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The L -Fuzzy Prime Filter Degrees on Lattices and Its Induced L -Fuzzy Convex Structure

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Abstract: The aim of this paper is to examine the L -fuzzy prime filter degrees on lattices and their induced L -fuzzy convex structure. Firstly, the notion of L -fuzzy prime filter degrees on lattices is established using the implication operator when L is a completely distributive lattice. Secondly, an equivalent characterization of L -fuzzy prime filter degrees on lattices is provided. The equivalence relation, through the definitions of reflexivity, symmetry, and transitivity, provides a method for partitioning subsets within a lattice that possesses the prime filter property. Finally, the L -fuzzy convex structure induced by the L -fuzzy prime filter degrees on lattices is examined. Simultaneously, the properties of L -fuzzy prime filter degrees on lattices in relation to images and preimages under homomorphic mappings are discussed.

Keywords: L -fuzzy prime filter degrees; image and preimage; implication operator; homomorphic mappings



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1. Introduction

Non-classical mathematical logic theories serve as significant tools in the field of artificial intelligence for handling uncertain information. Among these, lattices are one of the many important and widely applied algebraic systems within non-classical logical algebraic systems, with filters and prime filters being two crucial concepts within lattices [1]. As is well known, filters and prime filters are important instrumental concepts in the study of the structure of logical algebras. Moreover, filters and prime filters are also widely applied in the domain of uncertainty inference. Subsequently, many scholars have pursued the study of filters and prime filters on lattices, as well as their fuzzification. The fuzzy prime filter on lattices was first introduced by Yuan and Wu [2] in 1990. Following this, Thomas et al. [3–6] developed the fuzzy theory of lattices, defined a fuzzy lattice as a type of fuzzy algebra and provided a description of its characteristics. Zhang et al. [7] reported on fuzzy filters and fuzzy ideals of BL-algebras. In 1971, Rosenfeld [8] proposed the notion of fuzzy subgroups, and it has since seen rapid development. In 2010, Shi [9] first introduced the concept of measures for fuzzy subgroups. Providing a new tool for the study of fuzzy algebraic system theory. In 2018, Huang Fei et al. [10] proposed the concept of measures of fuzzy filters and fuzzy prime filters on lattices.

Convexity theory has become more and more important in the field of mathematics. The in-depth exploration and advancement of convexity theory have given rise to the development of numerous branches of mathematics, such as convex optimization [11], convex geometry [12], and convex analysis [13], among others. Due to the widespread presence of convex sets, convex structures are closely related to many mathematical structures. In particular, with the development of fuzzy mathematics theory, the combination of convex space theory and fuzzy mathematics theory has formed a new research direction, namely, fuzzy convex space theory. Rosa [14] introduced the concept of fuzzy convex structure in 1994. In 2009, Maruyama [15] generalized the concept to M fuzzy sets. In 2014, Shi and Xiu [16] generalized it from a completely different perspective and presented the notion

of M -fuzzifying convex structures. In 2022, Wang and An [17] proposed a definition of L -fuzzy subrings measures under completely distributive lattice conditions and its induced L -fuzzy convex structure. In the same year, Wang and Zeng [18] proposed a novel approach to the fuzzification of fields. In 2024, Wang and Xu [19] also proposed a study on the related problems of L -fuzzy vector subspace.

The main purpose of this paper is to examine the L -fuzzy prime filter degrees on lattices under completely distributive lattice conditions and its induced L -fuzzy convex structure. Also, under the condition of lattice homomorphism, the properties of L -fuzzy convex to convex mappings and L -fuzzy convex preserving mappings from the perspective of category theory are studied.

2. Preliminaries

Throughout this paper, L is a completely distributive lattice and S is a non-empty set and denotes the set of all L -fuzzy sets on S by L^S . The largest element and smallest element in L are denoted by $\top$ and $\perp$, respectively. For $a, b \in L$, we say that $a \prec b$, if forever subset $D \subseteq L$, $b \leq \bigvee D$ implies $a \leq d$ for some $d \in D$. The set $\{a \in L \mid a \prec b\}$ denoted by $\beta(b)$ is called the greatest minimal family of b [20]; In addition, For $a, b \in L$, we say that $a \not\prec b$, if forever subset $E \subseteq L$, $b \geq \bigwedge E$ implies $a \geq e$ for some $e \in E$. The set $\{a \in L \mid b \not\prec a\}$ denoted by $\alpha(b)$ is called the greatest maximal family of b [20]. An element a in L is called a co-prime element if $b \vee c \geq a$ implies $b \geq a$ or $c \geq a$ for any $b, c \in L$ [20]; an element a in L is called a prime element if $a \leq b \wedge c$ implies $a \leq b$ or $a \leq c$ for any $b, c \in L$ [21]. The set of all nonzero co-prime elements of L is denoted by $J(L)$, and the set of all non-unit prime elements of L is denoted by $P(L)$.

