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The Efficiency of the New Extended EWMA Control Chart for Detecting Changes Under an Autoregressive Model and Its Application

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Abstract: Control charts are frequently used instruments for process quality monitoring. Another name for the NEEWMA control chart is the new extended exponentially weighted moving average (new extended EWMA) control chart. The lower control limit (LCL) and upper control limit (UCL) are equally spaced from the center line, giving it a symmetrical design. Because of its symmetry, the NEEWMA chart is very good at identifying even the tiniest changes in operation by detecting deviations from the target in both upward and downward directions. This study derives an explicit formula for the average run length (ARL) of the NEEWMA control chart based on the autoregressive (AR) model with exponential white noise. The focus is on the zero-state performance of the NEEWMA control chart, which is derived using explicit formulas. Banach's fixed-point theorem was used to prove existence and uniqueness of this formula. The accuracy of this formula is validated by comparing it to the numerical integral equation (NIE) method using percentage accuracy (%Acc). The results show that the NEEWMA control chart is more efficient than the ARL evaluated by the NIE method, particularly regarding computation time. The performance of the NEEWMA control chart is compared with the EWMA and extended EWMA control charts by evaluating both the ARL and standard deviation run length (SDRL). The NEEWMA control chart outperforms the others in detection performance, followed by the extended EWMA and EWMA control charts. Further verification of its superior performance is provided through comparisons using the average extra quadratic loss (AEQL) and the performance comparison index (PCI), which confirm that it outperforms both the EWMA and extended EWMA control charts across various parameters and shift sizes. Finally, an illustrative example using real-life economic data demonstrates its efficiency.

Keywords: average run length; autoregressive model; control chart; numerical integral equation; statistical process control



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1. Introduction

Statistical process control (SPC) is a quality control approach that uses statistical methods to detect, monitor, regulate, and improve a process. SPC enables operations to run smoothly and consistently produce goods that meet quality standards. It is widely used in manufacturing and can also be applied to any process where quality and change monitoring are priorities, including in service industries, environmental management, economics, healthcare, and more. The main objective of a control chart is to promptly

identify any assignable causes that lead to defective products. Control charts confirm that the manufacturing process is under control when assignable causes are absent; when they are present, they indicate that the process is out of control. These fundamental concepts are applicable across various fields [1,2]. Walter A. Shewhart was the first to propose the concept of control charts [3]. Shewhart's control charts can analyze process performance using only current sample data. While they are highly sensitive to large shifts, they are also simple to construct and analyze. A statistical technique used in quality control to track and identify minor to moderate shifts or changes in the process mean is the cumulative sum (CUSUM) control chart, which was suggested by Page [4]. A popular statistical process control tool for tracking a process's stability over time is the exponentially weighted moving average (EWMA) control chart, which was first put forth by Roberts [5] in 1959. After that, an expansion of the standard EWMA control chart that adds more details about the process under observation is the extended exponentially weighted moving average (extended EWMA) control chart, denoted as EEWMA control chart, which was proposed by Neveed et al. [6]. The key distinction between the EWMA and EEWMA control charts lies in how the smoothing parameter, or weight, is calculated. In the EEWMA control chart, this parameter is adjusted based on the time series length, or a second smoothing parameter is added. Recently, Javed et al. developed a new variation of the EWMA control chart, called the new extended exponentially weighted moving average (NEEWMA) control chart [7]. It was created by adding a third smoothing parameter to the EEWMA control chart, which enhances its effectiveness in tracking changes and enables it to adapt more easily to various data patterns and trends. Consequently, the NEEWMA control chart has been the focus of interest in this study.

Control charts with symmetric two-sided upper and lower control limits, such as EWMA, EEWMA, and NEEWMA, are essential tools for process monitoring. However, depending on the nature of the process and the monitoring objectives, some real-world applications may require a one-sided control chart with either an upper or lower limit. Despite this, control charts are typically designed with two-sided upper and lower control limits. Moreover, control charts are generally intended to track and evaluate processes using identically distributed (i.i.d.) independent data. However, in certain cases, processes may exhibit autocorrelation, where previous data points influence the current data. In such cases, traditional control charts may not perform effectively. Therefore, specialized control charts or modifications to account for autocorrelation are required to ensure accurate process monitoring.

The ARMA stands for autoregressive moving average, which consists of autoregressive (AR) and moving average (MA) components and is the most often used model for autocorrelated processes. It has demonstrated a comparatively high level of forecasting accuracy. Both internal and external factors that affect the time series being modeled can have an impact on autocorrelated processes. Moreover, a large portion of the data in this study follows an exponential distribution, as shown in [8].

One statistical metric used in statistical process control (SPC) to assess the efficacy of a control chart is the average run length (ARL). It is defined as the average number of observations or samples taken from a process before the control chart signals that the process is out of control.

The ARL is made up of two main components: ARL_1 is the average run length while the process is in control. For a well-designed control chart and steady process, a high ARL_0 is ideal. ARL_1 denotes the average run length when the process is out of control. If there is a shift or variation in the process, the control chart should notice it quickly and signal the need for corrective action or further study. Several methods for estimating the ARL

have been proposed in the literature (see [9–11]), including the Markov chain, Monte Carlo simulation, numerical integral equation (NIE) approaches, and the explicit formula.

Building on the initial work referenced below, Zhao H. et al. [12] utilized the Markov chain model to analyze the performance of the EWMA chart, extending its application from monitoring Poisson processes under step shifts to those with linear drift. Numerous studies have increasingly utilized the NIE method and explicit formula for ARL calculations. Notably, the explicit formula has been widely recognized for its ability to reduce computation time for ARL. For instance, the analytical solution for the ARL of the CUSUM control chart for first-order autoregressive with an explanatory variable running on the CUSUM chart was presented by Paichit [13]. The accuracy of the results produced from precise equations with NIE by the Gauss–Legendre rule was checked. Phanyaem [14] established explicit formulas for the ARL on the CUSUM control chart under observations based on the seasonal AR(p) models with exogenous variables and compared them with the ARL that was obtained by NIE methods via the midpoint rule, the trapezoidal rule, Simpson’s rule, and the Gaussian rule. Moreover, these methods for calculating ARL were extended to be used with real-life data to confirm their efficiency. The ARL was evaluated by Paichit and Peerajit [15], who developed the NIE methods using four quadrature rules: the Gauss–Legendre rule for a modified EWMA control chart based on when the observations are from continuous distributions like gamma or Weibull, and the composite midpoint, trapezoidal, and Simpson’s rules. Additionally, they illustrated the efficacy of this approach using actual data. Karoon et al. [16] presented the explicit formulas of ARL for the double EWMA control chart for AR(p) and applied them to a dataset of economic time series to showcase the effectiveness of the detection of changes process. Afterward, Phanthuna et al. [17] reported the ARL derived from the explicit formula of the double modified EWMA control chart based on the AR(p) model, with actual COVID-19 data provided to illustrate its application. Recently, Karoon and Areepong [18] published an explicit solution for deriving ARL on the adjusted modified EWMA control chart using the AR(p) model with trend and quadratic trend, as well as real-world stock price data.

The formulation of an explicit formula for ARL on a new extended EWMA control chart for observations from the AR(p) model in the situation of exponential white noise has not been examined in prior research. Deriving the ARL explicit formula on the new extended EWMA control chart for the AR(p) model is the aim of this study. Additionally, an evaluation of the performance of EWMA control charts and extended EWMA was carried out with both simulated data and real-life data in a variety of economic time series.

2. Structures of Control Charts

The control charts in this study, such as the traditional EWMA chart, extended EWMA chart, and new extended EWMA chart, may identify even slight changes in the process, regardless of which way they occur, because the UCL and LCL are symmetrically positioned around the goal value. When the data points surpass the control limitations, the chart will sound an alert if the process mean deviates in either direction.

2.1. The Traditional EWMA Control Chart

Let X_t represent the data sequence at $t = 1, 2, \dots$ with mean (μ) and variance (σ^2) based on the AR(p) process with exponential white noise. At $t = 0$, X_t , or X_0 , it represents the initial value of the control chart statistics. This value is a constant, $X_0 = u$.

Initially, the EWMA control chart was created by Roberts in 1959 [5]. It is possible to keep track of the control chart and frequently use this technique to detect small changes. The statistics of the traditional EWMA control chart, denoted as Z_t , are explained below.

$$Z_t = \lambda X_t + (1 - \lambda)Z_{t-1}, t = 1, 2, 3, \dots, \quad (1)$$

where λ is the exponential smoothing parameter with $0 < \lambda \leq 1$. If $\lambda = 1$ in the EWMA statistics (Z_t) from (1), it simplifies to $Z_t = X_t$. This means that the current value Z_t is entirely determined by the most recent observation X_t , with no influence from previous values Z_{t-1} . Additionally, the solutions for the mean ($E(Z_t)$) and the variance ($Var(Z_t)$) are expressed as follows.

$$E(Z_t) = \mu$$

$$Var(Z_t) = \left(\frac{\lambda}{2 - \lambda} \right) \sigma^2.$$

For the control limits, the upper control limit (UCL) and lower control limit (LCL) are given by the following formulas for monitoring the process:

$$UCL = \mu + K_1 \sigma \sqrt{\frac{\lambda}{2 - \lambda}}, \quad (2)$$

$$LCL = \mu - K_1 \sigma \sqrt{\frac{\lambda}{2 - \lambda}}, \quad (3)$$

where K_1 represents the width of the control limit of the EWMA control chart. These formulas, (2) and (3), are used to define the control limits for the process. The EWMA stopping time (τ_b) can then be expressed as follows:

$$\tau_b = \inf\{t \geq 0 : Z_t < a \text{ or } Z_t > b\},$$

where a and b represent the lower control (LCL) and upper control (UCL) limits, respectively.

