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Investigating Oscillations in Higher-Order Half-Linear Dynamic Equations on Time Scales

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Abstract: This study presents novel and generalizable sufficient conditions for determining the oscillatory behavior of solutions to higher-order half-linear neutral delay dynamic equations on time scales. Utilizing the Riccati transformation technique in combination with Taylor monomials, we derive new and comprehensive oscillation criteria that cover a wide range of cases, including super-linear, half-linear, and sublinear equations. These results extend and improve upon existing oscillation criteria found in the literature by introducing more general conditions and providing a broader applicability to different types of dynamic equations. Furthermore, the study highlights the role of symmetry in the underlying equations, demonstrating how symmetry properties can be leveraged to simplify the analysis and provide additional insights into oscillatory behavior. To demonstrate the practical relevance of our findings, we include illustrative examples that show how these new criteria, along with symmetry-based perspectives, can be effectively applied to various time scales.

Keywords: higher order; nonlinear dynamic equations; oscillation; Riccati transformation



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1. Introduction

Time scales theory has attracted considerable interest since Hilger introduced it in his landmark paper [1]. This framework offers a unified approach to unifying, extending, and generalizing concepts from discrete, continuous, and quantum calculus to arbitrary time domains, fostering a rapidly expanding field of research.

The study of the oscillation of dynamic differential equations with time scales is great importance in understanding and analyzing continuous and discrete systems of various natural-life phenomena. This in turn enables us to unify the modeling of processes that evolve on continuous and discrete times and to bridge the gap between differential equations and difference equations. For example, these equations are used in modeling the cyclical behavior of economic systems using time scale calculation and in analyzing the oscillation and stability of the economic market; for more information, please see [2–4].

The $\mathbb{T}$ time scale is defined as a closed nonempty subset of $\mathbb{R}$ that inherits the standard topology. The intervals on a time scale, indicated with a subscript $\mathbb{T}$, show where the usual interval and $\mathbb{T}$ intersect. Specifically, when $\mathbb{T}$ equals $\mathbb{R}$ or $\mathbb{Z}$, it matches the established differential equations and difference equations theories, respectively.

For any $\zeta \in \mathbb{T}$, we define the forward and backward jump operators $\sigma : \mathbb{T} \rightarrow \mathbb{T}$, $\rho : \mathbb{T} \rightarrow \mathbb{T}$, respectively, as follows:

$$\sigma(\zeta) := \inf\{s \in \mathbb{T} : s > \zeta\}, \quad \text{and} \quad \rho(\zeta) := \sup\{s \in \mathbb{T} : s < \zeta\} \quad \text{for } \zeta \in \mathbb{T}.$$

A point $\zeta \in \mathbb{T}$ is classified as right-scattered if $\sigma(\zeta) > \zeta$, right-dense if $\sigma(\zeta) = \zeta$, left-scattered if $\rho(\zeta) < \zeta$, left-dense if $\rho(\zeta) = \zeta$, and dense if $\rho(\zeta) = \zeta = \sigma(\zeta)$.

The $\mathbb{T}$ time scale has a μ graininess function which is given by $\mu(\zeta) = \sigma(\zeta) - \zeta$ and the function $\chi^\sigma(\zeta)$ indicates $\chi(\sigma(\zeta))$ for any $\chi : \mathbb{T} \rightarrow \mathbb{R}$. We also define the rd-continuous function $k : \mathbb{T} \rightarrow \mathbb{R}$ as a function with right-dense continuous points and that has finite left-sided limits at left-dense points. Furthermore, the set $C_{rd}(\mathbb{T}, \mathbb{R})$ denotes the all rd-continuous functions and $C_{rd}^1(\mathbb{T}, \mathbb{R})$ is the set of all differential rd-continuous functions where $\chi : \mathbb{T} \rightarrow \mathbb{R}$.

The delta derivative of a function $\chi : \mathbb{T} \rightarrow \mathbb{R}$ is defined as

$$\chi^\Delta(\zeta) := \begin{cases} \frac{\chi^\sigma(\zeta) - \chi(\zeta)}{\mu(\zeta)}, & \mu(\zeta) > 0 \\ \lim_{s \rightarrow \zeta} \frac{\chi(\zeta) - \chi(s)}{\zeta - s}, & \mu(\zeta) = 0 \end{cases} \quad \text{for } \zeta \in \mathbb{T}.$$

The product rule for delta derivatives is

$$(\chi\psi)^\Delta(\zeta) = \chi^\Delta(\zeta)\psi(\zeta) + \chi^\sigma(\zeta)\psi^\Delta(\zeta) \quad \text{for } \zeta \in \mathbb{T}.$$

The delta derivative of the quotient χ/k is given by

$$\left(\frac{\chi}{\psi}\right)^\Delta(\zeta) = \frac{\psi(\zeta)\chi^\Delta(\zeta) - \chi(\zeta)\psi^\Delta(\zeta)}{\psi^\sigma(\zeta)\psi(\zeta)} \quad \text{for } \zeta \in \mathbb{T}.$$

If a function Ξ satisfies $\Xi^\Delta = \chi$ on $\mathbb{T}^\kappa$, it is called an antiderivative of χ . The integral of χ is defined as

$$\int_s^\zeta \chi(\tau) \Delta\tau = \Xi(\zeta) - \Xi(s), \quad \text{where } s, \zeta \in \mathbb{T}.$$

The past two decades have witnessed substantial research efforts dedicated to the study of the oscillatory and nonoscillatory behavior of solutions to dynamic equations on time scales; see [5–12]. Several papers are devoted to the study of oscillatory behavior in the case of second-order dynamic equations. While some studies have addressed the oscillation of solutions of higher-order equations, they primarily focus on the half-linear dynamic equation without a neutral term. For example, in [13], the authors explore oscillatory behavior under these conditions. Similarly, Grace et al., in [14], investigated the oscillation of higher-order nonlinear dynamic equations with a negative nonlinear neutral term. Their findings were derived using an integral criterion and a comparison theorem.

In contrast, the current work examines the oscillatory behavior of solutions to higher-order dynamic equations with a neutral term. By employing Riccati substitution, this study considers three distinct cases—superlinear, half-linear, and sublinear forms—to address gaps in the existing literature.

Although higher-order dynamic equations on time scales have garnered significant attention in recent research, many critical aspects remain unresolved within the broader theory of dynamic equations on time scales. In this paper, we focus on specific higher-order half-linear neutral delay dynamic equations of the following form:

$$\left[r(\zeta) \Phi_\alpha \left([x(\zeta) + p(\zeta)x(\tau(\zeta))]^{\Delta^{n-1}} \right) \right]^\Delta + q(\zeta) \Phi_\beta(x(\delta(\zeta))) = 0 \quad \text{for } \zeta \in [\zeta_0, \infty)_{\mathbb{T}}, \quad (1)$$

where $\Phi_\gamma(v) := \text{sgn}(v)|v|^\gamma$ for $v \in \mathbb{R}$ and $\gamma \in \mathbb{R}^+$, $\alpha, \beta \in \mathbb{R}^+$, $n \in \mathbb{N}$ with n even, and $n \geq 2$.

For any arbitrary time scale $\mathbb{T}$ with $\sup \mathbb{T} = \infty$, and a fixed point ζ_0 in $\mathbb{T}$, we assume the following conditions:

- (H1) $r \in C_{rd}^1([\zeta_0, \infty)_{\mathbb{T}}, (0, \infty))$ such that $r^\Delta(\zeta) \geq 0$ for all $\zeta \in [\zeta_0, \infty)_{\mathbb{T}}$ with $\int_{\zeta_0}^\infty \frac{\Delta \zeta}{r^{\frac{1}{\alpha}}(\zeta)} = \infty$.
- (H2) $p, q \in C_{rd}([\zeta_0, \infty)_{\mathbb{T}}, \mathbb{R})$ such that $0 \leq p(\zeta) < 1$ and $q(\zeta) \geq 0$ for all $\zeta \in [\zeta_0, \infty)_{\mathbb{T}}$.
- (H3) $\tau \in C_{rd}([\zeta_0, \infty)_{\mathbb{T}}, \mathbb{T})$ such that $\tau(\zeta) \leq \zeta$ for all $\zeta \in [\zeta_0, \infty)_{\mathbb{T}}$ with $\lim_{\zeta \rightarrow \infty} \tau(\zeta) = \infty$.
- (H4) $\delta \in C_{rd}^1([\zeta_0, \infty)_{\mathbb{T}}, \mathbb{T})$ such that $\delta(\zeta) \leq \zeta$ and $\delta^\Delta(\zeta) > 0$ for all $\zeta \in [\zeta_0, \infty)_{\mathbb{T}}$, with $\delta([\zeta_0, \infty)_{\mathbb{T}}) = [\delta(\zeta_0), \infty)_{\mathbb{T}}$ and $\lim_{\zeta \rightarrow \infty} \delta(\zeta) = \infty$.

