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Solutions of Cauchy Problems for the Gardner Equation in Three Spatial Dimensions

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Abstract: In this paper, we generalize the 2 + 1-dimensional Gardner (2DG) equation to three spatial dimensions, i.e., 3 + 1 and 3 + 2 dimensions, and construct the solutions of the Cauchy problems and Lax pairs for the Gardner equation in three spatial dimensions via a novel non-local $\bar{d}$ -bar formalism. Several new long derivative operators D_x , D_y and D_t are introduced to study the initial value problems for the Gardner equation in three spatial dimensions. It follows that Propositions 1 and 3 summarize the main results of this paper.

Keywords: Gardner equation; multi-dimensional; $\bar{d}$ -bar; Cauchy problem

1. Introduction

The study of initial value problems for integrable nonlinear evolution equations (NLEEs) [1,2] has been an important research topic in mathematical physics. In 1967, Gardner, Greene, Kruskal, and Miura proposed the inverse scattering transform method for the study of Cauchy problems associated with the well-known Korteweg–de Vries (KdV) equation [3]. The inverse scattering transform method plays a great role in solving the initial value problem of the nonlinear evolution equations [4–7]. Many experts and scholars have used this method to investigate different equations. The problems of solving the Schrödinger equation [8], modified KdV equation [9], Degasperis–Procesi equation [10], $n \times n$ AKNS [11] and other equations [12–14] are investigated. The role of the Riemann–Hilbert problem is now referred to as the $\bar{\partial}$ ($\bar{d}$ -bar)-problem [15,16]. In 1981, Beals and Coifman proposed a more powerful $\bar{\partial}$ -dressing method [17,18] for solving the $\bar{\partial}$ -problem. Ablowitz and A.S. Fokas utilized the $\bar{\partial}$ -method to solve the inverse scattering problem of Kadomtsev–Petviashvili II (KP II) and Davey–Stewartson-type II (DS II) [19]. The $\bar{\partial}$ -method is widely used to solve NLEEs [20–22] with initial value problems. In recent years, the study of extended high-dimensional equations [23–25] has become the focus of attention of experts and scholars at home and abroad, especially the higher-order and high-dimensional equations with more than two spatial dimensions. For example, D.N. Chalisajar et al. studied controllability of nonlinear integro-differential third order dispersion system and obtained controllability results using two standard types of nonlinearities, namely Lipschitzian and monotone [26,27]. In [28], Fokas et al. obtained a 4 + 2 generalization of KP and the DS-type systems, the solutions of the 4 + 2-dimensional KP and DS equations are constructed in [29,30] by a nonlocal $\bar{d}$ -bar formalism. Non-local $\bar{d}$ -bar adopts the form of a two-dimensional integral: k_R and k_I , $k = k_R + ik_I$. The nonlinear Fourier transform [31,32] depends on four real parameters l_R , l_I , k_R and k_I . Searching for the initial value solutions of high-dimensional equations has been an important topic in the field of soliton theory and integrable systems. Taking three-dimensional equations as an example, in [33], Zhang et al.



Academic Editor: Christodoulos Sophocleous

Received: 27 November 2024

Revised: 29 December 2024

Accepted: 8 January 2025

Published: 11 January 2025

Citation: Zhang, Y.; Gui, L.; Feng, B. Solutions of Cauchy Problems for the Gardner Equation in Three Spatial Dimensions. *Symmetry* **2025**, *17*, 102. <https://doi.org/10.3390/sym17010102>

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extended the 2DG equation to three dimensions and constructed the Cauchy solution of the Gardner equation in $3 + 2$. Fokas et al. studied the initial value problem for the associated three-dimensional extensions of the KP equation [34], which provides a good method to study the extension of low-dimensional equations to high-dimensional equations.

In this paper, we first consider the 2DG equation, which can be constructed by two types (see Section 3). One type is $(3 + 1)$ dimensions

$$u_t + \alpha' u_{xxx} + \beta' uu_x + \gamma' u^2 u_x + \delta' \partial_x^{-1} (u_{yy} - u_{zz} + 2iu_{yz}) + \epsilon' u_x \partial_x^{-1} (u_y + iu_z) = 0, \quad (1)$$

where $\alpha', \beta', \gamma', \delta', \epsilon'$ are arbitrary constants. We need know that ∂_x^{-1} is an inverse operator of ∂_x , $\partial_x^{-1} \partial_x = \partial_x \partial_x^{-1} = 1$. When $\partial_z = 0$, $\alpha', \beta', \gamma', \delta', \epsilon'$ are real, Equation (1) reduces to the $2 + 1$ -dimensional Gardner equation.

Another is $(3 + 2)$ dimensions, as follows:

$$u_t + iu_\rho + \alpha' u_{xxx} + \beta' uu_x + \gamma' u^2 u_x + \delta' \partial_x^{-1} (u_{yy} - u_{zz} + 2iu_{yz}) + \epsilon' u_x \partial_x^{-1} (u_y + iu_z) = 0, \quad (2)$$

where $u(x, y, z, t, \rho)$ is a complex-valued function, and ρ in Equation (2) is regarded as the second time-like variable. Both equations can extend the applications by the approach proposed by Fokas. The study of the $(3 + 1)$ - and $(3 + 2)$ -dimensional extended Gardner Equations (1) and (2) also adopts the tool called the nonlocal d -bar method, which is completely similar to that by Fokas. The difference between them the approach presented in this paper is extended to a higher dimensional than that of the KP and the DS-type systems investigated by Fokas.

