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On Certain Rough Marcinkiewicz Integral Operators with Grafakos-Stefanov Kernels on Product Spaces

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Abstract: In this paper, several classes of rough Marcinkiewicz integral operators along surfaces of revolution on product spaces are investigated. We prove the L^p boundedness of these operators when their kernels functions belong to a class of functions related to a class of functions introduced by Grafakos-Stefanov. The results in this paper extend and improve several known results on Marcinkiewicz integrals over a symmetric space.

Keywords: L^p bound; rough Marcinkiewiz integrals; surfaces of revolution; product domains; Grafakos-Stefanov class

1. Introduction

Let $\mathbb{R}^q$ ($q = m$ or n) be the $2 \leq q$-Euclidean space and $\mathbb{S}^{q-1}$ be the unit sphere in $\mathbb{R}^q$ equipped with the normalized Lebesgue surface measure $d\sigma_q(\cdot) \equiv d\sigma$. Also, let $z' = z/|z|$ for $z \in \mathbb{R}^q \backslash \{0\}$.

Let $\mho$ be a measurable function defined on $\mathbb{R}^m \times \mathbb{R}^n$, integrable over $\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}$ and satisfy the following:

$$\mho(rx, sy) = \mho(x, y), \ \forall r, s > 0, \tag{1}$$

$$\int_{\mathbb{S}^{m-1}} \mho(x, y) d\sigma(x) = \int_{\mathbb{S}^{n-1}} \mho(x, y) d\sigma(y) = 0. \tag{2}$$

For an appropriate mapping $\Phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, we consider the parametric Marcinkiewicz integral operator $\mathfrak{M}_{\mho,\Phi}$ along the surface of revolution $\Lambda_\Phi(u, v) = (u, v, \Phi(|u|, |v|))$ defined, initially for $g \in C_0^\infty(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})$, by

$$\mathfrak{M}_{\mho,\Phi}(g)(u, v, w) = \left(\iint_{\mathbb{R}_+ \times \mathbb{R}_+} |F_{r,s}(g)(u, v, w)|^2 \frac{dr ds}{rs} \right)^{1/2}, \tag{3}$$

where

$$F_{r,s}(g)(u, v, w) = \frac{1}{rs} \int_{|y| \leq s} \int_{|x| \leq r} f(u - x, v - y, w - \Phi(|x|, |y|)) \frac{\mho(x, y)}{|x|^{m-1} |y|^{n-1}} dx dy.$$

We point out her that the Marcinkiewicz integral $\mathfrak{M}_{\mho,\Phi}$ is a natural generalization of the Marcinkiewicz inegral $\mathfrak{M}_\mho^\phi$ along surface of revolution $\Lambda_\Phi(u) = (u, \Phi(|u|))$ in the one parameter setting which is given by

$$\mathfrak{M}_\mho^\phi(g)(u, u_{m+1}) = \left(\int_{\mathbb{R}_+} \left| \frac{1}{r} \int_{|x| \leq r} g(u - x, u_{m+1} - \phi(|x|)) \frac{\mho(x)}{|x|^{m-1}} dx \right|^2 \frac{dr}{r} \right)^{1/2}. \tag{4}$$

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The investigation of the L^p boundedeness of $\mathfrak{M}_{\upsilon}^{\phi}$ under certain assumptions on υ and ϕ has received a large amount of attention by many mathematicians: when $\phi \equiv 0$ [1–4] and when ϕ satisfies some certain conditions [5–9].

Let $G_{\alpha}\left(\mathbb{S}^{m-1}\right)$ (for $\alpha > 0$) be the class of all functions $\upsilon \in L^1(\mathbb{S}^{m-1})$ satisfying the condition

$$\sup_{\xi \in \mathbb{S}^{m-1}} \int_{\mathbb{S}^{m-1}} |\upsilon(x)| \log^{\alpha}(|\xi \cdot x|^{-1}) d\sigma(x) < \infty. \tag{5}$$

It is worth mentioning that the class $G_{\alpha}\left(\mathbb{S}^{m-1}\right)$ was defined by Walsh in [10] and developed by Grafakos and Stefanov in [11].

An interesting result related to our work in the one parameter setting was obtained in [12] as described in the following theorem:

Theorem 1. *Let $\upsilon \in G_{\alpha}\left(\mathbb{S}^{m-1}\right)$ for some $\alpha > 1/2$. Then $\mathfrak{M}_{\upsilon}^{\phi}$ is bounded on $L^p(\mathbb{R}^m \times \mathbb{R})$ for all $p \in (\frac{1+2\alpha}{2\alpha}, 1+2\alpha)$ if one of the following conditions is satisfied:*

(a) $\phi \in C^1(\mathbb{R}_+)$, ϕ' is increasing and convex function with either $m = 2$ or $m \geq 3$ and $\phi'(0) = 0$.

