenous biharmonic equation with the following boundary conditions:

$$\nabla^2 \nabla^2 \tilde{\phi} = 0, \quad (34)$$

$$\begin{aligned} \frac{\partial^2 \tilde{\phi}}{\partial \tilde{x}^2} \Big|_{\tilde{y}=\pm 1} &= \psi_1(\tilde{x}), & \frac{\partial^2 \tilde{\phi}}{\partial \tilde{y}^2} \Big|_{\tilde{x}=\pm 1} &= \psi_2(\tilde{y}), \\ \frac{\partial^2 \tilde{\phi}}{\partial \tilde{x} \partial \tilde{y}} \Big|_{\tilde{y}=1} &= -\frac{\partial^2 \tilde{\phi}}{\partial \tilde{x} \partial \tilde{y}} \Big|_{\tilde{y}=-1} = \chi_1(\tilde{x}), & \frac{\partial^2 \tilde{\phi}}{\partial \tilde{x} \partial \tilde{y}} \Big|_{\tilde{x}=1} &= -\frac{\partial^2 \tilde{\phi}}{\partial \tilde{x} \partial \tilde{y}} \Big|_{\tilde{x}=-1} = \chi_2(\tilde{y}), \end{aligned} \quad (35)$$

where ψ_1 , ψ_2 , χ_1 , χ_2 are arbitrary square-integrable functions. The solution of this problem can be found in the form of a sum:

$$\tilde{\phi} = \tilde{\phi}_0 + \tilde{\phi}^*, \quad (36)$$

where $\tilde{\phi}^*$ meets inhomogeneous boundary conditions (35) but does not satisfy the homogeneous biharmonic Equation (34). Another summand $\tilde{\phi}_0$, instead, meets the homogeneous boundary condition and inhomogeneous equation:

$$\nabla^2 \nabla^2 \tilde{\phi}_0 = -\nabla^2 \nabla^2 \tilde{\phi}^*.$$

Thus, the sum (36) will represent the solution of the boundary problem (34) and (35).

Taking into account symmetry relation (12), it is possible to simplify boundary conditions (35):

$$\begin{aligned}\frac{\partial^2 \tilde{\phi}}{\partial \tilde{x}^2} \Big|_{\tilde{y}=\pm 1} &= \psi(\tilde{x}), & \frac{\partial^2 \tilde{\phi}}{\partial \tilde{y}^2} \Big|_{\tilde{x}=\pm 1} &= \psi(\tilde{y}), \\ \frac{\partial^2 \tilde{\phi}}{\partial \tilde{x} \partial \tilde{y}} \Big|_{\tilde{y}=1} &= -\frac{\partial^2 \tilde{\phi}}{\partial \tilde{x} \partial \tilde{y}} \Big|_{\tilde{y}=-1} = \chi(\tilde{x}), & \frac{\partial^2 \tilde{\phi}}{\partial \tilde{x} \partial \tilde{y}} \Big|_{\tilde{x}=1} &= -\frac{\partial^2 \tilde{\phi}}{\partial \tilde{x} \partial \tilde{y}} \Big|_{\tilde{x}=-1} = \chi(\tilde{y}),\end{aligned}$$

where ψ is an even function, while χ is an odd one. In this case, the function $\tilde{\phi}^*$ may be presented as follows:

$$\begin{aligned}\tilde{\phi}^* &= \alpha_0 \left(\frac{\tilde{x}^2}{4} + \frac{\tilde{y}^2}{4} \right) - \sum_{k=1}^{\infty} \frac{\alpha_k}{\pi^2 k^2} [\cos(\pi k \tilde{x}) + \cos(\pi k \tilde{y})] \\ &+ \sum_{k=1}^{\infty} \frac{2\gamma_k}{\pi^2 k} \left\{ \cos\left(\frac{\pi \tilde{y}}{2}\right) [\cos(\pi k \tilde{x}) - (-1)^k] + \cos\left(\frac{\pi \tilde{x}}{2}\right) [\cos(\pi k \tilde{y}) - (-1)^k] \right\}.\end{aligned}\quad (37)$$

Substitution of this function in (35) results in expansions of functions ψ and χ into the Fourier series:

$$\begin{aligned}\frac{\partial^2 \tilde{\phi}}{\partial \tilde{x}^2} \Big|_{\tilde{y}=\pm 1} &= \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} \alpha_k \cos(\pi k \tilde{y}), & \frac{\partial^2 \tilde{\phi}}{\partial \tilde{y}^2} \Big|_{\tilde{x}=\pm 1} &= \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} \alpha_k \cos(\pi k \tilde{x}), \\ \frac{\partial^2 \tilde{\phi}}{\partial \tilde{x} \partial \tilde{y}} \Big|_{\tilde{y}=\pm 1} &= \pm \sum_{k=1}^{\infty} \gamma_k \sin(\pi k \tilde{x}), & \frac{\partial^2 \tilde{\phi}}{\partial \tilde{x} \partial \tilde{y}} \Big|_{\tilde{x}=\pm 1} &= \pm \sum_{k=1}^{\infty} \gamma_k \sin(\pi k \tilde{y}).\end{aligned}$$

Thus, this solution allows one to satisfy any boundary conditions that may occur in the case of the loading function from (12). In (37), the sums are represented in the most natural form; however, for further derivation, it is convenient to separate the first term of the second sum and renumber the coefficients:

$$\begin{aligned}\tilde{\phi}^* &= \alpha_0 \left(\frac{\tilde{x}^2}{4} + \frac{\tilde{y}^2}{4} \right) - \sum_{k=1}^{\infty} \frac{\alpha_k}{\pi^2 k^2} [\cos(\pi k \tilde{x}) + \cos(\pi k \tilde{y})] \\ &+ \frac{2\gamma_0}{\pi^2} \left\{ \cos\left(\frac{\pi \tilde{y}}{2}\right) [\cos(\pi \tilde{x}) + 1] + \cos\left(\frac{\pi \tilde{x}}{2}\right) [\cos(\pi \tilde{y}) + 1] \right\} \\ &+ \sum_{k=1}^{\infty} \frac{2\gamma_k}{\pi^2 (k+1)} \left\{ \cos\left(\frac{\pi \tilde{y}}{2}\right) [\cos(\pi (k+1) \tilde{x}) + (-1)^k] + \cos\left(\frac{\pi \tilde{x}}{2}\right) [\cos(\pi (k+1) \tilde{y}) + (-1)^k] \right\}.\end{aligned}$$

It is now necessary to define the function $\tilde{\phi}_0$ in such a way that the full solution (36) satisfies the homogeneous biharmonic equation. In order to do this, this function will be expressed as a series over basis functions $\tilde{f}_k$:

$$\tilde{\phi}_0 = \sum_{k=1}^{\infty} \zeta_k \tilde{f}_k.$$

Substitution of this expression and the boundary solution (37) into the biharmonic equation leads to the following relation:

$$\begin{aligned} \sum_{k=1}^{\infty} \zeta_k \nabla^2 \nabla^2 \tilde{f}_k &= \sum_{k=1}^{\infty} \alpha_k \pi^2 k^2 [\cos(\pi k \tilde{x}) + \cos(\pi k \tilde{y})] \\ &\quad - \gamma_0 \frac{\pi^2}{8} \left[\cos\left(\frac{\pi y}{2}\right) (25 \cos \pi x + 1) + \cos\left(\frac{\pi x}{2}\right) (25 \cos \pi y + 1) \right] \\ &\quad - \sum_{k=1}^{\infty} \gamma_k \frac{\pi^2}{8(k+1)} \cos\left(\frac{\pi y}{2}\right) \left[\left(4(k+1)^2 + 1\right)^2 \cos(\pi(k+1)x) + (-1)^k \right] \\ &\quad - \sum_{k=1}^{\infty} \gamma_k \frac{\pi^2}{8(k+1)} \cos\left(\frac{\pi x}{2}\right) \left[\left(4(k+1)^2 + 1\right)^2 \cos(\pi(k+1)y) + (-1)^k \right]. \end{aligned}$$

As before, with projector operator $\mathcal{P}_q$ (27) and (30), one can transform the above relation to an infinite-dimensional system of algebraic equations with respect to ζ_k , α_k and γ_k :

$$\begin{aligned} \sum_{k=1}^{\infty} \zeta_k \left(\xi_q \delta_{qk} + \int_{-1}^1 \int_{-1}^1 2 \frac{\partial^4 \tilde{f}_k}{\partial \tilde{x}^2 \partial \tilde{y}^2} \tilde{f}_q d\tilde{x} d\tilde{y} \right) &= \sum_{k=1}^{\infty} \alpha_k \pi^2 k^2 \int_{-1}^1 \int_{-1}^1 [\cos(\pi k \tilde{x}) + \cos(\pi k \tilde{y})] \tilde{f}_q d\tilde{x} d\tilde{y} \\ &\quad - \gamma_0 \frac{\pi^2}{8} \int_{-1}^1 \int_{-1}^1 \left[\cos\left(\frac{\pi y}{2}\right) (25 \cos \pi x + 1) + \cos\left(\frac{\pi x}{2}\right) (25 \cos \pi y + 1) \right] \tilde{f}_q d\tilde{x} d\tilde{y} \\ &\quad - \sum_{k=1}^{\infty} \gamma_k \frac{\pi^2}{8(k+1)} \int_{-1}^1 \int_{-1}^1 \cos\left(\frac{\pi y}{2}\right) \left[\left(4(k+1)^2 + 1\right)^2 \cos(\pi(k+1)x) + (-1)^k \right] \tilde{f}_q d\tilde{x} d\tilde{y} \\ &\quad - \sum_{k=1}^{\infty} \gamma_k \frac{\pi^2}{8(k+1)} \int_{-1}^1 \int_{-1}^1 \cos\left(\frac{\pi x}{2}\right) \left[\left(4(k+1)^2 + 1\right)^2 \cos(\pi(k+1)y) + (-1)^k \right] \tilde{f}_q d\tilde{x} d\tilde{y}, \end{aligned}$$

and reduce it to a finite-dimensional one [92,93]. It is convenient to write the reduced system in a matrix form:

$$\boldsymbol{\zeta} = \boldsymbol{G}^{-1} \cdot \left[(\boldsymbol{A} + \boldsymbol{B}) \cdot \boldsymbol{\alpha} + (\boldsymbol{C}^0 + \boldsymbol{D}^0) \gamma_0 + (\boldsymbol{C} + \boldsymbol{D}) \cdot \boldsymbol{\gamma} \right], \quad (38)$$

where $\boldsymbol{\zeta}$, $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$ are vectors of corresponding Fourier coefficients, $\boldsymbol{C}^0$, $\boldsymbol{D}^0$ are auxiliary vectors and $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$ are auxiliary matrices. Recall that the components of the Galerkin matrix, $\boldsymbol{G}$, are presented by the relation (31). The components of other matrices can be tersely expressed through the corresponding auxiliary integrals $I_0(\dots), \dots, I_5(\dots)$ (see Appendix A):

$$\begin{aligned} A_{qk} &= \frac{\pi^2 k^2}{\|f_q\|} \left[I_4(\lambda_{S(q)}, k) - I_5(\lambda_{S(q)}, k) \right] I_0(\lambda_{C(q)}), \\ C_{qk} &= \frac{\pi^2}{8(k+1) \cosh(\frac{\pi}{2}) \|f_q\|} \left[I_2(\lambda_{C(q)}, \frac{\pi}{2}) - I_3(\lambda_{C(q)}, \frac{\pi}{2}) \right] \\ &\quad \times \left[\left(I_4(\lambda_{S(q)}, k+1) - I_5(\lambda_{S(q)}, k+1) \right) \left(4(k+1)^2 + 1 \right)^2 + (-1)^k I_0(\lambda_{S(q)}) \right]. \end{aligned}$$

Here, the components of only two matrices are shown. Matrices $\boldsymbol{B}$ and $\boldsymbol{D}$ can be written in a similar way to $\boldsymbol{A}$ and $\boldsymbol{C}$, respectively, using the replacement of eigenvalues $\lambda_{S(q)}$ by $\lambda_{C(q)}$ and vice versa. The components of vectors $\boldsymbol{C}^0$ and $\boldsymbol{D}^0$ can be obtained from the corresponding components of $\boldsymbol{C}$ and $\boldsymbol{D}$ by assuming k is equal to zero:

$$C_q^0 = C_{q0}, \quad D_q^0 = D_{q0}.$$

Thus, the solution (36) is fully defined. Further, for brevity, we will use the following notations:

$$\begin{aligned}\tilde{\phi}_{\alpha k}^* &:= -\frac{1}{\pi^2 k^2} \cos(\pi k \tilde{x}), & \tilde{\phi}_{\beta k}^* &:= -\frac{1}{\pi^2 k^2} \cos(\pi k \tilde{y}), \\ \tilde{\phi}_{\gamma 0}^* &:= \frac{2}{\pi^2} \cos\left(\frac{\pi \tilde{y}}{2}\right) [\cos(\pi \tilde{x}) + 1], & \tilde{\phi}_{\delta 0}^* &:= \frac{2}{\pi^2} \cos\left(\frac{\pi \tilde{x}}{2}\right) [\cos(\pi \tilde{y}) + 1], \\ \tilde{\phi}_{\gamma k}^* &:= \frac{2}{\pi^2 (k+1)} \cos\left(\frac{\pi \tilde{y}}{2}\right) [\cos(\pi (k+1) \tilde{x}) + (-1)^k], \\ \tilde{\phi}_{\delta k}^* &:= \frac{2}{\pi^2 (k+1)} \cos\left(\frac{\pi \tilde{x}}{2}\right) [\cos(\pi (k+1) \tilde{y}) + (-1)^k].\end{aligned}$$

With this notation, the solution (37) takes the following form:

$$\tilde{\phi}^* = \alpha_0 \left(\frac{\tilde{x}^2}{4} + \frac{\tilde{y}^2}{4} \right) + \gamma_0 (\tilde{\phi}_{\gamma 0}^* + \tilde{\phi}_{\delta 0}^*) + \sum_{k=1}^{\infty} \left[\alpha_k (\tilde{\phi}_{\alpha k}^* + \tilde{\phi}_{\beta k}^*) + \gamma_k (\tilde{\phi}_{\gamma k}^* + \tilde{\phi}_{\delta k}^*) \right].$$

As an example, consider boundary-value problem (35) and the equation above (37) with following boundary values (see Figure 7):

$$\psi(t) = (1 - t^2)^2, \quad \chi(t) = t(1 - t^2)^2.$$

The corresponding Airy function and stress distributions are shown in Figure 8.

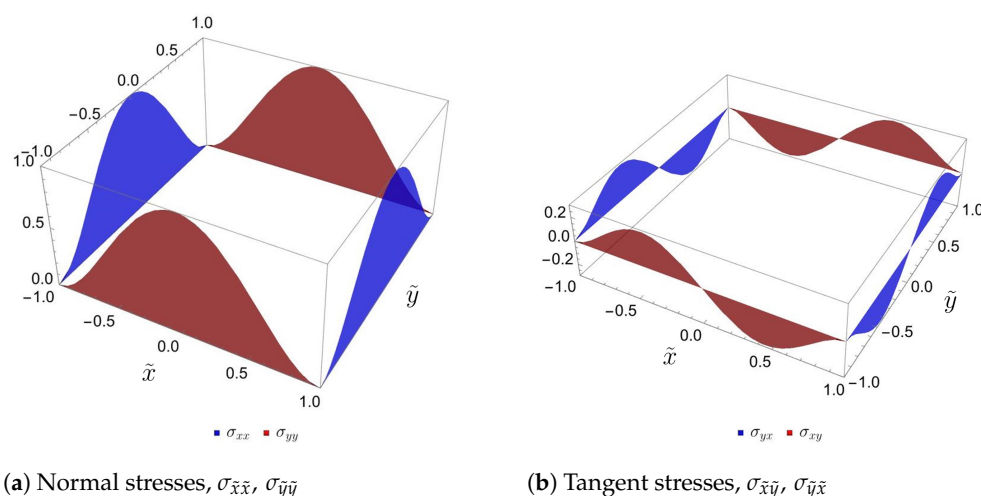


Figure 7. Stress functions.

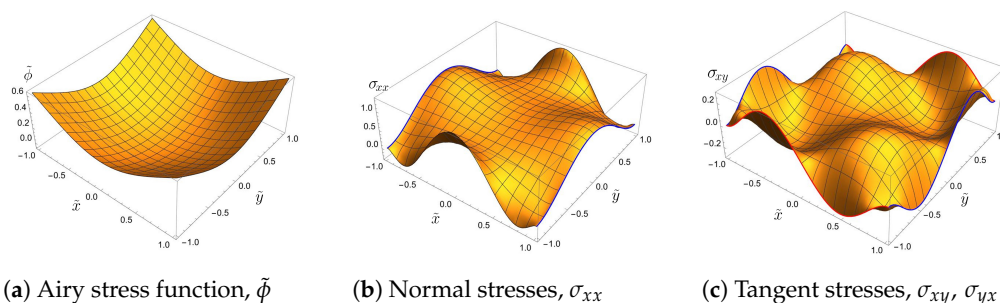


Figure 8. The Airy function and corresponding stress distributions.

4.2. Plate with Movable Edges

Consider now the first main boundary-value problem (14)–(17) for a square plate. We will seek the solution in the following form:

$$\tilde{w} = \sum_{k=1}^{\infty} g_k \tilde{f}_k, \quad \tilde{\phi} = \sum_{k=1}^{\infty} (s_k + \zeta_k) \tilde{f}_k + \tilde{\phi}^*. \quad (39)$$

Due to the specificity of boundary conditions (17), the solution of the auxiliary boundary problem can be significantly simplified:

$$\tilde{\phi}^* = \alpha_0 \left(\frac{\tilde{x}^2}{4} + \frac{\tilde{y}^2}{4} \right) + \sum_{k=1}^{\infty} \alpha_k (\tilde{\phi}_{\alpha k}^* + \tilde{\phi}_{\beta k}^*). \quad (40)$$

Expressions for the coefficients of the other part of the auxiliary solution (38) can be written in the following form:

$$\zeta = G^{-1} \cdot (A + B) \cdot \alpha. \quad (41)$$

Although Equation (14) are non-linear, the bilinearity of the Monge–Ampère form (3) allows us to use the expansions (39) and (40) for reducing them to the system of algebraic equations:

$$\begin{aligned} \sum_{k=1}^{\infty} g_k \left\{ \nabla^2 \nabla^2 \tilde{f}_k - \frac{\alpha_0}{2} \left(\frac{\partial^2 \tilde{f}_k}{\partial \tilde{y}^2} + \frac{\partial^2 \tilde{f}_k}{\partial \tilde{x}^2} \right) - \sum_{r=1}^{\infty} \left[(s_r + \zeta_r) L(\tilde{f}_k, \tilde{f}_r) + \alpha_r L(\tilde{f}_k, \tilde{\phi}_{\alpha r}^* + \tilde{\phi}_{\beta r}^*) \right] \right\} &= \tilde{p}, \\ \sum_{k=1}^{\infty} \left\{ s_k \nabla^2 \nabla^2 \tilde{f}_k + \kappa g_k \sum_{r=1}^{\infty} g_r L(\tilde{f}_k, \tilde{f}_r) \right\} &= 0. \end{aligned} \quad (42)$$

Boundary conditions (15) and (17) are satisfied automatically due to an appropriate choice of basis functions in (39) and (40). Apart from that, the symmetry of the problem allows us to consider boundary conditions (16) satisfied if at least one of them is met. Then, substituting (39) and (40) into, for example, the first condition from (16), we have

$$\sum_{k=1}^{\infty} \left\{ \int_0^1 \left[\frac{\partial^2}{\partial \tilde{y}^2} (s_k \tilde{f}_k + \zeta_k \tilde{f}_k + \alpha_k \tilde{\phi}_{\beta k}^*) - \kappa g_k \sum_{r=1}^{\infty} g_r \frac{\partial \tilde{f}_k}{\partial \tilde{x}} \frac{\partial \tilde{f}_r}{\partial \tilde{x}} \right] d\tilde{x} \right\} + \frac{\alpha_0}{2} (1 - \nu) = 0. \quad (43)$$