A solution of (1) is a nontrivial function $x \in C_{rd}([s_x, \infty)_{\mathbb{T}}, \mathbb{R})$, where $\zeta_x \in [\zeta_0, \infty)_{\mathbb{T}}$ and $s_x := \min\{\inf_{\zeta \in [\zeta_x, \infty)_{\mathbb{T}}} \{\tau(\zeta)\}, \inf_{\zeta \in [\zeta_x, \infty)_{\mathbb{T}}} \{\delta(\zeta)\}\}$. This function has the property that $r\Phi_\alpha([x + p \cdot x \circ \tau]^{\Delta^{n-1}}) \in C_{rd}^1([\zeta_x, \infty)_{\mathbb{T}}, \mathbb{R})$ and satisfies (1) identically on $[\zeta_x, \infty)_{\mathbb{T}}$. A solution x of Equation (1) is oscillatory if it changes sign infinitely often. Otherwise, it is nonoscillatory. Equation (1) is oscillatory if all its solutions are oscillatory.

In the following, we provide background details that motivate our study in this paper. Grace et al. [15] examined the oscillatory behavior of (1) in the specific case of $n = 2$, $p(\zeta) \equiv 0$, and $\delta(\zeta) = \zeta$; i.e.,

$$\left(r\Phi_\alpha(x^\Delta) \right)^\Delta(\zeta) + q(\zeta)x^\beta(\zeta) = 0 \quad \text{for } \zeta \in [\zeta_0, \infty)_{\mathbb{T}}, \quad (2)$$

where α and β are quotients of positive odd integers, and r and q are real-valued, positive, rd-continuous functions. They discussed all possible cases of $\beta < \alpha$, $\beta = \alpha$, and $\beta > \alpha$. They considered

$$\int_{\zeta_0}^\infty \frac{\Delta \zeta}{r^{\frac{1}{\alpha}}(\zeta)} = \infty$$

and

$$0 < Q(\zeta) := \int_{\zeta}^\infty q(\xi) \Delta \xi < \infty \quad \text{for } \zeta \in [\zeta_0, \infty)_{\mathbb{T}},$$

and determined sufficient criteria for oscillation of all solutions:

- $\limsup_{\zeta \rightarrow \infty} \int_s^\zeta \frac{1}{r^{\frac{1}{\alpha}}(\xi)} \left(Q^\sigma(\xi) + \varepsilon \int_{\sigma(\xi)}^\infty \frac{Q^{1+\frac{1}{\alpha}}(\sigma(\eta))}{r^{\frac{1}{\alpha}}(\eta)} \Delta \eta \right)^{\frac{1}{\alpha}} \Delta \xi = \infty$ for all large $s \in [\zeta_0, \infty)_{\mathbb{T}}$ and small $\varepsilon \in \mathbb{R}^+$ if $\beta > \alpha$.
- $\limsup_{\zeta \rightarrow \infty} \left(\int_s^\zeta \frac{\Delta \xi}{r^{\frac{1}{\alpha}}(\xi)} \right) \left(Q(\zeta) + \alpha \int_\zeta^\infty \frac{Q^{1+\frac{1}{\alpha}}(\sigma(\xi))}{r^{\frac{1}{\alpha}}(\xi)} \Delta \xi \right)^{\frac{1}{\alpha}} > 1$ for all large $s \in [\zeta_0, \infty)_{\mathbb{T}}$ if $\beta = \alpha$.
- $\limsup_{\zeta \rightarrow \infty} Q^{\frac{\alpha-\beta}{\beta\alpha}}(\zeta) \left(\int_s^\zeta \frac{\Delta \xi}{r^{\frac{1}{\alpha}}(\xi)} \right) \left(Q(\zeta) + \varepsilon \int_\zeta^\infty \frac{Q^{1+\frac{1}{\alpha}}(\sigma(\xi))}{r^{\frac{1}{\alpha}}(\xi)} Q^{\frac{\alpha-\beta}{\beta\alpha}}(\xi) \Delta \xi \right)^{\frac{1}{\alpha}} = \infty$ for all large $s \in [\zeta_0, \infty)_{\mathbb{T}}$ and small $\varepsilon \in \mathbb{R}^+$ if $\beta < \alpha$.

Yang et al. [16] considered another specific case, investigating the oscillation criteria of second-order quasi-linear dynamic equations of the form

$$\left[r(\zeta) \Phi_\alpha([x(\zeta) + p(\zeta)x(\tau(\zeta))]^\Delta) \right]^\Delta + q(\zeta)x^\beta(\delta(\zeta)) = 0 \quad \text{for } \zeta \in [\zeta_0, \infty)_{\mathbb{T}},$$

where $\int_{\zeta_0}^\infty \frac{\Delta \zeta}{r^{\frac{1}{\alpha}}(\zeta)} = \infty$, $\tau(\zeta) \leq \zeta$, $\delta(\zeta) \leq \zeta$, $0 \leq p(\zeta) < 1$, and $q(\zeta) \geq 0$ for all $\zeta \in [\zeta_0, \infty)_{\mathbb{T}}$ with $\lim_{\zeta \rightarrow \infty} \tau(\zeta) = \infty$ and $\lim_{\zeta \rightarrow \infty} \delta(\zeta) = \infty$.

In 2013, Grace [13] established new oscillation criteria for the higher-order dynamic equation

$$\left(r\Phi_\alpha(x^{\Delta^{n-1}})\right)^\Delta(\zeta) + q(\zeta)x^\alpha(\zeta) = 0 \quad \text{for } \zeta \in [\zeta_0, \infty)_{\mathbb{T}},$$

where α is the ratio of positive odd integers, and r and q are real-valued rd-continuous functions defined on $\mathbb{T}$.

On the other hand, many researchers have been interested in the oscillation of solutions of differential equations of different orders and classifications, especially functional ones, due to their importance in modeling systems that contain different times; see [17–22]. In refs. [23,24], they studied the oscillation of fractional differential equations and developed the well-known classical criteria. We also find some studies interested in partial differential equations, such as [25].

This motivates us to study the oscillatory behavior of Equation (1) for $\beta > \alpha$, $\beta = \alpha$, and $\beta < \alpha$, improving and extending the results of [15] and [16] using the Riccati technique.

The results in this paper extend some of the findings in [6,11,26–30]. The obtained results provide a unified approach for investigating the oscillation behavior of n th-order quasi-linear neutral delay differential and difference equations.

In what follows, we introduce some theorems that we will use in the investigation of our main results.

Theorem 1 ([3], Theorem 1.93). *Suppose that there exists a strictly increasing function $w : \mathbb{T} \rightarrow \mathbb{R}$ and the time scale $\tilde{\mathbb{T}} := w(\mathbb{T})$. Also, let the function $S : \tilde{\mathbb{T}} \rightarrow \mathbb{R}$. If there is $S^\Delta(w(\zeta))$ and $w^\Delta(\zeta)$ for $\zeta \in \mathbb{T}^\kappa$, then*

$$(S \circ w)^\Delta(\zeta) = S^\Delta(w(\zeta))w^\Delta(\zeta).$$

Theorem 2 ([31], Theorem 5). *Assume that $F \in C_{rd}^n(\mathbb{T}, \mathbb{R})$ is positive or negative with $\sup \mathbb{T} = \infty$ and $n \in \mathbb{N}$, and F^{Δ^n} is non-positive or non-negative, and not identically zero on $[\zeta_0, \infty)_{\mathbb{T}}$ for some $\zeta_0 \in \mathbb{T}$. Then, there is $\zeta_1 \in [\zeta_0, \infty)_{\mathbb{T}}$, $m \in [0, n]_{\mathbb{Z}}$ such that $(-1)^{n-m}F(\zeta)F^{\Delta^n}(\zeta) \geq 0$ for all $\zeta \in [\zeta_0, \infty)_{\mathbb{T}}$, with*

- (i) $F(\zeta)F^{\Delta^i}(\zeta) > 0$ for all $\zeta \in [\zeta_0, \infty)_{\mathbb{T}}$ and all $i \in [0, m]_{\mathbb{Z}}$;
- (ii) $(-1)^{m+i}F(\zeta)F^{\Delta^i}(\zeta) \geq 0$ for all $\zeta \in [\zeta_0, \infty)_{\mathbb{T}}$ and all $i \in [m, n]_{\mathbb{Z}}$.

Lemma 1 ([32], Lemma 2.8). *Let $n \in \mathbb{N}$, with $n \geq 2$, $\sup \mathbb{T} = \infty$, and $F \in C_{rd}^n(\mathbb{T}, \mathbb{R}_0^+)$, with $F^{\Delta^n} \leq 0$ on $[\zeta_0, \infty)_{\mathbb{T}}$. Let Theorem 2 hold with $m \in [1, n]_{\mathbb{N}}$. Then, there exists a sufficiently large $s \in \mathbb{T}$ such that*

$$F^\Delta(\zeta) \geq h_{m-1}(\zeta, s)F^{\Delta^m}(\zeta) \quad \text{for all } \zeta \in [s, \infty)_{\mathbb{T}}.$$

Lemma 2 ([33], Lemma 2.7). *Let $n \in \mathbb{N}$, $\sup \mathbb{T} = \infty$ and $F \in C_{rd}^n([\zeta_0, \infty), \mathbb{R}_0^+)$, with $F^{\Delta^n} \leq 0$ on $[\zeta_0, \infty)_{\mathbb{T}}$. Let Theorem 2 hold with $m \in [0, n]_{\mathbb{N}}$. Then, there exists a sufficiently large $s \in \mathbb{T}$ such that*

$$F(\zeta) \geq h_m(\zeta, s)F^{\Delta^m}(\zeta) \quad \text{for all } \zeta \in [s, \infty)_{\mathbb{T}}.$$

2. Main Results

Firstly, we will introduce some fundamental lemmas.