(b) ϕ is a polynomial with either $m = 2$ or $m \geq 3$ and $\phi'(0) = 0$.

Our focus will be on the operator $\mathfrak{M}_{\upsilon,\Phi}$. If $\Phi \equiv 0$, we denote $\mathfrak{M}_{\upsilon,\Phi}$ by $\mathfrak{M}_{\upsilon}$ which is the classical Marcinkiewicz integral on product domains. The study of the boundedness of $\mathfrak{M}_{\upsilon}$ started by Ding in [13] and then continued by many authors. For a sample of past studies and for more information about the applications and development of the operator $\mathfrak{M}_{\upsilon}$, we refer the readers to consult: For the case $\Phi \equiv 0$ [14–16] and for the case Φ satisfies specific conditions [17–19] and the references therein.

For $\alpha > 0$, let $G_{\alpha}\left(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}\right)$ be the set of all functions $\upsilon \in L^1(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ which satisfy Grafakos and Stefanov condition:

$$\sup_{(\xi,\zeta) \in \mathbb{S}^{m-1} \times \mathbb{S}^{n-1}} \iint_{\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}} |\upsilon(x,y)| \log^{\alpha}(|\xi \cdot x|^{-1}) \log^{\alpha}(|\zeta \cdot y|^{-1}) d\sigma(x) d\sigma(y) < \infty. \tag{6}$$

By following the same arguments as those in [11], we get

$$\bigcup_{\kappa > 1} L^{\kappa}\left(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}\right) \not\subseteq G_{\alpha}\left(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}\right) \text{ for any } \alpha > 0, \tag{7}$$

$$\bigcap_{\alpha > 0} G_{\alpha}\left(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}\right) \not\subseteq L(\log L)(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}) \not\subseteq \bigcup_{\alpha > 0} G_{\alpha}\left(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}\right). \tag{8}$$

Very recently, in [20] the authors studied the related singular integral operators along surfaces of revolution $\mathcal{T}_{\upsilon,\Phi}$ which are given by

$$\mathcal{T}_{\upsilon,\Phi}(g)(u,v,w) = \iint_{\mathbb{R}^m \times \mathbb{R}^n} f(u-x, v-y, w-\Phi(|x|,|y|)) \frac{\upsilon(x,y)}{|x|^m |y|^n} dx dy, \tag{9}$$

where $\Phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is an appropriate mapping. When $\upsilon \in G_{\alpha}\left(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}\right)$ for some $\alpha > 1/2$, the authors of [20] showed that $\mathcal{T}_{\upsilon,\Phi}$ is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})$ for all $p \in (\frac{1+2\alpha}{2\alpha}, 1+2\alpha)$ under various assumptions on Φ.

In light of the assumptions imposed on Φ in [20] and of the results in [12] concerning the boundedness of Marcinkiewicz operator $\mathfrak{M}_{\upsilon}^{\phi}$ in the one parameter setting whenever $\upsilon \in G_{\alpha}\left(\mathbb{S}^{m-1}\right)$ as well as of the results in [18] concerning the boundedness of Marcinkiewicz operator $\mathfrak{M}_{\upsilon,\Phi}$ whenever $\upsilon \in L(\log L)(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}) \cup B_{\kappa}^{(0,0)}(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$, a question arises naturally is the following:

Question: Under the same conditions on Φ in [20], does the operator $\mathfrak{M}_{\upsilon,\Phi}$ satisfy the L^p boundedness provided that $\upsilon \in G_{\alpha}\left(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}\right)$ for some $\alpha > 1/2$?

Our main purpose in this paper is on giving a positive answer to this question. Indeed, we obtain the following:

Theorem 2. *Assume that $\mho$ belongs to the space $G_\alpha\left(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}\right)$ for some $\alpha>1/2$ and satisfies the conditions (1)–(2). Assume that $\Phi\in C^1(\mathbb{R}_+\times\mathbb{R}_+)$. If one of the following conditions holds, then $\mathfrak{M}_{\mho,\Phi}$ is bounded on $L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ for all $p\in(\frac{1+2\alpha}{2\alpha},1+2\alpha)$.*

(*i*) $m=2=n$, $D_t\Phi(t,\ell)$ *and* $D_\ell\Phi(t,\ell)$ *are convex increasing functions,*
(*ii*) $m=2$, $n\geq 3$, $D_t\Phi(t,\ell)$ *and* $D_\ell\Phi(t,\ell)$ *are convex increasing functions with* $D_\ell(t,0)=0$,
(*iii*) $n=2$, $m\geq 3$, $D_t\Phi(t,\ell)$ *and* $D_\ell\Phi(t,\ell)$ *are convex increasing functions with* $D_t\Phi(0,\ell)=0$,
(*iv*) $m\geq 3$, $n\geq 3$, $\Phi_t(t,\ell)$ *and* $D_\ell\Phi(t,\ell)$ *are convex increasing functions with* $D_t\Phi(0,\ell)=0=D_\ell\Phi(t,0)$.

