

Article

The Formulae and Symmetry Property of Bernstein Type Polynomials Related to Special Numbers and Functions

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Abstract: The aim of this paper is to derive formulae for the generating functions of the Bernstein type polynomials. We give a PDE equation for this generating function. By using this equation, we give recurrence relations for the Bernstein polynomials. Using generating functions, we also derive some identities including a symmetry property for the Bernstein type polynomials. We give some relations among the Bernstein type polynomials, Bernoulli numbers, Stirling numbers, Dahee numbers, the Legendre polynomials, and the coefficients of the classical superoscillatory function associated with the weak measurements. We introduce some integral formulae for these polynomials. By using these integral formulae, we derive some new combinatorial sums involving the Bernoulli numbers and the combinatorial numbers. Moreover, we define Bezier type curves in terms of these polynomials.

Keywords: Bernstein polynomials; coefficients of the classical superoscillatory function; functional equation; generating function; Legendre polynomials; special functions; symmetry property



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1. Introduction

Special functions are one of the most important topics in mathematics, mathematical physics, and other applied sciences. In fact, special functions can be defined with the help of generating functions, infinite series, infinite multipliers, repeated differentiation, integral representations, differentials, differences, integrals, functional equations, trigonometric series, series involving orthogonal functions, etc. Similarly, special polynomials and special numbers are equally important topics in mathematics and other branches of science. Recently, the Bernstein polynomials with their symmetry properties have many applications in various areas involving approximations of functions, statistics, curved surfaces, Computer Aided Geometric Design, etc. By the aid of these polynomials, the Bezier curves and surfaces are constructed. It is well known that the Bezier curves and surfaces have very important geometric applications. In particular, the Bezier curves are used in the construction of many mathematical models in the theory of curves, the theory of splines, etc. The motivation of this article is to not only give new formulae, finite sum formulae, and relations that include the symmetry property by combining Bernstein type polynomials with other special numbers and polynomials, but also investigate some properties of generating functions for these polynomials.

In order to give the main results of this article, the following notations and definitions are needed.

The Bernoulli numbers, B_n , are defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \quad (1)$$

It is well known that both the Bernoulli numbers and their generating functions are used in many important subjects, including trigonometric functions, series expansions of hyperbolic functions, the Euler–Maclaurin sum formula, finding the value of the Riemann zeta function at negative integers, Fermat’s Last Theorem, etc. (cf. [1,2]).

The Stirling numbers of the first kind, $S_1(n, k)$, are defined by

$$\frac{(\ln(1+t))^k}{k!} = \sum_{n=0}^{\infty} S_1(n, k) \frac{t^n}{n!}.$$

In both mathematics and combinatorics, the Stirling numbers of the first kind are known to spring up in the problem of finding permutations (cf. [1–3]).

The Legendre polynomials are defined by

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n,$$

which gives

$$P_n(x) = \sum_{k=0}^n \binom{n}{k}^2 \left(\frac{x-1}{2}\right)^{n-k} \left(\frac{x+1}{2}\right)^k. \quad (2)$$

The Legendre polynomials are orthogonal polynomials with respect to constant function. These polynomials are widely used in the theory of orthogonal polynomials, in the wave functions of electrons in the orbits of an atom, in the determination of potential functions in the spherically symmetric geometry, etc. (cf. [2,4], (p. 162, Equation (9) [5])). We give a few values of Legendre polynomials given by Equation (2) as follows:

$$\begin{aligned} P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\ P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) \end{aligned}$$

The Bernstein polynomials $B_j^n(x)$ arises in the following the Bernstein operator:

$$B_n[f](x) = \sum_{j=0}^n B_j^n(x) f\left(\frac{j}{n}\right),$$

where

$$B_j^n(x) = \binom{n}{j} x^j (1-x)^{n-j}, \quad (3)$$

and $f : [0, 1] \rightarrow \mathbb{R}$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N} = \{1, 2, 3, \dots\}$, and $0 \leq j \leq n$ (cf. [6–15]).

Generating function for the Bernstein polynomials is defined by

$$\mathcal{F}_{B,k}(t, x) = \frac{(tx)^k}{k!} e^{(1-x)t} = \sum_{n=0}^{\infty} B_k^n(x) \frac{t^n}{n!} \quad (4)$$

(cf. [13,14]; see also [6] and the references cited in each of these earlier works).