2.2. The Extended EWMA Control Chart

The extended EWMA control chart, or EEWMA, was suggested by Neveed et al. in 2018 [6]. It can be used to fast monitor and detect slight to moderate changes in the process mean. An explanation of the EEWMA control chart's statistics can possibly be found using Equation (4) below.

$$Y_t = \lambda_1 X_t - \lambda_2 X_{t-1} + (1 - \lambda_1 + \lambda_2)Y_{t-1}, t = 1, 2, 3, \dots, \quad (4)$$

where λ_1 and λ_2 are the exponential smoothing parameters with $0 < \lambda_1 \leq 1$ and $0 \leq \lambda_2 < \lambda_1$, respectively. Next, the EEWMA (Y_t) must provide the solutions for the mean ($E(Y_t)$) and the variance ($Var(Y_t)$), which are expressed as follows:

$$E(Y_t) = \mu$$

$$Var(Y_t) = \left(\frac{\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2(1 - \lambda_1 + \lambda_2)}{2(\lambda_1 - \lambda_2) - (\lambda_1 - \lambda_2)^2} \right) \sigma^2.$$

According to the EEWMA structure, the upper control limit (UCL) and lower control limit (LCL) of the monitoring process are defined as follows:

$$UCL = \mu + K_2 \sigma \sqrt{\frac{\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2(1 - \lambda_1 + \lambda_2)}{2(\lambda_1 - \lambda_2) - (\lambda_1 - \lambda_2)^2}}, \quad (5)$$

$$LCL = \mu - K_2 \sigma \sqrt{\frac{\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2(1 - \lambda_1 + \lambda_2)}{2(\lambda_1 - \lambda_2) - (\lambda_1 - \lambda_2)^2}}, \quad (6)$$

where K_2 represents the width of the control limit of the EEWMA control chart, as shown in Equations (5) and (6). Then, the EEWMA stopping time (τ_h) can be given as follows:

$$\tau_h = \inf\{t \geq 0 : Y_t < g \text{ or } Y_t > h\},$$

where g and h represent the lower control (LCL) and upper control (UCL) limits, respectively.

2.3. The New Extended EWMA Control Chart

By adding an extra exponential smoothing parameter, Javed et al., in 2024 [7], created a new extended EWMA control chart called the NEEWMA control chart. The following is an expression for the NEEWMA statistic:

$$N_t = \lambda_1 X_t - \lambda_2 X_{t-1} - \lambda_3 X_{t-2} + AN_{t-1}, t = 1, 2, 3, \dots, \quad (7)$$

where A is defined as $(1 - \lambda_1 + \lambda_2 + \lambda_3)$, with λ_1 , λ_2 , and λ_3 being the exponential smoothing parameters, where $0 < \lambda_1 \leq 1$, $0 \leq \lambda_2 < \lambda_1$, and $0 \leq \lambda_3 < \lambda_2$, respectively. The NEEWMA (Y_t) can then provide the solutions for the mean ($E(N_t)$) and variance ($Var(N_t)$), as stated below.

$$E(N_t) = \mu$$

$$Var(N_t) = \left(\frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2A(\lambda_1\lambda_2 - \lambda_2\lambda_3) - 2A^2\lambda_1\lambda_3}{1 - A^2} \right) \sigma^2.$$

The upper control limit (UCL) and lower control limit (LCL) for the monitoring process are determined according to the NEEWMA structure as follows:

$$UCL = \mu + K_3 \sigma \sqrt{\frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2A(\lambda_1\lambda_2 - \lambda_2\lambda_3) - 2A^2\lambda_1\lambda_3}{1 - A^2}}, \quad (8)$$

$$LCL = \mu - K_3 \sigma \sqrt{\frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2A(\lambda_1\lambda_2 - \lambda_2\lambda_3) - 2A^2\lambda_1\lambda_3}{1 - A^2}}, \quad (9)$$

where K_3 represents the width of the upper control limit (UCL) in (8) and lower control limit (LCL) in (9) of the NEEWMA control chart. The NEEWMA stopping time (τ_s) can then be written as follows:

$$\tau_s = \inf\{t \geq 0 : N_t < r \text{ or } N_t > s\},$$

where r and s represent the lower control (LCL) and upper control (UCL) limits, respectively.

Furthermore, the NEEWMA statistic becomes the EEWMA statistic when $\lambda_3 = 0$, and the traditional EWMA statistic when both $\lambda_2 = 0$ and $\lambda_3 = 0$.

3. Evaluation of Average Run Length Under AR(p) Model

This section presents the derivation of the ARL using two methods: the explicit formula and the NIE method, both based on the AR(p) model. It also includes the proof of the existence and uniqueness of the ARL obtained through the explicit solution.

3.1. Explicit Formula of ARL Under AR(p) Model

Initially, the autoregressive model of order p (AR(p) model) was studied as a component of the autoregressive moving average (ARMA(p,q)) models used for time series data. It expresses the present value of a series as a linear mixture of its prior p values plus a random error term. The general form of an AR(p) model is

$$X_t = c + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + \epsilon_t, \quad (10)$$

where X_t is the value of the time series at time t , c is the constant of the model, ϕ_i represents the coefficients of autoregressive with $i = 1, 2, \dots, p$, and ϵ_t is a white noise error term of exponential distribution ($\epsilon_t \sim \text{Exp}(\gamma)$). Thus, the probability density function of the exponential distribution (ϵ_t) is expressed as $f(x, \gamma) = (1/\gamma)e^{-x(1/\gamma)}$, for $\gamma > 0$. Moreover, the AR(p) model's initial values represent $X_{t-1}, X_{t-2}, \dots, X_{t-p}$.

Next, the explicit formula of the ARL on the NEEWMA control chart with the AR(p) model is solved. The procedure for solving the explicit formula for the ARL is shown below:

Step 1: X_t from (10) is substituted into the NEEWMA statistic (N_t)

To begin, X_t from (10) is substituted into the NEEWMA statistic (N_t) shown in (7), and it can be rewritten as follows:

$$\begin{aligned} N_t &= \lambda_1 (c + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + \epsilon_t) - \lambda_2 X_{t-1} - \lambda_3 X_{t-2} + AN_{t-1} \\ &= \lambda_1 c + \lambda_1 \phi_1 X_{t-1} + \lambda_1 \phi_2 X_{t-2} + \cdots + \lambda_1 \phi_p X_{t-p} + \lambda_1 \epsilon_t - \lambda_2 X_{t-1} - \lambda_3 X_{t-2} + AN_{t-1}. \end{aligned}$$

Step 2: To define the initial time as $t = 1$ and perform variable changes, complete the following:

The initial time $t = 1$ is set, with $N_{t-1} = N_0$ being assigned the value ω , including the expression $\phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p}$, which is rewritten in summation notation as $\sum_{i=1}^p \phi_i X_{t-i}$. Thus, the equation is rearranged as follows:

$$N_1 = \lambda_1 c + \lambda_1 \epsilon_1 + \lambda_1 \sum_{i=1}^p \phi_i X_{t-i} - \lambda_2 X_{t-1} - \lambda_3 X_{t-2} + A\omega.$$

Step 3: Assume that ϵ_1 represents the control limit for N_1 when the process is in control.

The control limits for N_1 , represented by the UCL and the LCL, are defined by the interval $r < N_1 < s$ as described in Section 2.3. This control limit interval can be expressed as follows:

$$r < \lambda_1 c + \lambda_1 \epsilon_1 + \lambda_1 \sum_{i=1}^p \phi_i X_{t-i} - \lambda_2 X_{t-1} - \lambda_3 X_{t-2} + A\omega < s.$$

Step 4: The equation above was rewritten to express it in terms of the variable ϵ_1 .

To determine the variable B and C , they are defined as $\lambda_2 X_{t-1} + \lambda_3 X_{t-2}$ and $c + \sum_{i=1}^p \phi_i X_{t-i}$, respectively. The variable ϵ_1 was rewritten as follows:

$$\begin{aligned} r - \lambda_1(C) + B - A\omega &< \lambda_1 \epsilon_1 < s - \lambda_1(C) + B - A\omega \\ \frac{r - A\omega + B - \lambda_1 C}{\lambda_1} &< \epsilon_1 < \frac{s - A\omega + B - \lambda_1 C}{\lambda_1} \\ \frac{r - A\omega + B}{\lambda_1} - C &< \epsilon_1 < \frac{s - A\omega + B}{\lambda_1} - C. \end{aligned}$$

Step 5: The Fredholm integral equation [19], which was used to derive explicit formula for ARL.

Let $Q(\omega)$ be the explicit ARL solution for the NEEWMA control chart based on the AR(p) model. This can be rearranged as follows:

$$Q(\omega) = 1 + \frac{1}{\lambda_1} \int_{\frac{r-A\omega+B}{\lambda_1}-C}^{\frac{s-A\omega+B}{\lambda_1}-C} Q(N_1) f(\epsilon_1) d\epsilon_1. \quad (11)$$

Let ς be $N_1 = \lambda_1 c + \lambda_1 \epsilon_1 + \lambda_1 \sum_{i=1}^p \phi_i X_{t-i} - \lambda_2 X_{t-1} - \lambda_3 X_{t-2} + A\omega$. Thus $\frac{d\varsigma}{d\epsilon} = \lambda_1$, and by solving for $d\epsilon_1$, we get $d\epsilon_1 = \frac{1}{\lambda_1} d\varsigma$. After that, from (11), we changed the variable, and it became the following solution:

$$Q(\omega) = 1 + \frac{1}{\lambda_1} \int_r^s Q(\varsigma) f\left(\frac{\varsigma - A\omega + B}{\lambda_1} - C\right) d\varsigma. \quad (12)$$

Note that Equation (12) will be used to verify the solution of the explicit ARL, which has the characteristics of existence and uniqueness, by using the mathematical theorem—the fixed-point theorem in Section 3.3.