As before, one can act on these equations by projector operator and reduce them to an infinite-dimensional system of algebraic equations. For Equation (42) it is possible to use operator (27):

$$\begin{aligned} \sum_{k=1}^{\infty} g_k \left\{ \zeta_k \delta_{qk} + \int_{-1}^1 \int_{-1}^1 2 \frac{\partial^4 \tilde{f}_k}{\partial \tilde{x}^2 \partial \tilde{y}^2} \tilde{f}_q d\tilde{x} d\tilde{y} - \frac{\alpha_0}{2} \int_{-1}^1 \int_{-1}^1 \left(\frac{\partial^2 \tilde{f}_k}{\partial \tilde{y}^2} + \frac{\partial^2 \tilde{f}_k}{\partial \tilde{x}^2} \right) \tilde{f}_q d\tilde{x} d\tilde{y} \right. \\ \left. - \sum_{r=1}^{\infty} \left[(s_r + \zeta_r) \int_{-1}^1 \int_{-1}^1 L(\tilde{f}_k, \tilde{f}_r) \tilde{f}_q d\tilde{x} d\tilde{y} + \alpha_r \int_{-1}^1 \int_{-1}^1 L(\tilde{f}_k, \tilde{\phi}_{\alpha r}^* + \tilde{\phi}_{\beta r}^*) \tilde{f}_q d\tilde{x} d\tilde{y} \right] \right\} = \int_{-1}^1 \int_{-1}^1 \tilde{p} \tilde{f}_q d\tilde{x} d\tilde{y}, \\ \sum_{k=1}^{\infty} \left\{ s_k \left[\zeta_k \delta_{qk} + \int_{-1}^1 \int_{-1}^1 2 \frac{\partial^4 \tilde{f}_k}{\partial \tilde{x}^2 \partial \tilde{y}^2} \tilde{f}_q d\tilde{x} d\tilde{y} \right] + \kappa g_k \sum_{r=1}^{\infty} g_r \int_{-1}^1 \int_{-1}^1 L(\tilde{f}_k, \tilde{f}_r) \tilde{f}_q d\tilde{x} d\tilde{y} \right\} = 0. \end{aligned} \quad (44)$$

while boundary relation (43) should be subjected by the action of a one-dimensional projector operator. Since an auxiliary part of the Airy function is presented through cosines, it is convenient to use the projector operators with a cosine kernel:

$$\mathcal{F}_0(F(v)) = \int_{-1}^1 F dv, \quad \mathcal{F}_q^c(F(v)) = \int_{-1}^1 F \cos(\pi qv) dv, \quad q = 1 \dots \infty.$$

The action of these operators on the boundary relation results in

$$\begin{aligned} & -\kappa \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} \int_{-1}^1 \int_0^1 g_k g_r \frac{\partial \tilde{f}_k}{\partial \tilde{x}} \frac{\partial \tilde{f}_r}{\partial \tilde{x}} d\tilde{x} d\tilde{y} + \alpha_0(1 - \nu) = 0, \\ & \sum_{k=1}^{\infty} \left\{ \int_{-1}^1 \int_0^1 \left[(s_k + \zeta_k) \frac{\partial^2 \tilde{f}_k}{\partial \tilde{y}^2} \cos(\pi q\tilde{y}) - \kappa g_k \sum_{r=1}^{\infty} g_r \frac{\partial \tilde{f}_k}{\partial \tilde{x}} \frac{\partial \tilde{f}_r}{\partial \tilde{x}} \cos(\pi q\tilde{y}) \right] d\tilde{x} d\tilde{y} + \alpha_k \delta_{qk} \right\} = 0. \end{aligned} \quad (45)$$

For solving systems (44) and (45), it is necessary to reduce it to a finite-dimensional one. The application of the reduction method to an infinite system of linear equations is validated by well-developed theory rooted in early works of Koyalovich [92,93]. However, there is no rigorous foundation for its application to non-linear systems in the present. In spite of this fact, the reduction method is widely used [16,67,70] and provides results in good agreement with practice. “Why should I refuse a good dinner simply because I do not understand the digestive processes involved?” (Oliver Heaviside).

So, applying the reduction method, one can transform (44) and (45) to the following system:

$$\begin{aligned} & \left[G - \frac{\alpha_0}{2} (M^{\alpha 0} + M^{\beta 0}) - M \cdot (s + \zeta) - (M^{\alpha} + M^{\beta}) \cdot \alpha \right] \cdot g = b, \\ & G \cdot s + \kappa (M \cdot g) \cdot g = 0, \\ & -\kappa (U^{w0} \cdot g) \cdot g + \alpha_0(1 - \nu) = 0, \quad U \cdot (s + \zeta) + E \cdot \alpha - \kappa (U^w \cdot g) \cdot g = 0, \end{aligned} \quad (46)$$

where $M^{\alpha 0}$, $M^{\beta 0}$, U^{w0} , U are matrices $n \times n$ and M^{α} , M^{β} , U^w , M are cubic matrices $n \times n \times n$, the expression for the components of which are presented in Table 3.

Table 3. Components of auxiliary structures (movable edges case).

Structure	Components Expression
matrix $M^{\alpha 0}$	$M_{qk}^{\alpha 0} =$
	$-\frac{\lambda_{C(k)}^2 I_G(\lambda_{C(q)}, \lambda_{C(k)})}{\ f_k\ \ f_q\ } \left(\cos^2(\lambda_{S(k)}) + \cosh^2(\lambda_{S(k)}) \right) \delta_{S(q)S(k)}$
matrix $M^{\beta 0}$	$M_{qk}^{\beta 0} =$
	$-\frac{\lambda_{S(k)}^2 I_G(\lambda_{S(q)}, \lambda_{S(k)})}{\ f_k\ \ f_q\ } \left(\cos^2(\lambda_{C(k)}) + \cosh^2(\lambda_{C(k)}) \right) \delta_{C(q)C(k)}$
matrix U^{w0}	$U_{qk}^{w0} =$
	$\frac{\lambda_{S(k)} \lambda_{S(q)} I_U(\lambda_{S(q)}, \lambda_{S(k)})}{2\ f_k\ \ f_q\ } \left(\cos^2(\lambda_{C(k)}) + \cosh^2(\lambda_{C(k)}) \right) \delta_{C(q)C(k)}$
matrix U	$U_{qk} = -\frac{\lambda_{C(k)}^2 \left(I_4(\lambda_{C(k)}, q) + I_5(\lambda_{C(k)}, q) \right) I_0(\lambda_{S(k)})}{2\ f_k\ }$

Table 3. Cont.

Structure	Components Expression
cubic matrix $\mathbf{M}$	$M_{qkr} = \frac{I_M(\lambda_{S(q)}, \lambda_{S(k)}, \lambda_{S(r)}, \lambda_{C(q)}, \lambda_{C(k)}, \lambda_{C(r)})}{\ f_k\ \ f_r\ \ f_q\ }$
cubic matrix $\mathbf{M}^\alpha$	$M_{qkr}^\alpha = -\frac{\lambda_{C(k)}^2 I_{M3}(\lambda_{S(q)}, \lambda_{S(k)}, \pi r) I_G(\lambda_{C(q)}, \lambda_{C(k)})}{\ f_k\ \ f_q\ \cosh(\pi r)}$
cubic matrix $\mathbf{M}^\beta$	$M_{qkr}^\beta = -\frac{\lambda_{S(k)}^2 I_{M3}(\lambda_{C(q)}, \lambda_{C(k)}, \pi r) I_G(\lambda_{S(q)}, \lambda_{S(k)})}{\ f_k\ \ f_q\ \cosh(\pi r)}$
cubic matrix $\mathbf{U}^w$	$U_{qkr}^w = \frac{\lambda_{S(k)} \lambda_{S(r)} I_{M3}(\lambda_{C(k)}, \lambda_{C(r)}, \pi q) I_U(\lambda_{S(k)}, \lambda_{S(r)})}{2 \ f_k\ \ f_r\ \cosh(\pi q)}$

Turning now to the solution of the constructed finite-dimensional algebraic system. With account of (41), exclude the auxiliary vector ζ from the last equation of (46):

$$\mathbf{U} \cdot \mathbf{s} + [\mathbf{U} \cdot \mathbf{G}^{-1} \cdot (\mathbf{A} + \mathbf{B}) + \mathbf{E}] \cdot \boldsymbol{\alpha} - \kappa(\mathbf{U}^w \cdot \mathbf{g}) \cdot \mathbf{g} = \mathbf{0}. \quad (47)$$

Moreover, it is possible to express in turn vectors $\mathbf{s}$, $\boldsymbol{\alpha}$ and constant α_0 from system (46)–(47):

$$\begin{aligned} \mathbf{s} &= -\kappa(\mathbf{G}^{-1} \cdot \mathbf{M} \cdot \mathbf{g}) \cdot \mathbf{g}, & \alpha_0 &= \frac{\kappa}{1-\nu} (\mathbf{U}^{w0} \cdot \mathbf{g}) \cdot \mathbf{g}, \\ \boldsymbol{\alpha} &= \kappa [\mathbf{U} \cdot \mathbf{G}^{-1} \cdot (\mathbf{A} + \mathbf{B}) + \mathbf{E}]^{-1} \cdot [(\mathbf{U} \cdot \mathbf{G}^{-1} \cdot \mathbf{M} + \mathbf{U}^w) \cdot \mathbf{g}] \cdot \mathbf{g}, \end{aligned}$$

and substitute them into the first equation of (46), then the whole system can be reduced to the following system of cubic equations:

$$\mathbf{G} \cdot \mathbf{g} - \kappa \left[(\mathbf{S} \cdot \mathbf{g}) \cdot \mathbf{g} + \frac{1}{2(1-\nu)} ((\mathbf{U}^{w0} \cdot \mathbf{g}) \cdot \mathbf{g}) (\mathbf{M}^{\alpha 0} + \mathbf{M}^{\beta 0}) \right] \cdot \mathbf{g} = \mathbf{b}. \quad (48)$$

Here, $\mathbf{S}$ is the quadratic matrix with known components:

$$\mathbf{S} = -\mathbf{M} \cdot \mathbf{G}^{-1} \cdot \mathbf{M} + [\mathbf{M} \cdot \mathbf{G}^{-1} \cdot (\mathbf{A} + \mathbf{B}) + \mathbf{M}^\alpha + \mathbf{M}^\beta] \cdot \mathbf{R},$$

where $\mathbf{R}$ is an auxiliary cubic matrix deriving through known matrices:

$$\mathbf{R} = [\mathbf{U} \cdot \mathbf{G}^{-1} \cdot (\mathbf{A} + \mathbf{B}) + \mathbf{E}]^{-1} \cdot (\mathbf{U} \cdot \mathbf{G}^{-1} \cdot \mathbf{M} + \mathbf{U}^w).$$

Although calculation of even finite numbers of components of cubic matrix $\mathbf{S}$ requires some rather cumbersome calculations, it does not present any insurmountable difficulties.