Propositions 1 and 3 imply that the time dependence of the nonlinear Fourier transform associated with Equations (1) and (2) involves the exponents $\exp(i\omega t + iv\rho)$ and $\exp(i\omega t - vt)$, respectively, where ω, v are specific real functions of the spectral variables l_R, l_I, ξ . By adding the dimension of the n th complex value function, we are adding an exponential term of the time and frequency. For example, the nonlinear Fourier transformation (spectral transformation) can become the linear limit. The dependence of the time and frequency shows that the studies equations are dispersive and their Cauchy problems are well posed. It should be emphasized that we introduce some new long derivative operators to reduce the $4 + 2$ -dimensional equation. The solutions of the Cauchy problem for the Gardner equation in $3 + 2$ and $3 + 1$ dimensions are constructed in the Sections 5 and 6, respectively. To the best of our knowledge, these results, i.e., solutions of the three-dimensional Gardner equation, have not appeared elsewhere.

2. Methods

The main methods in this paper are in the following: (a). Complexification of independent variables: A way for extending the $(2 + 1)$ -dimensional Gardner equation to $4 + 2$ dimensions (see Section 3). (b). Non-local d -bar formalism: The solutions of the $(3 + 1)$ -dimensional and $(3 + 2)$ -dimensional Gardner equations are obtained (see Sections 5 and 6).

3. Gardner Equation in Three Dimensions

First, we study the three-dimensional Gardner equation, namely $3 + 1$ and $3 + 2$, and construct corresponding Lax pairs. The $(2 + 1)$ -dimensional Gardner equation is investigated in [35], which has received a surge of considerable attention in recent years, to a great extent due to its emerging relevance in various applications. This equation has been investigated by many mathematicians and physicists. For example, the quasi-periodic solutions of the 2GD equation have been studied in [36]. Moreover, Casorati and

Grammian determinant solutions of the equation have been constructed in [37]. Then, by complexifying the independent variables x, y, t of the 2DG equation, we can obtain

$$u_{\bar{t}} + u_{\bar{x}\bar{x}\bar{x}} + 6\beta uu_{\bar{x}} - \frac{3}{2}\alpha^2 u^2 u_{\bar{x}} + 3\partial_{\bar{x}}^{-1} u_{\bar{y}\bar{y}} - 3\alpha u_{\bar{x}} \partial_{\bar{x}}^{-1} u_{\bar{y}} = 0, \quad (3)$$

which is the Gardner equation in $4 + 2$ dimensions, and

$$x = x_1 + ix_2, \quad y = y_1 + iy_2, \quad t = t_1 + it_2, \quad x_1, x_2, y_1, y_2, t_1, t_2 \in \mathbb{R}. \quad (4)$$

Equation (4) is as follows:

$$\partial_{\bar{x}} = \frac{1}{2}(x_1 + ix_2), \quad \partial_{\bar{y}} = \frac{1}{2}(y_1 + iy_2), \quad \partial_{\bar{t}} = \frac{1}{2}(t_1 + it_2), \quad (5)$$

and

$$(\partial_{\bar{x}}^{-1} f)(x_1, x_2) = \frac{1}{\pi} \int \int_{\mathbb{R}^2} \frac{f(x'_1, x'_2)}{x - x'} dx'_1 dx'_2, \quad (6)$$

where α and $\beta \in \mathbb{R}$.

By means of the following transformations:

$$\xi = ax_1 + bx_2, \quad \tau = \tilde{a}t_1 + \tilde{b}t_2, \quad (7)$$

where $a, b, \tilde{a}, \tilde{b} \in \mathbb{R}$, we get

$$\partial_{\bar{x}} = A \frac{\partial}{\partial \xi}, \quad \partial_{\bar{t}} = \tilde{A} \frac{\partial}{\partial \tau}, \quad A = \frac{a + ib}{2}, \quad \tilde{A} = \frac{\tilde{a} + i\tilde{b}}{2}. \quad (8)$$

Renaming u, y as $Uu, Y/\bar{Y}$, where U and Y are complex constants, Equation (3) obtains the following form:

$$u_{\tau} + \alpha' u_{\xi\xi\xi} + \beta' uu_{\xi} + \gamma' u^2 u_{\xi} + \delta' \partial_{\xi}^{-1} u_{\bar{y}\bar{y}} + \epsilon' u_{\xi} \partial_{\xi}^{-1} u_{\bar{y}}, \quad (9)$$

where

$$\alpha' = \frac{A^3}{\tilde{A}}, \quad \beta' = \frac{6\beta AU}{\tilde{A}}, \quad \gamma' = -\frac{3\alpha^2 AU^2}{2\tilde{A}}, \quad \delta' = \frac{3Y^2}{A\tilde{A}}, \quad \epsilon' = -\frac{3\alpha Y}{\tilde{A}}. \quad (10)$$

Replacing $\tau, \xi, y_1, y_2, \frac{\delta'}{4}, \frac{\epsilon'}{2}$ with $t, x, y, z, \delta', \epsilon'$, Equation (9) with $\tilde{a}t_1 = 2t, t_2 = 0$, we obtain Equation (1). Likewise, Equation (9) with $\tilde{a}t_1 = 2t, t_2 = 2\rho$ forms Equation (2).

The lax pair for Equation (3) presents

$$[D_{\bar{y}} + D_{\bar{x}}^2 + \alpha u D_{\bar{x}} + \beta u] \psi = 0, \quad (11)$$

$$[D_{\bar{t}} + 4D_{\bar{x}}^3 + \alpha u D_{\bar{x}}^2 + (3\alpha u_{\bar{x}} + \frac{3}{2}\alpha^2 u^2 + 6\beta u - 3\alpha \partial_{\bar{x}}^{-1} u_{\bar{y}}) D_{\bar{x}} + 3\beta u_x + \frac{3}{2}\alpha \beta u^2 + 3\beta \partial_{\bar{x}}^{-1} u_{\bar{y}}] \psi = 0, \quad (12)$$

where $D_{\bar{x}} = \partial_{\bar{x}} + l, D_{\bar{y}} = \partial_{\bar{y}} - l^2, D_{\bar{t}} = \partial_{\bar{t}} - 4l^3, l = l_1 + il_2$. Equation (11) is rewritten as

$$[\frac{Y}{A^2} \partial_{\bar{y}} + \partial_{\xi}^2 + 2l \partial_{\xi} + \frac{\alpha U}{A} u (\partial_{\xi} + l) + \frac{\beta U}{A^2} u] \psi = 0, \quad (13)$$

substituting l for Al into the above equation yields the following form:

$$[(\frac{\delta'}{3\alpha'})^{\frac{1}{2}} \partial_{\bar{y}} + \partial_{\xi}^2 + 2l \partial_{\xi} + \alpha (\frac{-2\gamma'}{3\alpha'^2})^{\frac{1}{2}} u (\partial_{\xi} + l) + \frac{\beta'}{6\alpha'} u] \psi = 0. \quad (14)$$