The Bernstein polynomials satisfy the following well-known symmetry property:

$$B_k^j(x) = B_{j-k}^j(1-x)$$

(cf. [12]). By using (3), a few values of the Bernstein polynomials are given as follows:

$$\begin{aligned} B_0^1(x) &= 1 - x \\ B_1^1(x) &= x \\ B_0^2(x) &= (1 - x)^2 \\ B_1^2(x) &= 2x(1 - x) \\ B_2^2(x) &= x^2 \end{aligned}$$

Although the Bernstein polynomials are very simple and easy to understand and use, it is well known that they provide great convenience in mathematical modeling including real world problems and in fields such as polynomial theory, approximation theory, operator theory, geometry, probability theory, and the construction of curve families (cf. [7–23]).

Both the Bernstein polynomials and their generating functions are also related to generating functions for the coefficients of the classical superoscillatory function associated with weak measurements, which is given by

$$\left(\cos\left(\frac{y}{n}\right) + ix \sin\left(\frac{y}{n}\right) \right)^n = \sum_{k=0}^n c_k(n, x) e^{iy\left(1 - \frac{2k}{n}\right)},$$

where

$$c_k(n, a) = \binom{n}{k} \left(\frac{1+a}{2}\right)^{n-k} \left(\frac{1-a}{2}\right)^k, \quad (5)$$

$y \in \mathbb{R}$, $i^2 = -1$, and $x > 1$ (cf. [24,25]).

The Daehee numbers of the first kind, D_n , are defined by

$$\frac{\ln(1+t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!},$$

(cf. [26]).

The remainder of this article is summarized as follows:

In Section 2, we define a new family of polynomials in terms of Bernstein polynomials and give their generating function. By using this function, we give many formulae involving the Bernstein polynomials, the Legendre polynomials, and the coefficients of the classical superoscillatory function and these polynomials.

In Section 3, by applying the $\frac{\partial^m}{\partial t^m}$ derivative operator to generating function, we derive recurrence relations for the special polynomials in terms of the Bernstein polynomials. We also give some relations involving a symmetry property among the new family of polynomials, $c_k(n, x)$ and $P_n(x)$.

In Section 4, by using integral formulae, we derive some combinatorial finite sum formulae.

In Section 5, by using integral formulae, we obtain new formulae involving the new family of polynomials, the Bernoulli numbers, the Stirling numbers of the first kind, the Legendre polynomials, and the Daehee numbers.

In Section 6, we define Bezier type curves.

Finally, we complete this paper with the conclusion section.

2. Special Polynomials in Terms of Bernstein Polynomials

In this section, we define special polynomials in terms of the Bernstein polynomials with their generating function. We derive some new formulae for these polynomials, the Bernstein polynomials, the Legendre polynomials, and the coefficients of the classical superoscillatory function associated with weak measurements.

We define the following polynomials in terms of the Bernstein polynomials, which are called Bernstein type polynomials:

$$Y_k^n(x) = \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} B_k^j(x), \quad (6)$$

where $n \in \mathbb{N}_0$ and $0 \leq k \leq n$.

Since $B_k^j(x) = 0$ if $j < k$, we rewrite (6) as follows:

$$Y_k^n(x) = \sum_{j=k}^n (-1)^{n-j} \binom{n}{j} B_k^j(x). \quad (7)$$

For $n = 1, 2, 3$ and $k = 0, 1, 2, 3$ a few values of $Y_k^n(x)$ are given as follows, as shown in Figure 1:

$$\begin{aligned} Y_0^1(x) &= -x, & Y_1^1(x) &= x, & Y_0^2(x) &= x^2, & Y_1^2(x) &= -2x^2, & Y_2^2(x) &= x^2, \\ Y_0^3(x) &= -x^3, & Y_1^3(x) &= 3x^3, & Y_2^3(x) &= -3x^3, & Y_3^3(x) &= x^3. \end{aligned}$$

By implementing the computation formula, given by (7), in Mathematica [27] by the Wolfram programming language, we provide Figure 1, which depicts the Bernstein type polynomials $Y_k^n(x)$.