Remark 2. In general, the system of cubic equations allows multiple solutions, so it is usually necessary to analyze the branching of them [94]. However, if the in-plane stresses are of tensile type, the solution is unique [78,79].

To solve the cubic Equation (48), it is possible to use Newton's method for linearizing the equation at each iteration. Denote a non-linear algebraic operator in LHS of (48) as $\mathcal{A}$:

$$\mathcal{A}(\mathbf{g}) := \mathbf{G} \cdot \mathbf{g} - \kappa \left[(\mathbf{S} \cdot \mathbf{g}) \cdot \mathbf{g} + \frac{1}{2(1-\nu)} ((\mathbf{U}^{w0} \cdot \mathbf{g}) \cdot \mathbf{g}) (\mathbf{M}^{\alpha 0} + \mathbf{M}^{\beta 0}) \right] \cdot \mathbf{g}.$$

Let $\mathbf{g}_0$ be the initial value of the solution obtained from the linear problem (32) or as an expansion of some approximate solution. Then, the cubic equation can be reduced to a linear one in the neighbour of $\mathbf{g}_0$ as follows:

$$\mathcal{A}(\mathbf{g}) = \mathcal{A}(\mathbf{g}_0) + \left. \frac{\partial \mathcal{A}}{\partial \mathbf{g}} \right|_{\mathbf{g}=\mathbf{g}_0} \cdot (\mathbf{g} - \mathbf{g}_0) + o(\mathbf{g} - \mathbf{g}_0). \quad (49)$$

where

$$\begin{aligned} \left. \frac{\partial \mathcal{A}}{\partial \mathbf{g}} \right|_{\mathbf{g}=\mathbf{g}_0} &= \sum_{k=1}^n \mathbf{G} \cdot \mathbf{E}_{kk} - \kappa \sum_{k=1}^n \{[(\mathbf{S} \cdot \mathbf{E}_{kk}) \cdot \mathbf{g}_0] \cdot \mathbf{g}_0 + [(\mathbf{S} \cdot \mathbf{g}_0) \cdot \mathbf{E}_{kk}] \cdot \mathbf{g}_0 + [(\mathbf{S} \cdot \mathbf{g}_0) \cdot \mathbf{g}_0] \cdot \mathbf{E}_{kk}\} \\ &\quad - \frac{\kappa}{2(1-\nu)} \sum_{k=1}^n \left\{ \left[(\mathbf{U}^{w0} \cdot \mathbf{E}_{kk}) \cdot \mathbf{g}_0 + (\mathbf{U}^{w0} \cdot \mathbf{g}_0) \cdot \mathbf{E}_{kk} \right] (\mathbf{M}^{\alpha 0} + \mathbf{M}^{\beta 0}) \cdot \mathbf{g}_0 \right. \\ &\quad \left. + \left[(\mathbf{U}^{w0} \cdot \mathbf{g}_0) \cdot \mathbf{g}_0 \right] (\mathbf{M}^{\alpha 0} + \mathbf{M}^{\beta 0}) \cdot \mathbf{E}_{kk} \right\}. \end{aligned}$$

4.3. Plate with Immovable Edges

Consider the second main boundary-value problem, defined by boundary conditions (18). In this case the additional term appears in the right-hand side of (40):

$$\tilde{\phi}^* = [\dots] + \gamma_0 (\tilde{\phi}_{\gamma 0}^* + \tilde{\phi}_{\delta 0}^*) + \sum_{k=1}^{\infty} \gamma_k (\tilde{\phi}_{\gamma k}^* + \tilde{\phi}_{\delta k}^*).$$

Hereafter, we denote terms contained in corresponding equations in the previous case by the symbol $[\dots]$. The first and the third equations of system (42) are modified as follows:

$$\begin{aligned} [\dots] - \sum_{k=1}^{\infty} g_k \left[\gamma_0 L(f_k, \tilde{\phi}_{\gamma 0}^* + \tilde{\phi}_{\delta 0}^*) + \sum_{r=1}^{\infty} \gamma_r L(f_k, \tilde{\phi}_{\gamma r}^* + \tilde{\phi}_{\delta r}^*) \right] &= \tilde{p}, \\ [\dots] + \int_0^1 \left\{ \gamma_0 \left[\frac{\partial^2}{\partial \tilde{y}^2} (\tilde{\phi}_{\gamma 0}^* + \tilde{\phi}_{\delta 0}^*) - \nu \frac{\partial^2}{\partial \tilde{x}^2} \tilde{\phi}_{\delta 0}^* \right] + \sum_{k=1}^{\infty} \gamma_k \left[\frac{\partial^2}{\partial \tilde{y}^2} (\tilde{\phi}_{\gamma k}^* + \tilde{\phi}_{\delta k}^*) - \nu \frac{\partial^2}{\partial \tilde{x}^2} \tilde{\phi}_{\delta k}^* \right] \right\} d\tilde{x} &= 0, \end{aligned}$$

while the second equation remains unchangeable. Apart from that, this system must be added by one more boundary condition equation:

$$\left[\frac{\alpha_0}{2} (1-\nu) + \gamma_0 \frac{\partial^2 \tilde{\phi}_{\gamma 0}^*}{\partial \tilde{x}^2} + \sum_{k=1}^{\infty} \left((s_k + \zeta_k) \frac{\partial^2 \tilde{f}_k}{\partial \tilde{x}^2} + \alpha_k \frac{\partial^2 \tilde{\phi}_{\alpha k}^*}{\partial \tilde{x}^2} + \gamma_k \frac{\partial^2 \tilde{\phi}_{\gamma k}^*}{\partial \tilde{x}^2} - \nu \alpha_k \frac{\partial^2 \tilde{\phi}_{\beta k}^*}{\partial \tilde{y}^2} \right) \right] \Big|_{\tilde{x}=1} = 0.$$

As in the previous case, projecting into basis functions with subsequent reduction allows us to obtain a finite system of algebraic equations:

$$\begin{aligned} \left[\mathbf{G} - \frac{\alpha_0}{2} (\mathbf{M}^{\alpha 0} + \mathbf{M}^{\beta 0}) - \gamma_0 (\mathbf{M}^{\gamma 0} + \mathbf{M}^{\delta 0}) - \mathbf{M} \cdot (\mathbf{s} + \boldsymbol{\zeta}) \right. \\ \left. - (\mathbf{M}^{\alpha} + \mathbf{M}^{\beta}) \cdot \boldsymbol{\alpha} - (\mathbf{M}^{\gamma} + \mathbf{M}^{\delta}) \cdot \boldsymbol{\gamma} \right] \cdot \mathbf{g} = \mathbf{b}, \\ \mathbf{G} \cdot \mathbf{s} + \kappa (\mathbf{M} \cdot \mathbf{g}) \cdot \mathbf{g} = 0, \\ \mathbf{U} \cdot (\mathbf{s} + \boldsymbol{\zeta}) + \mathbf{E} \cdot \boldsymbol{\alpha} + (\mathbf{U}^{\gamma} + \mathbf{U}^{\delta 1} - \nu \mathbf{U}^{\delta 2}) \cdot \boldsymbol{\gamma} + \gamma_0 (\mathbf{u}^{\gamma} + \mathbf{u}^{\delta 1} - \nu \mathbf{u}^{\delta 2}) - \kappa (\mathbf{U}^w \cdot \mathbf{g}) \cdot \mathbf{g} = 0, \\ \mathbf{V}^{\gamma} \cdot \boldsymbol{\gamma} + \mathbf{V} \cdot (\mathbf{s} + \boldsymbol{\zeta}) + \gamma_0 \mathbf{v}^{\gamma} - \nu \mathbf{E} \cdot \boldsymbol{\alpha} = 0, \\ (\mathbf{u}^{\gamma 0} - \nu \mathbf{u}^{\delta 0}) \cdot \boldsymbol{\gamma} - \kappa (\mathbf{U}^{w0} \cdot \mathbf{g}) \cdot \mathbf{g} + \alpha_0 (1-\nu) - \frac{2\gamma_0}{\pi} (1-\nu) = 0, \\ \mathbf{v}^{\gamma 0} \cdot \boldsymbol{\gamma} + \mathbf{v}^0 \cdot (\mathbf{s} + \boldsymbol{\zeta}) + \mathbf{v}^{\alpha 0} \cdot \boldsymbol{\alpha} + \alpha_0 (1-\nu) + \frac{8\gamma_0}{\pi} = 0. \end{aligned}$$

Here, the additional vectors, matrices, and cubic matrices are used. The expressions for components of some of it are presented in Table 4. Components of vectors $\mathbf{u}^\gamma$, $\mathbf{u}^{\delta 1}$, $\mathbf{u}^{\delta 2}$, $\mathbf{v}^\gamma$ and matrices $\mathbf{M}^{\gamma 0}$, $\mathbf{M}^{\delta 0}$ can be found from corresponding matrices $\mathbf{U}^\gamma$, $\mathbf{U}^{\delta 1}$, $\mathbf{U}^{\delta 2}$, $\mathbf{V}^\gamma$ and cubic matrices $\mathbf{M}^\gamma$, $\mathbf{M}^\delta$ if we assume their last indexes as zero.

Table 4. Components of auxiliary structures (immovable edges case).