Theorem 3. *Assume that $\mho\in G_\alpha\left(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}\right)$ for some $\alpha>1/2$ and satisfies the conditions (1)–(2). Assume that $\Phi(t,\ell)=\sum\limits_{i=0}^{d_1}\sum\limits_{j=0}^{d_2}a_{j,i}t^{\alpha_i}\ell^{\beta_j}$ is a generalized polynomial on $\mathbb{R}^2$, where the exponents in each set $\{\alpha_i:0\leq i\leq d_1\}$ and $\{\beta_j:0\leq j\leq d_2\}$ are distinct positive numbers. Then $\mathfrak{M}_{\mho,\Phi}$ is bounded on $L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ for all $p\in(\frac{1+2\alpha}{2\alpha},1+2\alpha)$ whenever one of the following conditions holds:*

(*i*) $\alpha_i\neq 1$ *and* $\beta_j\neq 1$ *for all* $0\leq i\leq d_1$ *and* $0\leq j\leq d_2$,
(*ii*) *There is* $\alpha_{i_0}=1$ *for some* $0\leq i_0\leq d_1$ *and* $\beta_j\neq 1$ *for all* $0\leq j\leq d_2$*; and if* $m\geq 3$*, we need* $D_t\Phi(0,\ell)=0$,
(*iii*) *There is* $\beta_{j_0}=1$ *for some* $0\leq j_0\leq d_2$ *and* $\alpha_i\neq 1$ *for all* $0\leq i\leq d_1$*; and if* $n\geq 3$*, we need* $D_\ell\Phi(t,0)=0$,
(*iv*) *There exists* $\alpha_{i_0}=1$ *for some* $0\leq i_0\leq d_1$ *and there exists* $\beta_{j_0}=1$ *for some* $0\leq j_0\leq d_2$*. Moreover, whenever* $m\geq 3$ *we need* $D_t\Phi(0,\ell)=0$ *and whenever* $n\geq 3$ *we need* $D_\ell\Phi(t,0)=0$.

Theorem 4. *Assume that $\mho$ is given as in Theorem 2. Assume that $\Phi(t,\ell)=\phi(t)P(\ell)$, where $\phi(t)$ is in $C^1(\mathbb{R}_+)$, ϕ' is increasing and convex function; and P is a generalized polynomials given by $P(\ell)=\sum\limits_{j=0}^{d_2}a_j\ell^{\beta_j}$. Then $\mathfrak{M}_{\mho,\Phi}$ is bounded on $L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ for all $p\in(\frac{1+2\alpha}{2\alpha},1+2\alpha)$ if one of the following conditions is satisfied:*

(*i*) $\beta_j\neq 1$ *for all* $0\leq j\leq d_2$*, and if* $m\geq 3$ *we need* $\phi'(0)=0$,
(*ii*) *There is* $\beta_{j_0}=1$ *for some* $0\leq j_0\leq d_2$*, and if* $m,n\geq 3$ *we need* $\phi'(0)=0=P(0)$.

Theorem 5. *Assume $\Phi(t,\ell)=\phi_1(t)+\phi_2(\ell)$, where ϕ_j $(j=1,2)$ is either a generalized polynomial or is in $C^1(\mathbb{R}_+)$, ϕ_j' is increasing and convex function. If $\mho\in G_\alpha\left(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}\right)$ for some $\alpha>1/2$ and one of the following conditions is held, then $\mathfrak{M}_{\mho,\Phi}$ is bounded on $L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ for all $p\in(\frac{1+2\alpha}{2\alpha},1+2\alpha)$.*

Case 1: ϕ_j $(j=1,2)$ is in $C^1(\mathbb{R}_+)$, ϕ_j' is increasing and convex function.

(*i*) $m=2$, $n=2$,
(*ii*) $m=2$, $n\geq 3$ *and* $\phi_1'(0)$,
(*iii*) $n=2$, $m\geq 3$ *and* $\phi_2'(0)$,
(*iv*) $m\geq 3$, $n\geq 3$ *and* $\phi_1'(0)=0=\phi_2'(0)$.

