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Sharp Results for a New Class of Analytic Functions Associated with the q -Differential Operator and the Symmetric Balloon-Shaped Domain

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Abstract: In our current study, we apply differential subordination and quantum calculus to introduce and investigate a new class of analytic functions associated with the q -differential operator and the symmetric balloon-shaped domain. We obtain sharp results concerning the Maclaurin coefficients the second and third-order Hankel determinants, the Zalcman conjecture, and its generalized conjecture for this newly defined class of q -starlike functions with respect to symmetric points.

Keywords: differential subordination; analytic function; q -starlike function; symmetric points; Zalcman conjecture; Hankel determinant; balloon-shaped symmetric domain



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1. Introduction

Let $\mathcal{H}(\mathcal{D})$ denote the class of all of analytic functions ψ defined in the open unit disk $\mathcal{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, and let $\mathcal{A}$ be the subclass of $\mathcal{H}(\mathcal{D})$ containing all analytic functions $\psi(\zeta)$ provided by the Maclaurin series:

$$\psi(\zeta) = \zeta + \sum_{t=2}^{\infty} a_t \zeta^t. \quad (1)$$

By $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{C}$, $\mathcal{K}$, and $\mathcal{R}$, we refer to the well-known subclasses of $\mathcal{H}(\mathcal{D})$, which include all univalent, starlike, convex, close-to-convex functions, and functions with bounded turnings, respectively. These classes and their generalizations have been extensively studied using various techniques and tools over the past few decades. In Geometric Function Theory, subordination is an important tool for defining certain new subclasses of analytic functions and establishing several geometric properties of the subclasses. Let ψ and ξ be two analytic functions in $\mathcal{H}(\mathcal{D})$, where ψ is said to be subordinate to ξ , written as $\psi \prec \xi$, if a Schwarz function w exists in $\mathcal{H}(\mathcal{D})$ with $w(0) = 0$ and $|w(\zeta)| < 1$, such that $\psi = \xi(w(\zeta))$ (see [1]). Furthermore, if the function $\xi \in \mathcal{S}$, then

$$\psi \prec \xi \text{ if and only if } \psi(0) = \xi(0) = 0 \text{ and } \psi(\mathcal{D}) \subset \xi(\mathcal{D}). \quad (2)$$

For two analytic functions ψ and ξ in $\mathcal{A}$ with the series representation of ψ given in (1) and $\xi(\zeta) = \zeta + \sum_{t=2}^{\infty} b_t \zeta^t$, the convolution (Hadamard product) $\psi * \xi$ is defined by

$$(\psi * \xi)(\zeta) = \zeta + \sum_{t=2}^{\infty} a_t b_t \zeta^t.$$

By applying the principle of subordination, Shanmugam [2] and Padmanabhan et al. [3] studied the natural generalization of starlike and convex functions. Later, Ma and Minda [4] introduced unified classes $\mathcal{S}^*(\xi)$ and $\mathcal{C}(\xi)$, as follows:

$$\mathcal{S}^*(\xi) = \left\{ \psi \in \mathcal{A} : \frac{\xi \psi'(\zeta)}{\psi(\zeta)} \prec \xi(\zeta) \quad \zeta \in \mathcal{D} \right\}, \quad (3)$$

$$\mathcal{C}(\xi) = \left\{ \psi \in \mathcal{A} : 1 + \frac{\xi \psi''(\zeta)}{\psi'(\zeta)} \prec \xi(\zeta) \quad \zeta \in \mathcal{D} \right\}, \quad (4)$$

where ξ is an analytic, univalent, and starlike function whose real part is positive in $\mathcal{D}$ with $\xi(0) = 1$ and $\xi'(0) > 0$. The geometry of the function ξ is such that its image domain is symmetric about the real axis. Over time, numerous researchers have defined and investigated various new classes of analytic and univalent functions by applying certain operators and considering image domains of the diverse functions ξ . For detailed information, see [5–11].

Researchers have recently shifted their focus towards q -calculus, recognizing its applications and significance. This shift led them to define important and interesting subclasses of analytic functions in the framework of q -calculus. Q -calculus is like classical calculus, but it does not use the notation of limits. Its theory has attracted researchers because of its extensive applications in special functions, differential equations, mathematical analysis, number theory, hypergeometric series, applied physics, complex analysis, and related areas.

Jackson [12,13] was the first to introduce the q -analogs of differential and integral operators. Srivastava [14] initiated the application of q -calculus in Geometric Function Theory by introducing q -hypergeometric functions. Since then, many authors have significantly contributed by defining and exploring new subclasses of analytic functions using q -calculus, which have played a crucial role in developing Geometric Function Theory.

Ismail et al. [15] employed the q -difference operator to extend starlike functions to q -starlike functions, while Agarwal et al. [16] further extended this work. Furthermore, Ashis et al. [17] used q -hypergeometric series and trigonometric functions to derive recursion formulas. For more details, see [18–23].

Recently, an appealing and innovative approach to defining and studying new classes of analytic functions through image domain generalization and specific q -operators has become an attractive and effective tool. For example, Jabeen et al. [24] defined and studied a class of q -convex functions using the q -Ruscheweyh operator in conic regions. Khan et al. [25] addressed sharp coefficient problems for a subclass defined by the q -operator and a generalization of the lemniscate of Bernoulli. Mahmood et al. [26] proved the geometric properties of q -starlike functions associated with the q -integral operator, and Seoudy et al. [27] addressed certain coefficient problems for q -starlike and q -convex functions. Vijay et al. [28] studied Ozaki-type functions related to fractional operators and three-leaf-type functions. Sailu et al. [29] defined a new class of starlike functions using the q -version of Limaçon functions, and Shaba et al. [30] proved important inequalities for this generalized Limaçon domain. Furthermore, Srivastava et al. [31] established a subclass of q -starlike functions through the generalization of the exponential function and estimated Hankel determinants of different orders, while Swarup [32] introduced a new subclass of q -starlike functions associated with the q -extension of the tangent hyperbolic function. Zhang et al. [33] introduced a new subclass of the q -starlike functions associated with generalizing the conic domain.

Next, we present some basic q -calculus definitions that will aid our investigations. Also, see [34] for more applications of the q -difference operator.

Definition 1 ([12]). For $q \in (0, 1)$, the q -difference operator of a function ψ is defined as follows:

$$D_q \psi(\zeta) = \frac{\psi(q\zeta) - \psi(\zeta)}{\zeta(q - 1)} \quad (\zeta \in \mathcal{D}, \zeta \neq 0). \quad (5)$$

For example,

$$D_q \left\{ \sum_{t=1}^{\infty} a_t \zeta^t \right\} = \sum_{t=1}^{\infty} [t]_q a_t \zeta^{t-1},$$

where $[t]_q$ is the so called q -number, defined by:

$$[t]_q = \begin{cases} \frac{1-q^t}{1-q}, & t \in \mathbb{C}, \\ \sum_{k=0}^{n-1} q^k = 1 + q + q^2 + \dots + q^{n-1}, & t = n \in \mathbb{N}. \end{cases}$$

One can observe that $\lim_{q \rightarrow 1^-} D_q \psi(\zeta) = \psi'(\zeta)$. Thus, the classical derivative can be seen as the limit of the q -difference operators.

Definition 2 ([35]). For $t \geq 0$, the q -generalization of the factorial notation is defined as follows:

$$[t]_q! = \begin{cases} 1, & t = 0, \\ [1]_q [2]_q [3]_q \dots [t]_q, & t \in \mathbb{N}. \end{cases}$$

Definition 3 ([31]). For $q \in (0, 1)$, the q -exponential function is defined as

$$e_q^\zeta = \sum_{t=2}^{\infty} \frac{\zeta^t}{[t]_q!} = 1 + \frac{\zeta}{[1]_q!} + \frac{\zeta^2}{[2]_q!} + \frac{\zeta^3}{[3]_q!} + \dots$$

Now, we introduce the new class $\psi \in \mathcal{S}_B^{*s}(q)$ of q -starlike functions connected with the q -difference operator and the balloon-shaped domain, as initiated by Ahmed et al. initiated in [36]. This class is symmetric about the real axis.

Definition 4. Let $\psi \in \mathcal{S}_B^{*s}(q)$ denote the subclass of $\mathcal{A}$ that contains all ψ known as q -starlike functions with symmetric points, such that the following subordination holds

$$\frac{2\zeta D_q \psi(\zeta)}{\psi(\zeta) - \psi(-\zeta)} \prec \frac{2\sqrt{1+q\zeta}}{1+e_q^{-\zeta}}, \quad \zeta \in \mathcal{D}, \quad q \in (0, 1). \quad (6)$$

Geometrically, for the function $\psi \in \mathcal{S}_B^{*s}(q)$, the functional $\frac{2\zeta D_q \psi(\zeta)}{\psi(\zeta) - \psi(-\zeta)}$ maps the open unit disc $\mathcal{D}$ into the inflated balloon-shaped domain. This new class extends the notion of starlike functions with respect to symmetric points, initiated by Sakaguchi [37], to q -starlike functions with respect to symmetric points. We also extend the class established and investigated by Khan et al. in [38].