Lemma 3. *Let (H1)–(H4) hold. Assume that q is eventually not identically equal to zero. If x is a nonoscillatory eventually positive solution of (1), then*

$$z^\Delta(\zeta) > 0, \quad z^{\Delta^{n-1}}(\zeta) > 0 \quad \text{and} \quad z^{\Delta^n}(\zeta) \leq 0 \quad \text{for } \zeta \in [\zeta_0, \infty)_{\mathbb{T}},$$

where

$$z(\zeta) := x(\zeta) + p(\zeta)x(\tau(\zeta)) \quad \text{for } \zeta \in [\zeta_0, \infty)_{\mathbb{T}}. \quad (3)$$

Proof. Since x is an eventually positive solution of (1), then by (H3) and (H4) there exists $\zeta_1 \in [\zeta_0, \infty)_{\mathbb{T}}$ such that $x(\zeta) > 0$, $x(\delta(\zeta)) > 0$ and $x(\tau(\zeta)) > 0$ for all $\zeta \in [\zeta_1, \infty)_{\mathbb{T}}$. From (1) and (H2), we obtain $z(\zeta) \geq x(\zeta) > 0$ for all $\zeta \in [\zeta_1, \infty)_{\mathbb{T}}$. Consequently,

$$\left[r(\zeta) \Phi_{\alpha}(z^{\Delta^{n-1}}(\zeta)) \right]^{\Delta} = -q(\zeta)x^{\beta}(\delta(\zeta)) \leq 0 \quad \text{for all } \zeta \in [\zeta_1, \infty)_{\mathbb{T}}. \quad (4)$$

Hence, $r\Phi_{\alpha}(z^{\Delta^{n-1}})$ is monotone and of one sign eventually. We claim that $z^{\Delta^{n-1}}(\zeta) > 0$ for all $\zeta \in [\zeta_1, \infty)_{\mathbb{T}}$. If not, then there is $\zeta_2 \in [\zeta_1, \infty)_{\mathbb{T}}$ such that $z^{\Delta^{n-1}}(\zeta) \leq 0$ for all $\zeta \in [\zeta_2, \infty)_{\mathbb{T}}$. Since q is not eventually identically zero, we can assume that $z^{\Delta^{n-1}}(\zeta) < 0$ for all $\zeta \in [\zeta_2, \infty)_{\mathbb{T}}$. From (4), we have

$$r(\zeta)\Phi_{\alpha}(z^{\Delta^{n-1}}(\zeta)) \leq -c < 0 \quad \text{for all } \zeta \in [\zeta_2, \infty)_{\mathbb{T}},$$

where $c := -r(\zeta_2)\Phi_{\alpha}(z^{\Delta^{n-1}}(\zeta_2)) > 0$, then

$$z^{\Delta^{n-1}}(\zeta) \leq -\frac{c^{\frac{1}{\alpha}}}{r^{\frac{1}{\alpha}}(\zeta)} \quad \text{for all } \zeta \in [\zeta_2, \infty)_{\mathbb{T}}. \quad (5)$$

Integrating (5) on $[\zeta_2, \zeta] \subset [\zeta_2, \infty)_{\mathbb{T}}$, we obtain

$$z^{\Delta^{n-2}}(\zeta) \leq z^{\Delta^{n-2}}(\zeta_2) - c^{\frac{1}{\alpha}} \int_{\zeta_2}^{\zeta} \frac{\Delta \xi}{r^{\frac{1}{\alpha}}(\xi)} \quad \text{for all } \zeta \in [\zeta_2, \infty)_{\mathbb{T}}.$$

Letting $\zeta \rightarrow \infty$, then it follows from (H1) that $\lim_{\zeta \rightarrow \infty} z^{\Delta^{n-2}}(\zeta) = -\infty$. Consequently, $\lim_{\zeta \rightarrow \infty} z(\zeta) = -\infty$, which leads to a contradiction. Therefore, $z^{\Delta^{n-1}}(\zeta) > 0$ for all $\zeta \in [\zeta_1, \infty)_{\mathbb{T}}$. Now, since

$$\left[r(\zeta)(z^{\Delta^{n-1}}(\zeta))^{\alpha} \right]^{\Delta} = r^{\Delta}(\zeta)(z^{\Delta^{n-1}}(\zeta))^{\alpha} + r^{\sigma}(\zeta) \left[(z^{\Delta^{n-1}}(\zeta))^{\alpha} \right]^{\Delta} \quad \text{for all } \zeta \in [\zeta_1, \infty)_{\mathbb{T}}, \quad (6)$$

then, applying the Pötzsche chain rule ([3], Theorem 1.90), we obtain

$$\begin{aligned} \left[(z^{\Delta^{n-1}}(\zeta))^{\alpha} \right]^{\Delta} &= \left\{ \alpha \int_0^1 [z^{\Delta^{n-1}}(\zeta) + h\mu z^{\Delta^n}(\zeta)]^{\alpha-1} dh \right\} z^{\Delta^n}(\zeta) \\ &\geq \alpha \begin{cases} (z^{\Delta^{n-1}}(\zeta))^{\alpha-1}, & \alpha \in (1, \infty)_{\mathbb{R}} \\ (z^{\Delta^{n-1}\sigma}(\zeta))^{\alpha-1}, & \alpha \in (0, 1]_{\mathbb{R}} \end{cases} z^{\Delta^n}(\zeta) \end{aligned} \quad (7)$$

for all $\zeta \in [\zeta_1, \infty)_{\mathbb{T}}$. By (4), (6) and (7), we obtain

$$r^{\Delta}(\zeta)(z^{\Delta^{n-1}}(\zeta))^{\alpha} + \alpha r^{\sigma}(\zeta) \begin{cases} (z^{\Delta^{n-1}}(\zeta))^{\alpha-1}, & \alpha \in (1, \infty)_{\mathbb{R}} \\ (z^{\Delta^{n-1}\sigma}(\zeta))^{\alpha-1}, & \alpha \in (0, 1]_{\mathbb{R}} \end{cases} z^{\Delta^n}(\zeta) \leq 0 \quad \text{for all } \zeta \in [\zeta_1, \infty)_{\mathbb{T}},$$

which yields $z^{\Delta^n}(\zeta) \leq 0$ for all $\zeta \in [\zeta_1, \infty)_{\mathbb{T}}$. Then, by Theorem 2 and Lemma 1, we obtain

$$z^{\Delta}(\zeta) > 0, \quad z^{\Delta^{n-1}}(\zeta) > 0 \quad \text{and} \quad z^{\Delta^n}(\zeta) \leq 0 \quad \text{for all } \zeta \in [\zeta_1, \infty)_{\mathbb{T}},$$

which completes the proof. $\square$

The following result extends [16] (Theorem 2.1).

Theorem 3. Assume that

$$\int_{\zeta}^{\infty} q(\zeta)[1 - p(\delta(\zeta))]^{\beta} \Delta \zeta = \infty. \quad (8)$$

Then, every solution of (1) is oscillatory.

Proof. Suppose Equation (1) has a nonoscillatory solution x on $[\tau_0, \infty)$. To simplify, assume that $x(\zeta), x(\tau(\zeta)), x(\delta(\zeta)) > 0$ for all $\zeta \in [\zeta_1, \infty)$, where $\zeta_1 \in [\zeta_0, \infty)$. Then, z is defined by (3) and satisfies $z(\zeta) > 0$ for all $\zeta \in [\zeta_1, \infty)_{\mathbb{T}}$. By Lemma 3, there exists $\zeta_2 \in [\zeta_1, \infty)_{\mathbb{T}}$ such that $z^{\Delta}(\zeta) > 0$, $z^{\Delta^{n-1}}(\zeta) > 0$ and $z^{\Delta^n}(\zeta) \leq 0$ for all $\zeta \in [\zeta_2, \infty)_{\mathbb{T}}$. Thus,

$$x(\zeta) = z(\zeta) - p(\zeta)x(\tau(\zeta)) \geq z(\zeta) - p(\zeta)z(\tau(\zeta)) \quad \text{for all } \zeta \in [\zeta_2, \infty)_{\mathbb{T}}$$

and

$$x^{\beta}(\delta(\zeta)) \geq [1 - p(\delta(\zeta))]^{\beta} z^{\beta}(\delta(\zeta)) \quad \text{for all } \zeta \in [\zeta_2, \infty)_{\mathbb{T}}. \quad (9)$$

Using (4) and (9), we obtain

$$\left[r(\zeta) (z^{\Delta^{n-1}}(\zeta))^{\alpha} \right]^{\Delta} \leq -q(\zeta)[1 - p(\delta(\zeta))]^{\beta} z^{\beta}(\delta(\zeta)) \quad \text{for all } \zeta \in [\zeta_2, \infty)_{\mathbb{T}}.$$

Now, we define

$$\omega(\zeta) := \frac{r(\zeta) (z^{\Delta^{n-1}}(\zeta))^{\alpha}}{z^{\beta}(\delta(\zeta))} \quad \text{for all } \zeta \in [\zeta_2, \infty)_{\mathbb{T}}.$$

Then,

$$\omega^{\Delta}(\zeta) \leq -q(\zeta)[1 - p(\delta(\zeta))]^{\beta} - \frac{[z^{\beta}(\delta(\zeta))]^{\Delta}}{z^{\beta}(\delta(\zeta))} \omega^{\sigma}(\zeta) \quad \text{for all } \zeta \in [\zeta_2, \infty)_{\mathbb{T}}. \quad (10)$$