In a completely distributive lattice L , there exist $\beta(b)$ and $\alpha(b)$ for each $b \in L$ which have $b = \bigvee \beta(b) = \bigwedge \alpha(b)$. α is an $\bigwedge$ - $\bigcup$ mapping, β is a union-preserving mapping [22].

Definition 1 ([1]). S is a poset, if $\forall a, b \in S$, $\sup\{a, b\}$ and $\inf\{a, b\}$ always exist, then S is a lattice.

Definition 2 ([22]). L is complete lattice, $\{J_i \mid i \in I\}$ is a family of sets with I as the index set, L is called a completely distributive lattice, if any family of subsets $S_i = \{a_{ij} \mid j \in I\}$ and it satisfies the following conditions:

- (1) $\bigvee_{i \in I} \bigwedge_{j \in J} a_{ij} = \bigwedge_{f \in \prod_{i \in I} i \in I} (\bigvee_{j \in J} a_{if(i)})$;
- (2) $\bigwedge_{i \in I} \bigvee_{j \in J} a_{ij} = \bigvee_{f \in \prod_{i \in I} i \in I} (\bigwedge_{j \in J} a_{if(i)})$;

Definition 3 ([23]). For a nonempty set S . Let $M \in L^S$, $a \in L$, we define

- (1) $M_{[a]} = \{s \in S \mid a \leq M(s)\}$;
- (2) $M^{(a)} = \{s \in S \mid M(s) \not\leq a\}$;
- (3) $M_{(a)} = \{s \in S \mid a \in \beta(M(s))\}$;
- (4) $M^{[a]} = \{s \in S \mid a \notin \alpha(M(s))\}$.

In a completely distributive lattice L , there exists an implication operator $\mapsto$ in L , and for any $w, z, e \in L$, its operation is given by:

$$w \mapsto z = \bigvee \{e \in L \mid w \wedge e \leq z\}. \quad (1)$$

Next, we list some properties of the implication operation in the following Lemma.

Lemma 1 ([24]). Let $(L, \vee, \wedge)$ be a complete distribution lattice and let $\mapsto$ be the implication operator corresponding to $\wedge$, then for all $w, z, e \in L$, we have

- (1) $\top \mapsto w = w$;
- (2) $e \leq w \mapsto z \Leftrightarrow w \wedge e \leq z$;
- (3) $w \mapsto z = \top \Leftrightarrow w \leq z$.

Definition 4 ([1]). Let G be a non-empty set of S , and satisfy the following conditions:

- (1) $\forall s \in S, o \in G \Rightarrow s \vee o \in G$;
- (2) $\forall s, o \in G \Rightarrow s \wedge o \in G$;
- (3) $\forall s, o \in G, s \vee o \in G \Rightarrow s \in G$ or $o \in G$.

then G is a prime filter of S .

Definition 5 ([25]). A mapping $\zeta: L^S \rightarrow M$ is called an L -fuzzy convexity on S if it satisfies the following conditions:

- (LMC1) $\zeta(\chi_\phi) = \zeta(\chi_S) = \top_M$;
- (LMC2) If $\{\omega_q \mid q \in Q\} \subseteq L^S$ is non-empty, such that $\zeta(\bigwedge_{q \in Q} \omega_q) \geq \bigwedge_{q \in Q} \zeta(\omega_q)$;
- (LMC3) If $\{\omega_q \mid q \in Q\} \subseteq L^S$ is totally ordered and non-empty, such that $\bigwedge_{q \in Q} \zeta(\omega_q) \leq \zeta(\bigvee_{q \in Q} \omega_q)$.

Where Q is the index set, (S, ζ) is called an (L, M) -fuzzy convex space, and an (L, L) -fuzzy convex space is called an L -fuzzy convex space for short.

Definition 6 ([25]). Let (S, ζ) and $(Y, \mathcal{D})$ be an (L, M) -fuzzy convex space, we have

- (1) If for all $\xi \in L^Y$, exist $\mathcal{D}(\xi) \leq \zeta(f_L^+(\xi))$, then a mapping $f: S \rightarrow Y$ is an (L, M) -fuzzy convex preserving mapping.
- (2) If for all $\xi \in L^S$, exist $\zeta(\xi) \leq \mathcal{D}(f_L^+(\xi))$, then the mapping $f: S \rightarrow Y$ is an (L, M) -fuzzy convex-to-convex mapping;
- (3) If a mapping f is bijection, (L, M) -fuzzy convex preserving and (L, M) -fuzzy convex to convex, then the mapping $f: S \rightarrow Y$ is (L, M) -fuzzy isomorphism.