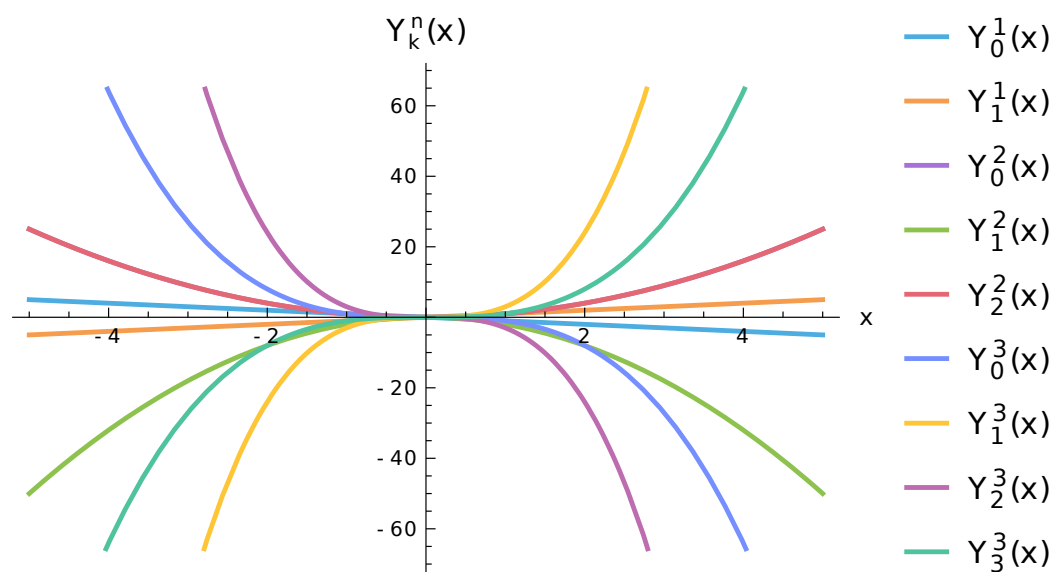


Figure 1. Depiction of the Bernstein type polynomials in the cases when $n \in \{0, 1, 2, 3\}$ and $k \in \{0, 1, 2, 3\}$.

The generating functions for $Y_k^n(x)$ are inspired by the following subdivision property of the Bernstein polynomials:

$$\mathcal{F}_{B,j}(xy, t) = \mathcal{F}_{B,j}(x, ty)e^{(1-y)t} \quad (8)$$

(cf. [28]). Therefore, we introduce the following generating functions for $Y_k^n(x)$.

Theorem 1. Let $n \in \mathbb{N}_0$ and $0 \leq k \leq n$. Then we have

$$\begin{aligned} G(t, x; k) &= \sum_{n=0}^{\infty} Y_k^n(x) \frac{t^n}{n!} \\ &= e^{-t} \mathcal{F}_{B,k}(t, x). \end{aligned} \quad (9)$$

Proof. Combining (9) and (6) yields

$$G(t, x; k) = \sum_{n=0}^{\infty} \sum_{j=0}^n \binom{n}{j} (-1)^{n-j} B_k^j(x) \frac{t^n}{n!}. \quad (10)$$

By the help of the Cauchy product rule for the series $\sum_{n=0}^{\infty} B_k^n(x) \frac{t^n}{n!}$ and $\sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!}$, Equation (10) reduces to

$$G(t, x; k) = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} \sum_{n=0}^{\infty} B_k^n(x) \frac{t^n}{n!}.$$

Therefore

$$G(t, x; k) = e^{-t} \sum_{n=0}^{\infty} B_k^n(x) \frac{t^n}{n!}.$$

Combining the above equation with (4), we obtain

$$G(t, x; k) = e^{-t} \mathcal{F}_{B,k}(t, x). \quad (11)$$

This complete proof of Theorem. $\square$

In [28], the following equation can be obtained from the subdivision property of the Bernstein basis functions.

$$\sum_{j=0}^n (-1)^{n-j} \binom{n}{j} B_k^j(y) = (-1)^{n-k} \binom{n}{k} y^n. \quad (12)$$

Comparing (6) with the (12), we obtain the following formula for $Y_k^n(x)$:

$$Y_k^n(x) = (-1)^{n-k} \binom{n}{k} y^n. \quad (13)$$

Remark 1. Substituting $n = k$ into (6), we obtain $Y_k^n(x) = x^n$, $n \in \{0, 1, 2, \dots\}$. The above monomial give a base $\{1, x, x^2, x^3, \dots\}$ for a class of polynomials.