Recalling the condition (H4) and the increasing fact of z on $[\zeta_2, \infty)_{\mathbb{T}}$, we can use Theorem 1 to obtain

$$\begin{aligned} [z^{\beta}(\delta(\zeta))]^{\Delta} &= \left\{ \beta \int_0^1 [z(\delta(\zeta)) + h\mu z^{\Delta}(\delta(\zeta))]^{\beta-1} dh \right\} [z(\delta(\zeta))]^{\Delta} \\ &\geq \beta \left\{ \begin{array}{ll} z^{\beta-1}(\delta(\zeta)), & \beta \in (1, \infty)_{\mathbb{R}} \\ z^{\beta-1}(\delta^{\sigma}(\zeta)), & \beta \in (0, 1]_{\mathbb{R}} \end{array} \right\} z^{\Delta}(\delta(\zeta)) \delta^{\Delta}(\zeta) \end{aligned}$$

for all $\zeta \in [\zeta_2, \infty)_{\mathbb{T}}$. Thus, we have

$$\begin{aligned} \frac{[z^{\beta}(\delta(\zeta))]^{\Delta}}{z^{\beta}(\delta(\zeta))} &\geq \beta \left\{ \begin{array}{ll} \frac{z^{\Delta}(\delta(\zeta))}{z(\delta(\zeta))}, & \beta \in (1, \infty)_{\mathbb{R}} \\ \frac{z^{\Delta}(\delta(\zeta))}{z(\delta^{\sigma}(\zeta))} \left(\frac{z(\delta^{\sigma}(\zeta))}{z(\delta(\zeta))} \right)^{\beta}, & \beta \in (0, 1]_{\mathbb{R}} \end{array} \right\} \delta^{\Delta}(\zeta) \\ &\geq \beta \frac{z^{\Delta}(\delta(\zeta))}{z(\delta^{\sigma}(\zeta))} \delta^{\Delta}(\zeta) \end{aligned}$$

for all $[\zeta_2, \infty)_{\mathbb{T}}$, using the increasing fact of z on $[\zeta_2, \infty)_{\mathbb{T}}$ and $\sigma(\zeta) \geq \zeta$. It follows from (10) that

$$\begin{aligned}\omega^\Delta(\zeta) &\leq -q(\zeta)[1 - p(\delta(\zeta))]^\beta - \beta\delta^\Delta(\zeta)\frac{z^\Delta(\delta(\zeta))}{z(\delta^\sigma(\zeta))}\omega^\sigma(\zeta) \\ &< -q(\zeta)[1 - p(\delta(\zeta))]^\beta\end{aligned}\quad (11)$$

for all $[\zeta_2, \infty)_{\mathbb{T}}$. Then, by integration, we obtain

$$\int_{\zeta_2}^{\zeta} q(\xi)[1 - p(\delta(\xi))]^\beta \Delta\xi = \omega(\zeta_1) - \omega(\zeta) < \omega(\zeta_1) < \infty \quad \text{for all } \zeta \in [\zeta_2, \infty)_{\mathbb{T}}.$$

This contradicts (8), and thus the proof is completed. $\square$

2.1. Super Half-Linear Case: $\beta > \alpha$

Now, unlike mentioned in Theorem 3, we consider the case

$$\int_{\zeta}^{\infty} q(\xi)[1 - p(\delta(\xi))]^\beta \Delta\xi < \infty. \quad (12)$$

To simplify notation, we let

$$Q(\zeta) := \int_{\zeta}^{\infty} q(\xi)[1 - p(\delta(\xi))]^\beta \Delta\xi \quad \text{for } \zeta \in [\zeta_0, \infty)_{\mathbb{T}}.$$

Theorem 4. Assume (12) and $\beta > \alpha$. Assume for all large $s \in [\zeta_0, \infty)_{\mathbb{T}}$ and for all small $\varepsilon \in \mathbb{R}^+$ that

$$\int_{\zeta}^{\infty} h_{n-2}(\delta(\xi), s) \frac{\delta^\Delta(\xi)}{r^{\frac{1}{\alpha}}(\delta(\xi))} H_\varepsilon(\sigma(\xi), s) \Delta\xi = \infty, \quad (13)$$

where

$$H_\lambda(\zeta, s) = Q(\zeta) + \lambda \int_{\zeta}^{\infty} h_{n-2}(\delta(\eta), s) \frac{\delta^\Delta(\eta)}{r^{\frac{1}{\alpha}}(\delta(\eta))} Q^{1+\frac{1}{\alpha}}(\sigma(\eta)) \Delta\eta \quad \text{for } s, \zeta \in [\zeta_0, \infty)_{\mathbb{T}} \text{ and } \lambda \in \mathbb{R}. \quad (14)$$

Then, every solution of (1) is oscillatory.

Proof. Suppose that (1) has a nonoscillatory solution x on $[\zeta_0, \infty)$ such that $x(\zeta)$, $x(\tau(\zeta))$, and $x(\delta(\zeta))$ are all positive for every $\zeta \in [\zeta_1, \infty)$, where $\zeta_1 \in [\zeta_0, \infty)$. By Lemma 3, there exists $\zeta_2 \in [\zeta_1, \infty)$ such that $z^\Delta(\zeta) > 0$, $z^{\Delta^{n-1}}(\zeta) > 0$, and $z^{\Delta^n}(\zeta) \leq 0$ for all $\zeta \in [\zeta_2, \infty)$. By Lemma 1, we have

$$z^\Delta(\delta(\zeta)) \geq h_{n-2}(\delta(\zeta), \zeta_3) z^{\Delta^{n-1}}(\delta(\zeta)) \quad \text{for all } \zeta \in [\zeta_3, \infty)_{\mathbb{T}}, \quad (15)$$

where $\zeta_3 \in [\zeta_2, \infty)$. Then, (11) leads to

$$\omega^\Delta(\zeta) \leq -q(\zeta)[1 - p(\delta(\zeta))]^\beta - \beta h_{n-2}(\delta(\zeta), \zeta_3) \delta^\Delta(\zeta) \frac{z^{\Delta^{n-1}}(\delta(\zeta))}{z(\delta^\sigma(\zeta))} \omega^\sigma(\zeta) \quad (16)$$

for all $\zeta \in [\zeta_3, \infty)_{\mathbb{T}}$. But since $r(\zeta)(z^{\Delta^{n-1}}(\zeta))^\alpha \leq 0$ for all $\zeta \in [\zeta_3, \infty)$, then from (H4) it is clear that

$$z^{\Delta^{n-1}}(\delta(\zeta)) \geq \frac{r^{\frac{1}{\alpha}}(\zeta) z^{\Delta^{n-1}}(\zeta)}{r^{\frac{1}{\alpha}}(\delta(\zeta))} \quad \text{for all } \zeta \in [\zeta_3, \infty)_{\mathbb{T}}. \quad (17)$$

Thus, from (16) and (17), we obtain

$$\omega^\Delta(\zeta) \leq -q(\zeta)[1 - p(\delta(\zeta))]^\beta - \beta h_{n-2}(\delta(\zeta), \zeta_3) \delta^\Delta(\zeta) r^{\frac{1}{\alpha}}(\zeta) \frac{z^{\Delta^{n-1}}(\zeta)}{r^{\frac{1}{\alpha}}(\delta(\zeta)) z(\delta^\sigma(\zeta))} \omega^\sigma(\zeta) \quad (18)$$

for all $\zeta \in [\zeta_3, \infty)_{\mathbb{T}}$. Since $[r(\zeta)(z^{\Delta^{n-1}}(\zeta))^\alpha]^\Delta \leq 0$ for all $\zeta \in [\zeta_3, \infty)$ and $\sigma(\zeta) \geq \zeta$, we have

$$r(\zeta)[z^{\Delta^{n-1}}(\zeta)]^\alpha \geq r^\sigma(\zeta)[z^{\Delta^{n-1}\sigma}(\zeta)]^\alpha \quad \text{for all } \zeta \in [\zeta_3, \infty)_{\mathbb{T}}.$$