In Geometric Function Theory, coefficient-related inequalities play a key role in understanding the behavior and convergence properties of analytic functions within specific domains, especially those with favorable geometric interpretations. For this purpose:

Zalcman proposed a conjecture for $\psi \in \mathcal{S}$ in the 1960s, which states

$$|a_t^2 - a_{2t-1}| \leq (t-1)^2, \quad \text{for } t \in \mathbb{N} - \{1\}. \quad (7)$$

In 1999, Ma [39] proposed a generalized Zalcman conjecture:

$$|a_t a_s - a_{t+s-1}| \leq (t-1)(s-1), \quad \text{for } s, t \in \mathbb{N} - \{1\}. \quad (8)$$

Despite several authors demonstrating these inequalities for specific values and coefficients of certain subclasses of univalent functions, these conjectures remain unsolved.

Let $\psi \in \mathcal{A}$, where the r^{th} Hankel determinant $\mathcal{H}_{r,n}$ is defined in [40] as follows:

$$\mathcal{H}_{r,n}(\psi) = \begin{vmatrix} a_n & a_{n+1} & \cdot & \cdot & \cdot & a_{n+r-1} \\ a_{n+1} & a_{n+2} & \cdot & \cdot & \cdot & a_{n+r} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n+r-1} & a_{n+r} & \cdot & \cdot & \cdot & a_{n+2(r-1)} \end{vmatrix},$$

where a_n are the coefficients of the Maclaurin series (1) for the analytic function ψ , with $a_1 = 1$. For example, the second and third Hankel determinants are given by.

$$\mathcal{H}_{2,1}(\psi) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2, \quad \mathcal{H}_{2,2}(\psi) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2, \quad (9)$$

$$\mathcal{H}_{3,1}(\psi) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2). \quad (10)$$

The primary goal of this investigation is to obtain sharp inequalities for the Maclaurin coefficient bounds, the upper limits of second and third-order Hankel determinants, the Zalcman conjecture, and its generalization conjecture for the newly defined class $\psi \in \mathcal{S}_B^{*s}(q)$ of q -starlike functions with respect to symmetric points.

2. A Set of Lemmas

The following lemmas will be instrumental in illustrating our main findings and providing further clarity.

Let $\mathcal{P}$ represent the subclass of $\mathcal{H}(\mathcal{D})$ containing all Carathéodory functions p , such that $p(0) = 1$, $\operatorname{Re}(p(\zeta)) > 0$, and with the following Maclaurin series.

$$p(\zeta) = 1 + \sum_{t=1}^{\infty} c_t \zeta^t. \quad (11)$$

Let $\psi \in \mathcal{A}$ and $p \in \mathcal{P}$ be functions with series representation given in (1) and (11), respectively. Then, class $\mathcal{W}(\phi, \varphi, p)$ is defined in [41], as

$$\frac{\phi * \psi}{\varphi * \psi} \prec p, \quad (12)$$

where

$$\varphi(\zeta) = \zeta + \sum_{t=2}^{\infty} u_t \zeta^t \text{ and } \phi(\zeta) = \zeta + \sum_{t=2}^{\infty} v_t \zeta^t.$$

Lemma 1 ([40]). Let $p \in \mathcal{P}$, then the following inequalities hold true

$$|c_t| \leq 2, \text{ for } t \geq 1, \quad (13)$$

$$|c_{t+k} - \nu c_t c_k| < 2, \text{ for } 0 \leq \nu \leq 1. \quad (14)$$

Lemma 2. Let $p \in \mathcal{P}$, then there exists k , $\delta \in \overline{\mathcal{D}} = \{\zeta \in \mathbb{C} : |\zeta| \leq 1\}$, such that

$$c_2 = \frac{1}{2} \left(c_1^2 + k(4 - c_1^2) \right), \quad (15)$$

and

$$c_3 = \frac{1}{4} \left(c_1^3 + 2c_1 k(4 - c_1^2) - (4 - c_1^2) c_1 k^2 + 2(4 - c_1^2) (1 - |k|^2) \delta \right). \quad (16)$$

The values provided in (15) and (16) are due to [40,42], respectively.

Lemma 3 ([42]). Let $p \in \mathcal{P}$, $0 \leq M \leq 1$ and $M(2M - 1) \leq N \leq M$. Then,

$$|c_3 - 2Mc_1c_2 + Nc_1^3| \leq 2.$$

Lemma 4 ([43]). Let $p \in \mathcal{P}$, and α, β, γ , and λ satisfy the inequalities $0 < \lambda < 1, 0 < \alpha < 1$, and

$$8\lambda(1 - \lambda) \left[(\alpha\beta - 2\gamma)^2 + (\alpha(\lambda + \alpha) - \beta)^2 \right] + \alpha(1 - \alpha)(\beta - 2\lambda\alpha)^2 \leq 4\alpha^2(1 - \alpha)^2\lambda(1 - \lambda). \quad (17)$$

Then, we have

$$\left| \gamma c_1^4 + \lambda c_2^2 + 2\alpha c_1 c_3 - \frac{3}{2} \beta c_1^2 c_2 - c_4 \right| \leq 2.$$

Lemma 5 ([41]). Let $\psi \in \mathcal{W}(\phi, \varphi, p)$ and $v_k \neq u_k$ for $(k=2,3)$. Then,

$$|a_2| \leq \frac{|p_1|}{|v_2 - u_2|}, \quad |a_3| \leq \frac{|p_1|}{|v_3 - u_3|} \max\{1, |\rho|\},$$

and

$$|a_3 - \mu a_2^2| \leq \frac{|p_1|}{|v_3 - u_3|} \max\{1, |\eta|\}, \quad \mu \in \mathbb{C},$$

where

$$\rho = \frac{p_2}{p_1} + \frac{u_2 p_1}{v_2 - u_2} \text{ and } \eta = \frac{v_3 - u_3}{(v_2 - u_2)^2} p_1 \mu - \rho.$$

The results are sharp.

3. Main Results

It is worth mentioning at the beginning of this section that if $\psi \in \mathcal{A}$, has the series representation given in (1), along with

$$\phi_1(\varsigma) = \varsigma D_q \left(\frac{\varsigma}{1 - \varsigma} \right) = \varsigma + \sum_{t=2}^{\infty} [t]_q \varsigma^t = \varsigma + (1+q)\varsigma + (1+q+q^2)\varsigma^2 + \dots \quad (18)$$

$$\varphi_1(\varsigma) = \frac{1}{2} \left(\frac{\varsigma}{1 - \varsigma} + \frac{\varsigma}{1 - \varsigma} \right) = \sum_{t=1}^{\infty} \varsigma^{2t-1} = \varsigma + \varsigma^3 + \varsigma^5 + \dots \quad (19)$$

and

$$p_q(\varsigma) = \frac{\sqrt{1+q\varsigma}}{1+e_q^{-\varsigma}} = 1 + p_1\varsigma + p_2\varsigma^2 + \dots$$

Then, for $a_1 = 1$,

$$\frac{\psi(\varsigma) - \psi(-\varsigma)}{2} = \sum_{t=1}^{\infty} a_{2t-1} \varsigma^{2t-1} = \left(\varsigma + \sum_{t=2}^{\infty} a_t \varsigma^t \right) * \left(\varsigma + \sum_{t=2}^{\infty} \varsigma^{2t-1} \right) = \psi(\varsigma) * \phi_1(\varsigma). \quad (20)$$

Furthermore, by applying the q -difference operator,

$$\varsigma D_q \psi(\varsigma) = \varsigma + \sum_{t=2}^{\infty} [t]_q a_t \varsigma^t = \left(\varsigma + \sum_{t=2}^{\infty} a_t \varsigma^t \right) * \left(\varsigma + \sum_{t=2}^{\infty} [t]_q \varsigma^t \right) = \psi(\varsigma) * \phi_1(\varsigma), \quad (21)$$

It follows from (20) and (21) that

$$\frac{2\varsigma D_q \psi(\varsigma)}{\psi(\varsigma) - \psi(-\varsigma)} = \frac{\psi(\varsigma) * \phi_1(\varsigma)}{\psi(\varsigma) * \phi_1(\varsigma)}.$$

Thus, we conclude that the newly defined class $\mathcal{S}_B^{*s}(q)$ is a particular case of the class $\mathcal{W}(\phi, \varphi, p)$, that is

$$\mathcal{S}_B^{*s}(q) = \mathcal{W}(\phi_1, \varphi_1, p_q). \quad (22)$$

Moreover, we can apply Lemma 5 from (22) to obtain the following corollary.