The same steps apply to Equation (11), which becomes

$$\begin{aligned} \{ \partial_\tau + \frac{4}{\alpha'} \partial_\xi^3 + \frac{12}{\alpha'} l \partial_\xi^2 + \frac{12}{\alpha'} l^2 \partial_\xi + (\frac{2\alpha^2 \gamma'}{3\alpha'^4})^{\frac{1}{2}} u (\partial_\xi + l)^2 + [(\frac{6\alpha^2 \gamma'}{\alpha'^4})^{\frac{1}{2}} u_\xi + \frac{\alpha^2 \beta'}{4\beta \alpha'^2} u^2 \\ + \frac{\beta'}{\alpha'^2} u - (\frac{-6\alpha^2 \gamma' \delta'}{\alpha'^5})^{\frac{1}{2}} \partial_\xi^{-1} u_y] u (\partial_\xi + l) + \frac{\beta'}{2\alpha'^2} u_x + (\frac{\gamma'^2 \beta'}{12\alpha'^5})^{\frac{1}{2}} \partial_\xi^{-1} u_y + \frac{\beta'}{4\alpha'^2} (\frac{-2\gamma'}{2\alpha'^2})^{\frac{1}{2}} u^2 \} \psi = 0. \end{aligned} \quad (15)$$

Renaming τ, ξ, y_1, y_2 as t, x, y, z , and renaming $\frac{\delta'}{4}, \frac{\epsilon'}{2}$ as δ', ϵ' , the lax pair of Equation (1) is constructed as follows:

$$[(\frac{\delta'}{3\alpha'})^{\frac{1}{2}} (\partial_y + i\partial_z) + \partial_x^2 + 2l\partial_x + \alpha (\frac{-2\gamma'}{3\alpha'^2})^{\frac{1}{2}} u (\partial_x + l) + \frac{\beta'}{6\alpha'} u] \psi = 0, \quad (16)$$

and

$$\begin{aligned} \{ \partial_\tau + \frac{4}{\alpha'} \partial_\xi^3 + \frac{12}{\alpha'} l \partial_\xi^2 + \frac{12}{\alpha'} l^2 \partial_\xi + (\frac{2\alpha^2 \gamma'}{3\alpha'^4})^{\frac{1}{2}} u (\partial_\xi + l)^2 + [(\frac{6\alpha^2 \gamma'}{\alpha'^4})^{\frac{1}{2}} u_\xi + \frac{\alpha^2 \beta'}{4\beta \alpha'^2} u^2 + \frac{\beta'}{\alpha'^2} u \\ - (\frac{-6\alpha^2 \gamma' \delta'}{\alpha'^5})^{\frac{1}{2}} \partial_\xi^{-1} (u_y + iu_z)] u (\partial_\xi + l) + \frac{\beta'}{2\alpha'^2} u_x + (\frac{\gamma'^2 \beta'}{12\alpha'^5})^{\frac{1}{2}} \partial_\xi^{-1} (u_y + iu_z) + \frac{\beta'}{4\alpha'^2} (\frac{-2\gamma'}{2\alpha'^2})^{\frac{1}{2}} u^2 \} \psi = 0. \end{aligned} \quad (17)$$

Similarly, Equation (2) has the following lax pair:

$$[(\frac{\delta'}{3\alpha'})^{\frac{1}{2}} (\partial_y + i\partial_z) + \partial_x^2 + 2l\partial_x + \alpha (\frac{-2\gamma'}{3\alpha'^2})^{\frac{1}{2}} u (\partial_x + l) + \frac{\beta'}{6\alpha'} u] \psi = 0, \quad (18)$$

and

$$\begin{aligned} \{ \partial_\tau + i\partial_\rho + \frac{4}{\alpha'} \partial_\xi^3 + \frac{12}{\alpha'} l \partial_\xi^2 + \frac{12}{\alpha'} l^2 \partial_\xi + (\frac{2\alpha^2 \gamma'}{3\alpha'^4})^{\frac{1}{2}} u (\partial_\xi + l)^2 + [(\frac{6\alpha^2 \gamma'}{\alpha'^4})^{\frac{1}{2}} u_\xi + \frac{\alpha^2 \beta'}{4\beta \alpha'^2} u^2 + \frac{\beta'}{\alpha'^2} u \\ - (\frac{-6\alpha^2 \gamma' \delta'}{\alpha'^5})^{\frac{1}{2}} \partial_\xi^{-1} (u_y + iu_z)] u (\partial_\xi + l) + \frac{\beta'}{2\alpha'^2} u_x + (\frac{\gamma'^2 \beta'}{12\alpha'^5})^{\frac{1}{2}} \partial_\xi^{-1} (u_y + iu_z) + \frac{\beta'}{4\alpha'^2} (\frac{-2\gamma'}{2\alpha'^2})^{\frac{1}{2}} u^2 \} \psi = 0. \end{aligned} \quad (19)$$