Therefore, (18) takes the form

$$\omega^\Delta(\zeta) \leq -q(\zeta)[1 - p(\delta(\zeta))]^\beta - \beta h_{n-2}(\delta(\zeta), \zeta_3) \frac{\delta^\Delta(\zeta)}{r^{\frac{1}{\alpha}}(\delta(\zeta))} z^{\frac{\beta}{\alpha}-1}(\delta^\sigma(\zeta)) \omega^{1+\frac{1}{\alpha}}(\sigma(\zeta)) \quad (19)$$

for all $\zeta \in [\zeta_3, \infty)_{\mathbb{T}}$. Integrating (19) on $[\zeta, s)_{\mathbb{T}} \subset [\zeta_3, \infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} \omega(s) - \omega(\zeta) &\leq - \int_{\zeta}^s q(\xi)[1 - p(\delta(\xi))]^\beta \Delta \xi \\ &\quad - \beta \int_{\zeta}^s h_{n-2}(\delta(\xi), \zeta_3) \frac{\delta^\Delta(\xi)}{r^{\frac{1}{\alpha}}(\delta(\xi))} z^{\frac{\beta}{\alpha}-1}(\delta^\sigma(\xi)) \omega^{1+\frac{1}{\alpha}}(\sigma(\xi)) \Delta \xi \end{aligned}$$

for all $s, \zeta \in [\zeta_3, \infty)_{\mathbb{T}}$, with $s \geq \zeta$. Again, since from (11), $\omega^\Delta \leq 0$ on $[\zeta_3, \infty)$, we have $\omega \geq Q$ on $[\zeta_3, \infty)$, then

$$\omega(\zeta) \geq Q(\zeta) + \beta \int_{\zeta}^{\infty} h_{n-2}(\delta(\xi), \zeta_3) \frac{\delta^\Delta(\xi)}{r^{\frac{1}{\alpha}}(\delta(\xi))} Q^{1+\frac{1}{\alpha}}(\sigma(\xi)) z^{\frac{\beta}{\alpha}-1}(\delta^\sigma(\xi)) \Delta \xi \quad (20)$$

for all $\zeta \in [\zeta_3, \infty)_{\mathbb{T}}$. But since z is increasing on $[\zeta_3, \infty)$, hence $\beta > \alpha$, so there exists a $\zeta_4 \in [\zeta_3, \infty)_{\mathbb{T}}$ and a small constant $\varepsilon_0 \in \mathbb{R}^+$ such that $z^{\frac{\beta}{\alpha}-1}(\delta(\zeta)) \geq \varepsilon_0$ for all $\zeta \in [\zeta_4, \infty)_{\mathbb{T}}$. Hence, (20) yields

$$\omega(\zeta) \geq Q(\zeta) + c \int_{\zeta}^{\infty} h_{n-2}(\delta(\xi), \zeta_3) \frac{\delta^\Delta(\xi)}{r^{\frac{1}{\alpha}}(\delta(\xi))} Q^{1+\frac{1}{\alpha}}(\sigma(\xi)) \Delta \xi = H_{\varepsilon_0}(\zeta, \zeta_3) \quad \text{for all } \zeta \in [\zeta_4, \infty)_{\mathbb{T}}, \quad (21)$$

where $\lambda := \beta\varepsilon$. Thus, from the definition of ω and (21), we have

$$\frac{[r(\zeta)(z^{\Delta^{n-1}}(\zeta))^\alpha]^{\frac{1}{\alpha}}}{z^\gamma(\delta(\zeta))} \geq H_{\varepsilon_0}(\zeta, \zeta_3) \quad \text{for all } \zeta \in [\zeta_4, \infty)_{\mathbb{T}}. \quad (22)$$

where $\gamma := \frac{\beta}{\alpha} > 1$. Moreover, since $r(z^{\Delta^{n-1}})^\alpha$ is nonincreasing on $[\zeta_4, \infty)$, then by (22), we have

$$\begin{aligned} \frac{r(\delta(\zeta))(z^{\Delta^{n-1}}(\delta(\zeta)))^\alpha}{z^\beta(\delta^\sigma(\zeta))} &\geq \frac{r^\sigma(\zeta)(z^{\Delta^{n-1}\sigma}(\zeta))^\alpha}{z^\beta(\delta^\sigma(\zeta))} \\ &= \omega^\sigma(\zeta) \geq H_{\varepsilon_0}(\sigma(\zeta), \zeta_3) \end{aligned}$$

for all $\zeta \in [\zeta_4, \infty)_{\mathbb{T}}$, which implies that

$$\frac{z^{\Delta^{n-1}}(\delta(\zeta))}{z^\gamma(\delta^\sigma(\zeta))} \geq \frac{1}{r^{\frac{1}{\alpha}}(\delta(\zeta))} H_{\varepsilon_0}(\sigma(\zeta), \zeta_3) \quad \text{for all } \zeta \in [\zeta_4, \infty)_{\mathbb{T}}. \quad (23)$$

Therefore, by (15), we obtain

$$\frac{z^\Delta(\delta(\zeta))}{z^\gamma(\delta^\sigma(\zeta))} \geq h_{n-2}(\delta(\zeta), \zeta_3) \frac{z^{\Delta^{n-1}}(\delta(\zeta))}{z^\gamma(\delta^\sigma(\zeta))} \quad \text{for all } \zeta \in [\zeta_4, \infty)_{\mathbb{T}}.$$

This, with (23), leads to

$$\frac{z^\Delta(\delta(\zeta))}{z^\gamma(\delta^\sigma(\zeta))} \geq h_{n-2}(\delta(\zeta), \zeta_3) \frac{1}{r^{\frac{1}{\alpha}}(\delta(\zeta))} H_{\varepsilon_0}^\alpha(\sigma(\zeta), \zeta_3) \quad \text{for all } \zeta \in [\zeta_4, \infty)_{\mathbb{T}}. \quad (24)$$

By the Pötzsche chain rule and Theorem 1, we find that

$$\begin{aligned} (z^{1-\gamma}(\delta(\zeta)))^\Delta &= (1-\gamma) \left\{ \int_0^1 [z(\delta(\zeta)) + h\mu[z(\delta(\zeta))]^\Delta]^{-\gamma} dh \right\} z^\Delta(\delta(\zeta)) \delta^\Delta(\zeta) \\ &= (1-\gamma) \left\{ \int_0^1 [(1-h)z(\delta(\zeta)) + h\mu z(\delta^\sigma(\zeta))]^{-\gamma} dh \right\} z^\Delta(\delta(\zeta)) \delta^\Delta(\zeta) \\ &\leq (1-\gamma) z^{-\gamma}(\delta^\sigma(\zeta)) z^\Delta(\delta(\zeta)) \delta^\Delta(\zeta) \end{aligned}$$

for all $\zeta \in [\zeta_3, \infty)_{\mathbb{T}}$. Thus,

$$\frac{(z^{1-\gamma}(\delta(\zeta)))^\Delta}{1-\gamma} \geq \frac{z^\Delta(\delta(\zeta))}{z^\gamma(\delta^\sigma(\zeta))} \delta^\Delta(\zeta) \quad \text{for all } \zeta \in [\zeta_3, \infty)_{\mathbb{T}}.$$

This, with (24), gives

$$\frac{(z^{1-\gamma}(\delta(\zeta)))^\Delta}{1-\gamma} \geq \frac{\delta^\Delta(\zeta)}{r^{\frac{1}{\alpha}}(\delta(\zeta))} h_{n-2}(\delta(\zeta), \zeta_3) H_{\varepsilon_0}(\sigma(\zeta), \zeta_3) \quad \text{for all } \zeta \in [\zeta_3, \infty)_{\mathbb{T}}.$$

Integrating on $[\zeta_3, \zeta]_{\mathbb{T}} \subset [\zeta_3, \infty)_{\mathbb{T}}$, we obtain

$$\int_{\zeta_3}^{\zeta} \frac{(z^{1-\gamma}(\delta(\xi)))^\Delta}{1-\gamma} \Delta \xi \geq \int_{\zeta_3}^{\zeta} h_{n-2}(\delta(\xi), \zeta_3) \frac{\delta^\Delta(\xi)}{r^{\frac{1}{\alpha}}(\delta(\xi))} H_{\varepsilon_0}(\sigma(\xi), \zeta_3) \Delta \xi \quad \text{for all } \zeta \in [\zeta_3, \infty)_{\mathbb{T}}.$$

Letting $\zeta \rightarrow \infty$, we obtain

$$\int_{\zeta_3}^{\zeta} h_{n-2}(\delta(\xi), \zeta_3) \frac{\delta^\Delta(\xi)}{r^{\frac{1}{\alpha}}(\delta(\xi))} H_{\varepsilon_0}(\sigma(\xi), \zeta_3) \Delta \xi \leq \frac{z^{1-\gamma}(\delta(\zeta_3))}{\gamma-1} \quad \text{for all } \zeta \in [\zeta_3, \infty)_{\mathbb{T}}.$$

This contradicts (13), and completes the proof. $\square$

2.2. Half-Linear Case: $\alpha = \beta$

Theorem 5. Assume (12) and $\beta = \alpha$. If

$$\limsup_{t \rightarrow \infty} H_\alpha(\sigma(\zeta), s) \left(h_{n-2}(\delta(\zeta), s) \int_s^{\delta(\zeta)} \frac{\Delta \xi}{r^{\frac{1}{\alpha}}(\delta(\xi))} \right)^\alpha > 1 \quad \text{for all large } s \in [\zeta_0, \infty)_{\mathbb{T}}, \quad (25)$$

where H is defined in (14). Then, every solution of (1) is oscillatory.