Corollary 1. Let $\psi \in \mathcal{S}_B^{*s}(q)$, provided in (1), then we have

$$|a_2| \leq \frac{1}{2}, \quad (23)$$

$$|a_3| \leq \frac{1}{2q}, \quad q \in (0, 1) \quad (24)$$

and

$$|a_3 - a_2^2| \leq \frac{1}{2q}, \quad q \in (0, 1). \quad (25)$$

Proof. Let $\psi \in \mathcal{S}_B^{*s}(q)$, then by the application of Lemma 5,

$$u_2 = 0, \quad u_3 = 1, \quad v_2 = 1 + q, \quad v_3 = 1 + q + q^2, \quad p_1 = \frac{1 + q}{2},$$

and

$$p_2 = \frac{1 + q}{4} - \frac{1}{2(1 + q)} - \frac{q^2}{8}.$$

Using Lemma 5, we obtain

$$|a_2| \leq \frac{1}{2}.$$

To prove (24) and (25), we consider

$$\rho = \frac{p_2}{p_1} + \frac{u_2 p_1}{v_2 - u_2} \quad \text{and} \quad \eta = \frac{(v_3 - u_3)}{(v_2 - u_2)^2} p_1 - \rho.$$

Applying the given values and simplifying,

$$|\rho| = \left| \frac{1}{4(1 + q)^2} (-q^3 + q^2 + 4q - 2) \right| \leq 1, \quad \text{for all } q \in (0, 1),$$

$$|\eta| = \left| \frac{1}{4(1 + q)^2} (3q^3 + 3q^2 - 2q + 2) \right| \leq 1, \quad \text{for all } q \in (0, 1).$$

Hence, Lemma 5 proves that

$$|a_3| \leq \frac{|p_1|}{|v_3 - u_3|} = \frac{1}{2q} \quad \text{and} \quad |a_3 - a_2^2| \leq \frac{|p_1|}{|v_3 - u_3|} = \frac{1}{2q}.$$

□

Theorem 1. Let $\psi \in \mathcal{S}_B^{*s}(q)$, as given in (1). Then,

$$|a_4| \leq \frac{1}{2(1 + q^2)}, \quad \text{for } q \in (0.535, 1), \quad (26)$$

$$|a_5| \leq \frac{1}{2q(1 + q^2)}, \quad \text{for } q \in (0.723, 1). \quad (27)$$

Proof. Let $\psi \in \mathcal{S}_B^{*s}(q)$, then by (6), we have

$$\frac{2\zeta D_q \psi(\zeta)}{\psi(\zeta) - \psi(-\zeta)} \prec \frac{2\sqrt{1+q\zeta}}{1+e_q^{-\zeta}}.$$

Thus, there is a Schwarz function $w(\zeta)$, such that

$$\frac{2\zeta D_q \psi(\zeta)}{\psi(\zeta) - \psi(-\zeta)} = \frac{2\sqrt{1+qw(\zeta)}}{1+e_q^{-w(\zeta)}}.$$

Let $p \in \mathcal{P}$, then we have

$$p(\zeta) = \frac{1+w(\zeta)}{1-w(\zeta)} = 1 + c_1\zeta + c_2\zeta^2 + c_3\zeta^3 + c_4\zeta^4 + \dots$$

It follows that

$$w(\zeta) = \frac{p(\zeta) - 1}{1 + p(\zeta)} = \frac{c_1\zeta + c_2\zeta^2 + c_3\zeta^3 + c_4\zeta^4 + \dots}{2 + c_1\zeta + c_2\zeta^2 + c_3\zeta^3 + c_4\zeta^4 + \dots},$$

which provided

$$\begin{aligned} w(\zeta) &= \frac{1}{2}c_1\zeta + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)\zeta^2 + \left(\frac{1}{8}c_1^3 - \frac{1}{2}c_1c_2 + \frac{1}{2}c_3\right)\zeta^3 \\ &+ \left(\frac{1}{2}c_4 - \frac{1}{2}c_1c_3 - \frac{1}{4}c_2^2 - \frac{1}{16}c_1^4 + \frac{3}{8}c_1^2c_2\right)\zeta^4 + \dots \end{aligned}$$

Substituting the above Maclaurin series of $w(\zeta)$ and simplifying, we obtain

$$\begin{aligned} \frac{2\sqrt{1+qw(\zeta)}}{1+e_q^{-w(\zeta)}} &= 1 + \frac{[2]_q c_1}{4}\zeta + \left(\frac{[2]_q c_2}{4} - \frac{(10-2q+5q^2+q^3)}{32[2]_q}c_1^2\right)\zeta^2 \\ &+ \left(\frac{[2]_q c_3}{4} - \frac{(5+4q+4q^2+q^3)}{16[2]_q}c_1c_2 + \frac{(6+2q+13q^2+8q^3+4q^4+q^5)}{128[3]_q}c_1^3\right)\zeta^3 \\ &+ \left(\frac{[2]_q c_4}{4} + \frac{(78+24q+45q^2+3q^3+36q^4+15q^5+3q^6)}{128[2]_q[3]_q}c_1^2c_2 - \frac{(216-136q+208q^2-76q^3+173q^4+118q^5+179q^6+95q^7+30q^8+5q^9)}{2048[3]_q[4]_q}c_1^4\right. \\ &\quad \left.- \frac{(4+4q+5q^2+q^3)}{16[2]_q}c_1c_3 - \frac{(4+4q+5q^2+q^3)}{32[2]_q}c_2^2\right)\zeta^4 + \dots \end{aligned} \quad (28)$$

On the other hand, taking the series form provided in (1) along with (5), we have

$$\begin{aligned} \frac{2\zeta D_q \psi(\zeta)}{\psi(\zeta) - \psi(-\zeta)} &= \frac{2\zeta \left(1 + \sum_{t=2}^{\infty} [t]_q a_t \zeta^{t-1}\right)}{\left(\zeta + \sum_{t=2}^{\infty} a_t \zeta^t\right) - \left((- \zeta) + \sum_{t=2}^{\infty} a_t (-\zeta)^t\right)}, \\ &= \frac{(1 + [2]_q a_2 \zeta + [3]_q a_3 \zeta^2 + [4]_q a_4 \zeta^3 + \dots)}{(1 + a_3 \zeta^2 + a_5 \zeta^4 + \dots)}. \end{aligned}$$

The simple computations provide

$$\begin{aligned} \frac{2\zeta D_q \psi(\zeta)}{\psi(\zeta) - \psi(-\zeta)} &= 1 + [2]_q a_2 \zeta + \left(([3]_q - 1) a_3 \right) \zeta^2 + \left([4]_q a_4 - [2]_q a_2 a_3 \right) \zeta^3 \\ &\quad + \left(([5]_q - 1) a_5 - ([3]_q - 1) a_3^2 \right) \zeta^4 + \dots \end{aligned} \quad (29)$$

From (28) and (29), comparing the coefficients of ζ , ζ^2 , ζ^3 , ζ^4 and simple computations provide

$$a_2 = \frac{c_1}{4}, \quad (30)$$

$$a_3 = \frac{1}{q[2]_q} \left(\frac{(1+q)}{4} c_2 - \frac{(10-2q+5q^2+q^3)}{32(1+q)} c_1^2 \right), \quad (31)$$

$$a_4 = \frac{1}{[4]_q} \left(\frac{(1+q)}{4} c_3 - \frac{(q^4+4q^3+3q^2+3q-1)}{16q(1+q)} c_1 c_2 + \frac{(q^7+5q^6+11q^5+15q^4+11q^3-5q^2-2q-10)}{128q(1+q)(1+q+q^2)} c_1^3 \right), \quad (32)$$

and

$$a_5 = \frac{1}{q[4]_q} \left(\frac{(1+q)}{4} c_4 + \frac{(3q^7+15q^6+34q^5-9q^4+37q^3-2q^2+62q-20)}{128q(q^3+2q^2+2q+1)} c_1^2 c_2 - \frac{(q^3+5q^2+4q+4)}{16(1+q)} c_1 c_3 - \frac{(q^4+5q^3+2q^2-2)}{32q(1+q)} c_2^2 - \frac{Y_1(q)}{2048q(q+1)^3(q^4+q^3+2q^2+q+1)} c_1^4 \right), \quad (33)$$

where

$$\begin{aligned} Y_1(q) &= 5q^{12} + 40q^{11} + 158q^{10} + 377q^9 + 505q^8 + 504q^7 + 74q^6 + 39q^5 \\ &\quad - 374q^4 - 96q^3 - 232q^2 + 96q - 200. \end{aligned}$$