4. Spectral Analysis of the t -Independent Part of the Lax Pair

Consider the spectral analysis of the following equation:

$$[\partial_y + i\partial_z + \partial_x^2 + 2l\partial_x + \alpha u (\partial_x + l) + \beta u] \psi = 0, \quad (20)$$

which can be generated by the equation formed from Equation (16) by letting $\beta' \rightarrow 6\beta\alpha'$, $\gamma' \rightarrow -\frac{3}{2}\alpha'^2$, $\delta' \rightarrow 3\alpha'$. Here, we suppose $u(x, y, z) \rightarrow 0$ sufficiently rapidly as $|x|, |y|, |z| \rightarrow \infty$. We obtain the solution ψ of Equation (20):

$$\begin{aligned} \psi(x, y, z, l) = 1 + \int_{R^3} G(x - x', y - y', z - z', l) [\alpha u(x', y', z') (\partial_{x'} + l) \\ + \beta u(x', y', z')] \psi(x', y', z', l) dx' dy' dz'. \end{aligned} \quad (21)$$

Then, we get that

$$\begin{aligned} \frac{\partial \psi}{\partial l}(x, y, z, l) = F(x, y, z, l) + \int_{R^3} G(x - x', y - y', z - z', l) \\ \times [\bar{\partial} [\alpha u(x', y', z') (\partial_{x'} + l) + \beta u(x', y', z')] \psi(x', y', z', l) dx' dy' dz', \end{aligned} \quad (22)$$

where $F(x, y, z, l)$ is defined by

$$\begin{aligned} F(x, y, z, l) = \int_{R^3} \bar{\partial} G(x - x', y - y', z - z', l) [\alpha u(x', y', z') (\partial_{x'} + l) \\ + \beta u(x', y', z')] \psi(x', y', z', l) dx' dy' dz'. \end{aligned} \quad (23)$$

By using the above equation, we have

$$G_y + iG_z + G_{xx} + 2lG_x = -\delta(x)\delta(y)\delta(z), \quad (24)$$

$$G(x, y, z, l) = \frac{1}{(2\pi)^3} \int_{R^3} \frac{e^{i(\xi x + \eta y + \zeta z)}}{\xi^2 - 2il\xi - i\eta + \zeta} d\xi d\eta d\zeta, \quad (25)$$

$$\hat{G}(x, y, z, l) = \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{\xi^2 - 2il\xi - i\eta + \zeta}, \quad (26)$$

with

$$\delta(x)\delta(y)\delta(z) = \frac{1}{(2\pi)^3} \int_{R^3} e^{i(\xi x + \eta y + \zeta z)} d\xi d\eta d\zeta. \quad (27)$$

To calculate $\partial G / \partial \bar{l}$, we find

$$\xi^2 - 2il\xi - i\eta + \zeta = -2i\zeta[l + \eta/2\xi + i(\zeta + \xi^2)/2\xi], \quad (28)$$

utilizing the following property:

$$\frac{\partial}{\partial \bar{l}} \left(\frac{1}{l - \bar{l}} \right) = \pi \delta(l_1 - \bar{l}_1) \delta(l_2 - \bar{l}_2), \quad l = l_1 + il_2, \bar{l} = \bar{l}_1 + i\bar{l}_2, \quad (29)$$

we obtain the equation

$$\frac{\partial G}{\partial \bar{l}}(x, y, z, l) = -\frac{1}{4i\pi^2} \int_R \xi e^{i[\xi x - 2l_1 \xi y - (\xi^2 + 2l_2 \xi)z]} d\xi. \quad (30)$$

We know that the expression $F(x, y, z, l)$ is as follows:

$$F(x, y, z, l) = \int_R \xi \hat{u}(\xi, l) e^{i[\xi x - 2l_1 \xi y - (\xi^2 + 2l_2 \xi)z]} d\xi, \quad (31)$$

where

$$\begin{aligned} \hat{u}(\xi, l) = & -\frac{1}{4i\pi^2} \int_{R^3} \xi e^{-i[\xi x' - 2l_1 \xi y' - (\xi^2 + 2l_2 \xi)z']} \\ & \times [\alpha u(x', y', z')(\partial_{x'} + l) + \beta u(x', y', z')] \psi(x', y', z', l) dx' dy' dz'. \end{aligned} \quad (32)$$

Letting $\psi(x, y, z, l) \rightarrow \psi(x, y, z, l + i\zeta)$ in Equation (21), and multiplying by $e^{i[\xi x - 2l_1 \xi y - (\xi^2 + 2l_2 \xi)z]}$, allows us to find

$$\begin{aligned} e^{i[\xi x - 2l_1 \xi y - (\xi^2 + 2l_2 \xi)z]} \psi(x, y, z, l + i\zeta) = & e^{i[\xi x - 2l_1 \xi y - (\xi^2 + 2l_2 \xi)z]} \\ & + \int_{R^3} \check{G}(x - x', y - y', z - z', l, \xi) [\alpha u(x', y', z')(\partial_{x'} + l) + \beta u(x', y', z')] \\ & \times e^{-i[\xi x' - 2l_1 \xi y' - (\xi^2 + 2l_2 \xi)z']} \psi(x', y', z', l + i\zeta) dx' dy' dz', \end{aligned} \quad (33)$$

where

$$\check{G}(x, y, z, l, \xi) = \frac{1}{(2\pi)^3} \int_{R^3} \frac{e^{i[(\check{\xi} + \xi)x + (\check{\eta} - 2l_1 \check{\xi})y + (\check{\zeta} - \xi^2 - 2l_2 \check{\xi})z]}}{\check{\xi}^2 - 2il\check{\xi} + 2\check{\zeta}\xi - i\check{\eta} + \xi} d\check{\xi} d\check{\eta} d\check{\zeta}. \quad (34)$$

Bringing the transformations

$$\check{\xi} = \xi + \zeta, \quad \check{\eta} = \eta - 2l_1 \zeta, \quad \check{\zeta} = \zeta - \xi^2 - 2l_2 \zeta, \quad (35)$$

into Equation (34) yields

$$\check{G}(x, y, z, l, \xi) = \frac{1}{(2\pi)^3} \int_{R^3} \frac{e^{i(\check{\xi}x + \check{\eta}y + \check{\zeta}z)}}{\check{\xi}^2 - 2il\check{\xi} - i\check{\eta} + \xi} d\check{\xi} d\check{\eta} d\check{\zeta} = G(x, y, z, l). \quad (36)$$