Since we determine $\epsilon_1 \sim \text{Exp}(\gamma)$, we now have

$$\begin{aligned} Q(\omega) &= 1 + \frac{1}{\lambda_1} \int_r^s Q(\varsigma) \frac{1}{\gamma} e^{-\left(\frac{\varsigma - A\omega + B}{\lambda_1 \gamma} - \frac{C}{\gamma}\right)} d\varsigma \\ &= 1 + \frac{1}{\lambda_1 \gamma} \int_r^s Q(\omega) e^{-\frac{\varsigma}{\lambda_1 \gamma}} e^{\frac{A\omega - B}{\lambda_1 \gamma}} e^{\frac{C}{\gamma}} d\varsigma \\ &= 1 + \frac{1}{\lambda_1 \gamma} e^{\frac{A\omega - B}{\lambda_1 \gamma} + \frac{C}{\gamma}} \int_r^s Q(\omega) e^{-\frac{\varsigma}{\lambda_1 \gamma}} d\varsigma. \end{aligned}$$

As mentioned above, we define the variable ψ as $\int_r^s Q(\omega) e^{-\frac{\varsigma}{\lambda_1 \gamma}} d\varsigma$ and set $\xi(\omega)$ to be $e^{\frac{A\omega - B}{\lambda_1 \gamma} + \frac{C}{\gamma}}$, resulting in the following expression:

$$Q(\omega) = 1 + \frac{\psi \cdot \xi(\omega)}{\lambda_1 \gamma}. \quad (13)$$

Afterward, the variable ψ is considered.

$$\begin{aligned} \psi &= \int_r^s Q(\omega) e^{-\frac{\varsigma}{\lambda_1 \gamma}} d\varsigma \\ &= \int_r^s \left[1 + \frac{\psi \cdot \xi(\omega)}{\lambda_1 \gamma} \right] e^{-\frac{\varsigma}{\lambda_1 \gamma}} d\varsigma \\ &= \int_r^s e^{-\frac{\varsigma}{\lambda_1 \gamma}} d\varsigma + \int_r^s \frac{\psi \cdot \xi(\omega)}{\lambda_1 \gamma} e^{-\frac{\varsigma}{\lambda_1 \gamma}} d\varsigma \\ &= -\lambda_1 \gamma \left(e^{-\frac{s}{\lambda_1 \gamma}} - e^{-\frac{r}{\lambda_1 \gamma}} \right) - \frac{\psi}{\lambda_1 - \lambda_2 - \lambda_3} e^{-\frac{B}{\lambda_1 \gamma} + \frac{C}{\gamma}} \left(e^{-\frac{(\lambda_1 - \lambda_2 - \lambda_3)s}{\lambda_1 \gamma}} - e^{-\frac{(\lambda_1 - \lambda_2 - \lambda_3)r}{\lambda_1 \gamma}} \right) \\ &= -\lambda_1 \gamma \left(e^{-\frac{s}{\lambda_1 \gamma}} - e^{-\frac{r}{\lambda_1 \gamma}} \right) - \frac{\psi e^{-\frac{B}{\lambda_1 \gamma} + \frac{C}{\gamma}}}{\eta} \left(e^{-\frac{\eta s}{\lambda_1 \gamma}} - e^{-\frac{\eta r}{\lambda_1 \gamma}} \right), \end{aligned}$$

where η represents $\lambda_1 - \lambda_2 - \lambda_3$.

Next, the integral equation of ψ can be shown as

$$\psi = \frac{-\lambda_1 \gamma \left(e^{-\frac{s}{\lambda_1 \gamma}} - e^{-\frac{r}{\lambda_1 \gamma}} \right)}{1 + \frac{1}{\eta} e^{-\frac{B}{\lambda_1 \gamma} + \frac{C}{\gamma}} \left(e^{-\frac{\eta s}{\lambda_1 \gamma}} - e^{-\frac{\eta r}{\lambda_1 \gamma}} \right)}. \quad (14)$$

Finally, (14) is replaced in (13), and the solution for $Q(\omega)$ is expressed in (15) as follows:

$$Q(\omega) = 1 - \frac{\eta e^{\frac{A\omega}{\lambda_1 \gamma}} \left(e^{-\frac{s}{\lambda_1 \gamma}} - e^{-\frac{r}{\lambda_1 \gamma}} \right)}{\eta e^{\frac{B}{\lambda_1 \gamma} - \frac{C}{\gamma}} + \left(e^{-\frac{\eta s}{\lambda_1 \gamma}} - e^{-\frac{\eta r}{\lambda_1 \gamma}} \right)}. \quad (15)$$

Step 6: γ is set to γ_0 and γ_1 in (15) for the process in control and out of control, respectively.

Thus, for the process in-control ($\gamma = \gamma_0$), the ARL_0 of the NEEWMA control chart can be represented as follows:

$$ARL_0 = 1 - \frac{\eta e^{\frac{A\omega}{\lambda_1 \gamma_0}} \left(e^{-\frac{s}{\lambda_1 \gamma_0}} - e^{-\frac{r}{\lambda_1 \gamma_0}} \right)}{\eta e^{\frac{B}{\lambda_1 \gamma_0} - \frac{C}{\gamma_0}} + \left(e^{-\frac{\eta s}{\lambda_1 \gamma_0}} - e^{-\frac{\eta r}{\lambda_1 \gamma_0}} \right)}. \quad (16)$$

Meanwhile, for the out-of-control process ($\gamma = \gamma_1$) and $\gamma_1 = \gamma_0(1 + \delta)$, the following can be shown:

$$ARL_1 = 1 - \frac{\eta e^{\frac{A\omega}{\lambda_1 \gamma_1}} \left(e^{-\frac{s}{\lambda_1 \gamma_1}} - e^{-\frac{r}{\lambda_1 \gamma_1}} \right)}{\eta e^{\frac{B}{\lambda_1 \gamma_1} - \frac{C}{\gamma_1}} + \left(e^{-\frac{\eta s}{\lambda_1 \gamma_1}} - e^{-\frac{\eta r}{\lambda_1 \gamma_1}} \right)}. \quad (17)$$

Therefore, the ARL_1 of the NEEWMA control chart can be represented in Equation (17) above.

3.2. NIE Method of ARL Under AR(p) Model

The NIE method with quadrature rules has the advantage of providing an accurate and computationally effective way to estimate the ARL [16]. In this part, the AR(p) model with exponential white noise is utilized to perform the analytical ARL using the NIE technique on the NEEWMA control chart. The procedure for solving the ARL using the NIE method is shown below:

Step 1: The Gauss–Legendre rule is used to calculate the ARL.

Let $\vartheta(\omega)$ be the ARL of the NEEWMA control chart for the AR(p) model, computed using the Gauss–Legendre rule. It was computed over the interval $[r, s]$ and determined by solving n linear systems of equations. The quadrature rule approximates an integral in the manner described as follows:

$$\int_r^s Q(\zeta) f(\zeta) d\zeta \approx \sum_{j=1}^m w_j f(v_j), \quad (18)$$

where v_j is a point, defined as $w_j \left(j - \frac{1}{2} \right) + r$, for $j = 1, 2, \dots, m$, and then $w_j = (s - r)/m$ is the weight function at the quadrature rule (as shown in Equation (18)).

Step 2: Using the quadrature rule, the ARL for NIE is expressed as

$$\vartheta(v_i) = 1 + \frac{1}{\lambda_1} \sum_{j=1}^m w_j \vartheta(v_j) f\left(\frac{v_j - Av_i + B}{\lambda_1} - C\right), i = 1, 2, \dots, m. \quad (19)$$

Rearranging Equation (19) to solve the system in terms of the n linear equation yields

$$\vartheta(v_1) = 1 + \frac{1}{\lambda_1} \sum_{j=1}^m w_j \vartheta(v_j) f\left(\frac{v_j - Av_1 + B}{\lambda_1} - C\right)$$

$$\begin{aligned}\vartheta(v_2) &= 1 + \frac{1}{\lambda_1} \sum_{j=1}^m w_j \vartheta(v_j) f\left(\frac{v_j - Av_2 + B}{\lambda_1} - C\right) \\ &\vdots \\ \vartheta(v_n) &= 1 + \frac{1}{\lambda_1} \sum_{j=1}^m w_j \vartheta(v_j) f\left(\frac{v_j - Av_n + B}{\lambda_1} - C\right).\end{aligned}$$

As mentioned above, this system can be expressed as

$$M_{m \times 1} = (I_m - R_{m \times m})^{-1} 1_{m \times 1},$$

$$\text{where } M_{m \times 1} = \begin{bmatrix} \vartheta(v)_1 \\ \vartheta(v)_2 \\ \vdots \\ \vartheta(v)_n \end{bmatrix}, I_m = \text{diag}(1, 1, \dots, 1) \text{ and } 1_{m \times 1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Step 3: $R_{m \times m}$ is shown as the element of R matrix.

Let $R_{m \times m}$ be the matrix and set the m to m^{th} as the element of R matrix. Thus, it can be expressed as

$$[R_{ij}] \approx \frac{1}{\lambda_1} \sum_{j=1}^m w_j f\left(\frac{v_j - Av_i + B}{\lambda_1} - C\right).$$

The numerical approximation of the integral equation, if $(I - R)^{-1}$ exists, is the matrix term,

$$M_{m \times 1} = (I_{m \times 1} - R_{m \times m})^{-1} 1_{m \times 1}.$$

Step 4: v_i is substituted with ω in $\vartheta(v_i)$.

Ultimately, by substituting v_i with ω , we obtain the following expression:

$$\vartheta(\omega) = 1 + \frac{1}{\lambda_1} \sum_{j=1}^m w_j \vartheta(v_j) f\left(\frac{v_j - A\omega + B}{\lambda_1} - C\right). \quad (20)$$

Hence, the numerical approximation of the integral equation for $\vartheta(\omega)$, referred to as the ARL by the NIE method, is shown in Equation (20) above.