From (32), we have

$$|a_4| = \frac{(1+q)}{4[4]_q} \left| c_3 - \frac{(q^4+4q^3+3q^2+3q-1)}{4q(1+q)^2} c_1 c_2 + \frac{(q^7+5q^6+11q^5+15q^4+11q^3-5q^2-2q-10)}{32q(1+q)^2(1+q+q^2)} c_1^3 \right|.$$

To apply Lemma 3, we consider

$$M = \frac{(q^4 + 4q^3 + 3q^2 + 3q - 1)}{8q(1+q)^2} \text{ and } N = \frac{(q^7 + 5q^6 + 11q^5 + 15q^4 + 11q^3 - 5q^2 - 2q - 10)}{32q(1+q)^2(1+q+q^2)}.$$

It can be easily seen that $0 < M < 1$ for $q \in (0.24926, 1)$ and $N < M$ for all $q \in (0, 1)$. Additionally, $M(2M - 1) - N < 0$ for $q \in (0.5353, 1)$. Therefore, by Lemma 3, we have

$$|a_4| \leq \frac{1}{2(1+q^2)}, \quad q \in (0.5353, 1).$$

From (33), we have

$$|a_5| = \frac{(1+q)}{4q[4]_q} \left| \frac{Y_1(q)}{512q(1+q)^4(q^4+q^3+2q^2+q+1)} c_1^4 + \frac{(q^4+5q^3+2q^2+2)}{8q(1+q)^2} c_2^2 + \frac{(q^3+5q^2+4q+4)}{4(1+q)^2} c_1 c_3 - \frac{(3q^7+15q^6+34q^5-9q^4+37q^3-2q^2+62q-20)}{32q(1+q)(q^3+2q^2+2q+1)} c_1^2 c_2 - c_4 \right|.$$

To apply Lemma 4, we consider,

$$\gamma = \frac{Y_1(q)}{512q(1+q)^4(q^4+q^3+2q^2+q+1)}, \quad \lambda = \frac{(q^4+5q^3+2q^2+2)}{8q(1+q)^2},$$

where

$$Y_1(q) = 5q^{12} + 40q^{11} + 158q^{10} + 377q^9 + 505q^8 + 504q^7 + 74q^6 + 39q^5 - 374q^4 - 96q^3 - 232q^2 + 96q - 200,$$

and

$$\alpha = \frac{(q^3 + 5q^2 + 4q + 4)}{8(1+q)^2}, \beta = \frac{(3q^7 + 15q^6 + 34q^5 - 9q^4 + 37q^3 - 2q^2 + 62q - 20)}{48q(1+q)(q^3 + 2q^2 + 2q + 1)}.$$

Clearly, $0 < \lambda < 1$ and $0 < \alpha < 1$ for $q \in (0, 1)$. Now, consider

$$8\lambda(1-\lambda) \left[(\alpha\beta - 2\gamma)^2 + (\alpha(\lambda + \alpha) - \beta)^2 \right] + \alpha(1-\alpha)(\beta - 2\lambda\alpha)^2 - 4\alpha^2(1-\alpha)^2\lambda(1-\lambda) = \Psi_1(q),$$

where

$$\Psi_1(q) = \frac{\Lambda_1(q)}{147456q^2(q+1)^{10}(q^2+q+1)^2},$$

and

$$\Lambda_1(q) = \begin{pmatrix} -111q^{20} - 672q^{19} + 2156q^{18} + 26552q^{17} + 65731q^{16} - 74150q^{15} - 884511q^{14} \\ -2936298q^{13} - 6360820q^{12} - 10237558q^{11} + 12723173q^{10} - 12062862q^9 \\ -8015668q^8 - 2679044q^7 + 1316924q^6 + 2479552q^5 + 1758320q^4 + 678208q^3 \\ + 129344q^2 + 17920q + 3328 \end{pmatrix}$$

After some simple calculations, one can see that $\Psi_1(q) \leq 0$, for all $q \in (0.723, 1)$. By using Lemma 4, we obtain

$$|a_5| \leq \frac{1}{2q(1+q^2)}.$$

□

Remark 1. The inequalities (23)–(27) in Corollary 1 and Theorem 1 are sharp. The equalities are obtained by the normalized extremal function ψ_n , respectively defined by

$$\frac{2\zeta D_q \psi_n(\zeta)}{\psi_n(\zeta) - \psi_n(-\zeta)} = \frac{2\sqrt{1+q\zeta^n}}{1+e_q^{-\zeta^n}}, \text{ for } n = 1, 2, 3, 4.$$

That is, the normalized extremal functions are provided by:

$$\frac{2\zeta D_q \psi_1(\zeta)}{\psi_1(\zeta) - \psi_1(-\zeta)} = \frac{2\sqrt{1+q\zeta}}{(1+e_q^{-\zeta})} = 1 + \frac{1}{2}(1+q)\zeta + \frac{(-q^3+q^2+4q-2)}{8(q+1)}\zeta^2 + \dots, \quad (34)$$

$$\frac{2\zeta D_q \psi_2(\zeta)}{\psi_2(\zeta) - \psi_2(-\zeta)} = \frac{2\sqrt{1+q\zeta^2}}{(1+e_q^{-\zeta^2})} = 1 + \frac{1}{2}(1+q)\zeta^2 + \frac{(-q^3+q^2+4q-2)}{8(q+1)}\zeta^4 + \dots, \quad (35)$$

$$\frac{2\zeta D_q \psi_3(\zeta)}{\psi_3(\zeta) - \psi_3(-\zeta)} = \frac{2\sqrt{1+q\zeta^3}}{(1+e_q^{-\zeta^3})} = 1 + \frac{1}{2}(1+q)\zeta^3 + \frac{(-q^3+q^2+4q-2)}{8(q+1)}\zeta^6 + \dots, \quad (36)$$

$$\frac{2\zeta D_q \psi_4(\zeta)}{\psi_4(\zeta) - \psi_4(-\zeta)} = \frac{2\sqrt{1+q\zeta^4}}{(1+e_q^{-\zeta^4})} = 1 + \frac{1}{2}(1+q)\zeta^4 + \frac{(-q^3+q^2+4q-2)}{8(q+1)}\zeta^8 + \dots. \quad (37)$$

To show that the function defined by (34) is the extremal function of the sharpness of (23), we compare the coefficients of the series in (29) with the coefficients of the series in (34), and then apply definition (1), we have

$$[2]_q a_2 = \frac{1}{2}(1+q), \quad (1+q)a_2 = \frac{1}{2}(1+q), \text{ which gives } a_2 = \frac{1}{2}.$$

$$([3]_q - 1)a_3 = \frac{(-q^3 + q^2 + 4q - 2)}{8(1+q)}, \text{ which gives } a_3 = \frac{(-q^3 + q^2 + 4q - 2)}{8q(1+q)^2}.$$

$$\vdots$$

Thus, the normalized extremal function defined by (34) is given below.

$$\psi(\varsigma) = \varsigma + \frac{1}{2}\varsigma^2 + \frac{(-q^3 + q^2 + 4q - 2)}{8q(1+q)^2}\varsigma^3 + \dots$$

In this manner, we can also obtain the extremal functions defined by (35)–(37).

The following example provides an extremal function of the sharpness (23) and confirms that the new class $\mathcal{S}_B^{*s}(q)$ is non-empty.

Example 1. Let

$$\psi(\varsigma) = \varsigma + 0.5\varsigma^2 + 0.064\varsigma^3 - 0.000137\varsigma^4 - 0.000355\varsigma^5,$$

then

$$\frac{2\varsigma D_{0.8}\psi(\varsigma)}{\psi(\varsigma) - \psi(-\varsigma)} = \frac{2\sqrt{1+0.8\varsigma}}{1+e_{0.8}^{-\varsigma}}, \quad \varsigma \in \mathcal{D}.$$

It implies $\psi \in \mathcal{S}_B^{*s}(0.8)$ with

$$\begin{aligned} |a_2| &= 0.5, \\ |a_3| &= 0.064 < \frac{1}{2 \times 0.8}, \\ |a_4| &= 0.000137 < \frac{1}{2(1+0.8^2)}, \\ |a_5| &= 0.000355 < \frac{1}{2 \times 0.8(1+0.8^2)}. \end{aligned}$$

Theorem 2. Let $\psi \in \mathcal{S}_B^{*s}(q)$ be provided in (1). Then,

$$|a_4 - a_2a_3| \leq \frac{1}{2(q^2 + 1)}, \text{ for } q \in (0.762, 1). \quad (38)$$

The result is sharply obtained by the function given in (36).

