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Solving a Novel System of Time-Dependent Nuclear Reactor Equations of Fractional Order

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Abstract: Building upon the previous research that solved neutron diffusion equations in simplified slab geometry, this study advances the field by addressing the more complex cylindrical geometry, focusing on neutron diffusion equations that are coupled with delayed neutrons in cylindrical reactors of fractional order. The method of solving used integrates the technique of residual power series (RPS) with the Laplace transform (LT) method. Anomalous neutron behavior is explained by examining the non-Gaussian scenario with various fractional parameters α. The LRPSM Laplace transform and residual power series method employed in this approach eliminates the complex difficulties. This simplicity makes the method particularly coherent with different fractional calculus applications. To validate the proposed method, numerical simulations are conducted with two different initial conditions representing distinct scenarios. The obtained results are presented in suitable tables and figures. It should be emphasized that this system is solved for the first time utilizing fractional calculus techniques. The outcomes are consistent with those achieved using the Adomian decomposition method.

Keywords: nuclear reactor equation; Laplace transform method; residual power series method; factional differential equations; cylindrical reactors

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1. Introduction

The utilization of nuclear reactor energy marks a pivotal aspect of a clean energy source. Presently, nuclear reactors contribute to 10% of the global energy demand. The inception of nuclear energy emerged from the concept of inducing fission in heavy nuclei through the bombardment of (thermal) slow neutrons.

While the neutron diffusion equation has been extensively studied in slab geometries, this research extends these concepts to the more intricate cylindrical reactor geometry [1–3]. Researchers in this field pursue scientific advancements by investigating various aspects. Different scenarios of nuclear reactor equations have been addressed using diverse methods. For instance, the homotopy perturbation method (HPM) has been employed to solve neutron diffusion equations for reactors exhibiting bare hemispherical and cylindrical symmetries for non-fractional systems [4,5]. Additionally, LRPSM has been used to tackle different branches of sciences [6–8].

The target of this work is solving the neutron diffusion equations coupled with delayed neutrons in cylindrical reactors fractionally using the Laplace residual power series method where the theoretical results are verified numerically.

Studying the nuclear reactor equation system with delayed neutrons provides a real application of fractional calculus in nuclear reactor theory. This application has historically been studied only for the slab reactor which can be considered as a simple example that is solved in Cartesian coordinates; in this work, the cylindrical reactor equations is solved in cylindrical coordinates which have an additional term in the equations of the system. This improvement of system has never been considered before.

This paper is prepared as follows. Section 2 introduces essential information about the studied phenomena. Section 3 presents the needed concepts and theories related to fractional calculus and the Laplace transforms. LRPSM details are provided in Section 4. Numerical examples of the theoretical study are considered in Section 5. The paper's conclusion is given in Section 6.

2. The Physical Phenomena

Neutron distribution in reactors is analogous to the dissolution of a solute in physical chemistry. This procedure is expressed using a continuity equation.

$$\frac{1}{v_c}\frac{\partial \varphi(\boldsymbol{r},t)}{\partial t} = s(\boldsymbol{r},t) - \Sigma_a \varphi(\boldsymbol{r},t) - \nabla\cdot\boldsymbol{J}(\boldsymbol{r},t), \tag{1}$$

In the context where $\varphi(r,t)$ represents the neutron flux, $\boldsymbol{J}(\boldsymbol{r},t)$ denotes the neutron current density, Σ_a is the absorption cross-section in the macroscopic scale, v_c represents the velocity of the neutrons, and $s(\boldsymbol{r},t)$ is the head of the neutron's idiom, the relationship between $\varphi(r,t)$ and $\boldsymbol{J}(\boldsymbol{r},t)$ can be expressed through Fick's law.

$$\boldsymbol{J}(\boldsymbol{r},\ t) = -D\nabla\varphi(\boldsymbol{r},\ t) \tag{2}$$

The neutrons are diffused according to the following equation which can be stated when D is designated as the neutron diffusion coefficient:

$$\frac{1}{v_c}\frac{\partial \varphi(\boldsymbol{r},t)}{\partial t} = s(\boldsymbol{r},\ t) - \Sigma_a\ \varphi(\boldsymbol{r},t) + D\nabla^2\varphi(\boldsymbol{r},\ t) \tag{3}$$

The source term $s(\boldsymbol{r},\ t)$ can be written in nuclear reactor physics as $\nu\Sigma_f\ \varphi(\boldsymbol{r},t)$ where ν is the number of neutrons produced per fission and Σ_f is the fission cross-section in the macroscopic scale.