In Equation (33), replacing $\check{G}(x - x', y - y', z - z', l, \xi)$ with $G(x - x', y - y', z - z', l)$, we obtain a d -bar problem as follows:

$$\frac{\partial \psi}{\partial l}(x, y, z, l) = \int_R \xi \hat{u}(\xi, l) e^{i[\xi x - 2l_1 \xi y - (\xi^2 + 2l_2 \xi)z]} \psi(x, y, z, l + i\xi) d\xi, \quad (37)$$

where $\psi(x, y, z, l)$ is to 1 as l is to infinity. The solution of (33) is constructed as

$$\psi(x, y, z, l) = 1 - \frac{1}{\pi} \int_{R^3} \frac{d\xi dk_R dk_I}{k - l} \xi e^{i[\xi x - 2k_1 \xi y - (\xi^2 + 2k_2 \xi)z]} \hat{u}(\xi, k) \psi(x, y, z, k + i\xi), \quad (38)$$

and $l \in \mathbb{C}, k = k_1 + ik_2$.

Equation (38) implies

$$\psi(x, y, z, l) = 1 + \frac{\psi_1(x, y, z)}{l} + O(l^{-2}), \quad l \rightarrow \infty, \quad (39)$$

where

$$\psi_1(x, y, z) = \frac{1}{\pi} \int_{R^3} \xi e^{i[\xi x - 2k_1 \xi y - (\xi^2 + 2k_2 \xi)z]} \hat{u}(\xi, k) \psi(x, y, z, k + i\xi) d\xi dk_R dk_I. \quad (40)$$

Substituting Equation (39) into Equation (20) gives

$$u(x, y, z) = -\frac{2}{\alpha} \partial_x \ln(\beta + \alpha \psi_1(x, y, z)). \quad (41)$$

and

$$u(x, y, z) = -\frac{2}{\alpha} \partial_x \ln(\beta + \alpha \frac{1}{\pi} \int_{R^3} \xi e^{i[\xi x - 2k_1 \xi y - (\xi^2 + 2k_2 \xi)z]} \hat{u}(\xi, k) \psi(x, y, z, k + i\xi) d\xi dk_R dk_I). \quad (42)$$

In summary, based on the above spectral analysis, we obtain two transformations—the direct transform (32) and inverse transform (42)—which form the nonlinear Fourier transform pair. In the linear limit of $u, \psi \rightarrow 1$, we can obtain

$$\hat{u}(\xi, l) = -\frac{1}{4i\pi^2} \int_{R^3} \xi e^{i[\xi x - 2l_1 \xi y - (\xi^2 + 2l_2 \xi)z]} [\alpha u(x, y, z)(\partial_x + l) + \beta u(x, y, z)] dx dy dz, \quad (43)$$

$$u(x, y, z) = -\frac{2}{\alpha} \partial_x \ln(\beta + \alpha \frac{1}{\pi} \int_{R^3} \xi e^{i[\xi x - 2l_1 \xi y - (\xi^2 + 2l_2 \xi)z]} \hat{u}(\xi, l) d\xi dl_R dl_I). \quad (44)$$

By variable transformations $l_1 = \xi, l_2 = -2l_1 \xi$ and $l_3 = -\xi^2 - 2l_2 \xi$, we can obtain the following three-dimensional Fourier transform pair:

$$\hat{u}(\xi, l) = -\frac{1}{4i\pi^2} \int_{R^3} \xi e^{i(l_1 x + l_2 y + l_3 z)} [\alpha u(x, y, z)(\partial_x + l) + \beta u(x, y, z)] dx dy dz, \quad (45)$$

$$u(x, y, z) = -\frac{2}{\alpha} \partial_x \ln(\beta + \alpha \frac{1}{\pi} \int_{R^3} \xi e^{i(l_1 x + l_2 y + l_3 z)} \hat{u}(\xi, l) d\xi dl_R dl_I). \quad (46)$$

5. Solution of Gardner Equation in 3 + 2

In this section, the initial value problem of the three spatial dimensions Gardner equation with two spatial dimensions is studied.

Proposition 1. Given the complex-valued function $u(x, y, z) \in S(\mathbb{R}^3)$, whose L_1 and L_2 norms are sufficiently small, let $\psi_0(x, y, z, l)$ denote the solution of Equation (21), where u is replaced by u_0 . Assuming $\hat{u}_0(\xi, l)$ is defined by Equation (32) by letting $u \rightarrow \psi$ and $u_0 \rightarrow \psi_0$, suppose $\psi(x, y, z, t, \rho, l)$ to be the solution of Equation (38), and allow $\hat{u}(\xi, l)$ to be replaced by

$$\hat{u}(\xi, l, t, \rho) = \hat{u}_0(\xi, l) e^{i(\omega t + v\rho)} \quad (47)$$

and

$$\begin{aligned} \omega &= 4\xi^3 + 12(l_1^2 + l_2^2)\xi^2 + 6(2l_2^2 - 2l_1^2 - \alpha l_1)\xi, \\ v &= -3\xi(4l_1 + \alpha)(\xi + 2l_2). \end{aligned} \quad (48)$$

Let $u(x, y, z, t, \rho)$ be defined in Equation (42) by substituting $\hat{u}(\xi, l) \rightarrow \psi(x, y, z, l)$ and $\hat{u}(\xi, l, t, \rho) \rightarrow \psi(x, y, z, t, \rho, l)$. $u(x, y, z, t, \rho)$ solves

$$\begin{aligned} u_t + iu_\rho + u_{xxx} + 6\beta uu_x - \frac{3}{2}\alpha^2 u^2 u_x \\ + 3\partial_x^{-1}(u_{yy} - u_{zz} + 2iu_{yz}) - 3\alpha u_x \partial_x^{-1}(u_y + iu_z) = 0. \end{aligned} \quad (49)$$

and

$$u(x, y, z, 0, 0) = u_0(x, y, z). \quad (50)$$

Proof. The equation met by u is equivalent to

$$\frac{\partial \psi}{\partial l}(l) = \int_{\mathbb{R}} \xi E(l, \xi) \psi(l + i\xi) \hat{u}_0(\xi, l) d\xi, \quad (51)$$

and

$$\psi(l) = 1 + \frac{\psi_1}{l} + O(l^{-2}), \quad l \rightarrow \infty, \quad (52)$$

where $E(l, \xi)$ reads

$$E(l, \xi) = e^{i[\xi x - 2l_1 \xi y - (\xi^2 + 2l_2 \xi)z + \omega t + v\rho]}. \quad (53)$$