3. L-Fuzzy Prime Filter Degrees on Lattices

In this section, we first proceed to define the concept L -fuzzy prime filter degrees on lattices. In addition, the equivalence characterizations of the L -fuzzy prime filter degrees on lattices are proposed.

Definition 7. $\omega \in L^S$, the L -fuzzy prime filter degrees of ω is defined as

$$I(\omega) = \bigwedge_{s, o \in S} \{(\omega(s) \wedge \omega(o) \mapsto \omega(s \wedge o)) \wedge (\omega(s) \mapsto \omega(s \vee o)) \wedge (\omega(s \vee o) \mapsto (\omega(s) \vee \omega(o)))\}. \quad (2)$$

This means that ω is an L -fuzzy prime filter of S if and only if $I(\omega) = \top$. For convenience, we always agree that $\emptyset$ is a prime filter of S

The theorem of the L -fuzzy prime filter when $I(\omega) = \top$ can be expressed as follows.

Theorem 1. $\omega \in L^S$, if $\forall s, o \in S$, the following conditions are satisfied:

- (1) $\omega(s \wedge o) \geq \omega(s) \wedge \omega(o)$;
- (2) $\omega(s \vee o) \geq \omega(s)$;
- (3) $\omega(s) \vee \omega(o) \geq \omega(s \vee o)$. then ω is L -fuzzy prime filter of S .

Example 1. Let $S = \{0, a, 1\}$ be a lattice, $L = [0, 1]$, define operation $\wedge, \vee$ and $\mu, \kappa, \iota \in L^S$, inside:

$\wedge$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1
$\vee$	0	a	1
0	0	a	1
a	a	a	1
1	1	1	1

Let $\mu = \frac{0.2}{0} + \frac{0.4}{a} + \frac{0.9}{1}$, then

(s, o)	$\mu(s)$	$\mu(o)$	$\mu(s \wedge o)$	$\mu(s \vee o)$
$(0, a)$	0.2	0.4	0.2	0.4
$(0, 1)$	0.2	0.9	0.2	0.9
$(a, 0)$	0.4	0.2	0.2	0.4
$(a, 1)$	0.4	0.9	0.4	0.9
$(1, 0)$	0.9	0.2	0.2	0.9
$(1, a)$	0.9	0.4	0.4	0.9
$(0, 0)$	0.2	0.2	0.2	0.2
(a, a)	0.4	0.4	0.4	0.4
$(1, 1)$	0.9	0.9	0.9	0.9

$I(\mu) = \top$.

Let $\kappa = \frac{0.4}{0} + \frac{0.6}{a} + \frac{0.2}{1}$, then

(s, o)	$\kappa(s)$	$\kappa(o)$	$\kappa(s \wedge o)$	$\kappa(s \vee o)$
$(0, a)$	0.4	0.6	0.4	0.6
$(0, 1)$	0.4	0.2	0.4	0.2
$(a, 0)$	0.6	0.4	0.4	0.6
$(a, 1)$	0.6	0.2	0.6	0.2
$(1, 0)$	0.2	0.4	0.4	0.2
$(1, a)$	0.2	0.6	0.6	0.2
$(0, 0)$	0.4	0.4	0.4	0.4
(a, a)	0.6	0.6	0.6	0.6
$(1, 1)$	0.2	0.2	0.2	0.2

we can obtain the L-fuzzy prime filter degrees of κ : $I(\kappa) = 0.2$.

Let $\iota = \frac{0.4}{0} + \frac{0.3}{a} + \frac{0}{1}$, then

(s, o)	$\iota(s)$	$\iota(o)$	$\iota(s \wedge o)$	$\iota(s \vee o)$
$(0, a)$	0.4	0.3	0.4	0.3
$(0, 1)$	0.4	0	0.4	0
$(a, 0)$	0.3	0.4	0.4	0.3
$(a, 1)$	0.3	0	0.3	0
$(1, 0)$	0	0.4	0.4	0
$(1, a)$	0	0.3	0.3	0
$(0, 0)$	0.4	0.4	0.4	0.4
(a, a)	0.3	0.3	0.3	0.3
$(1, 1)$	0	0	0	0

$I(\iota) = 0$.