3.3. The Existence and Uniqueness of ARL

The validity of the ARL equation is theoretically supported by Banach's fixed-point theorem [20], which guarantees that the integral equation for explicit formulations has a unique solution. For any continuous function defined by T , let T be an operation on the class of all, as illustrated by the following equation:

$$T(Q(\omega)) = 1 + \frac{1}{\lambda_1} \int_r^s Q(\varsigma) f\left(\frac{\varsigma - A\omega + B}{\lambda_1} - C\right) d\varsigma. \quad (21)$$

If an operator T is a contraction, then the fixed-point theorem $T(Q(\omega)) = Q(\omega)$ has a unique solution, according to Banach's fixed-point theorem. The theorem can be applied in the manner described below to demonstrate that (21) exists and has a unique solution.

Theorem 1. *Banach's Fixed-Point Theorem. Let $T : X \rightarrow X$ be a contraction mapping with contraction constant $0 \leq \kappa < 1$. Let (X, d) be a full metric space. Then, $\|T(Q(\omega)_1) - T(Q(\omega)_2)\| \leq \kappa \|Q(\omega)_1 - Q(\omega)_2\|$, for all $Q(\omega)_1, Q(\omega)_2 \in X$ has to show a unique fixed point $x \in X$ (such that $T(Q(\omega)) = Q(\omega)$), i.e., a unique fixed point in X .*

Proof. To present that, T in (21) denotes a mapping of a contraction for $Q(\omega)_1, Q(\omega)_2 \in [r, s]$ by showing that $\|T(Q(\omega)_1) - T(Q(\omega)_2)\| \leq \kappa \|Q(\omega)_1 - Q(\omega)_2\|$, for all $Q(\omega)_1, Q(\omega)_2 \in \kappa[r, s]$ with $0 \leq \kappa < 1$ under the norm $\|Q(\omega)\|_\infty = \sup_{\omega \in [r, s]} |Q(\omega)|$.

Consider the following:

$$\begin{aligned} \|T(Q(\omega)_1) - T(Q(\omega)_2)\|_\infty &= \sup_{\omega \in [r, s]} |Q(\omega)_1 - Q(\omega)_2| \\ &= \sup_{\omega \in [r, s]} \left| \frac{\xi(\omega)}{\lambda_1 \gamma} \int_r^s \left((Q(\omega)_1 - Q(\omega)_2) e^{-\frac{\xi}{\lambda_1 \gamma}} \right) d\xi \right| \\ &\leq \sup_{\omega \in [r, s]} \left| \|T(Q(\omega)_1) - T(Q(\omega)_2)\|_\infty \xi(\omega) \left(e^{-\frac{\bar{s}}{\lambda_1 \gamma}} - e^{-\frac{\bar{r}}{\lambda_1 \gamma}} \right) \right| \\ &= \|T(Q(\omega)_1) - T(Q(\omega)_2)\|_\infty \sup_{\omega \in [r, s]} |\xi(\omega)| \left| e^{-\frac{\bar{s}}{\lambda_1 \gamma}} - e^{-\frac{\bar{r}}{\lambda_1 \gamma}} \right| \\ &\leq \kappa \|T(Q(\omega)_1) - T(Q(\omega)_2)\|_\infty \end{aligned}$$

where $\kappa = \sup_{\omega \in [r, s]} |\xi(\omega)| \left| e^{-\frac{\bar{s}}{\lambda_1 \gamma}} - e^{-\frac{\bar{r}}{\lambda_1 \gamma}} \right|$; $0 \leq \kappa < 1$. $\square$

4. Results and Discussion

4.1. Experimental Results

Here, the NIE method and the explicit formula's efficiency for the AR(p), such as the AR(1), AR(2), and AR(3) models, are compared in simulation studies on the NEEWMA control chart. The following outlines the procedure for comparing the NIE method with the explicit formula:

Step 1: Compute the control limit for the AR(p) model and set ARL_0 for the in-control process.

The first step is to determine the ARL for the in-control process, which is set to 370 in this study. Then, identify the exponential white noise (γ_0), determine the smoothing parameters, and set $\delta = 0$. Next, calculate the UCL that provides the explicit ARL based on Equation (16) and $ARL_0 = 370$.

Step 2: Compute ARL_1 for the out-of-control process.

For the out-of-control process, set $\delta = 0.001, 0.003, 0.005, 0.01, 0.03, 0.05, 0.1, 0.5$, and 1. Next, compute ARL_1 for various shift sizes using $\gamma_1 = \gamma_0(1 + \delta)$ from (17), along with the UCL value obtained from Step 1.

Step 3: Approximate ARL_1 using the NIE method, as given in (20).

The division point amount, or $m = 500$, is used to create the NIE for ARL. The computer specifications include an Intel(R) Core(TM) i5-8250U CPU and 4.00 GB of RAM to run MATHEMATICA (Version 8.0) by Wolfram Research (Champaign, IL, USA), and the computation time of CPU_{Exp} and CPU_{NIE} is shown in terms of seconds.

Step 4: Compare the ARL obtained by the explicit formula with the NIE method.

The first metric for this evaluation is the percentage accuracy (%Acc), which compares the ARL values obtained from the explicit solution, denoted as $Q(\omega)$ in (15), with those derived using the NIE method, denoted as $\vartheta(\omega)$ in (20). The percentage accuracy (%Acc) is defined as follows:

$$\%Acc = \left(1 - \left| \frac{Q(\omega) - \vartheta(\omega)}{Q(\omega)} \right| \right) \times 100. \quad (22)$$

As demonstrated in the literature by Phanyaem in 2021 [21], accuracy is represented by the absolute percentage difference (%Diff), which explains it in a similar manner as shown in (22).

In Table 1, the interval control limits were determined for both one-sided and two-sided control limits, including $[0, UCL]$ and $[0.01, UCL]$. The coefficients representing the relationship with past values (ϕ_i) were set to 0.1, and ARL_0 was also set. The results of the ARL values under AR(1), AR(2), and AR(3) models on the NEEWMA control chart,

obtained from both the explicit formula and NIE, showed equivalent values and were measured for accuracy using %Acc. The accuracy was excellent, with 100%Acc in all situations. However, the explicit formula takes less than 0.001 s, which is much faster to compute compared to the NIE, which takes about 2–3 s. As a result, it makes sense to use the explicit formula for estimating ARL to compare various control charts going forward.

Table 1. The ARL of the explicit formula and the NIE method on the NEEWMA control chart under the AR(p) model with the interval control limit $[0, s]$ and $[0.01, s]$ at $c = 0, \phi_1 = \phi_2 = \phi_3 = 0.1$.

Model	[LCL, s]		[0, s] ¹				[0.01, s] ²				
	δ	$Q(\omega)$	CPU _{Exp}	$\vartheta(\omega)$	CPU _{NIE}	%Acc	$Q(\omega)$	CPU _{Exp}	$\vartheta(\omega)$	CPU _{NIE}	%Acc
AR(1)	0	370.024	<0.001	370.024	2.390	100	370.050	<0.001	370.050	2.188	100
	0.001	308.276	<0.001	308.276	2.171	100	297.808	<0.001	297.808	2.141	100
	0.003	230.727	<0.001	230.727	2.250	100	213.814	<0.001	213.814	2.156	100
	0.005	184.007	<0.001	184.007	2.454	100	166.479	<0.001	166.479	2.187	100
	0.01	121.508	<0.001	121.508	2.251	100	106.637	<0.001	106.637	2.281	100
	0.03	49.916	<0.001	49.916	2.202	100	42.542	<0.001	42.542	2.516	100
	0.05	30.485	<0.001	30.485	2.188	100	25.896	<0.001	25.896	2.218	100
	0.1	14.574	<0.001	14.574	2.281	100	12.471	<0.001	12.471	2.204	100
	0.5	2.328	<0.001	2.328	2.187	100	2.170	<0.001	2.170	2.156	100
	1	1.386	<0.001	1.386	2.265	100	1.351	<0.001	1.351	2.204	100
AR(2)	0	370.009	<0.001	370.009	2.312	100	370.035	<0.001	370.035	2.265	100
	0.001	315.594	<0.001	315.594	2.359	100	306.127	<0.001	306.127	2.375	100
	0.003	243.513	<0.001	243.513	2.375	100	227.212	<0.001	227.212	2.36	100
	0.005	197.926	<0.001	197.926	2.36	100	180.376	<0.001	180.376	2.376	100
	0.01	134.203	<0.001	134.203	2.343	100	118.528	<0.001	118.528	2.406	100
	0.03	57.042	<0.001	57.042	2.344	100	48.765	<0.001	48.765	2.406	100
	0.05	35.261	<0.001	35.261	2.313	100	29.999	<0.001	29.999	2.296	100
	0.1	17.108	<0.001	17.108	2.375	100	14.630	<0.001	14.630	2.313	100
	0.5	2.734	<0.001	2.734	2.359	100	2.528	<0.001	2.528	2.344	100
	1	1.542	<0.001	1.542	2.453	100	1.493	<0.001	1.493	2.719	100
AR(3)	0	370.031	<0.001	370.031	2.345	100	370.020	<0.001	370.020	2.422	100
	0.001	314.388	<0.001	314.388	2.374	100	303.332	<0.001	303.332	2.329	100
	0.003	241.335	<0.001	241.335	2.406	100	222.626	<0.001	222.626	2.469	100
	0.005	195.514	<0.001	195.514	2.407	100	175.562	<0.001	175.562	2.578	100
	0.01	131.954	<0.001	131.954	2.406	100	114.344	<0.001	114.344	2.375	100
	0.03	55.746	<0.001	55.746	2.376	100	46.536	<0.001	46.536	2.296	100
	0.05	34.386	<0.001	34.386	2.406	100	28.521	<0.001	28.521	2.313	100
	0.1	16.640	<0.001	16.640	2.421	100	13.848	<0.001	13.848	2.344	100
	0.5	2.657	<0.001	2.657	2.467	100	2.395	<0.001	2.395	2.359	100
	1	1.512	<0.001	1.512	2.421	100	1.440	<0.001	1.440	2.312	100

¹ $s = 0.00329871$ for AR(1), $s = 0.0060152$ for AR(2), $s = 0.00544192$ for AR(3). ² $s = 0.01331644$ for AR(1), $s = 0.01604703$ for AR(2), $s = 0.01494955$ for AR(3).