Case 2: ϕ_1 is in $C^1(\mathbb{R}_+)$, ϕ_2 is increasing and convex function; ϕ_2 is a generalized polynomial on $\mathbb{R}$ given by $\phi_2(\ell)=\sum\limits_{j=0}^{d_2}a_j\ell^{\beta_j}$, where the exponents in the set $\{\beta_j:0\leq j\leq d_2\}$ are distinct positive numbers.

(*i*) $\beta_j\neq 1$ *for all* $0\leq j\leq d_2$*, and if* $n\geq 3$ *we need* $\phi_1'(0)=0$,
(*ii*) *There is* $\beta_{j_0}=1$ *for some* $0\leq j_0\leq d_2$*, and if* $m,n\geq 3$ *we need* $\phi_1'(0)=0=\phi_2'(0)$.

Theorem 6. *Assume that* $\Phi(t,\ell) = \phi(t\ell)$ *for all* $t,\ell > 0$, *where* $\phi \in C^1(\mathbb{R}_+)$. *If* $\mho \in G_\alpha\left(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}\right)$ *for some* $\alpha > 1/2$ *and one of the following conditions is satisfied, then* $\mathfrak{M}_{\mho,\Phi}$ *is bounded on* $L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ *for all* $p \in (\frac{1+2\alpha}{2\alpha}, 1+2\alpha)$.

(i) $m = 2 = n$ *and* ϕ' *is an increasing and convex function,*
(ii) *If* m *or* $n \geq 3$ *and* ϕ' *is an increasing and convex function with* $\phi'(0) = 0$.

Remark

(1) The authors of [16] confirmed the boundedness of $\mathfrak{M}_{\mho,\Phi}$ whenever $\Phi \equiv 0$ and $\mho \in L^q(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}) \subsetneq G_\alpha\left(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}\right)$. Therefore, our findings extend the results in [16].
(2) For the special case $\Phi \equiv 0$, our results prove the L^p boundedness of $\mathfrak{M}_{\mho,\Phi}$ for $p \in (\frac{1+2\alpha}{2\alpha}, 1+2\alpha)$, which is the result established in [21]. Hence, our results are essential generalization and improvement to the results in [21].
(3) The authors of [18] showed that $\mathfrak{M}_{\mho,\Phi}$ is bounded on $L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$ provided that $\mho \in L(logL)(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}) \cup B_q^{(0,0)}(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}) \nsubseteq G_\alpha\left(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}\right)$, for any $\alpha > 0$, $q > 1$.
(4) The surfaces of revolution $\Lambda_\Phi(u,v) = (u,v,\Phi(|u|,|v|))$ which are considered in Theorems 2–6 cover several important natural classical surfaces. For example, our theorems allow surfaces of the type Λ_Φ, where $\Phi(t,\ell) = t^2\ell^2(e^{-1/t}+e^{-1/\ell}), t,\ell > 0$; $\Phi(t,\ell) = t^\alpha\ell^\beta$ with $\alpha,\beta > 0$; $\Phi(t,\ell) = P(t,\ell)$ is a polynomial; $\Phi(t,\ell) = \phi_1(t)\phi_2(\ell)$, where for $i \in \{1,2\}$, each $\phi_i \in C^2(\mathbb{R}_+)$ is a convex increasing function with $\phi_i(0) = 0$.

Henceforward, the constant C denotes a positive real constant which is not necessary the same at each occurrence but it is independent of the essential variables.

2. Some Lemmas

For a suitable mapping $\Phi(r,s)$ on $\mathbb{R}_+\times\mathbb{R}_+$, we define the family of measures $\{\lambda^{k,j}_{\mho,\Phi,r,s} := \lambda^{k,j}_{r,s} : k,j \in \mathbb{Z}, r,s \in \mathbb{R}_+\}$ and its related maximal operator λ^* on $\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}$ by

$$\iiint_{\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}} g\,d\lambda^{k,j}_{r,s} = \frac{1}{2^{k+j}rs}\int_{2^j s\leq|u|\leq 2^{j+1}s}\int_{2^k r\leq|v|\leq 2^{k+1}r} g(v,u,\Phi(|v|,|u|))K_{\mho,h}(v,u)\,dv\,du$$

and

$$\lambda^*(g)(x,y,w) = \sup_{k,j\in\mathbb{Z}}\sup_{r,s\in\mathbb{R}_+}||\lambda^{k,j}_{r,s}| * g(x,y,w)|,$$

where $|\lambda^{k,j}_{r,s}|$ is defined in the same way as $\lambda^{k,j}_{r,s}$ but with replacing $\mho$ by $|\mho|$.