A relation between the Bernstein polynomials $B_k^n(x)$ and Bernstein type polynomials $Y_k^n(x)$ is given by the following theorem:

Theorem 2. Let $n, k \in \mathbb{N}_0$. Then we have

$$B_k^n(x) = \sum_{j=0}^n \binom{n}{j} Y_k^j(x). \quad (14)$$

Proof. By using Equation (11), we have

$$\mathcal{F}_{B,k}(t, x) = e^t G(t, x; k).$$

Thus,

$$\sum_{n=0}^{\infty} B_k^n(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{n=0}^{\infty} Y_k^n(x) \frac{t^n}{n!}.$$

By using the Cauchy product on the right-hand side of the above equation, we obtain

$$\sum_{n=0}^{\infty} B_k^n(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^n \binom{n}{j} Y_k^j(x) \frac{t^n}{n!}.$$

Comparing coefficient $\frac{t^n}{n!}$ on the both sides of the above equation, we arrive at the desired result. $\square$

3. Recurrence Relation and Derivative Formulae for $G(t, x; k)$ and $Y_k^n(x)$

In this section, by applying the $\frac{\partial^m}{\partial t^m}$ derivative operator to $G(t, x; k)$, we derive recurrence relations for $Y_k^n(x)$ and partial derivative equation for $G(t, x; k)$ with the aid of the Bernstein polynomials and their generating function.

The derivative equation of order m for $G(t, x; k)$ is given by the following theorem.

Theorem 3. Let $k, m \in \mathbb{N}_0$. Then we have

$$\frac{\partial^m}{\partial t^m} \{G(t, x; k)\} = \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} \sum_{c=0}^j B_c^j(x) G(t, x; k - c). \quad (15)$$

Proof. By applying $\frac{\partial^m}{\partial t^m}$ to Equation (11), we obtain

$$\frac{\partial^m}{\partial t^m} \{G(t, x; k)\} = \frac{\partial^m}{\partial t^m} \{e^{-t} \mathcal{F}_{B,k}(t, x)\}.$$

Applying Leibnitz's formula to the right-hand side of the above equation, with respect to t , we obtain

$$\frac{\partial^m}{\partial t^m} \{G(t, x; k)\} = \sum_{j=0}^m \binom{m}{j} \frac{\partial^j}{\partial t^j} \{\mathcal{F}_{B,k}(t, x)\} \frac{\partial^{m-j}}{\partial t^{m-j}} \{e^{-t}\}.$$

Combining the above equation with the following formula (which was proved in Theorem 3.10, [13]):

$$\frac{\partial^j}{\partial t^j} \{\mathcal{F}_{B,k}(t, x)\} = \sum_{c=0}^j B_c^j(x) \mathcal{F}_{B,k-c}(x, t), \quad (16)$$

we obtain

$$\frac{\partial^m}{\partial t^m} \{G(t, x; k)\} = \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} \sum_{c=0}^j B_c^j(x) e^{-t} \mathcal{F}_{B,k-c}(x, t). \quad (17)$$

Combining the above equation with (11) yields the desired theorem. $\square$

Theorem 4. Let $k, n, m \in \mathbb{N}_0$. Then we have

$$Y_k^{n+m}(x) = \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} \sum_{c=0}^j B_c^j(x) Y_{k-c}^n(x). \quad (18)$$

Proof. By joining (15) with the right-hand side of (9), we obtain

$$\sum_{n=m}^{\infty} Y_k^n(x) \frac{t^{n-m}}{(n-m)!} = \sum_{j=0}^m (-1)^{m-j} \sum_{c=0}^j B_c^j(x) \sum_{n=0}^{\infty} B_{k-c}^n(x) \frac{t^n}{n!}. \quad (19)$$