Next, the efficiency of the NEEWMA control chart under the AR(p) model, using the explicit formula to calculate ARL, is compared with the EWMA and EEWMA control charts based on various scenarios. Along with the ARL, performance metrics like the standard

deviation of the run length (SDRL) are utilized to evaluate how effectively control charts detect out-of-control conditions [22]. The SDRL for an in-control process is computed as follows:

$$ARL_0 = \frac{1}{\alpha}, SDRL_0 = \sqrt{\frac{(1-\alpha)}{\alpha^2}}, \quad (23)$$

where α denotes a type I error. In this study, ARL_0 was set at 370, and the $SDRL_0$ was evaluated using (23), which approximates to 370 as well. Moreover, the SDRL for an out-of-control process was calculated as follows:

$$ARL_1 = \frac{1}{1-\beta}, SDRL_1 = \sqrt{\frac{\beta}{(1-\beta)^2}}, \quad (24)$$

where β denotes a type II error. In this study, ARL_1 was used to calculate the $SDRL_1$ by (24). It is possible to assess how well a control chart identifies different kinds of process changes, which helps with decision making when using it. Better performance will be indicated by a chart with lower $SDRL_1$ and ARL_1 values, which can identify changes in the mean of the process more rapidly.

Additionally, the average extra loss caused by an out-of-control situation is known as the average extra quadratic loss (AEQL) [23]. During out-of-control periods, it is computed as the average difference between the observed values and the target or desired values. The AEQL is calculated using the formula as follows:

$$AEQL = \frac{1}{\Delta} \sum_{\delta_i=\delta_{\min}}^{\delta_{\max}} (\delta_i^2 \times ARL(\delta_i)), \quad (25)$$

where δ_i represents the value of the shift change for order i in the process, and Δ is the sum of shift change values from $\delta_{\min}$ to $\delta_{\max}$. In this research, we set Δ equal to 9 from the values $\delta_{\min} = 0.001$ to $\delta_{\max} = 1$. In the case where AEQL is used as a performance metric, as shown in (25), the control chart with the lowest AEQL value is considered the most effective tool in detecting process changes.

Another metric is the performance comparison index (PCI) [23], which is used to evaluate the performance of different control charts. The PCI is a statistical measure that helps assess and rank control charts based on their ability to detect process variations while maintaining control. It is calculated as the ratio of the AEQL of a given control chart to that of the most effective control chart, which is the one with the lowest AEQL. The mathematical formula for the PCI is shown below:

$$PCI = \frac{AEQL}{AEQL_{\text{lowest}}}. \quad (26)$$

In general, a higher efficiency in terms of quickly detecting out-of-control conditions and reducing false alarms is indicated by a PCI score of 1, which is calculated from (26).

Tables 2 and 3 show a comparison of the efficiency of the EWMA, EEWMA, and NEEWMA control charts in the case of different parameters under AR(1) and AR(3) models, respectively. Then, set the values of λ for each chart as follows: for EWMA, set $\lambda = \lambda_1 = 0.05$ and 0.10, based on commonly used values in the literature; for EEWMA, set λ_1 and λ_2 to 0.05, 0.04 and 0.10, 0.08, in accordance with the typical parameter settings for the EEWMA control chart; and for NEEWMA, which uses the same λ_1, λ_2 values as EEWMA, add two values of λ_3 , namely 0.04 and 0.06, to reflect the differences in the changes. In addition, the coefficient parameters were set to various values, while ARL_1 was fixed at 370. Table 2 presents the results under the control limit $[0, UCL]$, whereas

Table 3 presents the results under the control limit $[0.01, \text{UCL}]$. In all cases, the results indicated that the NEEWMA control chart had lower ARL_1 and SDRL_1 than the EEWMA and EWMA control charts. Furthermore, this study's results show that the NEEWMA control chart, which is defined as having a high value of λ_3 , yields an ARL_1 value lower than the NEEWMA control chart set with a lower value of λ_3 for all situations. To confirm the performance of the control charts, the excellent efficiency of the NEEWMA control chart was demonstrated by the AEQL and PCI, which were computed from the ARL values in (25) and (26). Thus, the NEEWMA control chart ($\lambda_3 = 0.06$) now shows the lowest AEQL and demonstrates PCI values equal to 1. In addition, the ($\lambda_1 = 0.10$) was given ARL_1 and SDRL_1 lower than $\lambda_1 = 0.05$. Thus, $\lambda_1 = 0.10$ was alternative to compare the efficiency of various control charts for the application of real-life datasets in this research.

Table 2. Comparing the efficiency of simulated data under the AR(1) model running on various control charts with the interval control limit $[0, \text{UCL}]$ at $c = 1$, $\phi_1 = \pm 0.2$, and $\text{ARL}_0 = 370$.

ϕ_1	Control Chart	EWMA		EEWMA		NEEWMA			
		$\lambda = \lambda_1 = 0.05$		(λ_1, λ_2) (0.05, 0.04)		$(\lambda_1, \lambda_2, \lambda_3)$ (0.05, 0.04, 0.02)		$(\lambda_1, \lambda_2, \lambda_3)$ (0.05, 0.04, 0.03)	
0.2	UCL	$b = 0.0336317$		$h = 0.00298453$		$s = 0.000270314$		$s = 0.000081382$	
	δ	ARL_1	SDRL_1	ARL_1	SDRL_1	ARL_1	SDRL_1	ARL_1	SDRL_1
	0.001	334.559	334.059	307.054	306.554	281.873	281.373	271.052	270.552
	0.003	280.564	280.064	228.672	228.171	190.298	189.797	175.880	175.379
	0.005	241.396	240.895	181.820	181.319	143.173	142.672	129.670	129.169
	0.01	178.516	178.015	119.572	119.071	87.670	87.169	77.471	76.969
	0.03	86.023	85.522	48.868	48.365	32.810	32.306	28.130	27.626
	0.05	55.761	55.258	29.790	29.286	19.337	18.831	16.367	15.859
	0.1	28.696	28.192	14.210	13.701	8.815	8.300	7.326	6.807
	0.5	4.977	4.449	2.272	1.700	1.501	0.867	1.326	0.658
	1	2.522	1.959	1.365	0.706	1.104	0.338	1.056	0.244
	AEQL	0.477		0.246		0.184		0.171	
	PCI	2.789		1.436		1.077		1.000	
−0.2	UCL	$b = 0.00998054$		$h = 0.00089839$		$s = 0.000081416$		$s = 0.000024512$	
	δ	ARL_1	SDRL_1	ARL_1	SDRL_1	ARL_1	SDRL_1	ARL_1	SDRL_1
	0.001	318.572	318.072	292.766	292.266	269.697	269.197	259.935	259.435
	0.003	248.968	248.467	205.980	205.479	174.123	173.622	162.126	161.625
	0.005	204.060	203.559	158.478	157.977	128.068	127.567	117.266	116.765
	0.01	140.096	139.595	99.820	99.319	76.316	75.815	68.512	68.010
	0.03	60.703	60.201	38.703	38.200	27.661	27.156	24.224	23.719
	0.05	37.874	37.371	23.153	22.647	16.097	15.589	13.938	13.429
	0.1	18.659	18.153	10.787	10.275	7.217	6.698	6.148	5.626
	0.5	3.091	2.542	1.775	1.173	1.322	0.653	1.213	0.508
	1	1.702	1.093	1.191	0.476	1.056	0.243	1.031	0.178

Table 2. Cont.

ϕ_1	Control Chart	EWMA		EEWMA		NEEWMA			
		$\lambda = \lambda_1 = 0.05$		(λ_1, λ_2) (0.05, 0.04)		$(\lambda_1, \lambda_2, \lambda_3)$ (0.05, 0.04, 0.02)		$(\lambda_1, \lambda_2, \lambda_3)$ (0.05, 0.04, 0.03)	
	AEQL	0.315		0.206		0.171		0.163	
	PCI	1.935		1.265		1.050		1.000	
ϕ_1	Control Chart	$\lambda = \lambda_1 = 0.10$		(λ_1, λ_2) (0.10, 0.08)		$(\lambda_1, \lambda_2, \lambda_3)$ (0.10, 0.08, 0.04)		$(\lambda_1, \lambda_2, \lambda_3)$ (0.10, 0.08, 0.06)	
	UCL	$b = 0.0690283$		$h = 0.00604232$		$s = 0.000547523$		$s = 0.0001648872$	
0.2	δ	ARL ₁	SDRL ₁	ARL ₁	SDRL ₁	ARL ₁	SDRL ₁	ARL ₁	SDRL ₁
	0.001	290.147	289.647	239.421	238.920	201.857	201.356	187.548	187.047
	0.003	202.648	202.147	140.264	139.763	105.606	105.105	94.261	93.759
	0.005	155.706	155.205	99.131	98.630	71.428	70.926	62.858	62.356
	0.01	98.627	98.126	57.123	56.621	39.381	38.877	34.184	33.680
	0.03	40.056	39.552	21.078	20.572	13.944	13.434	11.945	11.434
	0.05	25.186	24.681	12.880	12.369	8.420	7.904	7.184	6.666
	0.1	13.156	12.646	6.535	6.014	4.240	3.707	3.616	3.075
	0.5	3.126	2.578	1.663	1.051	1.261	0.573	1.170	0.446
	1	1.955	1.366	1.224	0.524	1.064	0.260	1.035	0.190
	AEQL	0.331		0.196		0.162		0.155	
	PCI	2.133		1.264		1.046		1.000	
−0.2	UCL	$b = 0.02027145$		$h = 0.00181888$		$s = 0.000164915$		$s = 0.000049664$	
	δ	ARL ₁	SDRL ₁	ARL ₁	SDRL ₁	ARL ₁	SDRL ₁	ARL ₁	SDRL ₁
	0.001	259.271	258.771	217.248	216.747	185.745	185.244	173.716	173.215
	0.003	162.146	161.645	118.890	118.389	92.898	92.396	84.115	83.614
	0.005	117.942	117.441	81.775	81.273	61.853	61.351	55.399	54.896
	0.01	70.119	69.617	45.848	45.345	33.597	33.094	29.778	29.274
	0.03	26.721	26.216	16.508	16.000	11.743	11.232	10.300	9.788
	0.05	16.512	16.004	10.025	9.512	7.071	6.552	6.182	5.660
	0.1	8.493	7.977	5.072	4.545	3.569	3.028	3.123	2.575
	0.5	2.097	1.517	1.404	0.753	1.168	0.443	1.111	0.351
	1	1.434	0.789	1.117	0.362	1.034	0.189	1.019	0.139
	AEQL	0.236		0.174		0.155		0.151	
	PCI	1.561		1.154		1.028		1.000	

Table 3. Comparing the efficiency of simulated data under the AR(3) model running on various control charts with the interval control limit [0.01, UCL] at $c = 1$, $\phi_1 = 0.1$, $\phi_2 = 0.2$, $\phi_3 = \pm 0.2$, and $ARL_0 = 370$.