Proof. From (30)–(32), using the definition of q -number, we have

$$|a_4 - a_2a_3| = \left| \left(\frac{(q+1)}{4(q^3+q^2+q+1)}c_3 - \frac{(q^4+4q^3+3q^2+3q-1)}{16q(q+1)(q^3+q^2+q+1)}c_1c_2 + \frac{(q^7+5q^6+11q^5+15q^4+11q^3-5q^2-2q-10)}{128q(q+1)(q^2+q+1)(q^3+q^2+q+1)}c_1^3 \right) - \left(\frac{c_1}{4} \right) \left(\frac{1}{q(q+1)} \left(\frac{(q+1)}{4}c_2 - \frac{(q^3+5q^2-2q+10)}{32(q+1)}c_1^2 \right) \right) \right|.$$

Collecting the like terms, we obtain

$$|a_4 - a_2a_3| = \left| \frac{(q+1)}{4(1+q+q^2+q^3)}c_3 - \frac{(2q^3+6q^2+5q+5)}{16q(q+1)(q^3+q^2+q+1)}c_1c_2 + \frac{(q^8+7q^7+10q^6+30q^5+27q^4+34q^3+3q^2+8q-10)}{128q(q+1)(q^2+1)(q^3+q^2+q+1)}c_1^3 \right|.$$

Some simple calculations yield

$$|a_4 - a_2 a_3| = \frac{(q+1)}{4(q^3 + q^2 + q + 1)} \left| c_3 - \frac{(2q^3 + 6q^2 + 5q + 5)}{4q(q+1)^2} c_1 c_2 + \frac{(q^8 + 7q^7 + 10q^6 + 30q^5 + 27q^4 + 34q^3 + 3q^2 + 8q - 10)}{32q(q+1)^2(q^2 + 1)} c_1^3 \right|.$$

Assume

$$M = \frac{(2q^3 + 6q^2 + 5q + 5)}{8q(q+1)^2} \text{ and } N = \frac{(q^8 + 7q^7 + 10q^6 + 30q^5 + 27q^4 + 34q^3 + 3q^2 + 8q - 10)}{32q(q+1)^2(q^2 + 1)}.$$

Clearly, $0 < M < 1$ for $q \in (0, 1)$ and $N < M$ for $q \in (0, 1)$ also $M(2M - 1) - N \leq 0$ for all $q \in (0.762, 1)$. Then, by Lemma 3, we have

$$|a_4 - a_2 a_3| \leq \frac{1}{2(q^2 + 1)}, \quad q \in (0.762, 1).$$

□

Theorem 3. Let $\psi \in \mathcal{S}_B^{*s}(q)$, as in (1). Then,

$$|a_3^2 - a_5| \leq \frac{1}{2q(q^2 + 1)}, \text{ for } q \in (0.628, 1). \quad (39)$$

The result is sharply obtained by the function given in (37).

Proof. From (31) and (33), applying the definition of q -number, we have

$$|a_3^2 - a_5| = \left| \left(\frac{1}{q(q+1)} \left(\frac{(q+1)}{4} c_2 - \frac{(q^3 + 5q^2 - 2q + 10)}{32(q+1)} c_1^2 \right) \right)^2 - \left(\frac{(q+1)}{4q(q^3 + q^2 + q + 1)} c_4 + \frac{(3q^7 + 15q^6 + 34q^5 - 9q^4 + 37q^3 - 2q^2 + 62q - 20)}{128q^2(q^3 + q^2 + q + 1)(q^3 + 2q^2 + 2q + 1)} c_1^2 c_2 - \frac{(q^3 + 5q^2 + 4q + 4)}{16q(q^3 + q^2 + q + 1)(q+1)} c_1 c_3 - \frac{(q^4 + 5q^3 + 2q^2 - 2)}{32q^2(q+1)(q^3 + q^2 + q + 1)} c_2^2 - \frac{(5q^{12} + 40q^{11} + 158q^{10} + 377q^9 + 505q^8 + 504q^7 + 74q^6 + 39q^5 - 374q^4 - 96q^3 - 232q^2 + 96q - 200)}{2048q^2(q+1)^3(q^4 + q^3 + 2q^2 + q + 1)(q^3 + q^2 + q + 1)} c_1^4 \right) \right|.$$

Collecting the like terms,

$$|a_3^2 - a_5| = \left| \frac{(7q^{11} + 62q^{10} + 226q^9 + 483q^8 + 885q^7 + 778q^6 + 966q^5 + 477q^4 + 732q^3 + 272q^2 + 496q + 216)}{2048q(q^2 + 1)^2(q+1)^4(q^2 + q + 1)} c_1^4 - \frac{(1+q)}{4q(q^3 + q^2 + q + 1)} c_4 + \frac{(q^3 + 5q^2 + 4q + 4)}{16q(1+q)(q^3 + q^2 + q + 1)} c_1 c_3 + \frac{(3q^3 + 9q^2 + 6q + 4)}{32q(q^2 + 1)(q+1)^2} c_2^2 - \frac{(5q^6 + 27q^5 + 44q^4 + 29q^3 + 61q^2 + 44q + 78)}{128q(q+1)^2(q^4 + q^3 + 2q^2 + q + 1)} c_1^2 c_2 \right|,$$

We can rewrite,

$$|a_3^2 - a_5| = \frac{(1+q)}{4q(q^3 + q^2 + q + 1)} \left| \frac{Y_2(q)}{512(q^2 + 1)(q+1)^4(q^2 + q + 1)} c_1^4 + \frac{(q^3 + 5q^2 + 4q + 4)}{4(1+q)^2} c_1 c_3 + \frac{(3q^3 + 9q^2 + 6q + 4)}{8(q+1)^2} c_2^2 - \frac{(5q^6 + 27q^5 + 44q^4 + 29q^3 + 61q^2 + 44q + 78)}{32(q+1)^2(q^2 + q + 1)} c_1^2 c_2 - c_4 \right|,$$

where

$$Y_2(q) = 7q^{11} + 62q^{10} + 226q^9 + 483q^8 + 885q^7 + 778q^6 + 966q^5 + 477q^4 + 732q^3 + 272q^2 + 496q + 216.$$

Now, by Lemma 4,

$$\gamma = \frac{Y_2(q)}{512(q^2 + 1)(q+1)^4(q^2 + q + 1)}, \quad \lambda = \frac{(3q^3 + 9q^2 + 6q + 4)}{8(q+1)^2}$$

and

$$\alpha = \frac{(q^3 + 5q^2 + 4q + 4)}{8(1+q)^2}, \beta = \frac{(5q^6 + 27q^5 + 44q^4 + 29q^3 + 61q^2 + 44q + 78)}{48(q+1)^2(q+q^2+1)}.$$

where $0 < \lambda < 1$ and $0 < \alpha < 1$ for $q \in (0, 1)$. Now, we consider

$$8\lambda(1-\lambda) \left[(\alpha\beta - 2\gamma)^2 + (\alpha(\lambda + \alpha) - \beta)^2 \right] + \alpha(1-\alpha)(\beta - 2\lambda\alpha)^2 - 4\alpha^2(1-\alpha)^2\lambda(1-\lambda) = \Psi_2(q),$$

where

$$\Psi_2(q) = -\frac{\Lambda_2(q)}{4718592(q+1)^{12}(q^4 + q^3 + 2q^2 + q + 1)^2}$$

and

$$\Lambda_2(q) = \begin{cases} 1089q^{28} + 19866q^{27} + 176309q^{26} + 982794q^{25} + 3770070q^{24} + 9974594q^{23} \\ + 16186303q^{22} + 7211406q^{21} - 36456810q^{20} - 127230918q^{19} - 277137857q^{18} \\ - 499537850q^{17} - 825273602q^{16} - 1101540386q^{15} - 904180239q^{14} \\ + 340893602q^{13} + 2946316773q^{12} + 6160911116q^{11} + 8878393692q^{10} \\ + 9404597248q^9 + 7766676656q^8 + 4432890112q^7 + 1351376128q^6 \\ - 655503488q^5 - 1192592192q^4 - 891531520q^3 - 411837184q^2 \\ - 110432256q - 13151232 \end{cases}$$

A calculation shows that $\Psi(q) \leq 0$, for all $q \in (0.628, 1)$. Hence, by Lemma 4, we obtain

$$|a_3^2 - a_5| \leq \frac{1}{2q(q^2 + 1)}.$$

□

Theorem 4. Let $\psi \in \mathcal{S}_B^{*s}(q)$ be given in (1). Then,

$$|a_2a_4 - a_5| \leq \frac{1}{2q(q^2 + 1)}, \quad \text{for } q \in (0, 0.514). \quad (40)$$

The result is sharply obtained by the function given in (37).