The following lemma can be proved directly by the results in [22,23], see also [20].

Lemma 1. *Let* $P : \mathbb{R}_+ \to \mathbb{R}^q$ *be a function given by* $P(t) = (c_1t^{\gamma_1},\cdots,c_nt^{\gamma_q})$, *where* $c_j's$ *are real numbers and* $\gamma_j's$ *are distinct positive exponents (not necessarily integer). Define the maximal function related to* P *by*

$$\mathcal{M}_P g(z) = \sup_{h>0}\frac{1}{h}\int_0^h |g(z-P(t))|dt.$$

Then, for all $1 < p \leq \infty$, *there exists a constant* $C_p > 0$ *(independent of* g *and* $c_j's$*) such that*

$$\|\mathcal{M}_P(g)\|_{L^p(\mathbb{R}^q)} \leq C_p|g\|_{L^p(\mathbb{R}^q)}.$$

We shall need the following estimate from [24].

Lemma 2. *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be a C^1 function such that φ' is convex and increasing with $\varphi'(0) = 0$. Then, there exists a positive constant C such that the inequality*

$$\left| \int_1^b e^{-i(2^k tr + \eta\varphi(2^k t))} dt \right| \le C \left| 2^k r \right|^{-1/2}$$

holds for all $b \ge 1$, $r, \eta \in \mathbb{R}$ and $k \in \mathbb{Z}$.

Lemma 3 ([25])**.** *Let $\Gamma(x) = \sum_{|\alpha| \le d} b_\alpha x^\alpha$, where $b_\alpha \in \mathbb{R}$. Then*

$$\left| \int_{[0,1]^q} e^{-i\Gamma(x)} dx \right| \le C_{d,q} \left(\sum_{0<|\alpha|\le d} |b_\alpha| \right)^{-1/d}.$$

Now we need to prove the following lemma which will play a key role on proving our main results.

Lemma 4. *Let $\upsilon \in G_\alpha(\mathbb{S}^{m-1} \times \mathbb{S}^{n-1})$ for $\alpha > 1/2$ and satisfy the conditions (1) and (2). Suppose that Φ is given as in any of Theorems 2–6. Then there is a real number $C > 0$ such that the inequalities*

$$\left\| \lambda_{r,s}^{k,j} \right\| \le C, \tag{10}$$

$$\left| \hat{\lambda}_{r,s}^{k,j}(\xi, \zeta, \eta) \right| \le C \min\left\{ \left| 2^k \xi \right|, \left(\log |2^k \xi| \right)^{-\alpha} \right\}, \tag{11}$$

$$\left| \hat{\lambda}_{r,s}^{k,j}(\xi, \zeta, \eta) \right| \le C \min\left\{ \left| 2^j \zeta \right|, \left(\log |2^j \zeta| \right)^{-\alpha} \right\} \tag{12}$$

hold for all $k, j \in \mathbb{Z}$ and $r, s > 0$, where $\left\| \lambda_{r,s}^{k,j} \right\|$ stands for the total variation of $\lambda_{r,s}^{k,j}$.

Proof. By the definition of $\lambda_{r,s}^{k,j}$, it is easy to prove the inequality (10). Next, by Hölder's inequality, we have

$$\begin{aligned} \left| \hat{\lambda}_{r,s}^{k,j}(\xi, \zeta, \eta) \right| &\le C \int_1^2 \int_1^2 \left| \iint_{\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}} e^{-i\{2^k rtx\cdot\xi + 2^j s\ell y\cdot\zeta + \Phi(2^k rt, 2^j s\ell)\eta\}} \right. \\ &\times \left. \upsilon(x,y) d\sigma(x) d\sigma(y) \right| \frac{dt d\ell}{t\ell} \\ &\le C \iint_{\mathbb{S}^{m-1} \times \mathbb{S}^{n-1}} |\upsilon(x,y)| \int_1^2 I_k(\xi, \eta, x, t) \frac{d\ell}{\ell} d\sigma(x) d\sigma(y), \end{aligned} \tag{13}$$

where

$$I_k(\xi, \eta, x, t) = \int_1^2 e^{-i\{2^k rtx\cdot\xi + \Phi(2^k rt, 2^j s\ell)\eta\}} \frac{dt}{t}.$$

Hence, we have

$$I_k(\xi, \eta, x, t) = \left| \int_1^2 e^{-i\{2^k rtx\cdot\xi + \eta 2^k rt D_t\Phi(0, 2^j s\ell) + [\Phi(2^k rt, 2^j s\ell) - 2^k rt D_t\Phi(0, 2^j s\ell)]\eta\}} \frac{dt}{t} \right|.$$

Now we need to consider several cases.