After some calculations, (19) becomes

$$\sum_{n=0}^{\infty} Y_k^{n+m}(x) \frac{t^n}{n!} = \sum_{j=0}^m (-1)^{m-j} \sum_{c=0}^j B_c^j(x) \sum_{n=0}^{\infty} B_{k-c}^n(x) \frac{t^n}{n!}. \quad (20)$$

Combining coefficients of $\frac{t^n}{n!}$ in the above equation, we obtain the desired result. $\square$

By substituting $m = 1$ into (18), we obtain the following corollary:

Corollary 1. Let $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then we have

$$Y_k^{n+1}(x) = x(Y_{k-1}^n(x) - Y_k^n(x)).$$

Theorem 5. Let $k, n, m \in \mathbb{N}_0$. Then we have

$$Y_k^{n+m}(x) = \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} \sum_{c=0}^j \sum_{v=0}^n (-1)^{n-v} \binom{n}{v} B_c^j(x) B_{k-c}^v(x). \quad (21)$$

Proof. By using (17), we obtain

$$\begin{aligned} & \sum_{n=m}^{\infty} Y_k^n(x) \frac{t^{n-m}}{(n-m)!} \\ &= \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} \sum_{c=0}^j B_c^j(x) \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} \sum_{n=0}^{\infty} B_{k-c}^n(x) \frac{t^n}{n!}. \end{aligned}$$

By using the Cauchy product on the right-hand side of the above equation, we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} Y_k^{n+m}(x) \frac{t^n}{n!} \\ &= \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} \sum_{c=0}^j B_c^j(x) \sum_{n=0}^{\infty} \sum_{v=0}^n (-1)^{n-v} \binom{n}{v} B_{k-c}^v(x) \frac{t^n}{n!}. \end{aligned}$$

By comparing coefficient $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. $\square$

Relations Involving the Symmetry Property among $Y_k^n(x)$, $c_k(n, x)$, and $P_n(x)$

In this subsection, we give some relations among $Y_k^n(x)$, $c_k(n, x)$, and $P_n(x)$.

A relation between $c_k(n, x)$ and $B_k^n(x)$ is given by

$$c_k(n, x) = B_k^n\left(\frac{1-x}{2}\right)$$

(cf. [25]). By using the following symmetry property:

$$B_k^n\left(\frac{1-x}{2}\right) = B_{n-k}^n\left(1 - \frac{1-x}{2}\right),$$

we obtain the symmetry property of $c_k(n, x)$ as follows:

$$c_k(n, x) = c_{n-k}(n, -x).$$

Since

$$Y_k^n(x) = (-1)^n Y_{n-k}^n(x),$$

$Y_k^n(x)$ has no symmetry property.

Therefore,

$$\sum_{k=0}^n Y_k^n(x) = 0. \quad (22)$$

Remark 2. From Equation (22), we conclude that $Y_k^n(x)$ polynomials do not form a partition of unity.

Substituting $x = -y$ into (2), we have

$$P_n(-y) = \sum_{k=0}^n \binom{n}{k}^2 \left(\frac{1-y}{2}\right)^k \left(\frac{y+1}{2}\right)^{n-k}.$$

Combining the above equation with (5), we obtain

$$P_n(-y) = (-1)^n \sum_{k=0}^n \binom{n}{k} c_k(n, y).$$

Substituting $x = 2y - 1$ into (2), we obtain

$$P_n(2y - 1) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k}^2 (1-y)^{n-k} y^k. \quad (23)$$

Combining (23) with (3), we obtain

$$P_n(2y - 1) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} B_k^n(y). \quad (24)$$

Combining (23) with (14), we obtain

$$P_n(2y - 1) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \sum_{j=0}^n \binom{n}{j} Y_k^j(x).$$

4. Combinatorial Finite Sums Formulae

In this section, by using the same method introduced in [28], we derive some combinatorial finite sums formulae.