Proof. From (30), (32) and (33), along with the definition of q -number, we have

$$|a_2a_4 - a_5| = \left| \left(\frac{c_1}{4} \right) \left(\frac{(q+1)}{4(q^3+q^2+q+1)}c_3 - \frac{(q^4+4q^3+3q^2+3q-1)}{16q(q+1)(q^3+q^2+q+1)}c_1c_2 + \frac{(q^7+5q^6+11q^5+15q^4+11q^3-5q^2-2q-10)}{128q(q+1)(q^2+q+1)(q^3+q^2+q+1)}c_1^3 \right) \right. \\ \left. - \left(\frac{(q+1)}{4q(q^3+q^2+q+1)}c_4 + \frac{(3q^7+15q^6+34q^5-9q^4+37q^3-2q^2+62q-20)}{128q^2(q^3+q^2+q+1)(q^3+2q^2+2q+1)}c_1^2c_2 \right. \right. \\ \left. \left. - \frac{(q^3+5q^2+4q+4)}{16q(q^3+q^2+q+1)(q+1)}c_1c_3 - \frac{(q^4+5q^3+2q^2-2)}{32q^2(q+1)(q^3+q^2+q+1)}c_2^2 \right. \right. \\ \left. \left. - \frac{(5q^{12}+40q^{11}+158q^{10}+377q^9+505q^8+504q^7+74q^6+39q^5-374q^4-96q^3-232q^2+96q-200)}{2048q^2(q+1)^3(q^4+q^3+2q^2+q+1)(q^3+q^2+q+1)}c_1^4 \right) \right|,$$

Combining the coefficients of like terms,

$$|a_2a_4 - a_5| = \left| \frac{(9q^{12}+68q^{11}+250q^{10}+573q^9+801q^8+800q^7+278q^6+91q^5-466q^4-212q^3-320q^2+56q-200)}{2048q^2(q^2+1)^2(q+1)^4(q^2+q+1)}c_1^4 \right. \\ \left. + \frac{(q^4+5q^3+2q^2+2)}{32(1+q+q^2+q^3)q^2(1+q)}c_2^2 + \frac{(2q^3+7q^2+5q+4)}{16q(q^2+1)(q+1)^2}c_1c_3 \right. \\ \left. - \frac{(5q^7+25q^6+50q^5+11q^4+47q^3+2q^2+60q-20)}{128q^2(q+1)^2(q^4+q^3+2q^2+q+1)}c_1^2c_2 - \frac{(1+q)}{4q(1+q+q^2+q^3)}c_4 \right|.$$

It implies

$$|a_2a_4 - a_5| = \frac{(1+q)}{4q(1+q+q^2+q^3)} \left| \frac{Y_3(q)}{512q(q^2+1)(q+1)^4(q^2+q+1)} c_1^4 + \frac{(q^4+5q^3+2q^2+2)}{8q(1+q)^2} c_2^2 + \frac{(2q^3+7q^2+5q+4)}{4(q+1)^2} c_1c_3 - \frac{(5q^7+25q^6+50q^5+11q^4+47q^3+2q^2+60q-20)}{32q(q+1)^2(q+q^2+1)} c_1^2c_2 - c_4 \right|,$$

where

$$Y_3(q) = 9q^{12} + 68q^{11} + 250q^{10} + 573q^9 + 801q^8 + 800q^7 + 278q^6 + 91q^5 - 466q^4 - 212q^3 - 320q^2 + 56q - 200.$$

Consider

$$\gamma = \frac{Y_3(q)}{512q(q^2+1)(q+1)^4(q^2+q+1)}, \lambda = \frac{(q^4+5q^3+2q^2+2)}{8q(1+q)^2},$$

and

$$\alpha = \frac{(2q^3+7q^2+5q+4)}{8(q+1)^2}, \beta = \frac{(5q^7+25q^6+50q^5+11q^4+47q^3+2q^2+60q-20)}{48q(q+1)^2(q+q^2+1)}.$$

Clearly, $0 < \lambda < 1$ and $0 < \alpha < 1$ for $q \in (0, 1)$. Now,

$$8\lambda(1-\lambda) \left[(\alpha\beta - 2\gamma)^2 + (\alpha(\lambda + \alpha) - \beta)^2 \right] + \alpha(1-\alpha)(\beta - 2\lambda\alpha)^2 - 4\alpha^2(1-\alpha)^2\lambda(1-\lambda) = \Psi_3(q),$$

where

$$\Psi_3(q) = -\frac{\Lambda_3(q)}{4718592q^4(q+1)^{12}(q^4+q^3+2q^2+q+1)^2}$$

and

$$\Lambda_3(q) = \begin{cases} 49q^{32} + 574q^{31} + 2669q^{30} + 4330q^{29} - 34358q^{28} - 383594q^{27} - 2118409q^{26} \\ - 8583478q^{25} - 31105086q^{24} - 99548394q^{23} - 267222045q^{22} - 601142446q^{21} \\ - 1131536606q^{20} - 1773679062q^{19} - 2297680443q^{18} - 2421183786q^{17} - 2068071911q^{16} \\ - 1477288612q^{15} - 1124226544q^{14} - 1302948636q^{13} - 1851382648q^{12} - 2371066992q^{11} \\ - 2338803868q^{10} - 1735883520q^9 - 915866752q^8 - 241534464q^7 + 86898560q^6 \\ + 121530688q^5 + 57728256q^4 + 14253312q^3 + 7542528q^2 + 3678208q + 975104. \end{cases}$$

It can be observed that $\Psi(q) \leq 0$ for all $q \in (0, 0.514)$. Now, by Lemma 4, we have

$$|a_2a_4 - a_5| \leq \frac{1}{2q(q^2+1)}, q \in (0, 0.514).$$

□

Remark 2. (i) In Theorems 2 and 3, we prove the generalized Zalcman conjectures for $s = 2, t = 3$ and $s = 3, t = 3$, respectively.

(ii) In Theorem 4, we prove the generalized Zalcman's conjecture for $s = 2$ and $t = 4$.

Theorem 5. Let $\psi \in \mathcal{S}_B^{*s}(q)$, as given in (1). Then,

$$|\mathcal{H}_{2,2}(\psi)| = |a_2a_4 - a_3^2| \leq \frac{1}{4q^2}, \text{ for } q \in (0.119, 1). \quad (41)$$

The function in (35) sharply obtains the result.

Proof. From (30)–(32), utilizing the definition of the q -number, we have

$$\left| a_2 a_4 - a_3^2 \right| = \left| \left(\frac{c_1}{4} \right) \left(\frac{(q+1)}{4(q^3+q^2+q+1)} c_3 - \frac{(q^4+4q^3+3q^2+3q-1)}{16q(q+1)(q^3+q^2+q+1)} c_1 c_2 + \frac{(q^7+5q^6+11q^5+15q^4+11q^3-5q^2-2q-10)}{128q(q+1)(q^2+q+1)(q^3+q^2+q+1)} c_1^3 \right) - \left(\frac{1}{q(q+1)} \left(\frac{(q+1)}{4} c_2 - \frac{(q^3+5q^2-2q+10)}{32(q+1)} c_1^2 \right) \right)^2 \right|,$$

Collecting the like terms,

$$\left| a_2 a_4 - a_3^2 \right| = \left| \frac{(1+q)}{16(1+q+q^2+q^3)} c_1 c_3 - \frac{1}{16q^2} c_2^2 + \frac{(q^4-4q^3+12q^2-q+10)}{64q^2(q^2+1)(q+1)^2} c_1^2 c_2 - \frac{(-q^{10}-3q^9-11q^8-42q^7+53q^6+31q^5+291q^4+162q^3+308q^2+80q+100)}{1024q^2(q+1)^4(q^4+q^3+2q^2+q+1)} c_1^4 \right|.$$

By using Lemma 2, we have

$$\left| a_2 a_4 - a_3^2 \right| = \left| \frac{(q^4-4q^3+10q^2-5q+8)}{128q^2(q^2+1)(q+1)^2} c_1^4 + \frac{(q^4-4q^3+8q^2-9q+6)}{128q^2(q^2+1)(q+1)^2} (4-c_1^2) c_1^2 k - \left(\frac{(1+q)}{64(1+q+q^2+q^3)} \right) (4-c_1^2) c_1^2 k^2 - \frac{1}{64q^2} (4-c_1^2)^2 k^2 + \left(\frac{(1+q)}{32(1+q+q^2+q^3)} \right) c_1 (4-c_1^2) (1-|k|^2) \delta \right|. \quad (42)$$