Case 1. If Φ is given as in Theorem 2, then by Lemma 2, we deduce that

$$I_k(\xi, \eta, x, t) \le \left| 2^k r|\xi| |x \cdot \xi' + |\eta|\xi|^{-1} D_t\Phi(0, 2^j s\ell) \right|^{-1/2}. \tag{14}$$

Let $\delta = \min\Big\{2, \eta|\xi|^{-1}D_t\Phi(0,2^j s\ell)\Big\}sgn(\eta D_t\Phi(0,2^j s\ell))$. Combine the inequality (14) with the trivial estimate $|I_k(\xi,\eta,x,t)| \leq 1$ and use the fact that $(t/\log^\alpha t)$ is increasing over the interval $(2^\alpha,\infty)$, we get that

$$|I_k(\xi,\eta,x,t)| \leq C\frac{\log^\alpha(2|\xi'\cdot x+\delta|^{-1})}{\log^\alpha|2^k r\xi|} \quad if\ |2^k r\xi| > 2^\alpha. \tag{15}$$

Thus, if $n \geq 3$, then by the additional assumption $D_t\Phi(0,2^j s\ell) = 0$, we get $\delta = 0$. Hence,

$$|I_k(\xi,\eta,x,t)| \leq C\frac{\log^\alpha(2|\xi'\cdot x|^{-1})}{\log^\alpha|2^k r\xi|} \quad if\ |2^k r\xi| > 2^\alpha. \tag{16}$$

Therefore, by (13) we acquire that

$$\begin{aligned}\left|\hat{\lambda}^{k,j}_{r,s}(\xi,\zeta,\eta)\right| &\leq C\Big(\log\Big|2^k r\xi\Big|\Big)^{-\alpha}\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}}|\mho(x,y)|\log^\alpha(2\big|\xi'\cdot x\big|^{-1})d\sigma(x)d\sigma(y)\\ &\leq C\Big(\log\Big|2^k r\xi\Big|\Big)^{-\alpha} \quad if\ |2^k r\xi| > 2^\alpha.\end{aligned} \tag{17}$$

Now, if $n = 2$, then follow the argument similar to that in [26]; we may assume that $\delta > 0$. Set $\delta' = \min\{1,\delta\}$ and $\theta = arcsin(\delta')$, and let e_+ denote the vector obtained by rotating ξ' by angles θ and e_- denote the vector obtained by rotating ξ' by angles $-\theta$. Then we conclude that a constant $C_0 \in (0,1)$ exists such that

$$\xi'\cdot x + \delta' \geq C_0\min\{|x\cdot e_-|^2, |x\cdot e_+|^2\}$$

for $x \in \mathbb{S}^1$. So, the estimate in (16) holds and therefore (17) holds for the case $n = 2$.

We notice her that the conclusions of this lemma when $n = 2, m = 2$ are proved without the additional assumptions $D_t\Phi(0,\ell) = 0$ and $D_\ell\Phi(t,o) = 0$.

In the same manner, we can prove that

$$\left|\hat{\lambda}^{k,j}_{r,s}(\xi,\zeta,\eta)\right| \leq C\Big(\log\Big|2^j s\zeta\Big|\Big)^{-\alpha} \quad if\ |2^j s\zeta| > 2^\alpha. \tag{18}$$

Now by using the cancellation conditions on $\mho$, we get

$$\begin{aligned}\left|\hat{\lambda}^{k,j}_{r,s}(\xi,\zeta,\eta)\right| &\leq C\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}}|\mho(x,y)|\int_1^2|J_k(\xi,\eta,x,t)|\frac{d\ell}{\ell}d\sigma(x)d\sigma(y)\\ &\leq C\Big|2^k r\xi\Big|,\end{aligned} \tag{19}$$

where

$$J_k(\xi,\eta,x,t) = \int_1^2 e^{-i\{2^k rtx\cdot\xi+\Phi(2^k rt,2^j s\ell)\eta\}} - e^{\Phi(2^k rt,2^j s\ell)\eta}\frac{dt}{t}.$$

Similarly, we obtain that

$$\left|\hat{\lambda}^{k,j}_{r,s}(\xi,\zeta,\eta)\right| \leq C\Big|2^k r\xi\Big|. \tag{20}$$

Consequently, by (17)–(20), we finish the proof of this lemma for the first case.

We notice here that the proofs of (19)–(20) do not depend on Φ.