For convenience, we study Equation (49) when $\alpha = 0$ and we suppress the x, y, z, t, ρ dependence of ψ and E . Then, we know that if ψ is defined by Equation (51) and u has the expression

$$u(x, y, z, t, \rho) = -\frac{2}{\beta} \psi_{1x}(x, y, z, t, \rho), \quad (54)$$

or equivalently via Equation (42), respectively, then u solves Equation (49) by letting $\hat{u}(\xi, l) \rightarrow \psi(x, y, z, l)$ and $\hat{u}(\xi, l, t, \rho) \rightarrow \psi(x, y, z, t, \rho, l)$.

By defining the following long derivative operators D_x, D_y and D_t :

$$D_x = \partial_x + l, \quad D_y = \partial_y + i\partial_z - l^2, \quad D_t = \partial_t + i\partial_\rho - 4l^3, \quad (55)$$

we know that

$$\frac{\partial}{\partial l}(D_t \psi(l)) = \int_{\mathbb{R}} E(l, \xi) D_t \psi(l + i\xi) d\chi, \quad d\chi = \xi \hat{u}_0(\xi, l) d\xi, \quad (56)$$

with

$$i\omega - v - 4l^4 = -(l + i\xi)^4. \quad (57)$$

Then, we provide the proof procedure for Equation (56) as follows:

$$\frac{\partial}{\partial l}(\psi_t(l)) = \int_{\mathbb{R}} E(l, \xi) [i\omega \psi(l + i\xi) + \psi_t(l + i\xi)] d\chi, \quad (58)$$

$$\frac{\partial}{\partial l}(i\psi_\rho(l)) = \int_{\mathbb{R}} E(l, \xi) [-v\psi(l + i\xi) + i\psi_\rho(l + i\xi)] d\chi, \quad (59)$$

$$\frac{\partial}{\partial l}(l^4 \psi(l)) = \int_{\mathbb{R}} E(l, \xi) l^4 \psi(l + i\xi) d\chi. \quad (60)$$

Organizing Equations (58)–(60) leads to Equation (56). The following two properties can be derived:

$$\frac{\partial}{\partial \bar{l}}(D_x \psi(l)) = \int_R E(l, \xi) D_x \psi(l + i\xi) d\xi, \quad (61)$$

$$\frac{\partial}{\partial \bar{l}}(D_y \psi(l)) = \int_R E(l, \xi) D_y \psi(l + i\xi) d\xi. \quad (62)$$

For the validity of the Property (62), we find that

$$i\omega - v - l^2 = -(l + i\xi)^2. \quad (63)$$

Hence, we introduce the lax pair of Equation (49) and define $Y_1(l)$ to be a (t, ρ) -independent part of the lax pair

$$Y_1(l) = (D_y + D_x^2 + \beta u)\psi, \quad (64)$$

and $Y_2(l)$ to be the (t, ρ) -dependent part

$$Y_2(l) = (D_t + 4D_x^3 + P_1 D_x + P_2)\psi, \quad (65)$$

where $p_i = p_i(x, y, z, t, \rho)$, $(i = 1, 2)$.

The term of $O(l^{-1})$ of Equation (64) presents the following:

$$\psi_{1y} + i\psi_{1z} + \psi_{1xx} + 2\psi_{2xx} + \beta u\psi_1 = 0. \quad (66)$$

and

$$\psi_{2x} = \frac{\beta}{4}\partial_x^{-1}u_y - \frac{i\beta}{4}\partial_x^{-1}u_z + \frac{\beta}{4}u_x + \frac{\beta^2}{4}u\partial_x^{-1}u. \quad (67)$$

Then, we know that

$$\begin{aligned} P_1 + 12\psi_{1x} &= 0, \\ P_2 + P_1\psi + 12\psi_{1xx} + 12\psi_{2x} &= 0. \end{aligned} \quad (68)$$

Organizing the above Equations (66)–(68) yields

$$P_1 = 6\beta u, \quad P_2 = 3\beta u_x - 3\beta\partial_x^{-1}(u_y + iu_z). \quad (69)$$

Equation (49) possesses the operators $Y_1(l)$ and $Y_2(l)$. The commutativity condition $Y_1[Y_2(\Phi)] = Y_2[Y_1(\Phi)]$ for Operators (64) and (65) is equivalent to Equation (49). Systems (64) and (65) can be regarded as the lax pair of Equation (49). The Cauchy problem for the Gardner equation in $(3 + 2)$ dimensions appears in Equations (49) and (50). The Cauchy problems could be similarly constructed for other $(3 + 2)$ -dimensional equations. $\square$

Remark 1. ω and v (48), which were defined in Equation (47), are obtained by substituting $u = \epsilon + i\lambda$ into the linear form of Equation (49), such that the real and imaginary parts of the equation are zero. And the exponential $\exp[i\xi x - 2il_1\xi y - i(\xi^2 + 2l_2\xi)z + i\omega t + iv\rho]$ satisfies the linear form of Equation (49)