Theorem 6. Let $n, k \in \mathbb{N}_0$ with $n \geq k$. Then we have

$$\int_0^1 Y_k^n(x) dx = (-1)^{n-k} \binom{n}{k} \frac{1}{n+1}. \quad (25)$$

Proof. Taking the integral of Equation (7) from 0 to 1, we obtain

$$\int_0^1 Y_k^n(x) dx = \sum_{j=k}^n \binom{n}{j} (-1)^{n-j} \int_0^1 B_k^j(x) dx.$$

Combining the above equation with the following known formula:

$$\int_0^1 B_k^j(x) dx = \frac{1}{j+1}, \quad (26)$$

(cf. [12,28]), we obtain

$$\int_0^1 Y_k^n(x) dx = \sum_{j=k}^n (-1)^{n-j} \binom{n}{j} \frac{1}{j+1}. \quad (27)$$

Recalling that (cf. (Equation (37) [28]))

$$\sum_{j=k}^n (-1)^{n-j} \binom{n}{j} \frac{1}{j+1} = (-1)^{n-k} \binom{n}{k} \frac{1}{n+1},$$

and the right-hand side of (27), we obtain the desired result. $\square$

If we replace k by n and n by $2n$ in (25), we find the following relation for integral formulae of the polynomials $Y_k^n(x)$:

Corollary 2. Let $n \in \mathbb{N}_0$. Then we have

$$\int_0^1 Y_n^{2n}(x) dx = \frac{(-1)^n}{2n+1} \binom{2n}{n}.$$

By using integral of the polynomial $Y_n^k(x)$, we derive the following new finite combinatorial sums.

Theorem 7. Let $m, n \in \mathbb{N}_0$. Then we have

$$\sum_{j=0}^m \sum_{c=0}^j \sum_{v=0}^n (-1)^{k-v-j} \frac{\binom{n}{v} \binom{v}{k-c} \binom{m}{j} \binom{j}{c}}{(j+v+1) \binom{j+v}{k}} = \frac{1}{n+m+1} \binom{n+m}{k}.$$

Proof. By taking integral of both sides of Equation (21) from 0 to 1, we obtain

$$\int_0^1 Y_k^{n+m}(x) dx = \sum_{j=0}^m (-1)^{m-j} \binom{m}{j} \sum_{c=0}^j \sum_{v=0}^n (-1)^{n-v} \binom{n}{v} \int_0^1 B_c^j(x) B_{k-c}^v(x). \quad (28)$$

Since

$$\int_0^1 B_k^n(x) dx = \frac{1}{n+1}$$

Ref. [12], we obtain

$$\int_0^1 Y_k^{n+m}(x) dx = (-1)^{n+m-k} \binom{n+m}{k} \frac{1}{n+m+1}. \quad (29)$$

By combining (28) with (29) and the following known formula in Refs. [12,29]:

$$\int_0^1 B_c^j(x) B_{k-c}^v(x) dx = \frac{\binom{v}{k-c} \binom{j}{c}}{(j+v+1) \binom{j+v}{k}},$$

we obtain the desired result. $\square$

Theorem 8. Let $n \in \mathbb{N}_0$. Then we have

$$\sum_{j=0}^n \binom{n}{j} \sum_{v=0}^j \binom{j}{v} \frac{(-1)^{j-v}}{v+1} = \frac{1}{n+1}.$$

Proof. By combining (6) and (14), we obtain

$$B_k^n(x) = \sum_{j=0}^n \binom{n}{j} \sum_{v=0}^j \binom{j}{v} (-1)^{j-v} B_k^v(x).$$

By taking integral of both sides of the above equation from 0 to 1, and using (26), we arrived the desired result. $\square$

From Remark 3.2 in [22], we can write

$$\sum_{j=0}^n \binom{n}{j} \frac{x^j}{j+1} = \frac{(x+1)^{n+1} - 1}{x(n+1)}.$$

If we take $x = -1$ in the above equation, we obtain

$$\sum_{j=0}^n \binom{n}{j} \frac{(-1)^j}{j+1} = \frac{1}{n+1}.$$

5. Relations among the Integral of the Polynomials $Y_k^n(x)$ and Other Special Numbers

In this section, we give some new formulae involving the integral of the polynomials $Y_k^n(x)$, the Bernoulli numbers, the Stirling numbers of the first kind, the Legendre polynomials, and the Dahehee numbers.