Case 2. If Φ is given as in Theorem 3. Notice that

$$\Phi(t,\ell) = \sum_{j=1}^{d_2}\sum_{i=1}^{d_1}a_{j,i}t^{\alpha_i}\ell^{\beta_j}.$$

Hence, $\Phi(t,\ell)$ can be written as

$$\Phi(t,\ell) = P_\ell(t) = \sum_{i=0}^{d_1} b_i(\ell) t^{\alpha_i}$$

and

$$\Phi(t,\ell) = Q_t(\ell) = \sum_{j=0}^{d_2} c_j(t) \ell^{\beta_j},$$

where

$$b_i(\ell) = \sum_{j=0}^{d_2} a_{j,i} \ell^{\beta_j} \quad and \quad c_j(t) = \sum_{i=0}^{d_1} a_{j,i} t^{\alpha_i}.$$

If $\alpha_i \neq 1$ for all $1 \leq i \leq d_1$, then by Lemma 3, we get that

$$I_k(\xi,\eta,x,t) \leq \left|2^k|\xi|(\xi'\cdot x)\right|^{-1/d_1}. \tag{21}$$

So, by following the same argument as above, we get

$$\left|\hat{\lambda}^{k,j}_{r,s}(\xi,\zeta,\eta)\right| \quad \leq \quad \left(\log\left|2^k r\xi\right|\right)^{-\alpha} \quad if\ |2^k r\xi| > 2^\alpha. \tag{22}$$

Consider $\alpha_{i_0} = 1$ for some $1 \leq i_0 \leq d_1$. Since

$$I_k(\xi,\eta,x,t) = \left|\int_1^2 e^{-i2^k rt|\xi|\{x\cdot\xi'+\eta|\xi|^{-1}b_{i_0}(2^j s\ell)+\eta\sum\limits_{i=0,i\neq i_0}^{d_1} b_i(2^j s\ell)(2^k rt))^{\alpha_i}\}}\frac{dt}{t}\right|,$$

then by invoking Lemma 3 we obtain

$$|I_k(\xi,\eta,x,t)| \leq C\left|2^k r|\xi|\left(x\cdot\xi' + \eta|\xi|^{-1}b_{i_0}(2^j s\ell)\right)\right|^{-1/D_t}. \tag{23}$$

Set $Y = \min\left\{2, \eta|\xi|^{-1}b_{i_0}(2^j s\ell)\right\} sgn(b_{i_0}(2^j s\ell)\eta)$. By combining (23) with the trivial estimate $|I_k(\xi,\eta,x,t)| \leq 1$ and using the fact that $(t/\log^\alpha t)$ is increasing over the interval $(2^\alpha,\infty)$, we get

$$|I_k(\xi,\eta,x,t)| \leq C\frac{\log^\alpha(2|\xi'\cdot x + Y|^{-1})}{\log^\alpha\left|2^k r\xi\right|} \quad if\ |2^k r\xi| > 2^\alpha. \tag{24}$$

When $n \geq 3$, by the additional condition $D_t\Phi(0,2^j s\ell) = 0$, we deduce that $Y = 0$; which yields that

$$|I_k(\xi,\eta,x,t)| \leq C\frac{\log^\alpha(2|\xi'\cdot x|^{-1})}{\log^\alpha\left|2^k r\xi\right|} \quad if\ |2^k r\xi| > 2^\alpha. \tag{25}$$

By the last estimate and (13) we get

$$\begin{aligned}\left|\hat{\lambda}^{k,j}_{r,s}(\xi,\zeta,\eta)\right| \quad &\leq \quad C\left(\log\left|2^k r\xi\right|\right)^{-\alpha}\iint_{\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}}|\mho(x,y)|\log^\alpha(2|\xi'\cdot x|^{-1})d\sigma(x)d\sigma(y)\\ &\leq \quad C\left(\log\left|2^k r\xi\right|\right)^{-\alpha} \quad if\ |2^k r\xi| > 2^\alpha.\end{aligned} \tag{26}$$

When $n = 2$, we follow the same arguments similar to those in Case 1, we obtain (26).

Similarly, we have that

$$\left|\hat{\lambda}^{k,j}_{r,s}(\xi,\zeta,\eta)\right| \leq C\Big(\log\left|2^j s\zeta\right|\Big)^{-\alpha} \quad if\ |2^j s\zeta| > 2^{\alpha}. \tag{27}$$

Case 3. This lemma can be proved for the other cases of Φ by following the same argument as in the proof adoptedfor the cases 1 and 2. We omit the details. □

Lemma 5. *Suppose that* $\mho \in L^1\left(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}\right)$ *and satisfies the conditions* (1) *and* (2). *Let* Φ *be given as in any one of Theorems 2–6. Then there exists* $C_p > 0$ *such that*

$$\|\lambda^*(g)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \leq C_p\|g\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}).} \tag{28}$$

We can easily prove this lemma by using iterated integration and using at least one of results in Corollary 5.3 in [24] or Lemma 1.