ϕ_3	Control Chart	EWMA		EEWMA		NEEWMA			
		$\lambda = \lambda_1 = 0.05$		(λ_1, λ_2) (0.05, 0.04)		$(\lambda_1, \lambda_2, \lambda_3)$ (0.05, 0.04, 0.02)		$(\lambda_1, \lambda_2, \lambda_3)$ (0.05, 0.04, 0.03)	
0.2	UCL	$b = 0.080001$		$h = 0.0160471$		$s = 0.01054526$		$s = 0.010163846$	
	δ	ARL ₁	SDRL ₁	ARL ₁	SDRL ₁	ARL ₁	SDRL ₁	ARL ₁	SDRL ₁
	0.001	342.687	342.187	306.323	305.823	276.423	275.923	263.637	263.137
	0.003	298.228	297.728	227.321	226.820	183.031	182.530	166.765	166.264
	0.005	263.855	263.355	180.445	179.944	136.440	135.939	121.549	121.048
	0.01	204.506	204.005	118.558	118.057	82.787	82.286	71.826	71.325
	0.03	106.406	105.905	48.771	48.268	30.982	30.478	26.066	25.561
	0.05	71.064	70.562	30.001	29.497	18.416	17.909	15.305	14.797
	0.1	37.859	37.356	14.631	14.122	8.598	8.082	7.030	6.510
	0.5	6.972	6.453	2.528	1.965	1.573	0.950	1.369	0.710
	1	3.467	2.925	1.494	0.859	1.135	0.392	1.073	0.280
	AEQL	0.655		0.268		0.189		0.173	
	PCI	3.779		1.545		1.092		1.000	
−0.2	UCL	$b = 0.1165162$		$h = 0.01902763$		$s = 0.010813445$		$s = 0.010244428$	
	δ	ARL ₁	SDRL ₁	ARL ₁	SDRL ₁	ARL ₁	SDRL ₁	ARL ₁	SDRL ₁
	0.001	356.647	356.147	311.720	311.220	281.385	280.885	268.128	267.628
	0.003	332.377	331.877	236.711	236.210	189.756	189.255	172.335	171.834
	0.005	311.037	310.537	190.551	190.050	142.787	142.286	126.577	126.076
	0.01	267.491	266.991	127.618	127.117	87.618	87.116	75.463	74.961
	0.03	168.494	167.993	53.767	53.265	33.210	32.706	27.660	27.156
	0.05	120.609	120.108	33.348	32.844	19.831	19.324	16.303	15.795
	0.1	67.395	66.893	16.422	15.914	9.310	8.796	7.523	7.005
	0.5	11.117	10.605	2.841	2.287	1.670	1.058	1.428	0.781
	1	4.962	4.434	1.623	1.006	1.167	0.441	1.090	0.313
	AEQL	0.990		0.294		0.197		0.178	
	PCI	5.566		1.656		1.107		1.000	
ϕ_3	Control Chart	$\lambda = \lambda_1 = 0.10$		(λ_1, λ_2) (0.10, 0.08)		$(\lambda_1, \lambda_2, \lambda_3)$ (0.10, 0.08, 0.04)		$(\lambda_1, \lambda_2, \lambda_3)$ (0.10, 0.08, 0.06)	
		$b = 0.156038$		$h = 0.02220982$		$s = 0.01110113$		$s = 0.010330956$	
0.2	δ	ARL ₁	SDRL ₁	ARL ₁	SDRL ₁	ARL ₁	SDRL ₁	ARL ₁	SDRL ₁
	0.001	313.733	313.233	245.953	245.452	203.355	202.854	187.107	186.606
	0.003	240.315	239.814	147.184	146.683	106.881	106.380	93.990	93.489
	0.005	194.752	194.251	104.997	104.496	72.445	71.944	62.704	62.202
	0.01	132.143	131.642	61.145	60.643	40.071	39.568	34.160	33.656
	0.03	57.872	57.370	22.875	22.369	14.306	13.797	12.031	11.520

Table 3. Cont.

ϕ_3	Control Chart	$\lambda = \lambda_1 = 0.10$		(λ_1, λ_2) (0.10, 0.08)		$(\lambda_1, \lambda_2, \lambda_3)$ (0.10, 0.08, 0.04)		$(\lambda_1, \lambda_2, \lambda_3)$ (0.10, 0.08, 0.06)	
0.2	0.05	37.104	36.601	14.078	13.568	8.696	8.180	7.288	6.769
	0.1	19.654	19.147	7.228	6.710	4.434	3.902	3.718	3.179
	0.5	4.537	4.006	1.850	1.254	1.318	0.648	1.205	0.496
	1	2.676	2.118	1.318	0.648	1.087	0.308	1.047	0.222
	AEQL	0.464		0.213		0.167		0.158	
	PCI	2.937		1.351		1.058		1.000	
	UCL	$b = 0.2367163$		$h = 0.02822904$		$s = 0.01164263$		$s = 0.010493715$	
	δ	ARL ₁	SDRL ₁	ARL ₁	SDRL ₁	ARL ₁	SDRL ₁	ARL ₁	SDRL ₁
−0.2	0.001	346.629	346.129	254.851	254.351	209.943	209.442	192.710	192.209
	0.003	307.518	307.018	157.040	156.539	112.483	111.982	98.310	97.809
	0.005	276.138	275.638	113.474	112.973	76.777	76.276	65.934	65.432
	0.01	219.465	218.964	66.992	66.490	42.753	42.250	36.098	35.594
	0.03	118.317	117.816	25.385	24.880	15.350	14.842	12.766	12.256
	0.05	79.514	79.013	15.675	15.167	9.341	8.827	7.738	7.221
	0.1	42.116	41.613	8.070	7.554	4.761	4.231	3.943	3.406
	0.5	7.602	7.085	2.025	1.441	1.372	0.715	1.237	0.542
	1	3.868	3.331	1.402	0.751	1.108	0.345	1.058	0.247
	AEQL	0.725		0.229		0.171		0.160	
	PCI	4.521		1.428		1.069		1.000	

4.2. Application to Real-Life Data

The datasets for GDP in agriculture; manufacturing; and electricity, gas, steam, and air conditioning supply—were used with a sample of 85 data points from quarterly data, ranging from Q1 2003 to Q1 2024, which can be found at https://www.nesdc.go.th/nesdb_en/ewt_news.php?nid=4521&filename=index, accessed on 18 September 2024. These datasets were fitted as AR(p) models using the t-statistics test from the Box–Jenkins method in SPSS, and the white noise of each model was tested for exponential mean using the one-sample Kolmogorov–Smirnov test. First, Dataset 1, which shows the quarterly data of the GDP agriculture dataset (unit: 100 billions), was fitted as an AR(1) model, expressed as $X_t = 2.92 + 0.756X_{t-1} + \epsilon_t$, where $\epsilon_t \sim \text{Exp}(\gamma_0 = 0.692)$. Secondly, Dataset 2, which shows the quarterly data of the GDP manufacturing dataset (unit: 100 billions), was fitted as an AR(2) model, expressed as $X_t = 8.146 + 0.719X_{t-1} + 0.262X_{t-2} + \epsilon_t$, where $\epsilon_t \sim \text{Exp}(\gamma_0 = 0.4891)$. Thirdly, Dataset 3, which shows the quarterly data of the GDP electricity, gas, steam and air conditioning supply dataset (unit: 10 billions), was fitted as an AR(3) model, expressed as $X_t = 0.856X_{t-1} - 0.603X_{t-2} + 0.743X_{t-3} + \epsilon_t$, where $\epsilon_t \sim \text{Exp}(\gamma_0 = 0.8946)$. The performance results of the NEEWMA, EWMA, and EEWMA control charts, as shown in Table 4, were evaluated with $\lambda = \lambda_1 = 0.10$. The on-control process was set with ARL₀ as 370. The findings indicate that the NEEWMA control chart, defined by $(\lambda_1, \lambda_2, \lambda_3)$ as (0.10, 0.08, 0.04) and (0.10, 0.08, 0.06), achieved lower ARL₁ and SDRL₁ values compared to the EWMA and EEWMA control charts across all scenarios and models when shifts occurred in the out-of-control process. Based on the AEQL and PCI values, the results show that the NEEWMA control chart with a lower λ_3 value exhibited the lowest AEQL

values and achieved PCI values of 1 in all scenarios, indicating that the NEEWMA control chart outperformed both the EWMA and EEWMA control charts across all settings. The NEEWMA control chart, with λ_3 closer to λ_2 , was more efficient than when λ_3 was farther from λ_2 . Furthermore, the EEWMA control chart outperformed the EWMA control chart. Findings were consistent across all cases studied. These results align with the ARL_1 and $SDRL_1$ evaluations, further confirming the superior performance of the NEEWMA chart on real-life data modeled as AR(1), AR(2), and AR(3), which showed results similar to those from the simulated data.