For the convenience of notation, let $c_1 = c \in [0, 2]$, and

$$\begin{aligned} \alpha_1 &= \left| \frac{(q^4-4q^3+10q^2-5q+8)}{128q^2(q^2+1)(q+1)^2} \right| = \frac{(q^4-4q^3+10q^2-5q+8)}{128q^2(q^2+1)(q+1)^2} > 0, \\ \alpha_2 &= \left| \frac{(q^4-4q^3+8q^2-9q+6)}{128q^2(q^2+1)(q+1)^2} \right| = \frac{(q^4-4q^3+8q^2-9q+6)}{128q^2(q^2+1)(q+1)^2} > 0, \\ \alpha_3 &= \left| -\frac{(1+q)}{64(1+q+q^2+q^3)} \right| = \frac{(1+q)}{64(1+q+q^2+q^3)} > 0, \\ \alpha_4 &= \left| -\frac{1}{64q^2} \right| = \frac{1}{64q^2} > 0, \\ \alpha_5 &= \left| \frac{(1+q)}{32(1+q+q^2+q^3)} \right| = \frac{(1+q)}{32(1+q+q^2+q^3)} > 0 \end{aligned}$$

Now, applying the modulus and using the triangular inequality along with $|k| = u$, $|c| = t$ and $|\delta| \leq 1$, we have

$$\left| a_2 a_4 - a_3^2 \right| \leq \left\{ \begin{array}{l} \alpha_1 t^4 + \alpha_2 (4-t^2) t^2 u + \alpha_3 (4-t^2) t^2 u^2 \\ + \alpha_4 (4-t^2)^2 u^2 + \alpha_5 t (4-t^2) (1-u^2) \end{array} \right\} = \mathcal{L}(t, u) \quad (43)$$

Now, we suppose that the upper bound for (43) exists in the interior of rectangle $[0, 2] \times [0, 1]$. Differentiating $\mathcal{L}(t, u)$ with respect to u , we have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial u} &= \alpha_2 (4-t^2) t^2 + 2\alpha_3 (4-t^2) t^2 u + 2\alpha_4 (4-t^2)^2 u - 2\alpha_5 t (4-t^2) u, \\ &= (4-t^2) \left((\alpha_2 + (2\alpha_3 - 2\alpha_4)u) t^2 + 8\alpha_4 u - 2\alpha_5 t u \right). \end{aligned}$$

Setting $\frac{\partial \mathcal{L}}{\partial u} = 0$ implies either $t = 2$ or $((\alpha_2 + (2\alpha_3 - 2\alpha_4)u) t^2 + 8\alpha_4 u - 2\alpha_5 t u) = 0$. The points (t, u) that satisfy these conditions are not interior points of the rectangle $[0, 2] \times [0, 1]$. Consequently, the function $\mathcal{L}(t, u)$ cannot achieve its maximum value within the interior of

the rectangle. Thus, the maximum value must occur at the boundary of the rectangle. For this, we consider the following cases

(i) On the line $t = 0, 0 \leq u \leq 1$, we have $\mathcal{L}(0, y) = 16\alpha_4 y^2$ and

$$\max \mathcal{L}(0, y) = \mathcal{L}(0, 1) = 16\alpha_4 = \frac{1}{4q^2}.$$

(ii) On the line $t = 2, 0 \leq u \leq 1$, we have $\mathcal{L}(2, y) = 16\alpha_1$ and

$$\max \mathcal{L}(2, y) = 16\alpha_1 = \frac{(q^4 - 4q^3 + 10q^2 - 5q + 8)}{8q^2(q^2 + 1)(q + 1)^2}.$$

(iii) On the line $u = 0, 0 \leq t \leq 1$, we have $\mathcal{L}(t, 0) = \alpha_1 t^4 + \alpha_5 t(4 - t^2) = g_1(t)$ and

$$g_1'(t) = 4\alpha_1 t^3 - 3\alpha_5 t^2 + 4\alpha_5 > 0 \text{ for all } 0 \leq t \leq 2,$$

this shows that $g_1(t)$ is an increasing function on $0 \leq t \leq 2$, therefore

$$\max \mathcal{L}(t, 0) = \mathcal{L}(2, 0) = 16\alpha_1 = \frac{(q^4 - 4q^3 + 10q^2 - 5q + 8)}{8q^2(q^2 + 1)(q + 1)^2}.$$

(iv) On the line $u = 1, 0 \leq t \leq 1$, we have

$$\mathcal{L}(t, 1) = \left\{ (\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4)t^4 + 4(\alpha_2 + \alpha_3 - 2\alpha_4)t^2 + 16\alpha_4 \right\} = g_2(t),$$

as $g_2'(0) = 0$ and

$$g_2''(0) = 8(\alpha_2 + \alpha_3 - 2\alpha_4) = -\frac{(q^4 + 8q^3 - 2q^2 + 17q - 2)}{16q^2(q^2 + 1)(q + 1)^2} \leq 0 \text{ for } q \in (0.119, 1),$$

so

$$\max \mathcal{L}(t, 1) = g_2(0) = 16\alpha_4 = \frac{1}{4q^2}.$$

Hence, on the rectangle $[0, 2] \times [0, 1]$, the $\max \mathcal{L}(t, y) = \frac{1}{4q^2}$, so

$$|\mathcal{H}_{2,2}(\psi)| = |a_2 a_4 - a_3^2| \leq \frac{1}{4q^2}, q \in (0.119, 1).$$

□

Theorem 6. Let $\psi \in \mathcal{S}_B^{*s}(q)$, as given in (1). Then,

$$|\mathcal{H}_{3,1}(\psi)| \leq \frac{(q^4 + 4q^3 + 2q^2 + 2q + 1)}{8q^3(q^2 + 1)^2}, \text{ for } q \in (0.762, 1).$$

Proof. From (10), we have

$$\begin{aligned} |\mathcal{H}_{3,1}(\psi)| &= \left| a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2) \right| \\ &\leq |a_3| |a_2 a_4 - a_3^2| + |a_4| |a_4 - a_2 a_3| + |a_5| |a_3 - a_2^2|. \end{aligned}$$

Using (24)–(27) and (38), (41), we have

$$|\mathcal{H}_{3,1}(\psi)| \leq \left(\frac{1}{2q} \right) \left(\frac{1}{4q^2} \right) + \left(\frac{1}{2(1+q^2)} \right) \left(\frac{1}{2(q^2+1)} \right) + \left(\frac{1}{2q(1+q^2)} \right) \left(\frac{1}{2q} \right).$$

Through simple calculations, we obtain the required result. $\square$

4. Conclusions

By applying the principle of subordination and quantum calculus, we introduced and explored a new class $\psi \in \mathcal{S}_B^{*s}(q)$ of analytic functions connected with the q -differential operator and the symmetric balloon-shaped domain. This class extended the notion of starlike functions with respect to symmetric points to q -starlike functions with respect to symmetric points. This study established sharp results for the Maclaurin coefficients, Hankel determinants, the Zalcman conjecture, and its generalization conjecture for the newly defined class.