Structure	Components Expression
vectors $\mathbf{u}^{\gamma 0}$, $\mathbf{u}^{\delta 0}$	$\mathbf{u}_q^{\gamma 0} = \mathbf{u}_q^{\delta 0} = -\frac{2(-1)^q}{\pi(q+1)}$
vector $\mathbf{v}^{\gamma 0}$	$v_q^{\gamma 0} = \frac{8(-1)^q(q+1)}{\pi}$
vector $\mathbf{v}^{\alpha 0}$	$v_q^{\alpha 0} = 2(-1)^q$
vector $\mathbf{v}^0$	$v_q^0 = -\frac{2\lambda_{S(q)}^2 \cos(\lambda_{S(q)}) \cosh(\lambda_{S(q)}) I_0(\lambda_{C(q)})}{\ f_q\ }$
matrix $\mathbf{U}^\gamma$	$U_{qk}^\gamma = \frac{2(-1)^{q+k}}{\pi(k+1)(4q^2-1)}$
matrix $\mathbf{U}^{\delta 1}$	$U_{qk}^{\delta 1} = -4q\pi^{-1}\delta_{q(k+1)}$
matrix $\mathbf{U}^{\delta 2}$	$U_{qk}^{\delta 2} = -(\pi q)^{-1}\delta_{q(k+1)}$
matrix $\mathbf{V}^\gamma$	$V_{qk}^\gamma = -\frac{8(k+1)(-1)^{q+k}}{\pi(4q^2-1)}$
matrix $\mathbf{V}$	$V_{qk} = -\frac{2\lambda_{S(k)}^2 \cos(\lambda_{S(k)}) \cosh(\lambda_{S(k)}) (I_4(\lambda_{C(k)}, q) - I_5(\lambda_{C(k)}, q))}{\ f_k\ }$
cubic matrix $\mathbf{M}^\gamma$	$M_{qkr}^\gamma = \frac{I_B(\lambda_{S(q)}, \lambda_{S(k)}, \lambda_{C(q)}, \lambda_{C(k)}, r+1)}{\ f_k\ \ f_q\ }$
cubic matrix $\mathbf{M}^\delta$	$M_{qkr}^\delta = \frac{I_B(\lambda_{C(q)}, \lambda_{C(k)}, \lambda_{S(q)}, \lambda_{S(k)}, r+1)}{\ f_k\ \ f_q\ }$

Turn to solve this system. First, scalars γ^0 can be expressed through other variables as follows:

$$\begin{aligned} \gamma_0 = & - \left\{ \left[\mathbf{v}^{\alpha 0} + \mathbf{v}^0 \cdot \mathbf{G}^{-1} \cdot (\mathbf{A} + \mathbf{B}) \right] \cdot \boldsymbol{\alpha} + \left[\mathbf{v}^{\gamma 0} + \mathbf{v}^0 \cdot \mathbf{G}^{-1} \cdot (\mathbf{C} + \mathbf{D}) - \mathbf{u}^{\gamma 0} + \nu \mathbf{u}^{\delta 0} \right] \cdot \boldsymbol{\gamma} \right. \\ & \left. + \kappa \left[\left(\mathbf{U}^{w0} - \mathbf{v}^0 \cdot \mathbf{G}^{-1} \cdot \mathbf{M} \right) \cdot \mathbf{g} \right] \cdot \mathbf{g} \right\} \left[\frac{2(5-\nu)}{\pi} + \mathbf{v}^0 \cdot \mathbf{G}^{-1} \cdot (\mathbf{C}^0 + \mathbf{D}^0) \right]^{-1} \\ = &: \mathcal{S}_{\gamma\alpha}^0 \cdot \boldsymbol{\alpha} + \mathcal{S}_{\gamma\gamma}^0 \cdot \boldsymbol{\gamma} + \left(\mathcal{S}_{\gamma g}^0 \cdot \mathbf{g} \right) \cdot \mathbf{g}. \end{aligned}$$

Further, one can express vector $\boldsymbol{\gamma}$:

$$\begin{aligned} \boldsymbol{\gamma} = & - \left[\mathbf{V}^\gamma + \mathbf{V} \cdot \mathbf{G}^{-1} \cdot (\mathbf{C} + \mathbf{D}) + \left(\mathbf{v}^\gamma + \mathbf{V} \cdot \mathbf{G}^{-1} \cdot (\mathbf{C}^0 + \mathbf{D}^0) \right) \otimes \mathcal{S}_{\gamma\gamma}^0 \right]^{-1} \\ & \cdot \left\{ \left[\left(\mathbf{v}^\gamma + \mathbf{V} \cdot \mathbf{G}^{-1} \cdot (\mathbf{C}^0 + \mathbf{D}^0) \right) \otimes \mathcal{S}_{\gamma\alpha}^0 + \mathbf{V} \cdot \mathbf{G}^{-1} \cdot (\mathbf{A} + \mathbf{B}) - \nu \mathbf{E} \right] \cdot \boldsymbol{\alpha} \right. \\ & \left. + \left(\mathbf{v}^\gamma + \mathbf{V} \cdot \mathbf{G}^{-1} \cdot (\mathbf{C}^0 + \mathbf{D}^0) \right) \left(\left(\mathcal{S}_{\gamma g}^0 \cdot \mathbf{g} \right) \cdot \mathbf{g} \right) - \kappa \left(\mathbf{V} \cdot \mathbf{G}^{-1} \cdot \mathbf{M} \cdot \mathbf{g} \right) \cdot \mathbf{g} \right\} \\ = &: \mathcal{S}_{\gamma\alpha} \cdot \boldsymbol{\alpha} + \left(\mathcal{S}_{\gamma g} \cdot \mathbf{g} \right) \cdot \mathbf{g} + \mathcal{T}_{\gamma g} \left(\mathcal{S}_{\gamma g}^0 \cdot \mathbf{g} \right) \cdot \mathbf{g}, \end{aligned}$$

where $\otimes$ is the Kronecker product for vectors, which assigns a matrix to a pair of vectors by the following rule:

$$\mathbf{a} \otimes \mathbf{b} := \mathbf{M}, \quad M_{ij} = a_i b_j.$$

Then α_0, α :

$$\begin{aligned} \alpha &= \left\{ \mathbf{U} \cdot \mathbf{G}^{-1} \cdot \left[(\mathbf{A} + \mathbf{B}) + (\mathbf{C} + \mathbf{D}) \cdot \mathcal{S}_{\gamma\alpha} + (\mathbf{C}^0 + \mathbf{D}^0) \otimes (\mathcal{S}_{\gamma\gamma}^0 \cdot \mathcal{S}_{\gamma\alpha} + \mathcal{S}_{\gamma\alpha}^0) \right] + \mathbf{E} \right. \\ &\quad \left. + (\mathbf{U}^\gamma + \mathbf{U}^{\delta 1} - \nu \mathbf{U}^{\delta 2}) \cdot \mathcal{S}_{\gamma\alpha} + (\mathbf{u}^\gamma + \mathbf{u}^{\delta 1} - \nu \mathbf{u}^{\delta 2}) \otimes (\mathcal{S}_{\gamma\gamma}^0 \cdot \mathcal{S}_{\gamma\alpha} + \mathcal{S}_{\gamma\alpha}^0) \right\}^{-1} \\ &\quad \cdot \left\{ \left[\left(\kappa \mathbf{U} \cdot \mathbf{G}^{-1} \cdot \mathbf{M} - \mathbf{U} \cdot \mathbf{G}^{-1} \cdot (\mathbf{C} + \mathbf{D}) \cdot \mathcal{S}_{\gamma g} - (\mathbf{U}^\gamma + \mathbf{U}^{\delta 1} - \nu \mathbf{U}^{\delta 2}) \cdot \mathcal{S}_{\gamma g} + \kappa \mathbf{U}^w \right. \right. \right. \\ &\quad \left. \left. - (\mathbf{U} \cdot \mathbf{G}^{-1} \cdot (\mathbf{C}^0 + \mathbf{D}^0) + (\mathbf{u}^\gamma + \mathbf{u}^{\delta 1} - \nu \mathbf{u}^{\delta 2})) \otimes \mathcal{S}_{\gamma\gamma}^0 \cdot \mathcal{S}_{\gamma g} \right) \cdot \mathbf{g} \right. \\ &\quad \left. - \left[(\mathbf{U} \cdot \mathbf{G}^{-1} \cdot (\mathbf{C} + \mathbf{D}) + (\mathbf{U}^\gamma + \mathbf{U}^{\delta 1} - \nu \mathbf{U}^{\delta 2})) \cdot \mathcal{T}_{\gamma g} \right. \right. \\ &\quad \left. \left. - (\mathbf{U} \cdot \mathbf{G}^{-1} \cdot (\mathbf{C}^0 + \mathbf{D}^0) + (\mathbf{u}^\gamma + \mathbf{u}^{\delta 1} - \nu \mathbf{u}^{\delta 2})) (1 + \mathcal{S}_{\gamma\gamma}^0 \cdot \mathcal{T}_{\gamma g}) \right] (\mathcal{S}_{\gamma g}^0 \cdot \mathbf{g}) \cdot \mathbf{g} \right\} \\ &= (\mathcal{S}_{\alpha g} \cdot \mathbf{g}) \cdot \mathbf{g} + \mathcal{T}_{\alpha g} (\mathcal{S}_{\gamma g}^0 \cdot \mathbf{g}) \cdot \mathbf{g}, \end{aligned}$$

$$\begin{aligned} \alpha_0 &= \frac{1}{1-\nu} \left\{ \left[\left(\frac{2}{\pi} (1-\nu) (\mathcal{S}_{\gamma\alpha}^0 \cdot \mathcal{S}_{\alpha g} + \mathcal{S}_{\gamma\gamma}^0 \cdot \mathcal{S}_{\gamma\alpha} \cdot \mathcal{S}_{\alpha g} + \mathcal{S}_{\gamma\gamma}^0 \cdot \mathcal{S}_{\gamma g} + \mathcal{S}_{\gamma g}^0) \right. \right. \right. \\ &\quad \left. \left. - (\mathbf{u}^{\gamma 0} - \nu \mathbf{u}^{\delta 0}) \cdot (\mathcal{S}_{\gamma\alpha} \cdot \mathcal{S}_{\alpha g} + \mathcal{S}_{\gamma g}) + \kappa \mathbf{U}^{w0} \right) \cdot \mathbf{g} \right] \cdot \mathbf{g} \\ &\quad + \left[\frac{2}{\pi} (1-\nu) (\mathcal{S}_{\gamma\alpha}^0 \cdot \mathcal{T}_{\alpha g} + \mathcal{S}_{\gamma\gamma}^0 \cdot \mathcal{S}_{\gamma\alpha} \cdot \mathcal{T}_{\alpha g} + \mathcal{S}_{\gamma\gamma}^0 \cdot \mathcal{T}_{\gamma g}) \right. \\ &\quad \left. - (\mathbf{u}^{\gamma 0} - \nu \mathbf{u}^{\delta 0}) \cdot (\mathcal{S}_{\gamma\alpha} \cdot \mathcal{T}_{\alpha g} + \mathcal{T}_{\gamma g}) \right] \cdot (\mathcal{S}_{\gamma g}^0 \cdot \mathbf{g}) \cdot \mathbf{g} \right\} =: (\mathcal{S}_{\alpha g}^0 \cdot \mathbf{g}) \cdot \mathbf{g} + \mathcal{T}_{\alpha g}^0 (\mathcal{S}_{\gamma g}^0 \cdot \mathbf{g}) \cdot \mathbf{g}. \end{aligned}$$