Lemma 2. Let $\omega \in L^S$, then for all $a \in L$, $I(\omega) \geq a$ if and only if for any $s, o \in S$,

$$\omega(s) \wedge \omega(o) \wedge a \leq \omega(s \wedge o), \omega(s) \wedge a \leq \omega(s \vee o), \omega(s \vee o) \wedge a \leq \omega(s) \vee \omega(o). \quad (3)$$

Proof. Necessity: let for any $a \in L$, have $a \leq I(\omega)$, i.e.,

$$a \leq \bigwedge_{s,o \in S} \{(\omega(s) \wedge \omega(o) \rightarrow \omega(s \wedge o)) \wedge (\omega(s) \rightarrow \omega(s \vee o)) \wedge (\omega(s \vee o) \rightarrow (\omega(s) \vee \omega(o)))\}. \quad (4)$$

it means that for any $s, o \in S$, we have $a \leq \omega(s) \wedge \omega(o) \rightarrow \omega(s \wedge o)$, $a \leq \omega(s) \rightarrow \omega(s \vee o)$ and $a \leq \omega(s \vee o) \rightarrow \omega(s) \vee \omega(o)$.

Furthermore, we can obtain

$$\omega(s) \wedge a \leq \omega(s \vee o), \omega(s) \wedge \omega(o) \wedge a \leq \omega(s \wedge o), \text{ and } \omega(s \vee o) \wedge a \leq \omega(s) \vee \omega(o).$$

Sufficiency: suppose that $a \in L$, such that for any $s, o \in S$, we have $\omega(s) \wedge a \leq \omega(s \vee o)$, $\omega(s) \wedge \omega(o) \wedge a \leq \omega(s \wedge o)$, and $\omega(s \vee o) \wedge a \leq \omega(s) \vee \omega(o)$.

Then

$$a \leq \omega(s) \wedge \omega(o) \rightarrow \omega(s \wedge o), a \leq \omega(s) \rightarrow \omega(s \vee o) \text{ and } a \leq \omega(s \vee o) \rightarrow \omega(s) \vee \omega(o). \text{ i.e.,}$$

$$a \leq (\omega(s) \wedge \omega(o) \rightarrow \omega(s \wedge o)) \wedge (\omega(s) \rightarrow \omega(s \vee o)) \wedge (\omega(s \vee o) \rightarrow (\omega(s) \vee \omega(o))). \quad (5)$$

Hence,

$$a \leq \bigwedge_{s,o \in S} \{(\omega(s) \wedge \omega(o) \rightarrow \omega(s \wedge o)) \wedge (\omega(s) \rightarrow \omega(s \vee o)) \wedge (\omega(s \vee o) \rightarrow (\omega(s) \vee \omega(o)))\}. \quad (6)$$

□

Through the above lemma and implication operation, we can obtain the following theorem.

Theorem 2. Let $\omega \in L^S$, then

$$I(\omega) = \bigvee \{a \in L \mid \omega(s) \wedge a \leq \omega(s \vee o), \omega(s) \wedge \omega(o) \wedge a \leq \omega(s \wedge o), \omega(s \vee o) \wedge a \leq \omega(s) \vee \omega(o), \forall s, o \in S\}. \quad (7)$$

An equivalent characterization of L -fuzzy prime filter degrees on lattices is given in the next theorem.

Theorem 3. Let $\omega \in L^S$, then

- (1) $I(\omega) = \bigvee \{a \in L \mid \forall c \in J(L), c \leq a, \omega_{[c]} \text{ is a prime filter of } S\};$
- (2) $I(\omega) = \bigvee \{a \in L \mid \forall c \in \beta(a), \omega_{(c)} \text{ is a prime filter of } S\}, \forall h, t \in L, \text{ exist } \beta(h \wedge t) = \beta(h) \cap \beta(t)$
- (3) $I(\omega) = \bigvee \{a \in L \mid \forall c \notin \alpha(a), \omega^{[c]} \text{ is a prime filter of } S\}, \forall h, t \in L, \text{ exist } \alpha(h \vee t) = \alpha(h) \cap \alpha(t)$
- (4) $I(\omega) = \bigvee \{a \in L \mid \forall c \in P(L), c \not\leq a, \omega^{(c)} \text{ is a prime filter of } S\};$

Only properties (2) and (3) are proved. The proofs of properties (1) and (4) are omitted since they are similar.