Table 4. Comparing the efficiency of real-life data fitted to an AR(p) model that was analyzed using various control charts with the interval control limit [0.01, UCL] at $ARL_0 = 370$.

Model	Control Chart	EWMA		EEWMA		NEEWMA			
		$\lambda_1 = 0.10$		$\lambda_1 = 0.10$ $\lambda_2 = 0.08$		$\lambda_1 = 0.10$ $\lambda_2 = 0.08$ $\lambda_3 = 0.04$		$\lambda_1 = 0.10$ $\lambda_2 = 0.08$ $\lambda_3 = 0.06$	
AR(1)	UCL	$b = 0.037713$		$h = 0.0108353$		$s = 0.0100258703$		$s = 0.0100045533$	
	δ	ARL_1	$SDRL_1$	ARL_1	$SDRL_1$	ARL_1	$SDRL_1$	ARL_1	$SDRL_1$
	0.001	293.364	292.864	236.412	235.911	196.868	196.367	182.865	182.364
	0.003	207.138	206.637	136.815	136.314	101.319	100.818	90.425	89.924
	0.005	160.014	159.513	96.094	95.593	68.004	67.502	59.837	59.335
	0.01	101.880	101.379	54.863	54.361	37.059	36.556	32.142	31.638
	0.03	41.268	40.765	19.794	19.288	12.730	12.220	10.855	10.342
	0.05	25.738	25.233	11.881	11.370	7.504	6.986	6.353	5.832
	0.1	13.162	12.652	5.816	5.293	3.619	3.079	3.054	2.505
	0.5	2.857	2.304	1.417	0.769	1.121	0.369	1.068	0.269
	1	1.757	1.153	1.109	0.348	1.019	0.139	1.008	0.090
	AEQL	0.302		0.175		0.152		0.149	
	PCI	2.035		1.181		1.026		1.000	
AR(2)	UCL	$b = 0.01000589$		$h = 0.010000042753$		$s = 0.01000000031319$		$s = 0.01000000026805$	
	δ	ARL_1	$SDRL_1$	ARL_1	$SDRL_1$	ARL_1	$SDRL_1$	ARL_1	$SDRL_1$
	0.001	215.758	215.257	192.076	191.575	172.295	171.794	164.842	164.341
	0.003	116.867	116.366	97.031	96.530	82.270	81.769	77.058	76.557
	0.005	79.769	79.268	64.429	63.927	53.484	52.981	49.695	49.192
	0.01	43.988	43.485	34.410	33.907	27.861	27.356	25.626	25.121
	0.03	14.941	14.433	11.087	10.575	8.590	8.074	7.741	7.224
	0.05	8.619	8.103	6.195	5.673	4.684	4.154	4.174	3.640
	0.1	3.944	3.408	2.736	2.179	2.057	1.475	1.841	1.244
	0.5	1.102	0.335	1.021	0.147	1.004	0.066	1.002	0.044
	1	1.013	0.113	1.001	0.034	1.000	0.010	1.000	0.005
	AEQL	0.152		0.146		0.144		0.143	
	PCI	1.061		1.019		1.004		1.000	

Table 4. Cont.

Model	Control Chart	EWMA		EEWMA		NEEWMA			
		$\lambda_1 = 0.10$		$\lambda_1 = 0.10$ $\lambda_2 = 0.08$		$\lambda_1 = 0.10$ $\lambda_2 = 0.08$ $\lambda_3 = 0.04$		$\lambda_1 = 0.10$ $\lambda_2 = 0.08$ $\lambda_3 = 0.06$	
AR(3)	UCL	$b = 0.0221977$		$h = 0.0108203$		$s = 0.010055813$		$s = 0.0100145594$	
	δ	ARL ₁	SDRL ₁	ARL ₁	SDRL ₁	ARL ₁	SDRL ₁	ARL ₁	SDRL ₁
	0.001	250.769	250.269	206.748	206.247	174.998	174.497	163.151	162.650
	0.003	152.379	151.878	109.608	109.107	85.001	84.499	76.824	76.323
	0.005	109.416	108.915	74.503	74.001	56.045	55.542	50.143	49.641
	0.01	64.147	63.645	41.289	40.786	30.163	29.659	26.727	26.222
	0.03	24.120	23.615	14.713	14.204	10.455	9.942	9.172	8.658
	0.05	14.849	14.340	8.911	8.396	6.283	5.761	5.496	4.971
	0.1	7.612	7.094	4.506	3.974	3.179	2.632	2.788	2.232
	0.5	1.911	1.320	1.310	0.638	1.119	0.365	1.075	0.285
	1	1.344	0.680	1.082	0.298	1.021	0.147	1.011	0.104
	AEQL	0.219		0.166		0.151		0.148	
	PCI	1.475		1.122		1.022		1.000	

Based on the findings mentioned above, the performance of the control charts in terms of their sensitivity to detecting shifts in the process mean was confirmed, demonstrating their ability to detect changes from a sample of 85 data points under the AR(p) model. The UCL and LCL of the control charts were symmetrically positioned around the center line, allowing for equal sensitivity in detecting process alterations in both directions. This symmetry ensures balanced performance and facilitates effective process monitoring, particularly when detecting subtle fluctuations. The LCL and UCL were designed to identify when a shift or change in the process occurs, indicating that the process is moving out of its intended state. Additionally, to detect changes with a two-sided control limit, K_1 , K_2 , and K_3 were determined to be equivalent for all variables in this study. The control charts—EWMA ($\lambda = \lambda_1 = 0.10$), EEWMA ($\lambda_1 = 0.10, \lambda_2 = 0.08$), and NEEWMA ($\lambda_1 = 0.10, \lambda_2 = 0.08, \lambda_3 = 0.04$), as shown in Figures 1–3 for AR(1), AR(2), and AR(3) models, respectively—were generated by calculating the LCL and UCL from (2), (3) for EWMA, (5), (6) for EEWMA, and (8), (9) for NEEWMA, as well as by calculating the mean and variance computed from each dataset. The effectiveness of change detection was demonstrated. In this study, the sensitivity of detecting changes in the process of the control chart under the zero state was examined when the process first shifted from a stable (in-control) state to an out-of-control state. The results show that for the data modeled by the AR(1) process, the NEEWMA control chart detects changes for the first time at the 5th sample, the EEWMA control chart identifies changes for the first time at the 10th sample, and the EWMA control chart is unable to detect changes with a sample size of 85 data points, as shown in Figure 1. For the AR(2) model shown in Figure 2, the EWMA control chart did not identify a change, as was the case with the AR(1) model. However, the NEEWMA and EEWMA control charts did identify the changes for the first time at the 8th and 19th samples, respectively. The NEEWMA control chart identified the change for the AR(3) model in Figure 3 for the first time at the 9th sample, while the EEWMA control chart did so at the 20th sample. Similar to the AR(1) and AR(2) models, the EWMA control chart was unable to identify the change. These findings indicate that the NEEWMA control chart can detect changes faster than both the EEWMA and EWMA control charts when the data used follow the AR(p) model. Therefore,

the NEEWMA control chart outperforms the EWMA and EEWMA control charts in detecting changes in the process when using data that follow the AR(p) time series model.

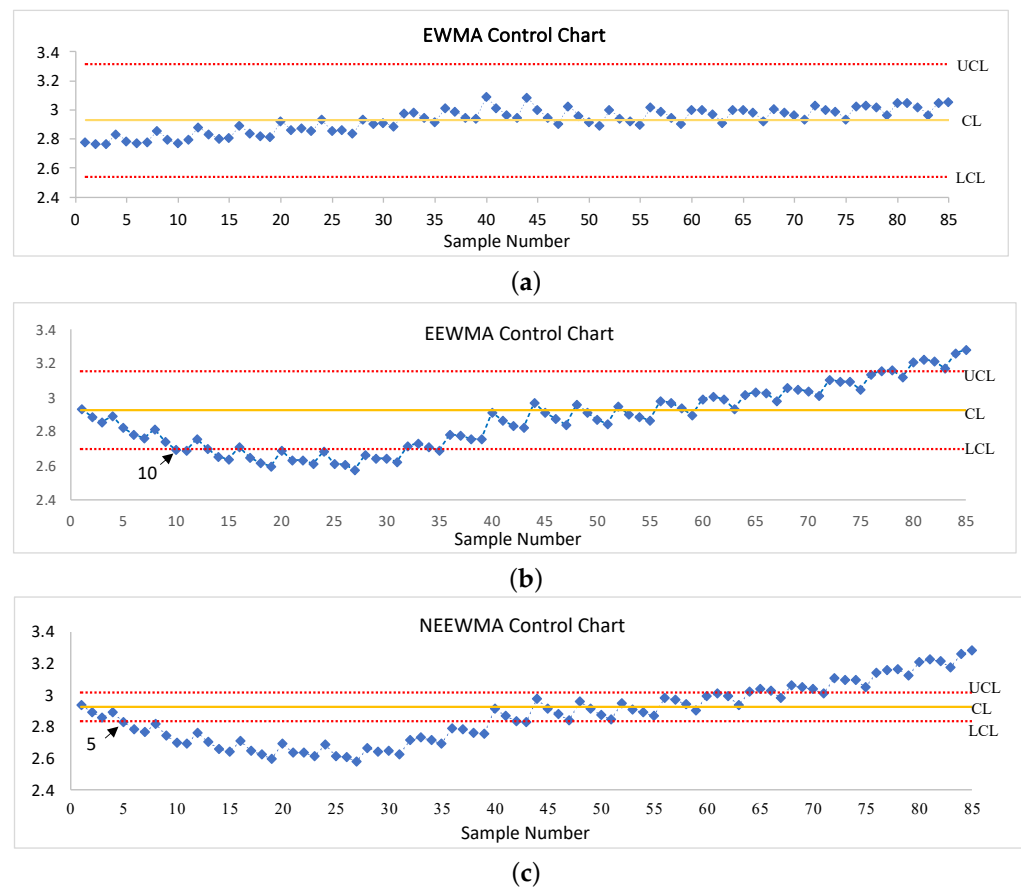


Figure 1. The efficiency of detection for the GDP Agriculture dataset fitted to an AR(1) model that was analyzed using (a) EWMA, (b) EEWMA, and (c) NEEWMA control charts.