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References

1. Miller, S.S.; Mocanu, P.T. Differential subordination and univalent functions. *Mich. Math. J.* **1981**, *28*, 157–171. [\[CrossRef\]](#)
2. Shanmugam, T.N. Convolution and Differential subordination. *Int. J. Math. Math. Sci.* **1989**, *12*, 333–340. [\[CrossRef\]](#)
3. Padmanabhan, K.S.; Parvatham, R. Some applications of differential subordination. *Bull. Aust. Math. Soc.* **1985**, *32*, 321–330. [\[CrossRef\]](#)
4. Ma, W.C.; Minda, D. A unified treatment of some special classes of univalent functions. In *Proceedings of the Conference on Complex Analysis at the Nankai Institute of Mathematics*; Li, Z., Ren, F., Yang, L., Zhang, S., Eds.; International Press: New York, NY, USA, 1994; pp. 157–169.
5. Bano, K.; Raza, M. Starlike functions associated with cosine function. *Bull. Iran. Math. Soc.* **2021**, *47*, 1513–1532. [\[CrossRef\]](#)
6. Cho, N.E.; Kumar, V.; Kumar, S.S.; Ravichandran, V. Radius problems for starlike functions associated with the sine function. *Bull. Iran. Math. Soc.* **2019**, *45*, 213–232. [\[CrossRef\]](#)
7. Geol, P.; Kumar, S.S. Certain class of starlike functions associated with modified sigmoid function. *Bull. Malays. Math. Sci. Soc.* **2020**, *43*, 957–991. [\[CrossRef\]](#)
8. Khan, M.G.; Khan, B.; Gong, J.; Tchier, F.; Tawfiq, F.M.O. Applications of first-order differential subordination for subfamilies of analytic functions related to symmetric image domains. *Symmetry* **2023**, *15*, 2004. [\[CrossRef\]](#)
9. Kumar, S.S.; Arora, K. Starlike functions associated with a petal shaped domain. *arXiv* **2020**, arXiv:2010.10072.
10. Wani, L.A.; Swaminathan, A. Starlike and convex functions associated with a Nephroid domain. *Bull. Malays. Math. Sci. Soc.* **2021**, *44*, 79–104. [\[CrossRef\]](#)
11. Khan, M.G.; Ahmad, B.; Murugusundaramoorthy, G.; Chinram, R.; Mashwani, W.K. Applications of modified sigmoid functions to a class of starlike functions. *J. Funct. Spaces* **2020**, *2020*, 8844814. [\[CrossRef\]](#)
12. Jackson, F.H. On q -functions and certain difference operator. *Trans. R. Soc. Edinb.* **1909**, *46*, 253–281. [\[CrossRef\]](#)
13. Jackson, F.H. On q -definite integrals. *Quart. J. Pure Appl. Math.* **1910**, *41*, 193–203.
14. Srivastava, H.M. Univalent functions, fractional calculus, and associated generalized hypergeometric functions. In *Univalent Functions, Fractional Calculus and Their Applications*; Srivastava, H.M., Owa, S., Eds.; John Wiley Sons: New York, NY, USA, 1989.
15. Ismail, M.E.H.; Merkes, E.; Styer, D. A generalization of starlike functions. *Complex Var.* **1990**, *14*, 77–84. [\[CrossRef\]](#)
16. Agrawal, S.; Sahoo, S.K. A generalization of starlike functions of order α . *Hokkaido Math. J.* **2017**, *46*, 15–27. [\[CrossRef\]](#)
17. Ashish, V.; Sarasvati, Y. Recursion formulas for Srivastava's general triple q -hypergeometric series. *Afr. Mat.* **2020**, *31*, 869–885.
18. Al-Shbeil, I.; Gong, J.; Khan, S.; Khan, N.; Khan, A.; Khan, M.F.; Goswami, A. Hankel and symmetric Toeplitz determinants for a new subclass of q -starlike functions. *Fractal Fract.* **2022**, *6*, 658. [\[CrossRef\]](#)
19. Al-Shaikh, S.B.; Abubaker, A.A.; Matarneh, K.; Khan, M.F. Some new applications of the q -analogous of differential and integral operators for new subclasses of q -starlike and q -convex functions. *Fractal Fract.* **2023**, *7*, 411. [\[CrossRef\]](#)
20. Khan, N.; Shafiq, M.; Darus, M.; Khan, B.; Ahmad, Q.Z. Upper bound of the third Hankel determinant for a subclass of q -starlike functions associated with lemniscate of Bernoulli. *J. Math. Inequal.* **2020**, *14*, 51–63. [\[CrossRef\]](#)
21. Srivastava, H.M.; Khan, N.; Darus, M.; Khan, S.; Ahmad, Q.Z.; Hussain, S. Fekete-Szegő type problems and their applications for a subclass of q -starlike functions with respect to symmetrical Points. *Mathematics* **2020**, *8*, 842. [\[CrossRef\]](#)
22. Taj, Y.; Zainab, S.; Ferdous, Q.X.; Tawfiq, M.O.; Raza, M.; Malik, S.N. Certain coefficient problems for q -starlike functions associated with q -analogue of sine function. *Symmetry* **2022**, *14*, 2200. [\[CrossRef\]](#)

23. Murugusundaramoorthy, K.R.K.G.; Purohit, S.D.; Suthar, D.L. Certain class of analytic functions with respect to symmetric points defined by q -calculus. *J. Math.* **2021**, *2021*, 8298848.
24. Jabeen, M.; Malik, S.N.; Mahmood, S.; Riaz, S.M.J.; Ali, M.S. On q -convex functions defined by the q -Ruscheweyh derivative operator in conic regions. *J. Math.* **2022**, *2022*, 2681789. [[CrossRef](#)]
25. Khan, M.F.; Khan, S.; Darus, M.; Hussain, S. Sharp coefficient inequalities for a class of analytic functions defined by q -difference operator associated with the q -lemniscate of Bernoulli. *Results Nonlinear Anal.* **2023**, *6*, 55–73.
26. Mahmood, S.; Raza, N.; Jarad, E.S.A.; Srivastava, G.; Srivastava, H.M.; Malik, S.N. Geometric properties of certain classes of analytic functions associated with a q -integral operator. *Symmetry* **2019**, *14*, 719. [[CrossRef](#)]
27. Seoudy, T.; Aouf, M. Coefficient estimates of new classes of q -starlike and q -convex functions of complex order. *J. Math. Inequal.* **2016**, *10*, 135–145. [[CrossRef](#)]
28. Vijaya, K.; Murugusundaramoorthy, G.; Breaz, D.; Oros, G.I.; El-Deeb, S.M. Ozaki-type bi-close-to-convex and bi-concave functions involving a modified caputo's fractional operator linked with a three-leaf function. *Fractal Fract.* **2024**, *8*, 220. [[CrossRef](#)]
29. Saliu, A.; Jabeen, K.; Al-Shbeil, I.; Aloraini, N.; Malik, S.N. On q -Limaçon functions. *Symmetry* **2022**, *14*, 242. [[CrossRef](#)]
30. Shaba, T.G.; Araci, S.; Adebesein, B.O.; Tchier, F.; Zainab, S.; Khan, B. Sharp bounds of the Fekete–Szegő problem and second Hankel determinant for certain Bi-univalent functions defined by a novel q -differential operator associated with q -Limaçon domain. *Fractal Fract.* **2023**, *7*, 506. [[CrossRef](#)]
31. Srivastava, H.M.; Khan, B.; Khan, N.; Tahir, M.; Ahmad, S.; Khan, N. Upper bound of the third Hankel determinant for a subclass of q -starlike functions associated with the q -exponential function. *Bull. Sci. Math.* **2021**, *167*, 102942. [[CrossRef](#)]
32. Swarup, C. Sharp coefficient bounds for a new subclass of q -starlike functions associated with q -analogue of the hyperbolic tangent function. *Symmetry* **2023**, *15*, 763. [[CrossRef](#)]
33. Zhang, X.; Khan, S.; Hussain, S.; Tang, H.; Shareef, Z. New subclass of q -starlike functions associated with generalized conic domain. *Aims Math.* **2020**, *5*, 4830–4848. [[CrossRef](#)]
34. Khan, M.F.; Al-shbeil, I.; Khan, S.; Khan, N.; Haq, W.U.; Gong, J. Applications of a q -Differential Operator to a Class of Harmonic Mappings Defined by q -Mittag-Leffler Functions. *Symmetry* **2022**, *14*, 1905. [[CrossRef](#)]
35. Gasper, G.; Rahman, M. *Basic Hypergeometric Series*; Cambridge University Press: Cambridge, UK, 2004; Volume 96.
36. Ahmad, A.; Gong, J.; Al-Shbeil, I.; Rasheed, A.; Ali, A.; Hussain, S. Analytic functions related to a balloon-shaped domain. *Fractal Fract.* **2023**, *7*, 865. [[CrossRef](#)]
37. Sakaguchi, K. On a certain univalent mapping. *J. Math. Soc. Jpn.* **1959**, *11*, 72–75. [[CrossRef](#)]
38. Khan, B.; Gong, J.; Khan, M.G.; Tchier, F. Sharp Coefficient Bounds for a Class of Symmetric Starlike Functions Involving the Balloon Shape Domain. *Heliyon*, **2024**; to appear.
39. Ma, W. Generalized Zalcman conjecture for starlike and typically real functions. *J. Math. Anal. Appl.* **1999**, *234*, 328–339. [[CrossRef](#)]
40. Pommerenke, C.; Jensen, G. *Univalent Functions*; Vandenhoeck and Ruprecht: Gottingen, Germany, 1975.
41. Dziok, J. A general solution of the Fekete–Szegő problem. *Bound. Value Probl.* **2013**, *2013*, 98. [[CrossRef](#)]
42. Libera, R.J.; Zlotkiewicz, E.J. Coefficient bounds for the inverse of a function with derivative in P . *Proc. Am. Math. Soc.* **1983**, *87*, 251–257. [[CrossRef](#)]
43. Ravichandran, V.; Verma, S. Bound for the fifth coefficient of certain starlike functions. *Comptes Rendus Math. Acad. Sci. Paris* **2015**, *353*, 505–510. [[CrossRef](#)]

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