3. Proof of Main Theorems

Assume that $\mho \in G_\alpha\left(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}\right)$ for some $\alpha > 1/2$ and satisfies the conditions (1) and (2), and let Φ be given as in any of Theorems 2–6. Then by Minkowski's inequality, we have

$$\begin{aligned}
\mathfrak{M}_{\mho,\Phi}(g)(u,v,w) &= \left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\left|\sum_{j=-\infty}^{-1}\sum_{k=-\infty}^{-1}2^{j+k}\left(\lambda^{k,j}_{r,s}*g\right)(u,v,w)\right|^2\frac{drds}{rs}\right)^{1/2}\\
&\leq \sum_{j=-\infty}^{-1}\sum_{k=-\infty}^{-1}2^{j+k}\left(\iint_{\mathbb{R}_+\times\mathbb{R}_+}\left|\left(\lambda^{0,0}_{r,s}*g\right)(u,v,w)\right|^2\frac{drds}{rs}\right)^{1/2}\\
&= \left(\int_1^2\int_1^2\left|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left(\lambda^{k,j}_{r,s}*g\right)(u,v,w)\right|^2\frac{drds}{rs}\right)^{1/2}\\
&\leq \left(\int_1^2\int_1^2\left|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left(\lambda^{k,j}_{r,s}*g\right)(u,v,w)\right|^2 drds\right)^{1/2}.
\end{aligned} \tag{29}$$

Choose two radial Schwartz functions $\Psi_1 \in \mathcal{S}(\mathbb{R}^m)$ and $\Psi_2 \in \mathcal{S}(\mathbb{R}^n)$ which satisfy the following properties:

(i) $0 \leq \Psi_1 \leq 1,\ 0 \leq \Psi_2 \leq 1,$
(ii) $\text{supp}(\Psi_1) \subseteq \{u \in \mathbb{R}^m :\ 1/4 \leq |u| \leq 4\},\ \text{supp}\ (\Psi_2) \subseteq \{v \in \mathbb{R}^n :\ 1/4 \leq |v| \leq 4\}.$
(iii) $\sum\limits_{a_1\in\mathbb{Z}}(\Psi_1(2^{a_1}r))^2 = 1,\ \sum\limits_{a_2\in\mathbb{Z}}(\Psi_2(2^{a_2}s))^2 = 1.$

Define the multiplier operators $\{T_{a_1,a_2}\}$ on $\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}$ by

$$\widehat{T_{a_1,a_2}(g)}\ (\xi,\zeta,\eta) = \Psi_1(2^{a_1}|\xi|)\Psi_2(2^{a_2}|\zeta|)\hat{g}(\xi,\zeta,\eta).$$

Thus for any $g \in C_0^\infty(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})$, we have

$$g(u,v,w) = \sum_{a_2\in\mathbb{Z}}\sum_{a_1\in\mathbb{Z}}T^2_{a_1,a_2}(g)(u,v,w),$$

which in turn by (29) leads to

$$\mathfrak{M}_{\mho,\Phi}(g)(u,v,w) \leq \mathcal{I}(g)(u,v,w), \tag{30}$$

where

$$\mathcal{I}(g)(u,v,w)=\left(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\int_1^2\int_1^2\left|\sum_{a_1\in\mathbb{Z}}\sum_{a_2\in\mathbb{Z}}T_{j-a_1,k-a_2}\left(\lambda_{r,s}^{k,j}*\left(T_{j-a_1,k-a_2}(g)\right)\right)(u,v,w)\right|^2 drds\right)^{1/2}.$$

Therefore, by Lemmas 4 and 5 and employing the same argument as in ([19], p. 301), we obtain

$$\|\mathcal{I}(g)\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})}\leq C\|f\|_{L^p(\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R})} \tag{31}$$

for all $p\in(1+1/(2\alpha),1+2\alpha)$. Now the proof is complete by the last inequality and using (30).

4. Conclusions

In this paper, we proved the L^p estimates for several types of functions of multiple Marcinkiewicz operators along surfaces of revolution. Precisely, we confirmed the boundedness of the aforementioned operators under very weak assumption on the kernels that is $\Omega\in G_\alpha\left(\mathbb{S}^{m-1}\times\mathbb{S}^{n-1}\right)$ for some $\alpha>1/2$. The method employed in this paper is a combination of arguments and ideas taken from [8,12,19,20]. Our results cover several types of natural surfaces that were considered by many mathematicians in their studying singular integrals and double Hilbert transforms on product domains.

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