Proof. Assume that (1) has a nonoscillatory solution x on $[\zeta_0, \infty)_{\mathbb{T}}$ such that $x(\zeta), x(\tau(\zeta)), x(\delta(\zeta)) > 0$ for all $\zeta \in [\zeta_1, \infty)$, where $\zeta_1 \in [\zeta_0, \infty)_{\mathbb{T}}$. Putting $\beta = \alpha$ in (20), we directly obtain

$$\omega(\zeta) \geq Q(\zeta) + \alpha \int_{\zeta}^{\infty} h_{n-2}(\delta(\xi), \zeta_1) \frac{\delta^\Delta(\xi)}{r^{\frac{1}{\alpha}}(\delta(\xi))} Q^{1+\frac{1}{\alpha}}(\sigma(\xi)) \Delta \xi \quad \text{for all } \zeta \in [\zeta_1, \infty)_{\mathbb{T}}. \quad (26)$$

Moreover, we have

$$\frac{1}{\omega(\zeta)} = \frac{1}{r(\zeta)} \left(\frac{z(\delta(\zeta))}{z^{\Delta^{n-1}}(\zeta)} \right)^\alpha \quad \text{for all } \zeta \in [\zeta_1, \infty)_{\mathbb{T}}.$$

Since $\lim_{\zeta \rightarrow \infty} \delta(\zeta) = \infty$, we can choose $\zeta_2 \in [\zeta_1, \infty)_{\mathbb{T}}$, so that $\delta(\zeta) \geq \zeta_1$ for all $\zeta \in [\zeta_2, \infty)_{\mathbb{T}}$. Hence,

$$\begin{aligned} z^{\Delta^{n-2}}(\delta(\zeta)) &> z^{\Delta^{n-2}}(\delta(\zeta)) - z^{\Delta^{n-2}}(\zeta_2) = \int_{\zeta_2}^{\delta(\zeta)} \frac{1}{r^{\frac{1}{\alpha}}(\xi)} \left[r(\xi) (z^{\Delta^{n-1}}(\xi))^\alpha \right]^{\frac{1}{\alpha}} \Delta \xi \\ &\geq \left[r(\delta(\zeta)) (z^{\Delta^{n-1}}(\delta(\zeta)))^\alpha \right]^{\frac{1}{\alpha}} \int_{\zeta_2}^{\delta(\zeta)} \frac{\Delta \xi}{r^{\frac{1}{\alpha}}(\xi)} \end{aligned}$$

for all $\zeta \in [\zeta_2, \infty)_{\mathbb{T}}$. But since $r(z^{\Delta^{n-1}})^\alpha$ is decreasing and $\delta(\zeta) \leq \zeta$, we obtain

$$z^{\Delta^{n-2}}(\delta(\zeta)) \geq \left[r(\zeta) (z^{\Delta^{n-1}}(\zeta))^\alpha \right]^{\frac{1}{\alpha}} \int_{\zeta_2}^{\delta(\zeta)} \frac{\Delta \xi}{r^{\frac{1}{\alpha}}(\xi)} \quad \text{for all } \zeta \in [\zeta_2, \infty)_{\mathbb{T}}. \quad (27)$$

From Lemma 2, we have

$$z^{\Delta^{n-2}}(\delta(\zeta)) \leq \frac{z(\delta(\zeta))}{h_{n-2}(\delta(\zeta), \zeta_2)} \quad \text{for all } \zeta \in [\zeta_3, \infty)_{\mathbb{T}},$$

where $\delta(\zeta) > \zeta_2$ for all $\zeta \in [\zeta_3, \infty)_{\mathbb{T}}$. From this, with (27), we obtain

$$\frac{z(\delta(\zeta))}{h_{n-2}(\delta(\zeta), \zeta_2)} \geq \left[r(\zeta) (z^{\Delta^{n-1}}(\zeta))^\alpha \right]^{\frac{1}{\alpha}} \int_{\zeta_2}^{\delta(\zeta)} \frac{\Delta \xi}{r^{\frac{1}{\alpha}}(\xi)} \quad \text{for all } \zeta \in [\zeta_3, \infty)_{\mathbb{T}}.$$

Hence,

$$\frac{1}{\omega(\zeta)} \geq \left(h_{n-2}(\delta(\zeta), \zeta_2) \int_{\zeta_2}^{\delta(\zeta)} \frac{\Delta \xi}{r^{\frac{1}{\alpha}}(\xi)} \right)^\alpha \quad \text{for all } \zeta \in [\zeta_3, \infty)_{\mathbb{T}},$$

i.e.,

$$\omega(\zeta) \left(h_{n-2}(\delta(\zeta), \zeta_2) \int_{\zeta_2}^{\delta(\zeta)} \frac{\Delta \xi}{r^{\frac{1}{\alpha}}(\xi)} \right)^\alpha \leq 1 \quad \text{for all } \zeta \in [\zeta_3, \infty)_{\mathbb{T}}.$$

From (26), we have $\omega^\Delta < 0$ on $[\zeta_3, \infty)_{\mathbb{T}}$. Given that $\sigma(\zeta) \geq \zeta$, it follows that

$$\omega^{\frac{1}{\alpha}}(\zeta) \geq \omega^{\frac{1}{\alpha}}(\sigma(\zeta)) \geq H_\alpha(\sigma(\zeta), \zeta_3) \quad \text{for all } \zeta \in [\zeta_3, \infty)_{\mathbb{T}},$$

i.e.,

$$H_\alpha(\sigma(\zeta), \zeta_3) \left(h_{n-2}(\delta(\zeta), \zeta_2) \int_{\zeta_2}^{\delta(\zeta)} \frac{\Delta \xi}{r^{\frac{1}{\alpha}}(\xi)} \right)^\alpha \leq 1 \quad \text{for all } \zeta \in [\zeta_3, \infty)_{\mathbb{T}}.$$

This contradicts (25), and completes the proof. $\square$

Corollary 1. Let (H1)–(H4) hold and $\beta = \alpha$. If

$$\liminf_{\zeta \rightarrow \infty} \frac{\alpha}{Q(\zeta)} \int_{\zeta}^{\infty} h_{n-2}(\delta(\xi), s) \frac{\delta^\Delta(\xi)}{r^{\frac{1}{\alpha}}(\delta(\xi))} Q^{1+\frac{1}{\alpha}}(\sigma(\xi)) \Delta \xi > \frac{\alpha}{(\alpha+1)^{1+\frac{1}{\alpha}}}, \quad (28)$$

then every solution of (1) is oscillatory.

Proof. Since $\beta = \alpha$, then (20) takes the form

$$\omega(\zeta) \geq Q(\zeta) + \alpha \int_{\zeta}^{\infty} h_{n-2}(\delta(\xi), \zeta_1) \frac{\delta^{\Delta}(\xi)}{r^{\frac{1}{\alpha}}(\delta(\xi))} Q^{1+\frac{1}{\alpha}}(\sigma(\xi)) \Delta \xi.$$

Now, since from (28), there exists $\ell > \frac{\alpha}{(\alpha+1)^{1+\frac{1}{\alpha}}}$ such that

$$\frac{\alpha}{Q(\zeta)} \int_{\zeta}^{\infty} h_{n-2}(\delta(\xi), \zeta_1) \frac{\delta^{\Delta}(\xi)}{r^{\frac{1}{\alpha}}(\delta(\xi))} Q^{1+\frac{1}{\alpha}}(\sigma(\xi)) \Delta \xi \geq \ell,$$

then

$$\frac{\omega(\zeta)}{Q(\zeta)} \geq 1 + \frac{\alpha}{Q(\zeta)} \int_{\zeta}^{\infty} h_{n-2}(\delta(\xi), \zeta_1) \frac{\delta^{\Delta}(\xi)}{r^{\frac{1}{\alpha}}(\delta(\xi))} \left(\frac{\omega^{\sigma}(\xi)}{Q^{\sigma}(\xi)} \right)^{1+\frac{1}{\alpha}} \Delta \xi.$$

Let $\lambda := \inf_{\zeta \in [\zeta_1, \infty)_{\mathbb{T}}} \frac{\omega(\zeta)}{Q(\zeta)}$, then $\lambda \geq 1 + \ell \lambda^{1+\frac{1}{\alpha}}$ and then by simple calculation we obtain $\lambda - \ell \lambda^{1+\frac{1}{\alpha}} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha}} \frac{1}{\ell^{\alpha}}$. This contradicts the fact that $\ell > \frac{\alpha^{\alpha}}{(\alpha+1)^{1+\frac{1}{\alpha}}}$; the proof is completed. $\square$

2.3. Sub Half-Linear Case: $\beta < \alpha$

Theorem 6. Let (12) hold and $\beta < \alpha$. Assume that

$$\begin{aligned} \limsup_{\zeta \rightarrow \infty} h_{n-2}(\zeta, s) Q^{\frac{\alpha-\beta}{\alpha\beta}} \left(\int_s^{\delta(\zeta)} r^{\frac{1}{\alpha}}(\xi) \Delta \xi \right) \\ \times \left(Q(\zeta) + \varepsilon \int_{\zeta}^{\infty} h_{n-2}(\delta(\xi), s) \frac{\delta^{\Delta}(\xi)}{r^{\frac{1}{\alpha}}(\delta(\xi))} Q^{1+\frac{1}{\beta}}(\sigma(\xi)) \Delta \xi \right)^{\frac{1}{\alpha}} = \infty \end{aligned} \quad (29)$$

for all large $s \in [\zeta_0, \infty)_{\mathbb{T}}$ and all small $\varepsilon \in \mathbb{R}^+$. Then, every solution of (1) is oscillatory.