Now, it is important to introduce buckling B^2 which is an essential nuclear reactor concept defined as $\frac{\nu\Sigma_f\ \varphi(\boldsymbol{r},t)-\Sigma_a}{D}$.

This signifies the widely recognized equation for time-dependent neutron diffusion in nuclear reactors.

Taking a step further, the fractional analysis of the time-dependent form of this equation adds more detail by examining the physical impact of delayed neutrons. The investigation of a system that involves coupled neutron diffusion equations with delayed neutrons in fractional form is central to this study, which is important in nuclear reactor dynamics.

Delayed neutrons are crucial due to their role in influencing the operational lifespan of neutrons in reactor environments.

The effective lifetime of neutrons encompasses the immediate lifetime and an extended period marked by precursor elements. This additional delay far exceeds the prompt neutron lifetime, thereby prolonging the reactor's time constant, which facilitates improved control.

Incorporating the effect of delayed neutrons into the $s(\boldsymbol{r},t)$ term in Equation (3). The source term will be

$$s(\boldsymbol{r},t) = \gamma\sum\nolimits_f\ \varphi(\boldsymbol{r},t) + \lambda\omega(\boldsymbol{r},t) \tag{4}$$

A group of time-variant equations functions within a single-dimensional framework.

$$\frac{1}{v_c}\frac{\partial}{\partial t}\varphi(x,t) = D\frac{\partial^2}{\partial x^2}\varphi(x,t) + \left(-\sum\nolimits_a + \gamma\sum\nolimits_f\right)\varphi(x,t) + \lambda\omega(x,t), \tag{5}$$

$$\frac{\partial}{\partial t}\omega(x,t) = \beta\nu\sum_f \varphi(x,t) - \lambda\omega(x,t), \tag{6}$$

where the following

$$\varphi(x,0) = \varphi_0(x), \;\; \omega(x,0) = \frac{\beta\gamma\sum_f}{\lambda}\varphi_0(x), \tag{7}$$

are needed conditions.

Here, the belated neutron intensity is symbolized by $\varphi(x,t)$, the average neutron yield per fission is γ, the decay constant is λ, and the proportion of delayed fission neutrons is β.

The term $\nu\Sigma_f \; \varphi(\boldsymbol{r},t) - \Sigma_a$ is interpreted as the overall cross-section Σ.

The dynamic equation highlights how neutron distribution changes over space and time within reactor settings.

Fractional calculus becomes notably important in the time-varying diffusion equations, especially when considering the delayed neutrons.

This aspect is termed anomalous diffusion, denoting a departure from the Gaussian distribution characteristic of regular diffusion.

Irregular cases of study are described by a non-linear, asymptotic association, especially one that follows a power-law pattern, between time and mean-squared displacement.

$$\left\langle x^2(t)\right\rangle \sim t^{\alpha}$$

The α parameter is instrumental in categorizing the fractional nuclear reactor equation for the diffused neutrons, where super-diffusion ($\alpha > 1$) and sub-diffusion ($\alpha < 1$) are the divisions of irregular diffusion ($\alpha \neq 1$).

When $\alpha = 1$, the diffusion process adheres to a Gaussian model, represented mathematically by the integer-order neutron diffusion equation.

When neutrons are confined, their movement becomes slower, leading to what is termed sub-diffusion. Conversely, fast neutron movement occurs when they travel extensively in one direction without encountering collisions, a phenomenon referred to as super-diffusion [9,10].