Theorem 9. Let $n \in \mathbb{N}_0$. Then we have

$$\int_0^1 Y_k^n(x) dx = \frac{(-1)^k}{n!} \binom{n}{k} \sum_{j=0}^n S_1(n, l) B_l.$$

Proof. Combining Equation (25) with the following known formula (cf. [3,26,30]):

$$\frac{(-1)^n}{n+1} = \frac{1}{n!} \sum_{j=0}^n S_1(n, l) B_l$$

after some elementary calculations, we obtain the desired result. $\square$

Combining (25) with the following formula (cf. [26]):

$$\frac{(-1)^n}{n+1} = \frac{D_n}{n!}$$

we obtain the following theorem.

Theorem 10. Let $n \in \mathbb{N}$. Then we have

$$\int_0^1 Y_k^n(x) dx = \frac{(-1)^k}{n!} \binom{n}{k} D_n.$$

Theorem 11. Let $n \in \mathbb{N}$. Then we have

$$\int_0^1 P_n(2y-1) dy = 0.$$

Proof. By taking the integral of Equation (24) from 0 to 1, we obtain

$$\int_0^1 P_n(2y-1) dy = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \int_0^1 B_k^n(y) dy.$$

Therefore,

$$\int_0^1 P_n(2y-1) dy = \frac{1}{n+1} \sum_{k=0}^n \binom{n}{k} (-1)^{n-k}.$$

Since

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = 0,$$

we arrive at the desired result. $\square$

6. Bezier Type Curves

Bezier curves are defined by the help of the Bernstein basis functions and control points $P_0, P_1, \dots, P_n$ as follows:

$$B_n(x) = \sum_{k=0}^n P_k B_k^n(x). \quad (30)$$

By using the same method of the above curve, we define the following Bezier type curves in terms of polynomials $Y_k^n(x)$ and control points $P_0, P_1, \dots, P_n$:

$$Q_n(x) = \sum_{k=0}^n P_k Y_k^n(x). \quad (31)$$

Using (31) and (7), we obtain

$$Q_n(x) = \sum_{k=0}^n P_k \sum_{j=k}^n (-1)^{n-j} \binom{n}{j} B_k^j(x).$$

Bezier type curves in terms of polynomials $Y_k^n(x)$ of degree n take the following values at $x = 0$ and $x = 1$:

$$\begin{aligned} Q_n(0) &= 0, \\ Q_n(1) &= \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} P_i. \end{aligned}$$

Remark 3. Since $Y_k^n(x)$ polynomials are monomials, the geometric family of the curves which are constructed in terms of Bernstein type polynomials may form a subclass of the Bezier curves.

Here, the applications of the curve $Q_n(x)$ are not given theoretically. The curve $Q_n(x)$ may be used to manipulate the curve to obtain shape properties. It is possible to investigate implementation for a real world problem such as human facial expression recognition and deformation hands with the help of the curvature of the Bezier type curves, whose machine learning is supported by statistical evaluations on feature vectors using the machine learning algorithm by the same method in [11,12,14,19–21].

Investigating the applications of graphs of Bezier type curves and examining the properties and deformation structures of these curves geometrically are outside the scope of this article.

7. Conclusions

We gave a generating function for a new family of polynomials, which are given in terms of Bernstein polynomials. By using this function, we derived some formulae involving the Bernstein polynomials, the Legendre polynomials, and the coefficients of the classical superoscillatory function and these polynomials. By applying the $\frac{\partial^n}{\partial t^n}$ derivative operator to a generating function, we obtained recurrence relations for the new family of polynomials in terms of the Bernstein polynomials. We gave some relations involving a symmetry property among $Y_k^n(x)$, $c_k(n, x)$, and $P_n(x)$. Using integral formulae for the new family of polynomials, we derived some combinatorial finite sums formulae.

In Section 5, by using integral formulae, we obtained new formulae involving the new family of polynomials, the Bernoulli numbers, the Stirling numbers of the first kind, the Legendre polynomials, and the Dahehee numbers. Finally, we defined the Bezier type curves.

As we mentioned in the introduction and other sections, the results obtained in this article with the newly found results have a high potential to be used in applied sciences,

control system theory, neural networks, machine learning, etc. This may provide us with new projects on the research and application of these issues in the future.

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