After introducing a set of auxiliary structures $\mathcal{S}$ and $\mathcal{T}$, one can simply reduce the full system to the cubic system:

$$\mathbf{G} \cdot \mathbf{g} - \left[(\tilde{\mathbf{S}} \cdot \mathbf{g}) \cdot \mathbf{g} + \left((\mathcal{S}_{\gamma g}^0 \cdot \mathbf{g}) \cdot \mathbf{g} \right) \tilde{\mathbf{T}} \right] \cdot \mathbf{g} = \mathbf{b},$$

where $\tilde{\mathbf{S}}$ is a quadric matrix:

$$\begin{aligned} \tilde{\mathbf{S}} &= -\kappa \mathbf{M} \cdot \mathbf{G}^{-1} \cdot \mathbf{M} + \frac{1}{2} (\mathbf{M}^{\alpha 0} + \mathbf{M}^{\beta 0}) \cdot \mathcal{S}_{\alpha g}^0 + \left[\mathbf{M} \cdot \mathbf{G}^{-1} \cdot (\mathbf{A} + \mathbf{B}) + \mathbf{M}^\alpha + \mathbf{M}^\beta \right] \cdot \mathcal{S}_{\alpha g} \\ &\quad + \left[\mathbf{M} \cdot \mathbf{G}^{-1} \cdot (\mathbf{C} + \mathbf{D}) + \mathbf{M}^\gamma + \mathbf{M}^\delta \right] \cdot (\mathcal{S}_{\gamma\alpha} \cdot \mathcal{S}_{\alpha g} + \mathcal{S}_{\gamma g}) \\ &\quad + \left[\mathbf{M} \cdot \mathbf{G}^{-1} \cdot (\mathbf{C}^0 + \mathbf{D}^0) + \mathbf{M}^{\gamma 0} + \mathbf{M}^{\delta 0} \right] \cdot (\mathcal{S}_{\gamma\alpha}^0 \cdot \mathcal{S}_{\alpha g} + \mathcal{S}_{\gamma\gamma}^0 \cdot \mathcal{S}_{\gamma\alpha} \cdot \mathcal{S}_{\alpha g} + \mathcal{S}_{\gamma\gamma}^0 \cdot \mathcal{S}_{\gamma g}) \end{aligned}$$

and $\tilde{\mathbf{T}}$ is a vector:

$$\begin{aligned} \tilde{\mathbf{T}} &= \frac{1}{2} \mathcal{T}_{\alpha g}^0 (\mathbf{M}^{\alpha 0} + \mathbf{M}^{\beta 0}) + \left[\mathbf{M} \cdot \mathbf{G}^{-1} \cdot (\mathbf{A} + \mathbf{B}) + \mathbf{M}^\alpha + \mathbf{M}^\beta \right] \cdot \mathcal{T}_{\alpha g} \\ &\quad + \left[\mathbf{M} \cdot \mathbf{G}^{-1} \cdot (\mathbf{C} + \mathbf{D}) + \mathbf{M}^\gamma + \mathbf{M}^\delta \right] \cdot (\mathcal{S}_{\gamma\alpha} \cdot \mathcal{T}_{\alpha g} + \mathcal{T}_{\gamma g}) \\ &\quad + \left[\mathbf{M} \cdot \mathbf{G}^{-1} \cdot (\mathbf{C}^0 + \mathbf{D}^0) + \mathbf{M}^{\gamma 0} + \mathbf{M}^{\delta 0} \right] (\mathcal{S}_{\gamma\alpha}^0 \cdot \mathcal{T}_{\alpha g} + \mathcal{S}_{\gamma\gamma}^0 \cdot \mathcal{S}_{\gamma\alpha} \cdot \mathcal{T}_{\alpha g} + \mathcal{S}_{\gamma\gamma}^0 \cdot \mathcal{T}_{\gamma g} + 1) \end{aligned}$$

As in the previous case, the obtained cubic equation can be linearized by a formula similar to (49) and solved by Newton's method.

5. Numerical Results and Discussion

In order to verify the proposed solution, mechanical tests with thin, square, completely clamped plates were carried out. The essence of tests is as follows. A plate initially undergoes finite strain under given pressure p_0 and then reloads by small increments of additional pressure δp . Corresponding increments of normal deflections δw are determined with holographic interferometry [95]. To this end, the plates, fixed in a square, thick-walled steel chamber, were employed. The initial deformed state was considered a reference and recorded on a reference hologram. Superimposing on it another hologram, recorded after the increase in pressure, results in a fringe pattern that identifies contour lines of the deflection increment field δw . Repeating these experiments for various initial pressures and a number of incremental steps, one can obtain a comprehensive description of the non-linear deformation at different levels of finite strains.

The physical and geometric parameters of the plate are given in Table 5.

Table 5. Plate parameters.

Length, cm	Thickness, μm	Young Modulus, GPa	Poisson's Ratio
6	184	128	0.35

Experimentally obtained fields of deflection increments were compared with the results of calculations according to the proposed solution. Intermediate theoretical results for initial pressure $p_0 = 3$ kPa are shown in Figure 9. A comparison of theoretical and experimental fields of deflection increments for three different initial pressures, $p_0 = 0$, $p_0 = 3$ kPa, and $p_0 = 6$ kPa, and additional loading with equal increments in 10 Pa, is shown on the chart in Figure 10. On this chart, one can see experimentally and theoretically obtained maximal deflections (in the center of the plate), represented in terms of contour line quantities. This representation of the displacements is convenient for comparison with the holographic experimental method, because in the latter we directly obtain contour lines of the displacement with a step equal to half the measuring laser wavelength.

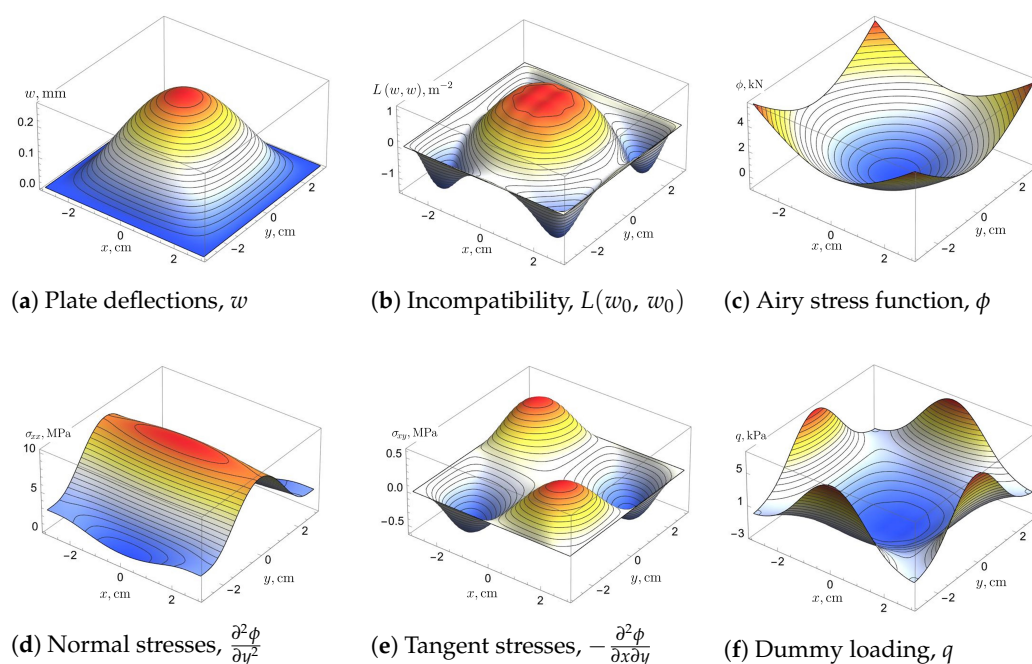


Figure 9. Distributions for the elements of the solution at initial pressure 3 kPa without incremental loading.

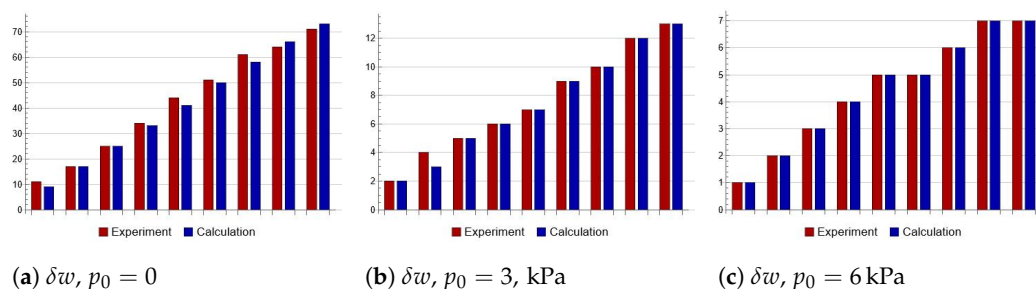


Figure 10. Experimental and theoretical deflections of the plate central point in terms of contour line order (order increases by 1 when displacements increase by 266 nm).

A more detailed juxtaposition of the theoretical and experimental contours for the 5th and 8th steps of incremental loading is presented in Figure 11. The above results demonstrate a good agreement between theoretical results, obtained with the Föppl-von Kármán model and proposed solution, and the experimental data.