Numerous studies have investigated the non-integer-order diffusion equation, particularly focusing on delayed neutrons. This includes research on the matrix representation of the fractional two-group in-reactor dynamics for slab reactors when the external source is applied [11], addressing neutron diffusion equations where the delayed neutrons are present in slab reactors [12], exploring a solution of kinetic nuclear reactor equation with a single delayed precursor concentration in for the one-dimensional slab reactors [13]. Within this paradigm, the fractional form of diffusion equations for neutrons with a single delayed neutron group may be articulated as follows:

$$\frac{1}{v_c}\mathfrak{D}_t^{\alpha}\varphi(x,t) = D\nabla^2\varphi(x,t) + \left(\gamma\sum_f - \sum_a\right)\varphi(x,t) + \lambda\omega(x,t), \tag{8}$$

$$\mathfrak{D}_t^{\alpha}\omega(x,t) = \beta\gamma\sum_f \varphi(x,t) - \lambda\omega(x,t), \tag{9}$$

$$\varphi(x,0) = \varphi_0(x), \omega(x,0) = \frac{\beta\gamma\sum_f}{\lambda}\varphi_0(x), \tag{10}$$

A wide array of computational and theoretical techniques including LTM, HPM, RPSM, and other methods are formulated to tackle fractional differential equations (FDEs) [6,7,14–26].

These techniques frequently face constraints, such as the need for extensive and intricate calculations.

To overcome these challenges, a connection to a transform operator is often necessary. In a previous work [24], the authors introduced a novel approach by combining RPSM with LTM to provide practical and series solutions for not only linear FDEs but also nonlinear FDEs.

Historically, all cases studying nuclear reactor equations have considered slab reactors in Cartesian geometry, representing the simplest case. This work, however, marks a significant advancement by exploring a more complex geometry, namely the cylindrical geometry, for cylindrical reactors.

Contrary to RPSM, LRPSM does not depend on fractional derivation for determining series coefficients but rather employs the concept of limits.

Consequently, LRPSM requires fewer calculations for the coefficient generation of RPSM. This technique is distinguished by its speed, low computer memory usage, and resistance to codes' round-off errors. Furthermore, quicker convergence is achieved since LRPSM uses a set of equations involving several variables to obtain the power series coefficients.

This study utilizes LRPSM to effectively resolve the interconnected fractional system of prompt and delayed neutrons.

3. Preliminaries

The definition of fractional derivatives using integrals makes them non-local operators. As a result, a time-fractional derivative displays a memory effect by encapsulating associations about the function from previous times. These derivatives provide a critical improvement for a more accurate and thorough explanation of dynamic and complex system behavior as they account for non-local and historical dispersed impacts.

This section introduces the Laplace transform and fractional calculus within the framework of Caputo's method. The development of the Laplace residual power series (LRPS) method, which is employed to solve fractional neutron diffusion equations involving a single delayed neutron group, is largely dependent on these components.

The basic concepts and ideas associated with these mathematical tools are presented.

Definition 1. *The Caputo operator of order α for $\varphi(\chi,t)$is given by [6]*

$$\mathfrak{D}_t^{\alpha}\varphi(\chi,t) = J_t^{n-\alpha}\partial_t^n\varphi(\chi,t),\ n-1<\alpha\leq n,\ \chi\in I.\ t>0, \tag{11}$$

$\mathfrak{D}_t^{\alpha}$ represents the Caputo derivative operator with a fractional order of α, $n\in\mathbb{N}$, and I denotes an interval. Additionally, J_t^{β} is defined as the Riemann–Liouville integral operator with a fractional time order of β.

$$J_t^{\beta}\varphi(\chi,t) = \begin{cases} \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\varphi(\chi,\tau)d\tau, & \beta>0,\ t>\tau\geq 0 \\ \varphi(\chi,t), & \beta=0 \end{cases} \tag{12}$$

Essential characteristics of the fractional derivative $\mathfrak{D}_t^{\alpha}$ will be introduced in the next lemma. Additional features can be found in Reference [26].