Proof. Assume that (1) has a nonoscillatory solution x on $[\zeta_0, \infty)_{\mathbb{T}}$ such that $x(\zeta), x(\tau(\zeta)), x(\delta(\zeta)) > 0$ for all $\zeta \in [\zeta_1, \infty)$, where $\zeta_1 \in [\zeta_0, \infty)_{\mathbb{T}}$. By Lemma 3, there exists $\zeta_2 \in [\zeta_1, \infty)_{\mathbb{T}}$ such that $z^{\Delta^{n-1}}(\zeta) > 0$, $z^{\Delta^n}(\zeta) \leq 0$, and $z^{\Delta}(\zeta) > 0$ for all $\zeta \in [\zeta_2, \infty)_{\mathbb{T}}$. From (20), we have $\omega(\zeta) \geq Q(\zeta)$ and

$$r^{\frac{1}{\alpha}}(\zeta) z^{\Delta^{n-1}}(\zeta) \geq z^{\frac{\beta}{\alpha}}(\delta(\zeta)) Q^{\frac{1}{\alpha}}(\zeta) \quad \text{for all } \zeta \in [\zeta_2, \infty)_{\mathbb{T}}.$$

But since $r(z^{\Delta^{n-1}})^{\alpha}$ is decreasing on $[\zeta_2, \infty)_{\mathbb{T}}$, then there exists a constant $\varepsilon_0 \in \mathbb{R}^+$ and $\zeta_3 \in [\zeta_2, \infty)_{\mathbb{T}}$ such that

$$\varepsilon_0 \geq r^{\frac{1}{\alpha}}(\zeta) z^{\Delta^{n-1}}(\zeta) \geq z^{\frac{\beta}{\alpha}}(\delta(\zeta)) Q^{\frac{1}{\alpha}}(\zeta) \quad \text{for all } \zeta \in [\zeta_3, \infty)_{\mathbb{T}},$$

which yields

$$z^{1-\frac{\alpha}{\beta}}(\delta^{\sigma}(\zeta)) \geq \varepsilon_0^{1-\frac{\alpha}{\beta}} Q^{\frac{\alpha-\beta}{\alpha\beta}}(\sigma(\zeta)) \quad \text{for all } \zeta \in [\zeta_3, \infty)_{\mathbb{T}}.$$

But since $\beta < \alpha$, then by (20), we obtain

$$\omega(\zeta) \geq Q(\zeta) + \beta \int_{\zeta}^{\infty} h_{n-2}(\delta(\xi), \zeta_3) \frac{\delta^{\Delta}(\xi)}{r^{\frac{1}{\alpha}}(\delta(\xi))} Q^{1+\frac{1}{\alpha}}(\sigma(\xi)) z^{\frac{\beta}{\alpha}-1}(\delta^{\sigma}(\xi)) \Delta \xi \quad (30)$$

for all $\zeta \in [\zeta_4, \infty)_{\mathbb{T}}$, where $\delta(\zeta) \geq \zeta_3$ for all $\zeta \in [\zeta_4, \infty)_{\mathbb{T}}$, which yields

$$z^{1-\frac{\beta}{\alpha}}(\delta(\zeta)) \frac{r^{\frac{1}{\alpha}}(\zeta) z^{\Delta^{n-1}}(\zeta)}{z(\delta(\zeta))} \geq Q(\zeta) + \varepsilon_1 \int_{\zeta}^{\infty} h_{n-2}(\delta(\xi), \zeta_3) \frac{\delta^{\Delta}(\xi)}{r^{\frac{1}{\alpha}}(\delta(\xi))} Q^{1+\frac{1}{\beta}}(\sigma(\xi)) \Delta \xi \quad (31)$$

for all $\zeta \in [\zeta_4, \infty)_{\mathbb{T}}$, where $\varepsilon_1 := \beta \varepsilon_0^{1-\frac{\alpha}{\beta}}$. From Lemma 1 and (30), (31) takes the form

$$\begin{aligned} \varepsilon_0^{\frac{\alpha}{\beta}-1} &\geq h_{n-2}(\zeta, \zeta_3) Q^{\frac{\alpha-\beta}{\alpha\beta}} \left(\int_{\zeta_3}^{\delta(\zeta)} r^{\frac{1}{\alpha}}(\xi) \Delta \xi \right) \\ &\quad \times \left(Q(\zeta) + \varepsilon_1 \int_{\zeta}^{\infty} h_{n-2}(\delta(\xi), \zeta_3) \frac{\delta^{\Delta}(\xi)}{r^{\frac{1}{\alpha}}(\delta(\xi))} Q^{1+\frac{1}{\beta}}(\sigma(\xi)) \Delta \xi \right)^{\frac{1}{\alpha}} \end{aligned}$$

for all $\zeta \in [\zeta_4, \infty)_{\mathbb{T}}$, which contradicts (29). $\square$

3. Applications and Examples

In the case of $n = 2$, we have the second-order quasi-linear neutral delay dynamic equation

$$\left[r(\zeta) \Phi_{\alpha}([x(\zeta) + p(\zeta)x(\tau(\zeta))]^{\Delta}) \right]^{\Delta} + q(\zeta)(x(\delta(\zeta)))^{\beta} = 0 \quad \text{for } \zeta \in [\zeta_0, \infty)_{\mathbb{T}}, \quad (32)$$

where $\alpha, \beta, p, q, \tau$, and δ are as mentioned previously for (1).

In this case, our results reduce to the following one.

Theorem 7. Let (H1)–(H4) hold. Suppose that one of the following conditions holds:

- (i) $\int_s^{\infty} q(\xi)[1 - p(\delta(\xi))]^{\beta} \Delta \xi = \infty$.
- (ii) $\int_s^{\infty} \frac{\delta^{\Delta}(\xi)}{r^{\frac{1}{\alpha}}(\xi)} H_{\varepsilon}(\sigma(\xi), s) \Delta \xi = \infty$ if $\beta > \alpha$.
- (iii) $\limsup_{\zeta \rightarrow \infty} H_{\alpha}(\zeta, s) \left(\int_s^{\delta(\zeta)} \frac{\delta^{\Delta}(\xi)}{r^{\frac{1}{\alpha}}(\delta(\xi))} \Delta \xi \right) > 1$ if $\beta = \alpha$.
- (iv) $\limsup_{\zeta \rightarrow \infty} \left(Q^{\frac{\alpha-\beta}{\alpha\beta}}(\zeta) \int_s^{\delta(\zeta)} \frac{\Delta \xi}{r^{\frac{1}{\alpha}}(\xi)} \right) \left(Q(\zeta) + \varepsilon \int_{\zeta}^{\infty} \frac{\delta^{\Delta}(\xi)}{r^{\frac{1}{\alpha}}(\delta(\xi))} Q^{1+\frac{1}{\beta}}(\sigma(\xi)) \Delta \xi \right)^{\frac{1}{\alpha}} = \infty$ if $\beta < \alpha$.

Then, every solution of (32) is oscillatory.

Remark 1. The above conclusions are included in those of the results in [16].

According to Theorems 3–6, we can obtain oscillation criteria for (1) on any time scale. For example, when $\mathbb{T} = \mathbb{R}$, (1) reduces to

$$\left[r(\zeta) \Phi_{\alpha}([x(\zeta) + p(\zeta)x(\tau(\zeta))]^{(n-1)}) \right]' + q(\zeta)(x(\delta(\zeta)))^{\beta} = 0 \quad \text{for } \zeta \in [\zeta_0, \infty)_{\mathbb{R}}, \quad (33)$$

where α and β are quotients of positive odd integers, and $\int_{\zeta_0}^{\infty} \frac{d\xi}{r^{\frac{1}{\alpha}}(\xi)} = \infty$. Note that when

$\mathbb{T} = \mathbb{R}$, $\sigma(\zeta) = \zeta$ and $h_k(\zeta, s) = \frac{(\zeta-s)^k}{k!}$ for $s, \zeta \in \mathbb{R}$ and $k \in \mathbb{N}$, by Theorems 3–6, we establish the following theorems which extend those of [6,11,26–30].

Theorem 8. Suppose that one of the following conditions holds:

- (i) $\int_{\zeta}^{\infty} q(\xi)[1 - p(\delta(\xi))]^{\beta} d\xi = \infty$.
- (ii) $\int_s^{\infty} \frac{(\delta(\xi)-s)^{n-2}}{(n-2)!} \frac{\delta'(\xi)}{r^{\frac{1}{\alpha}}(\xi)} H_{\varepsilon}(\xi, s) d\xi = \infty$ if $\beta > \alpha$.
- (iii) $\limsup_{\zeta \rightarrow \infty} H_{\alpha}(\delta(\zeta), s) \left(\int_s^{\infty} \frac{(\xi-s)^{n-2}}{(n-2)!} \frac{\delta'(\xi)}{r^{\frac{1}{\alpha}}(\delta(\xi))} d\xi \right) > 1$ if $\beta = \alpha$.
- (iv) $\limsup_{\zeta \rightarrow \infty} \left(Q^{\frac{\alpha-\beta}{\alpha\beta}}(\zeta) \int_s^{\delta(\zeta)} \frac{(\delta(\xi)-s)^{n-2}}{(n-2)!} \frac{d\xi}{r^{\frac{1}{\alpha}}(\xi)} \right) \left(Q(\zeta) + \varepsilon \int_{\zeta}^{\infty} \frac{(\delta(\xi)-s)^{n-2}}{(n-2)!} \frac{\delta'(\xi)}{r^{\frac{1}{\alpha}}(\delta(\xi))} Q^{1+\frac{1}{\beta}}(\xi) d\xi \right)^{\frac{1}{\alpha}} = \infty$ if $\beta < \alpha$.

where

$$Q(\zeta) := \int_{\zeta}^{\infty} q(\xi)[1 - p(\delta(\xi))]^{\beta} ds \quad \text{and} \quad H_{\lambda}(\zeta, s) := Q(\zeta) + \lambda \int_{\zeta}^{\infty} \frac{(\delta(\xi) - s)^{n-2}}{(n-2)!} \frac{1}{r^{\frac{1}{\alpha}}(\delta(\xi))} Q^{1+\frac{1}{\alpha}}(\xi) d\xi.$$

Then, every solution of (33) is oscillatory.