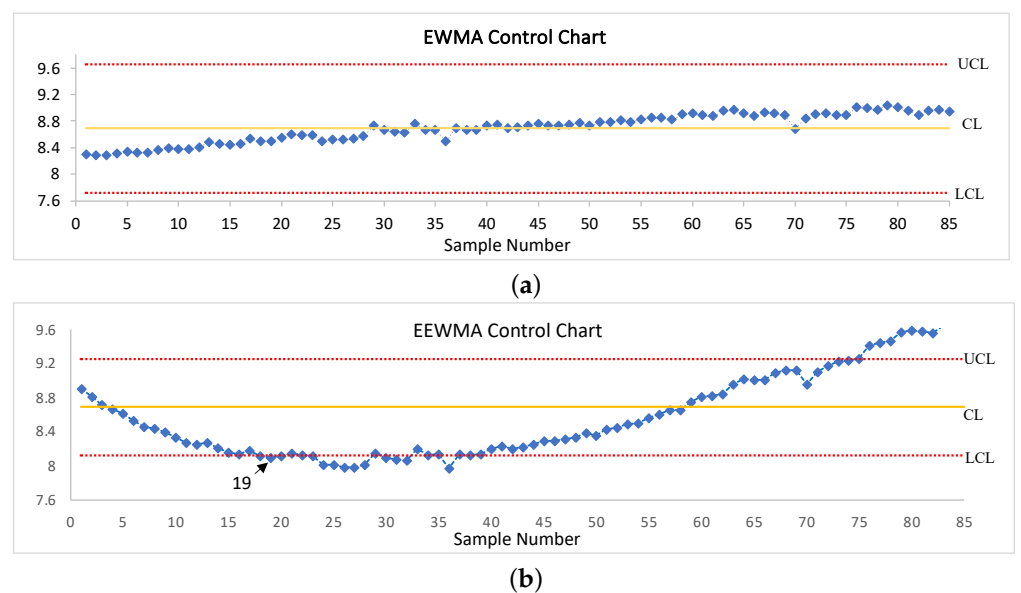


Figure 2. Cont.

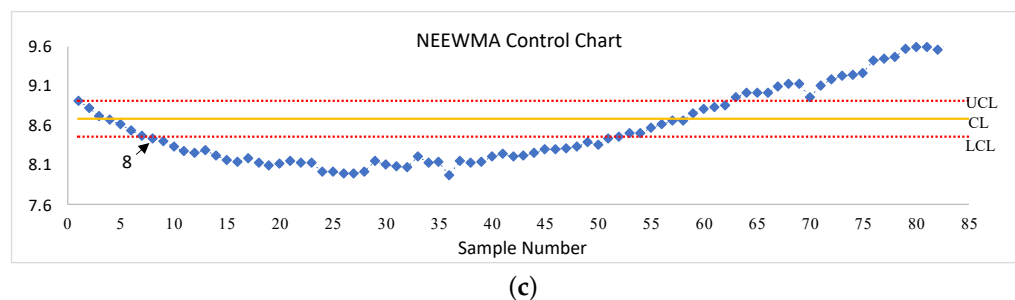


Figure 2. The efficiency of detection for the GDP Manufacturing dataset fitted to an AR(2) model that was analyzed using (a) EWMA, (b) EEWMA, and (c) NEEWMA control charts.

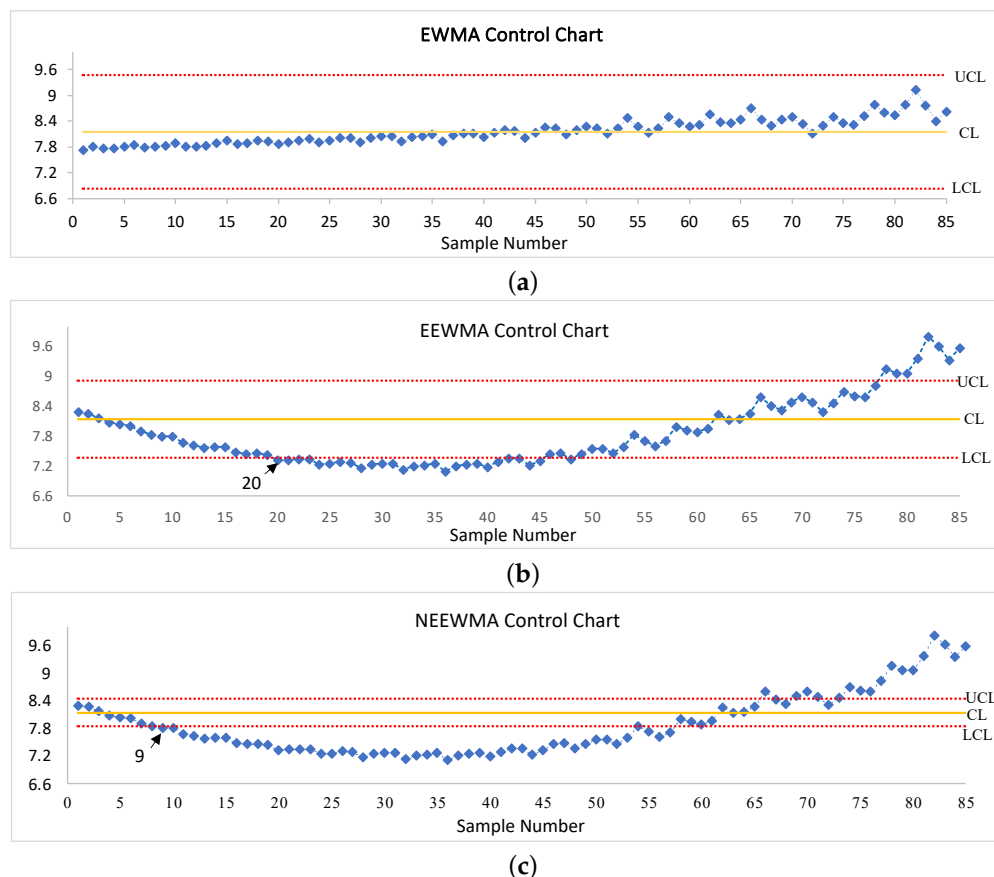


Figure 3. The efficiency of detection for the GDP electricity, gas, steam and air conditioning supply dataset fitted to an AR(3) model that was analyzed using (a) EWMA, (b) EEWMA, and (c) NEEWMA control charts.

5. Conclusions

By evaluating the ARL computation, we focused on deriving the explicit formula and NIE method based on the zero-state ARL for the AR(p) model with exponential white noise, running on the NEEWMA control chart with both symmetric one-sided and two-sided control limits. In the current study, the ARL on the NEEWMA control chart was determined by tracking the AR(p) model with exponential white noise using both approaches. Analytical ARL works better and can be computed instantly in comparison to numerical ARL, which necessitates significantly longer computation periods. Following that, the NEEWMA control chart, which was produced using an explicit formula, served as a substitute for comparing the EWMA and EEWMA control charts using the determined ARL. For all experiments conducted under the condition's investigation, it demonstrates more capability detection than the EEWMA and EWMA control charts. Based on the results,

the NEEWMA control chart is a useful tool for identifying slight to moderate shift changes in simulated and real data using the AR(p) model, especially when set to $\lambda = \lambda_1 = 0.10$. The control chart with $\lambda = \lambda_1 = 0.05$ produced worse results than this configuration, indicating that $\lambda = \lambda_1 = 0.10$ is a superior option for identifying changes in the scenarios under study. The findings reveal that the NEEWMA control chart achieved lower ARL_1 and $SDRL_1$ values compared to the EWMA and EEWMA control charts across all scenarios and models when shifts occurred in the out-of-control process. Furthermore, the results indicate that the NEEWMA control chart with a higher λ_3 value yields a lower ARL_1 value than the NEEWMA control chart with a lower λ_3 value in all scenarios. Moreover, the NEEWMA control chart with a lower λ_3 value exhibited the lowest AEQL values and achieved PCI values equal to 1 across all scenarios. This is because incorporating a lower λ_3 value enhanced the performance of the NEEWMA control chart in detecting changes, both in simulated and real-life data. Lastly, using real-life data from Thailand's quarterly GDP dataset, the effectiveness of the NEEWMA control chart under the AR(p) model in detecting process changes was confirmed to be superior to that of the EWMA and EEWMA control charts. The results show that by using time series data under the AR(p) model and analyzing it under the zero state, also known as the stable state, the focus is on the first instance when the process shows a signal indicating it is out of control. This occurs when the sample moves outside the LCL (lower control limit) and UCL (upper control limit) for the first time. The results above show that the NEEWMA control chart detected a change in the process faster than both the EEWMA control chart and the EWMA control chart, which was unable to detect changes with a sample size of 85 data points across all AR model types used in this study. As a result, the NEEWMA control chart is more effective than both the EWMA and EEWMA control charts in identifying changes in a process when the data follow the AR(p) time series model. Additionally, future observations can be tracked using the control chart, which was created from past data. The chart creates a baseline for detecting changes in subsequent data by establishing control limits based on historical data. Future data that depart from these control boundaries might suggest that the process is no longer under statistical control, which would call for additional research or remedial measures. However, the explicit ARL formulas of the NEEWMA control chart in this study can only be effectively used with data that are fitted to an AR model. Future studies should explore other models, such as SAR, MA, SMA, and ARIMA, to develop an explicit formula for the ARL of the NEEWMA control chart, making it applicable to real-world data with diverse characteristics.

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