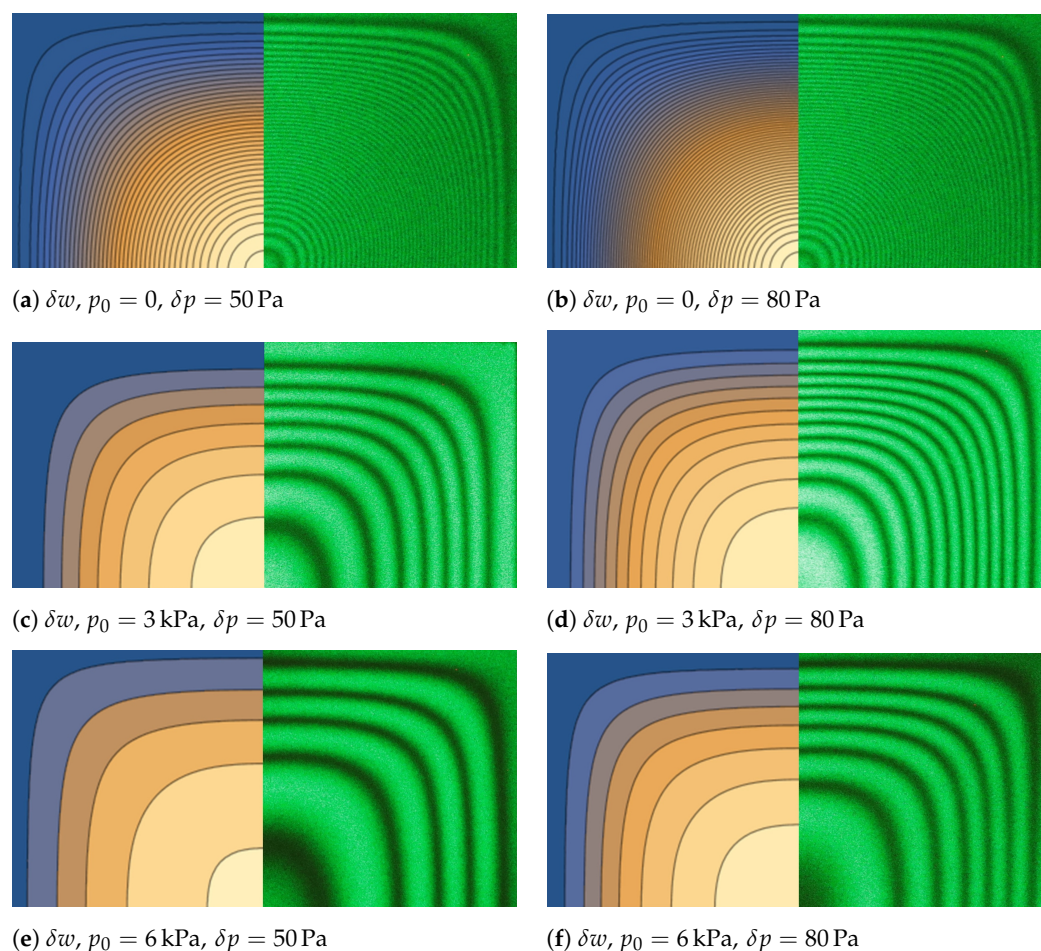


Figure 11. Juxtaposition of theoretical and experimental deflections under different initial and incremental pressure. Step of contour lines is taken to be equal to half laser wavelength (266 nm).

6. Conclusions

To sum up, we would like to highlight the key advantages of the proposed solution:

- The proposed solution, which reduces the boundary-value problem to the infinite-dimensional system of cubic algebraic equations with coefficients determined analytically via finite formula, can be an effective tool for modeling and design of flexible plates.

- The theoretical model accurately reproduces the tendency of the plate effective stiffness increasing with the increasing of in-plane tensile stresses state. Note that compressive in-plane stresses may decrease the effective stiffness even up to zero.
- The theoretical modeling shows a good agreement with the experiment. The comparison analysis demonstrates that the proposed solution allow us to describe plates bending both quantitatively and qualitatively.

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Abbreviations

The following abbreviations are used in this manuscript:

FEM finite element method
LHS left hand side
RHS right hand side

Appendix A

The list of the auxiliary integrals using in the article is provided below (note that parameter k is integer and ρ, ν, ζ satisfy transcendental equation $\tan(\cdot) + \tanh(\cdot) = 0$):

$$\begin{aligned}
 I_0(\rho) &= \int_{-1}^1 (\cos(\rho v) \cosh(\rho) - \cos(\rho) \cosh(\rho v)) dv = \frac{4 \sin(\rho) \cosh(\rho)}{\rho} \\
 I_1(\rho, \nu) &= \cos(\rho) \cos(\nu) \int_{-1}^1 \cosh(\rho v) \cosh(\nu v) dv \\
 &= \begin{cases} \frac{2 \cosh(\rho) \cosh(\nu) (\rho \sin(\rho) \cos(\nu) - \nu \sin(\nu) \cos(\rho))}{\nu^2 - \rho^2} & \rho \neq \nu \\ \frac{\rho \cos^2(\rho) - \sin(\rho) \cos(\rho) \cosh^2(\rho)}{\rho} & \rho = \nu \end{cases} \\
 I_2(\rho, \nu) &= \cos(\rho) \cosh(\nu) \int_{-1}^1 \cosh(\rho v) \cos(\nu v) dv \\
 &= \frac{2 \cos(\rho) \cosh(\nu) (\rho \sinh(\rho) \cos(\nu) + \nu \sin(\nu) \cosh(\rho))}{\nu^2 + \rho^2}
 \end{aligned}$$

$$\begin{aligned}
I_3(\rho, v) &= \cosh(\rho) \cosh(v) \int_{-1}^1 \cos(\rho v) \cos(vv) dv \\
&= \begin{cases} -\frac{2 \cosh(\rho) \cosh(v) (\rho \sin(\rho) \cos(v) - v \sin(v) \cos(\rho))}{v^2 - \rho^2} & \rho \neq v \\ \frac{\rho \cosh^2(\rho) + \sin(\rho) \cos(\rho) \cosh^2(\rho)}{\rho} & \rho = v \end{cases} \\
I_4(\rho, k) &= \cosh(\rho) \int_{-1}^1 \cos(\rho v) \cos(\pi k v) dv = \frac{2(-1)^k \rho \sin(\rho) \cosh(\rho)}{\rho^2 - \pi^2 k^2} \\
I_5(\rho, k) &= \cos(\rho) \int_{-1}^1 \cosh(\rho v) \cos(\pi k v) dv = -\frac{2(-1)^k \rho \sin(\rho) \cosh(\rho)}{\rho^2 + \pi^2 k^2} \\
I_6(\rho, v) &= \cos(\rho) \cos(v) \int_{-1}^1 \sinh(\rho v) \sinh(vv) dv \\
&= \begin{cases} \frac{2 \cosh(\rho) \cosh(v) (\rho \cos(\rho) \sin(v) - v \cos(v) \sin(\rho))}{v^2 - \rho^2} & \rho \neq v \\ -\frac{\rho \cos^2(\rho) + \sin(\rho) \cos(\rho) \cosh^2(\rho)}{\rho} & \rho = v \end{cases} \\
I_7(\rho, v) &= \cos(\rho) \cosh(v) \int_{-1}^1 \sinh(\rho v) \sin(vv) dv \\
&= \frac{2 \cosh(\rho) \cosh(v) (\rho \cos(\rho) \sin(v) + v \cos(v) \sin(\rho))}{v^2 + \rho^2} \\
I_8(\rho, v) &= \cosh(\rho) \cosh(v) \int_{-1}^1 \sin(\rho v) \sin(vv) dv \\
&= \begin{cases} \frac{2 \cosh(\rho) \cosh(v) (\rho \cos(\rho) \sin(v) - v \cos(v) \sin(\rho))}{v^2 - \rho^2} & \rho \neq v \\ \frac{\rho \cosh^2(\rho) - \sin(\rho) \cos(\rho) \cosh^2(\rho)}{\rho} & \rho = v \end{cases} \\
I_9(\rho, v, \xi) &= \cosh(\rho) \cosh(v) \cosh(\xi) \int_{-1}^1 \cos(\rho v) \cos(vv) \cos(\xi v) dv \\
&= \frac{1}{2} \left[\frac{\sin(\rho - v - \xi)}{\rho - v - \xi} + \frac{\sin(\rho + v - \xi)}{\rho + v - \xi} + \frac{\sin(\rho - v + \xi)}{\rho - v + \xi} + \frac{\sin(\rho + v + \xi)}{\rho + v + \xi} \right] \\
&\quad \times \cosh(\rho) \cosh(v) \cosh(\xi) \\
I_{10}(\rho, v, \xi) &= \cos(\rho) \cosh(v) \cosh(\xi) \int_{-1}^1 \cosh(\rho v) \cos(vv) \cos(\xi v) dv \\
&= -\frac{2 \cosh(\rho) \cosh(v) \cosh(\xi)}{(\rho^2 + (v - \xi)^2)(\rho^2 + (v + \xi)^2)} \\
&\quad \times \left[2\rho v \xi \sin(\rho) \sin(v) \sin(\xi) + \rho \sin(\rho) \cos(v) \cos(\xi) (\rho^2 + v^2 + \xi^2) \right. \\
&\quad \left. - v \cos(\rho) \sin(v) \cos(\xi) (\rho^2 + v^2 - \xi^2) - \xi \cos(\rho) \cos(v) \sin(\xi) (\rho^2 - v^2 + \xi^2) \right]
\end{aligned}$$

$$\begin{aligned}
I_{11}(\rho, \nu, \xi) &= \cos(\rho) \cos(\nu) \cosh(\xi) \int_{-1}^1 \cosh(\rho\nu) \cosh(\nu\nu) \cos(\xi\nu) d\nu \\
&= -\frac{2 \cosh(\rho) \cosh(\nu) \cosh(\xi)}{(\rho^2 - \nu^2)^2 + \xi^2(2\rho^2 + 2\nu^2 + \xi^2)} \\
&\quad \times \left[2\rho\nu\xi \sin(\rho) \sin(\nu) \sin(\xi) + \rho \sin(\rho) \cos(\nu) \cos(\xi) (\rho^2 - \nu^2 + \xi^2) \right. \\
&\quad \left. + \nu \cos(\rho) \sin(\nu) \cos(\xi) (-\rho^2 + \nu^2 + \xi^2) - \xi \cos(\rho) \cos(\nu) \sin(\xi) (\rho^2 + \nu^2 + \xi^2) \right]
\end{aligned}$$