Proof. (2) Let $\forall s, o \in S$,

$$\omega(s) \wedge \omega(o) \wedge a \leq \omega(s \wedge o), \quad (8)$$

then $\forall c \in \beta(a), s, o \in \omega_{(c)}$, this means that

$$c \in \beta(\omega(s)) \cap \beta(\omega(o)) \cap \beta(a) = \beta(\omega(s) \wedge \omega(o) \wedge a) \subseteq \beta(\omega(s \wedge o)). \quad (9)$$

have $s \wedge o \in \omega_{(c)}$. Suppose that $\forall z \in S, \omega(s) \wedge a \leq \omega(s \vee z)$, can obtain

$$\beta(\omega(s) \wedge a) \subseteq \beta(\omega(s \vee z)), \quad (10)$$

it holds that

$$c \in \beta(\omega(s) \cap \beta(a) = \beta(\omega(s) \wedge a) \subseteq \beta(\omega(s \vee z)), \quad (11)$$

i.e., $s \vee z \in \omega_{(c)}$.

Similarly, by $\omega(s \vee o) \wedge a \leq \omega(s) \vee \omega(o)$. It can be proven that

$$\beta(\omega(s \vee o) \wedge a) \subseteq \beta(\omega(s) \vee \omega(o)) \Rightarrow \beta(\omega(s \vee o) \wedge a) \subseteq \beta(\omega(s)) \cup \beta(\omega(o)). \quad (12)$$

have $s \in \omega_{(c)}, o \in \omega_{(c)}$, this implies that $\omega_{(c)}$ is a prime filter of S . This is to say that

$$\begin{aligned} I(\omega) &= \bigvee \{a \in L \mid \omega(s) \wedge \omega(o) \wedge a \leq \omega(s \wedge o), \omega(s) \wedge a \leq \omega(s \vee o), \\ &\quad \omega(s \vee o) \wedge a \leq \omega(s) \vee \omega(o), \forall s, o \in S\} \\ &\leq \bigvee \{a \in L \mid \forall c \in \beta(a), \omega_{(c)} \text{ is a prime filter of } S\}. \end{aligned} \quad (13)$$

Conversely, let

$$a \in \left\{a \in L \mid \forall c \in \beta(a), \omega_{(c)} \text{ is a prime filter of } S\right\} \quad (14)$$

For any $a \in L, s, o \in S$, let $c \in \beta(\omega(s) \wedge \omega(o) \wedge a)$.

By

$$\beta(\omega(s) \wedge \omega(o) \wedge a) = \beta(\omega(s)) \cap \beta(\omega(o)) \cap \beta(a). \quad (15)$$

we can obtain $c \in \beta(a), s, o \in \omega_{(c)}$, due to the above assumptions, we have $s \wedge o \in \omega_{(c)}$, i.e., $c \in \beta(\omega(s \wedge o))$, this shows that

$$\omega(s) \wedge \omega(o) \wedge a \leq \omega(s \wedge o). \quad (16)$$

Similarly, it is proven that

$$\omega(s) \wedge a \leq \omega(s \vee o). \quad (17)$$

$$\omega(s \vee o) \wedge a \leq \omega(s) \vee \omega(o). \quad (18)$$

Therefore,

$$\begin{aligned} I(\omega) &= \bigvee \{a \in L \mid \omega(s) \wedge \omega(o) \wedge a \leq \omega(s \wedge o), \omega(s) \wedge a \leq \omega(s \vee o), \\ &\quad \omega(s \vee o) \wedge a \leq \omega(s) \vee \omega(o), \forall s, o \in S\} \\ &\geq \bigvee \{a \in L \mid \forall c \in \beta(a), \omega_{(c)} \text{ is a prime filter of } S\}. \end{aligned} \quad (19)$$

(3) Suppose that $a \in L$, and $\forall s, o \in S$,

$$\omega(s) \wedge \omega(o) \wedge a \leq \omega(s \wedge o). \quad (20)$$

Then, $\forall c \notin \alpha(a), s, o \in \omega^{[c]}$, we know $c \notin \alpha(\omega(s)) \cup \alpha(\omega(o)) \cup \alpha(a) = \alpha(\omega(s) \wedge \omega(o) \wedge a)$.

By the above assumptions, we know $\alpha(\omega(s) \wedge \omega(o) \wedge a) \supseteq \alpha(\omega(s \wedge o))$. Hence, $c \notin \alpha(\omega(s \wedge o))$, have $s \wedge o \in \omega^{[c]}$.

Suppose $a \in L$, and $\forall s, o \in S, \omega(s) \wedge a \leq \omega(s \vee o)$, then $\forall z \in S$, we know $\omega(s) \wedge a \leq \omega(s \vee z)$ and $\alpha(\omega(s) \wedge a) \supseteq \alpha(\omega(s \vee z))$. Hence, $c \notin \alpha(\omega(s \vee z))$, have $s \vee z \in \omega^{[c]}$.