Corollary 2. Assume that the conditions of Corollary 1 hold, and $\mathbb{T} = \mathbb{R}$. If

$$\liminf_{\zeta \rightarrow \infty} \frac{\alpha}{Q(\zeta)} \int_{\zeta}^{\infty} \frac{(\delta(\xi) - s)^{n-2}}{(n-2)!} \frac{\delta'(\xi)}{r^{\frac{1}{\alpha}}(\delta(\xi))} Q^{1+\frac{1}{\alpha}}(\xi) d\xi > \frac{\alpha}{(\alpha+1)^{1+\frac{1}{\alpha}}}, \quad (34)$$

then every solution of (33) is oscillatory.

Our results are considered more general to second-order nonlinear dynamic equations of the form

$$\left(r\Phi_{\alpha}(x^{\Delta}) \right)^{\Delta}(\zeta) + q(\zeta)\Phi_{\beta}(x(\zeta)) = 0 \quad \text{for } \zeta \in [\zeta_0, \infty)_{\mathbb{T}},$$

where α and β are quotients of positive odd integers, and r and q are real-valued, positive, and rd-continuous functions.

Remark 2. Theorems 4–6 include [15] (Theorems 3.1–3.3), respectively, in the case of $n = 2$ and $p(\zeta) \equiv 0$.

Example 1. Consider the second-order differential equation

$$\left[x(\zeta) + \left(1 - \frac{1}{\zeta^{\gamma}} \right) x(\zeta - \tau_0) \right]'' + \frac{q_0}{\zeta^{2-\gamma}} x(\delta_0 \zeta) = 0 \quad \text{for } \zeta \in [1, \infty)_{\mathbb{R}}, \quad (35)$$

where $q_0 > 0$, $\tau_0 \geq 0$, $0 < \gamma < 1$, $0 < \delta_0 < 1$. Here, $n = 2$, $r(\zeta) \equiv 1$, $p(\zeta) = 1 - \frac{1}{\zeta^{\gamma}}$, $\tau(\zeta) = \zeta - \tau_0$, $q(\zeta) = \frac{q_0}{\zeta^{2-\gamma}}$, $\alpha = \beta = 1$, and $\delta(\zeta) = \delta_0 \zeta$ for $\zeta \in [1, \infty)_{\mathbb{R}}$. Readily, (H1)–(H3) are satisfied. Also, (H4) is satisfied since $\delta(\zeta) = \delta_0 \zeta \leq \zeta$ and $\delta^{\Delta}(\zeta) = \delta_0 > 0$ for all $\zeta \in [1, \infty)_{\mathbb{R}}$, with $\delta_0[1, \infty)_{\mathbb{R}} = [\delta_0, \infty)_{\mathbb{R}}$ and $\lim_{\zeta \rightarrow \infty} (\delta_0 \zeta) = \infty$. Since

$$Q(\zeta) = \int_{\zeta}^{\infty} q(\xi)[1 - p(\delta(\xi))]^{\gamma} d\xi = \frac{q_0}{\delta_0^{\gamma} \zeta} < \infty \quad \text{for all } \zeta \in [1, \infty)_{\mathbb{R}},$$

then all the assumptions of Corollary 2 hold, so we obtain

$$\liminf_{\zeta \rightarrow \infty} \frac{\alpha}{Q(\zeta)} \int_{\zeta}^{\infty} \frac{(\delta(\xi) - s)^{n-2}}{(n-2)!} \frac{\delta'(\xi)}{r^{\frac{1}{\alpha}}(\delta(\xi))} Q^{1+\frac{1}{\alpha}}(\xi) d\xi = \frac{\theta^{\gamma+1} \zeta}{\lambda} \int_{\zeta}^{\infty} \frac{q_0^2}{\delta_0^{2\gamma} \xi^2} d\xi = \frac{q_0}{\delta_0^{1-\gamma}} \quad (36)$$

From (34) and (36), we obtain

$$\frac{q_0}{\delta_0^{1-\gamma}} > \frac{1}{4}.$$

Then, by Corollary 2, (35) is oscillatory for $q_0 > \frac{1}{4} \delta_0^{1-\gamma}$.

Remark 3. Applying [30] (Corollary 2.2) to Example 1, we find that (35) is oscillatory if $q_0 > \frac{1}{4}$. Thus, for Example 1, Corollary 2 gives weaker conditions for oscillation than [30] (Corollary 2.2).

Example 2. Consider the differential equation

$$\left(\Phi_{1/3}(x'(\zeta)) \right)' + \frac{q_0}{\zeta^{4/3}} \Phi_{1/3}(x(0.9\zeta)) = 0 \quad \text{for } \zeta \in [1, \infty)_{\mathbb{R}}, \quad (37)$$

where $q_0 > 0$. Note that $r(\zeta) \equiv 1$, $p(\zeta) \equiv 0$, $q(\zeta) = \frac{q_0}{\zeta^{4/3}}$, $\alpha = \beta = 1/3$, $\delta(\zeta) = 0.9\zeta$, $\int_{\zeta}^{\infty} \frac{d\zeta}{r^{1/3}(\zeta)} = \infty$. Thus, (H1)–(H4) are satisfied. Since

$$Q(\zeta) = \int_{\zeta}^{\infty} q(\xi)[1 - p(\delta(\xi))]^{\beta} d\xi = \frac{3q_0}{\zeta^{1/3}} \quad \text{for } \zeta \in [1, \infty)_{\mathbb{R}},$$

we apply Corollary 2 to obtain

$$\liminf_{\zeta \rightarrow \infty} \frac{\alpha}{Q(\zeta)} \int_{\zeta}^{\infty} \frac{\delta'(\xi)(\delta(\xi) - s)^{n-2}}{(n-2)!r^{1/\alpha}(\delta(\xi))} Q^{1+1/\alpha}(\xi) d\xi = \liminf_{\zeta \rightarrow \infty} \frac{\zeta^{1/3}}{9q_0} \int_{\zeta}^{\infty} \frac{3^4 q_0^4}{\xi^{4/3}} (0.9) d\xi \quad (38)$$

From (34) and (38), we obtain

$$\frac{3^5}{10} q_0^3 > \frac{3^3}{4^4}.$$

Then, every solution of (37) is oscillatory if $q_0 > 0.163119485$.

Remark 4. The established criteria related to the oscillation of (37) based on comparison with a first-order delay differential equation (see, e.g., [26], Theorem 2) gives $q_0 > 3.61643$ and condition (23) in [34] gives $q_0 > 1.92916$. This indicates that our results provide more effective oscillation criteria for (37). Figure 1 illustrates solutions of (37) for various q_0 values.

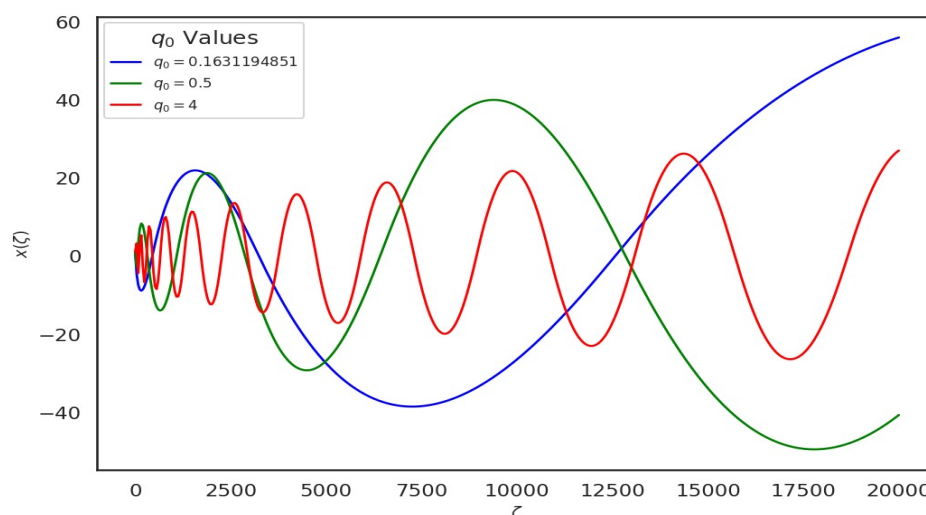


Figure 1. Some solutions of Equation (37).

4. Conclusions

This paper has presented the extensive development of new oscillation criteria for n th-order dynamic equations. Our comprehensive analysis led to the establishment of more effective criteria for oscillation in the case where $p(\zeta) = 0$. Notably, Corollary 2 enhances the results previously reported in [26] and provides improved criteria for delay differential equations compared to those in [30]. Furthermore, we introduced novel and significant findings specifically for the case where $\mathbb{T} = \mathbb{Z}$, contributing to the advancement of oscillation theory on time scales. These results not only refine existing methodologies but also expand the scope of applications within the field.

It would be interesting to extend this approach to the higher-order half-linear dynamic equation with negative neutral term

$$\left[r(\zeta) \Phi_{\alpha} \left([x(\zeta) - p(\zeta)x(\tau(\zeta))]^{\Delta^{n-1}} \right) \right]^{\Delta} + q(\zeta) \Phi_{\beta}(x(\delta(\zeta))) = 0 \quad \text{for } \zeta \in [\zeta_0, \infty)_{\mathbb{T}},$$

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