Proposition 2. If the initial condition u_0 has the following relation:

$$\overline{u_0(x, y, z)} = u_0(x, y, -z), \quad (70)$$

then the solution of Equation (49) satisfies

$$\overline{u(x, y, z, t, \rho)} = u(x, y, -z, t, -\rho). \quad (71)$$

Proof. Putting $\xi \rightarrow -\xi$, $\eta \rightarrow -\eta$ into (25) yields

$$G(x, y, -z, \bar{l}) = \frac{1}{(2\pi)^3} \int_{R^3} \frac{e^{i(-\xi x - \eta y - \xi z)}}{\xi^2 + 2i\bar{l}\xi + i\eta + \xi} d\xi d\eta d\xi = \overline{G(x, y, z, l)}. \quad (72)$$

By using Equations (70) and (72), as well as by putting $z \rightarrow -z$, $l \rightarrow \bar{l}$ into Equation (21), we obtain the following:

$$\begin{aligned} \psi(x, y, -z, \bar{l}) &= 1 + \int_{R^3} G(x - x', y - y', z' - z, \bar{l}) \psi(x', y', -z', \bar{l}) \\ &\quad \times [\alpha u_0(x', y', -z')(\partial_{x'} + \bar{l}) + \beta u_0(x', y', -z')] dx' dy' dz' \\ &= 1 + \int_{R^3} \overline{G(x - x', y - y', z - z', l)} \psi(x', y', -z', \bar{l}) \\ &\quad \times [\overline{\alpha u_0(x', y', -z')}(\partial_{x'} + l) + \overline{\beta u_0(x', y', -z')}] dx' dy' dz' = \overline{\psi(x, y, z, l)}. \end{aligned} \quad (73)$$

By substituting $\xi \rightarrow -\xi$, $z \rightarrow -z$, and $l \rightarrow \bar{l}$ into Equation (32), we have

$$\begin{aligned} -\hat{u}(-\xi, \bar{l}) &= \frac{1}{4i\pi^2} \int_{R^3} (-\xi) e^{-i[-\xi x' + 2l_1 \xi y' + (\xi^2 + 2l_2 \xi)z']} \\ &\quad \times [\alpha u_0(x', y', -z')(\partial_{x'} + \bar{l}) + \beta u_0(x', y', -z')] \psi(x', y', -z', \bar{l}) dx' dy' dz' = \overline{\hat{u}(\xi, l)}. \end{aligned} \quad (74)$$

Likewise, letting $\xi \rightarrow -\xi$, $z \rightarrow -z$, $l \rightarrow \bar{l}$, $\rho \rightarrow -\rho$ into Equation (51) gives

$$\frac{\partial \psi(x, y, -z, t, -\rho, \bar{l})}{\partial \bar{l}} = \int_R (-\xi) \hat{u}_0(-\xi, \bar{l}) e^{i[-\xi x + 2\bar{\xi} l_1 y + (\xi^2 + 2\bar{\xi} l_2)z - \omega t - v\rho]} \psi(x, y, -z, \bar{l} - i\xi) d\xi. \quad (75)$$

The conjugate of Equation (51) presents

$$\frac{\partial \overline{\psi(x, y, z, t, \rho, l)}}{\partial \bar{l}} = \int_R \bar{\xi} \hat{u}_0(\bar{\xi}, l) e^{-i[\bar{\xi} x - 2\bar{\xi} l_1 y - (\bar{\xi}^2 + 2\bar{\xi} l_2)z + \omega t + v\rho]} \overline{\psi(x, y, z, l + i\bar{\xi})} d\bar{\xi}. \quad (76)$$

We can obtain the following symmetry relation by organizing Equations (74)–(76):

$$\overline{\psi(x, y, z, t, \rho, l)} = \psi(x, y, -z, t, -\rho, \bar{l}). \quad (77)$$

Hence, the definition of u implies that u satisfies Equation (71). $\square$

6. Solution of Gardner Equation in 3 + 1

Based on the analysis presented in Section 5, we similarly study the following 3 + 1-dimensional Gardner equation:

$$u_t + u_{xxx} + 6\beta u u_x - \frac{3}{2}\alpha^2 u^2 u_x + 3\partial_x^{-1}(u_{yy} - u_{zz} + 2iu_{yz}) - 3\alpha u_x \partial_x^{-1}(u_y + iu_z) = 0. \quad (78)$$

After an analysis similar to that made in Proposition 1, we let

$$\hat{u}(\xi, l, t, \rho) = \hat{u}_0(\xi, l) e^{i\omega t + 3\xi(4l_1 + \alpha)(\xi + 2l_2)t}. \quad (79)$$

In order for the definition $\psi(x, y, z, t, l, \xi)$ to be valid and to hold, the following condition is necessary:

$$3\xi(4l_1 + \alpha)(\xi + 2l_2) \leq 0. \quad (80)$$

Then, we need to control the sign of ξ , $\xi + 2l_2$ and $4l_1 + \alpha$ simultaneously. For convenience, we study the case when $\alpha = 0$. Thus, the operator

$$\Delta_{l_1, l_2, \xi}[\cdot] = \int_{-\infty}^0 dl_1 \left[\int_{-\infty}^0 dl_2 \left(\int_{-\infty}^0 d\xi + \int_{-2l_2}^{\infty} d\xi \right) + \int_0^{\infty} dl_2 \left(\int_{-\infty}^{-2l_2} d\xi + \int_0^{\infty} d\xi \right) \right] \\ + \int_0^{\infty} dl_1 \left(\int_{-\infty}^0 dl_2 \int_0^{-2l_2} d\xi + \int_0^{\infty} dl_2 \int_{-2k_1}^0 d\xi \right) [\cdot], \quad (81)$$

is defined. The operator $\Delta_{l_1, l_2, \xi}$ satisfies the inequality in Equation (80) in the domain of integration. By using the defined operator $\Delta_{l_1, l_2, \xi}$ and following a similar procedure as in Proposition 1, we can obtain the following:

Proposition 3. Given the complex-valued function $u(x, y, z) \in S(\mathbb{R}^3)$, whose L_1 and L_2 norms are sufficiently small, let $\psi_0(x, y, z, l)$ denote the solution of Equation (21), where u is replaced by u_0 . Assuming that $\hat{u}_0(\xi, l)$ can be defined by Equation (32) by letting $u \rightarrow \psi$ and $u_0 \rightarrow \psi_0$, suppose $\psi(x, y, z, l, t)$ to be the solution of the linear integral equation

$$\psi(x, y, z, l) = 1 - \frac{1}{\pi} \Delta_{l_1, l_2, \xi} \left[\Lambda \frac{\xi \hat{u}_0(\xi, l)}{k - l} \psi(x, y, z, k + i\xi) \right], \quad (82)$$

where

$$\Lambda := \Lambda(x, y, z, t, k_1, k_2, \xi) \\ = \exp[i\xi x - 2ik_1 \xi y - i(\xi^2 + 2k_2 \xi)z + i\omega t + 12k_1 \xi(\xi + 2k_2)t]. \quad (83)$$

We know that $u(x, y, z, t)$ has the form

$$u(x, y, z, t) = -\frac{2}{\beta} \psi_{1x}(x, y, z, t), \quad (84)$$

where ψ presents

$$\psi(x, y, z, t) = 1 + \frac{\psi_1(x, y, z)}{l} + O(l^{-2}), \quad l \rightarrow \infty, \quad (85)$$

Then, u solves the 3 + 1-dimensional Equation (78), with

$$u(x, y, z, 0) = u_0(x, y, z). \quad (86)$$

In the linear limit $\psi \rightarrow 1$, and based on the above results, the following novel transformation pair can be obtained:

$$\hat{u}(\xi, l) = -\frac{1}{4i\pi^2} \int_{\mathbb{R}^3} \beta \xi e^{-i[\xi x - 2l_1 \xi y - (\xi^2 + 2l_2 \xi)z]} u(x, y, z) dx dy dz, \quad (87)$$

$$u(x, y, z) = -\frac{2}{\beta\pi} \Delta_{l_1, l_2, \xi} [\xi e^{i[\xi x - 2l_1 \xi y - (\xi^2 + 2l_2 \xi)z]} \hat{u}(\xi, l)]. \quad (88)$$

This pair is intrinsic to the linear version of solving Equation (78), i.e., the one obtained by neglecting the term with u . It implies that $u(\xi, l)$ automatically satisfies the following condition:

$$\hat{u}(\xi, l) = 0, \quad \text{for } 12l_1 \xi(\xi + 2l_2) > 0. \quad (89)$$

An important point to emphasize is that Equation (89) is the appropriate constraint to be satisfied by the initial data for the nonlinear evolution Equation (78).

Remark 2. If the initial condition u_0 has the following relation:

$$\overline{u_0(x, y, z)} = u_0(x, y, -z), \quad (90)$$

then the solution of Equation (78) satisfies

$$\overline{u(x, y, z, t)} = u(x, y, -z, t). \quad (91)$$

Similar considerations are valid for Equation (91). Equation (91) is obtained based on some symmetry relationships.

7. Results

In this section, we outline key results of this paper: (a). The $(4 + 2)$ -dimensional Gardner Equation (3), $(3 + 1)$ -dimensional and $(3 + 2)$ -dimensional Gardner Equations (1) and (2) and their corresponding Lax pairs (16)–(19), Fourier transformation pairs (43) and (44) were obtained. (b). The solutions of $(3 + 2)$ -dimensional and $(3 + 1)$ -dimensional Gardner equations were constructed in Propositions 1 and 3. (c). The approach presented in this paper can be used to other $(2 + 1)$ -dimensional equations. Hence, the method has extensive applicability.

8. Discussion

This paper was structured as follows: In Section 3, the $4 + 2$ integrable expansion of the Gardner Equation (3) was given. It possesses some rich mathematical structures. For example, the nonlocal commutators, d -bar equations for eigenfunction ψ , Green's function G , Fourier transformations, and so on. Then, the $4 + 2$ integrable extension of the equation was reduced to Equations (1) and (2) for three spatial dimensions through a series of transformations. Moreover, we constructed their corresponding Lax pairs, which can be used to obtain a richer mathematical structure. In Section 4, by performing the spectral analysis of the t -independent part (20) of the Lax pair, a classical d -bar problem (37) was defined, and we obtained a direct transform (32) and an inverse transform (42), i.e., the nonlinear Fourier transform pair. In Sections 5 and 6, the solutions of the Cauchy problems for the Gardner equation in $3 + 2$ and $3 + 1$ dimensions were constructed via a non-local d -bar formalism, which were proposed in Propositions 1 and 3, respectively. Several propositions given in this paper summarize the main results.

9. Conclusions

In this paper, solutions of Cauchy problems for the Gardner equation in three spatial dimensions were solved. The three-dimensional Gardner equations given in this paper have many potential applications in plasma physics, fluid dynamics, and other fields. For example, the implicit expression of a traveling wave solution, the Lie point symmetry and the group invariant solutions to the three-dimensional Gardner equation were investigated in ref. [38]. When $\partial_z = 0$ and $\partial_z = \partial_\rho = 0$ in Equations (1) and (2), respectively, they can be derived as the 2DG equation [35], which has physical significance and potential applications. Extending the low-dimensional equations to high-dimensional equations and investigating their various properties and solutions, such as soliton solutions, complexiton solutions, rational solutions, etc., via non-local d -bar method may be one of the future research topics. Various types of exact solutions of the Gardner equation in three spatial dimensions are being analyzed. The methods and results presented in this paper may provide a good inspiration for dealing with similar high-dimensional nonlinear equations [39–41].

Author Contributions: Methodology, B.F.; writing original draft, Y.Z. and L.G.; writing—review and editing, Y.Z. and L.G. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the National Natural Science Foundation of China, Grant No. 12371256, and the National Natural Science Foundation of China, Grant No. 11971475.

Data Availability Statement: No new data were created or analyzed in this study.

Conflicts of Interest: The authors declare no conflicts of interest.

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