Similarly, by $\forall s, o \in S$, if $s \vee o \in \omega^{[c]}, c \notin \alpha(\omega(s \vee o))$, we have $\forall a \in L, c \notin \alpha(a)$ and $c \notin \alpha(\omega(s \vee o) \cup \alpha(a)) = \alpha(\omega(s \vee o) \wedge a)$.

By the above assumptions, we know

$$\alpha(\omega(s) \vee \omega(o)) \subseteq \alpha(\omega(s \vee o) \wedge a) \Rightarrow \alpha(\omega(s)) \cap \alpha(\omega(o)) \subseteq \alpha(\omega(s \vee o) \wedge a). \quad (21)$$

Hence, let $\forall a \in L$, and $s, o \in S$, $\omega(s \vee o) \wedge a \leq \omega(s) \vee \omega(o)$, by $c \notin \alpha(\omega(s)) \cap \alpha(\omega(o))$, i.e., $s \in \omega^{[c]}, o \in \omega^{[c]}$, this means that $\omega^{[c]}$ is a prime filter of S and

$$a \in \left\{ a \in L \mid \forall c \notin \alpha(a), \omega^{[c]} \text{ is a prime filter of } S \right\} \quad (22)$$

It is proved that

$$\begin{aligned} I(\omega) &= \bigvee \{ a \in L \mid \omega(s) \wedge \omega(o) \wedge a \leq \omega(s \wedge o), \omega(s) \wedge a \leq \omega(s \vee o), \\ &\quad \omega(s \vee o) \wedge a \leq \omega(s) \vee \omega(o), \forall s, o \in S \} \\ &\leq \bigvee \left\{ a \in L \mid \forall c \notin \alpha(a), \omega^{[c]} \text{ is a prime filter of } S \right\}. \end{aligned} \quad (23)$$

On the contrary, $a \in \left\{ a \in L \mid \forall c \notin \alpha(a), \omega^{[c]} \text{ is a prime filter of } S \right\}$, now we prove $\forall a \in L$, and $\forall s, o \in S$,

$$\omega(s) \wedge \omega(o) \wedge a \leq \omega(s \wedge o). \quad (24)$$

Assume that $c \notin \alpha(\omega(s) \wedge \omega(o) \wedge a)$,

Due to

$$\alpha(\omega(s) \wedge \omega(o) \wedge a) = \alpha(\omega(s)) \cup \alpha(\omega(o)) \cup \alpha(a). \quad (25)$$

we have $c \notin \alpha(a)$, $s, o \in \omega^{[c]}$, since $\omega^{[c]}$ is a prime filter of S , it is concluded that $s \wedge o \in \omega^{[c]}$, $s \vee o \in \omega^{[c]}$, i.e., $c \notin \alpha(\omega(s \vee o))$, $c \notin \alpha(\omega(s \wedge o))$, we have

$$\omega(s) \wedge \omega(o) \wedge a \leq \omega(s \wedge o), \quad (26)$$

Similarly,

$$\omega(s) \wedge a \leq \omega(s \vee o), \quad (27)$$

$$\omega(s \vee o) \wedge a \leq \omega(s) \vee \omega(o). \quad (28)$$

It is proven that

$$\begin{aligned} I(\omega) &= \bigvee \{ a \in L \mid \omega(s) \wedge \omega(o) \wedge a \leq \omega(s \wedge o), \omega(s) \wedge a \leq \omega(s \vee o), \\ &\quad \omega(s \vee o) \wedge a \leq \omega(s) \vee \omega(o), \forall s, o \in S \} \\ &\geq \bigvee \left\{ a \in L \mid \forall c \notin \alpha(a), \omega^{[c]} \text{ is a prime filter of } S \right\}. \end{aligned} \quad (29)$$

□

Corollary 1. Let $\omega \in L^S$, the following statements are equivocal

- (1) ω is a prime filter of S ;
- (2) $\forall b \in J(L)$, $\omega_{[b]}$ is a prime filter of S ;
- (3) $\forall b, h \in L$, exist $\alpha(b \vee h) = \alpha(b) \cap \alpha(h)$, $\omega^{[b]}$ is a prime filter of S ;
- (4) $\forall b \in P(L)$, $\omega^{(b)}$ is a prime filter of S ;
- (5) $\forall b, h \in L$, exist $\beta(b \wedge h) = \beta(b) \cap \beta(h)$, $\omega_{(b)}$ is a prime filter of S .

4. L-Fuzzy Convex Structure Induced by L-Fuzzy Prime Filter Degrees on Lattices

In this section, we will induce the L -fuzzy convex structure based on L -fuzzy prime filter degrees on lattices. Additionally, the properties of L -fuzzy prime filter degrees on lattices between image and preimage under homomorphic mappings are discussed.