$$\begin{aligned}
I_{12}(\rho, \nu, \xi) &= \cos(\rho) \cos(\nu) \cos(\xi) \int_{-1}^1 \cosh(\rho\nu) \cosh(\nu\nu) \cosh(\xi\nu) d\nu \\
&= \frac{1}{2} \left[\frac{\sinh(\rho - \nu - \xi)}{\rho - \nu - \xi} + \frac{\sinh(\rho + \nu - \xi)}{\rho + \nu - \xi} + \frac{\sinh(\rho - \nu + \xi)}{\rho - \nu + \xi} + \frac{\sinh(\rho + \nu + \xi)}{\rho + \nu + \xi} \right] \\
&\quad \times \cos(\rho) \cos(\nu) \cos(\xi)
\end{aligned}$$

$$\begin{aligned}
I_{13}(\rho, \nu, \xi) &= \cosh(\rho) \cosh(\nu) \cosh(\xi) \int_{-1}^1 \cos(\rho\nu) \sin(\nu\nu) \sin(\xi\nu) d\nu \\
&= \frac{1}{2} \left[-\frac{\sin(\rho - \nu - \xi)}{\rho - \nu - \xi} + \frac{\sin(\rho + \nu - \xi)}{\rho + \nu - \xi} + \frac{\sin(\rho - \nu + \xi)}{\rho - \nu + \xi} - \frac{\sin(\rho + \nu + \xi)}{\rho + \nu + \xi} \right] \\
&\quad \times \cosh(\rho) \cosh(\nu) \cosh(\xi)
\end{aligned}$$

$$\begin{aligned}
I_{14}(\rho, \nu, \xi) &= \cos(\rho) \cosh(\nu) \cosh(\xi) \int_{-1}^1 \cosh(\rho\nu) \sin(\nu\nu) \sin(\xi\nu) d\nu \\
&= -\frac{2 \cosh(\rho) \cosh(\nu) \cosh(\xi)}{(\rho^2 + (\nu - \xi)^2)(\rho^2 + (\nu + \xi)^2)} \\
&\quad \times \left[2\rho\nu\xi \sin(\rho) \cos(\nu) \cos(\xi) + \rho \sin(\rho) \sin(\nu) \sin(\xi) (\rho^2 + \nu^2 + \xi^2) \right. \\
&\quad \left. + \nu \cos(\rho) \cos(\nu) \sin(\xi) (\rho^2 + \nu^2 - \xi^2) + \xi \cos(\rho) \sin(\nu) \cos(\xi) (\rho^2 - \nu^2 + \xi^2) \right]
\end{aligned}$$

$$\begin{aligned}
I_{15}(\rho, \nu, \xi) &= \cosh(\rho) \cos(\nu) \cosh(\xi) \int_{-1}^1 \cos(\rho\nu) \sinh(\nu\nu) \sin(\xi\nu) d\nu \\
&= -\frac{2 \cosh(\rho) \cosh(\nu) \cosh(\xi)}{(\rho^2 + \nu^2)^2 + \xi^2(-2\rho^2 + 2\nu^2 + \xi^2)} \\
&\quad \times \left[2\rho\nu\xi \sin(\rho) \cos(\nu) \cos(\xi) + \rho \sin(\rho) \sin(\nu) \sin(\xi) (\rho^2 + \nu^2 - \xi^2) \right. \\
&\quad \left. - \nu \cos(\rho) \cos(\nu) \sin(\xi) (\rho^2 + \nu^2 + \xi^2) - \xi \cos(\rho) \sin(\nu) \cos(\xi) (-\rho^2 + \nu^2 + \xi^2) \right]
\end{aligned}$$

$$\begin{aligned}
I_{16}(\rho, \nu, \xi) &= \cos(\rho) \cos(\nu) \cosh(\xi) \int_{-1}^1 \cosh(\rho\nu) \sinh(\nu\nu) \sin(\xi\nu) d\nu \\
&= -\frac{2 \cosh(\rho) \cosh(\nu) \cosh(\xi)}{(\rho^2 - \nu^2)^2 + \xi^2(2\rho^2 + 2\nu^2 + \xi^2)} \\
&\quad \times \left[2\rho\nu\xi \sin(\rho) \cos(\nu) \cos(\xi) - \rho \sin(\rho) \sin(\nu) \sin(\xi) (\rho^2 - \nu^2 + \xi^2) \right. \\
&\quad \left. - \nu \cos(\rho) \cos(\nu) \sin(\xi) (-\rho^2 + \nu^2 + \xi^2) - \xi \cos(\rho) \sin(\nu) \cos(\xi) (\rho^2 + \nu^2 + \xi^2) \right]
\end{aligned}$$

$$\begin{aligned}
I_{17}(\rho, \nu, \xi) &= \cosh(\rho) \cos(\nu) \cos(\xi) \int_{-1}^1 \cos(\rho\nu) \sinh(\nu\nu) \sinh(\xi\nu) d\nu \\
&= -\frac{2 \cosh(\rho) \cosh(\nu) \cosh(\xi)}{(\rho^2 + (\nu - \xi)^2)(\rho^2 + (\nu + \xi)^2)} \\
&\times \left[2\rho\nu\xi \sin(\rho) \cos(\nu) \cos(\xi) - \rho \sin(\rho) \sin(\nu) \sin(\xi) (\rho^2 + \nu^2 + \xi^2) \right. \\
&\quad \left. + \nu \cos(\rho) \cos(\nu) \sin(\xi) (\rho^2 + \nu^2 - \xi^2) + \xi \cos(\rho) \sin(\nu) \cos(\xi) (\rho^2 - \nu^2 + \xi^2) \right]
\end{aligned}$$

$$\begin{aligned}
I_{18}(\rho, \nu, \xi) &= \cos(\rho) \cos(\nu) \cos(\xi) \int_{-1}^1 \cosh(\rho\nu) \sinh(\nu\nu) \sinh(\xi\nu) d\nu \\
&= \frac{1}{2} \left[\frac{\sinh(\rho - \nu - \xi)}{\rho - \nu - \xi} - \frac{\sinh(\rho + \nu - \xi)}{\rho + \nu - \xi} - \frac{\sinh(\rho - \nu + \xi)}{\rho - \nu + \xi} + \frac{\sinh(\rho + \nu + \xi)}{\rho + \nu + \xi} \right] \\
&\times \cos(\rho) \cos(\nu) \cos(\xi)
\end{aligned}$$

$$I_G(\rho, \nu) = I_3(\rho, \nu) - I_2(\rho, \nu) + I_2(\nu, \rho) - I_1(\rho, \nu)$$

$$I_U(\rho, \nu) = I_6(\rho, \nu) + I_7(\rho, \nu) + I_7(\nu, \rho) + I_8(\rho, \nu)$$

$$\begin{aligned}
I_{M1}(\rho, \nu, \xi) &= I_9(\rho, \nu, \xi) - I_{10}(\rho, \nu, \xi) + I_{10}(\nu, \rho, \xi) - I_{10}(\xi, \rho, \nu) \\
&\quad - I_{11}(\rho, \nu, \xi) + I_{11}(\rho, \xi, \nu) - I_{11}(\nu, \xi, \rho) + I_{12}(\rho, \nu, \xi)
\end{aligned}$$

$$\begin{aligned}
I_{M2}(\rho, \nu, \xi) &= -I_{13}(\rho, \nu, \xi) + I_{14}(\rho, \nu, \xi) - I_{15}(\rho, \nu, \xi) - I_{15}(\rho, \xi, \nu) \\
&\quad + I_{16}(\rho, \nu, \xi) + I_{16}(\rho, \xi, \nu) - I_{17}(\rho, \nu, \xi) + I_{18}(\rho, \nu, \xi)
\end{aligned}$$

$$I_{M3}(\rho, \nu, \xi) = I_9(\rho, \nu, \xi) - I_{10}(\rho, \nu, \xi) - I_{10}(\nu, \rho, \xi) + I_{11}(\rho, \nu, \xi)$$

$$I_{M4}(\rho, \nu, \xi) = I_9(\rho, \nu, \xi) - I_{10}(\rho, \nu, \xi) + I_{10}(\nu, \rho, \xi) - I_{11}(\rho, \nu, \xi)$$

$$I_{M5}(\rho, \nu, \xi) = I_{13}(\rho, \nu, \xi) - I_{14}(\rho, \nu, \xi) + I_{15}(\rho, \nu, \xi) - I_{16}(\rho, \nu, \xi)$$

$$\begin{aligned}
I_M(\rho_1, \nu_1, \xi_1, \rho_2, \nu_2, \xi_2) &= \nu_1^2 \nu_2^2 I_{M1}(\rho_1, \nu_1, \xi_1) I_{M1}(\rho_2, \nu_2, \xi_2) \\
&\quad - 2\nu_1 \nu_2 \xi_1 \xi_2 I_{M2}(\rho_1, \nu_1, \xi_1) I_{M2}(\rho_2, \nu_2, \xi_2) \\
&\quad + \xi_1^2 \xi_2^2 I_{M1}(\rho_1, \xi_1, \nu_1) I_{M1}(\rho_2, \xi_2, \nu_2)
\end{aligned}$$

$$\begin{aligned}
I_B(\rho_1, \nu_1, \rho_2, \nu_2, k) &= \frac{1}{2k \cosh(\pi k) \cosh(\pi/2)} \\
&\times \left\{ 4k^2 \nu_2^2 I_{M3}(\rho_1, \nu_1, \pi k) I_{M4}\left(\rho_2, \nu_2, \frac{\pi}{2}\right) \right. \\
&\quad + \nu_1^2 I_{M4}(\rho_1, \nu_1, \pi k) I_{M3}\left(\rho_2, \nu_2, \frac{\pi}{2}\right) \\
&\quad - (-1)^k \nu_1^2 I_G(\rho_1, \nu_1) I_{M3}\left(\rho_2, \nu_2, \frac{\pi}{2}\right) \cosh(\pi k) \\
&\quad \left. - 4k \nu_1 \nu_2 I_{M5}(\rho_1, \nu_1, \pi k) I_{M5}\left(\rho_2, \nu_2, \frac{\pi}{2}\right) \right\}
\end{aligned}$$

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