Theorem 4. Let $\omega \in L^S$. Then, I is an L -fuzzy convex structure on S . It is called an L -fuzzy convex structure and is determined by an L -fuzzy prime filter on lattices.

Proof. (LMC1) $I(\chi_\emptyset) = I(\chi_S) = \top$ is obvious.

(LMC2) $\{\omega_q \mid q \in Q\} \subseteq L^S$ is nonempty, it remains to prove that

$$I(\bigwedge_{q \in Q} \omega_q) \geq \bigwedge_{q \in Q} I(\omega_q). \quad (30)$$

assume that $a \leq \bigwedge_{q \in Q} I(\omega_q)$, By Lemma 3, know $\forall s, o \in S$,

$$\omega_q(s) \wedge \omega_q(o) \wedge a \leq \omega_q(s \wedge o), \quad (31)$$

$$\omega_q(s) \wedge a \leq \omega_q(s \vee o), \quad (32)$$

$$\omega_q(s \vee o) \wedge a \leq \omega_q(s) \vee \omega_q(o). \quad (33)$$

where $\forall q \in Q$, this implies

$$\bigwedge_{q \in Q} \omega_q(s) \wedge \bigwedge_{q \in Q} \omega_q(o) \wedge a \leq \bigwedge_{q \in Q} \omega_q(s \wedge o); \quad (34)$$

$$\bigwedge_{q \in Q} \omega_q(s) \wedge a \leq \bigwedge_{q \in Q} \omega_q(s \vee o); \quad (35)$$

$$\bigwedge_{q \in Q} \omega_q(s \vee o) \wedge a \leq \bigwedge_{q \in Q} \omega_q(s) \vee \bigwedge_{q \in Q} \omega_q(o). \quad (36)$$

It means that $a \leq I(\bigwedge_{q \in Q} \omega_q)$, therefore, infer that

$$I(\bigwedge_{q \in Q} \omega_q) \geq \bigwedge_{q \in Q} I(\omega_q). \quad (37)$$

(LMC3) $\{\omega_q \mid q \in Q\} \subseteq L^S$ is totally ordered and non-empty. In order to prove $\bigwedge_{q \in Q} I(\omega_q) \leq I(\bigvee_{q \in Q} \omega_q)$, it needs to prove $\forall a \leq \bigwedge_{q \in Q} I(\omega_q)$; we have $a \leq I(\bigvee_{q \in Q} \omega_q)$. By Lemma 3, for any $q \in Q$, we can obtain

$$\omega_q(s) \wedge \omega_q(o) \wedge a \leq \omega_q(s \wedge o); \quad (38)$$

$$\omega_q(s) \wedge a \leq \omega_q(s \vee o); \quad (39)$$

$$\omega_q(s \vee o) \wedge a \leq \omega_q(s) \vee \omega_q(o). \quad (40)$$

for any $s, o \in S$, let $h \in J(L)$, such that

$$h \prec (\bigvee_{q \in Q} \omega_q(s)) \wedge (\bigvee_{q \in Q} \omega_q(o)) \wedge a, \quad (41)$$

then, we have

$$h \prec \bigvee_{q \in Q} \omega_q(s), h \prec \bigvee_{q \in Q} \omega_q(o), h \leq a. \quad (42)$$

there have $q, j \in Q$, it means that $h \leq \omega_q(s), h \leq \omega_j(o), h \leq a$.

since $\{\omega_q \mid q \in Q\}$ is ordered, there exists $k \in Q$, we assume $\omega_q \leq \omega_k$ and $\omega_j \leq \omega_k$, it follows that $h \leq \omega_k(s) \wedge \omega_k(o) \wedge a$. by $\omega_k(s) \wedge \omega_k(o) \wedge a \leq \omega_k(s \wedge o)$. We obtain $h \leq \omega_k(s \wedge o)$. Hence, $h \leq \bigvee_{q \in Q} \omega_q(s \wedge o)$. From the arbitrariness of h we have

$$(\bigvee_{q \in Q} \omega_q(s)) \wedge (\bigvee_{q \in Q} \omega_q(o)) \wedge a \leq \bigvee_{q \in Q} \omega_q(s \wedge o); \quad (43)$$

similarly,

$$\bigvee_{q \in Q} \omega_q(s) \wedge a \leq \bigvee_{q \in Q} \omega_i(s \vee o); \quad (44)$$

$$\bigvee_{q \in Q} \omega_i(s \vee o) \wedge a \leq \bigvee_{q \in Q} (\omega_q(s) \vee \omega_q(o)) \quad (45)$$

Combining Lemma 3, we have $I(\bigvee_{q \in Q} \omega_q) \geq a$. By the arbitrariness of a , we can obtain

$$\bigwedge_{q \in Q} I(\omega_q) \leq I(\bigvee_{q \in Q} \omega_q). \quad (46)$$

□

In what follows, the properties of L-fuzzy prime filter degrees on lattices, specifically between the image and preimage under homomorphic mappings are discussed.

Theorem 5. $f : S \rightarrow M$ is a lattice homomorphism, $\omega \in L^S$, $\rho \in L^M$, have

- (1) $I(f_L^{\leftarrow}(\rho)) \geq I(\rho)$, if f is surjective, $I(f_L^{\leftarrow}(\rho)) = I(\rho)$;
- (2) $I(f_L^{\rightarrow}(\omega)) \geq I(\omega)$, if f is injective, $I(f_L^{\rightarrow}(\omega)) = I(\omega)$;

Proof. (1) From Theorem 2, we can derive the following conclusion

$$\begin{aligned} I(f_L^{\leftarrow}(\rho)) &= \bigvee \left\{ a \in L \mid \forall c \in J(L), c \leq a, (f_L^{\leftarrow}(\rho))_{[c]} \text{ is a prime filter of } S \right\} \\ &= \bigvee \left\{ a \in L \mid \forall c \in J(L), c \leq a, f_L^{\leftarrow}(\rho_{[c]}) \text{ is a prime filter of } S \right\} \\ &\geq \bigvee \left\{ a \in L \mid \forall c \in J(L), c \leq a, \rho_{[c]} \text{ is a prime filter of } S \right\}. \\ &= I(\rho). \end{aligned} \quad (47)$$

The above " $\geq$ " can be replaced by " $=$ ", if f is subjective, therefore, $I(f_L^{\leftarrow}(\rho)) = I(\rho)$.

(2) Similarly, from Theorem 2, we can derive the following conclusion

$$\begin{aligned} I(f_L^{\rightarrow}(\omega)) &= \bigvee \left\{ a \in L \mid \forall c \in P(L), c \not\leq a, (f_L^{\rightarrow}(\omega))^{(c)} \text{ is a prime filter of } S \right\} \\ &= \bigvee \left\{ a \in L \mid \forall c \in P(L), c \not\leq a, f_L^{\rightarrow}((\omega)^{(c)}) \text{ is a prime filter of } S \right\} \\ &\geq \bigvee \left\{ a \in L \mid \forall c \in P(L), c \not\leq a, \omega^{(c)} \text{ is a prime filter of } S \right\}. \\ &= I(\omega). \end{aligned} \quad (48)$$

The above " $\geq$ " can be replaced by " $=$ ", if f is injective, therefore, $I(f_L^{\rightarrow}(\omega)) = I(\omega)$. □

Corollary 2. Let $f : S \rightarrow M$ be a lattice homomorphism, let $\omega \in L^S$, $\rho \in L^M$, we have

- (1) if ρ is an L-fuzzy prime filter of M , $f_L^{\leftarrow}(\rho)$ is a prime filter of S ;
- (2) if ω is a prime filter of S , $f_L^{\rightarrow}(\omega)$ is a prime filter of M .

Corollary 3. $f : S \rightarrow M$ is a lattice homomorphism, $\omega \in L^S$, $\rho \in L^M$, I_S and I_M is the L-fuzzy convex structure induced by L-fuzzy prime filter degree on S and M , respectively. Then, $f : (S, I_S) \rightarrow (M, I_M)$ is an L-fuzzy convex-to-convex mapping and L-fuzzy convex preserving mapping.

Corollary 4. $f : S \rightarrow M$ is a lattice isomorphism mapping, let I_S and I_M be the L-fuzzy convex structure induced by L-fuzzy prime filter degree on S and M , respectively. Then, $f : (S, I_S) \rightarrow (M, I_M)$ is an L-fuzzy isomorphism.

5. Conclusions

We first give the concept of the L -fuzzy prime filter degrees on lattices using the implication operator and provide the equivalent characterization of L -fuzzy prime filter degrees on lattices. Moreover, an L -fuzzy convex structure based on an L -fuzzy prime filter degrees on lattices is induced. Additionally, the properties of L -fuzzy prime filter degrees on lattices, specifically between the image and preimage under homomorphic mappings are discussed. In future research, we will attempt to induce L -fuzzy convex structures on a variety of algebraic structures, such as effect algebras and pseudo-effect algebras, and discuss the relationship between these algebraic structures and the L -fuzzy convex structures. Importantly, this concept is also expected to play a significant role in fuzzy inference, image processing. Looking to the future, we can apply the theories discussed in this paper to practical issues such as fuzzy inference for further in-depth research.

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