Lemma 1. *For $n-1<\alpha\leq n$,$\varepsilon>-1,\ t\geq 0$, and $\mu\in\mathbb{R}$, we have: [24]*

$$\begin{aligned} &(i)\ \mathfrak{D}_t^{\alpha}t^{\varepsilon} = \frac{\Gamma(\varepsilon+1)}{\Gamma(\varepsilon+1-\alpha)}t^{\varepsilon-\alpha}.\\ &(ii)\ \mathfrak{D}_t^{\alpha}\mu = 0.\\ &(iii)\ \mathfrak{D}_t^{\alpha}J_t^{\alpha}\varphi(\chi,t) = \varphi(\chi,t).\\ &(iv)\ J_t^{\alpha}\mathfrak{D}_t^{\alpha}\varphi(\chi,t) = \varphi(\chi,t) - \sum_{j=0}^{n-1}\partial_t^j\varphi(\chi,0^+)\frac{t^j}{j!}. \end{aligned} \tag{13}$$

Definition 2. *The following definition applies the Laplace transform of* $\varphi(\chi,t)$ *[6]:*

$$\Phi(\chi,s)=\mathcal{L}[\varphi(\chi,t)]:=\int_0^\infty e^{-st}\varphi(\chi,t)dt,\ s>\delta, \tag{14}$$

Then, the inverse Laplace transform of $\Psi(\chi,s)$ *will be applied*

$$\varphi(\chi,t)=\mathcal{L}^{-1}[\Phi(\chi,s)]:=\int_{c-i\infty}^{c+i\infty} e^{st}\Phi(\chi,s)ds, c=Re(s)>c_0. \tag{15}$$

In this context, c_0 is located within the right half-plane where the Laplace integral absolutely converges.

The next lemma delineates the fundamental properties of the LT and the Caputo sense of fractional derivative.

Lemma 2. *If* $\varphi(\chi,t)$*is a piecewise continuous function on the interval* $I\times[0,\infty)$*and has exponential orders* δ*, and if* $\Phi(\chi,s)$ *is the Laplace transform of* $\varphi(\chi,t)$*, then [24]*

$$\begin{aligned}
&(\mathrm{i})\lim_{s\to\infty} s\,\Phi(\chi,s)=\varphi(\chi,0), x\in I.\\
&(\mathrm{ii})\mathcal{L}[J_t^\alpha\varphi(\chi,t)]=s^{\alpha-1}\Phi(\chi,s), \alpha>0.\\
&(\mathrm{iii})\mathcal{L}[\mathfrak{D}_t^\alpha\varphi(\chi,t)]=s^\alpha\Phi(\chi,s)-\sum_{k=0}^{n-1}s^{\alpha-k-1}\partial_t^k\varphi(\chi,0), n-1<\alpha<n.\\
&(iv)\mathcal{L}[\mathfrak{D}_t^{m\alpha}\varphi(\chi,t)]=s^{m\alpha}\Phi(\chi,s)-\sum_{k=0}^{m-1}s^{(m-k)\alpha-1}\mathfrak{D}_t^{k\alpha}\varphi(\chi,0), 0<\alpha<1,
\end{aligned} \tag{16}$$

where $\mathfrak{D}_t^{m\alpha}=\mathfrak{D}_t^\alpha.\mathfrak{D}_t^\alpha\ldots\mathfrak{D}_t^\alpha$*(m -times).*

The next result we provide is the basis for developing an LRPS solution for the partial differential equations (PDEs). It is a unique fractional expansion.

Theorem 1. *Assume* $\varphi(\chi,t)$ *is a piecewise continuous function on* $I\times[0,\infty)$ *with exponential order* δ*. Consider that the function* $\Phi(\chi,s)$*, which is the Laplace transform* $\mathcal{L}[\varphi(\chi,t)]$*, exhibits the following fractional expansion [24]:*

$$\Phi(\chi,s)=\sum_{n=0}^{\infty}\frac{y_n(\chi)}{s^{n\alpha+1}},\ 0<\alpha\le 1,\ \chi\in I,\ s>\delta. \tag{17}$$

Then, $y_n(\chi)=\mathfrak{D}_t^{n\alpha}\varphi(\chi,0)$.

The next theorem describes the necessary and sufficient conditions for the series in expansion (17) to converge.

Theorem 2. *Assume* $\varphi(\chi,t)$*is a piecewise continuous function on the interval* $I\times[0,\infty)$*, and it has an exponential order of* δ*. Additionally,* $\Phi(\chi,s)$*is the Laplace transform of* $\varphi(\chi,t)$*. The fractional expansion in Theorem 1 can serve as a representation of it [24].*

It must be noted that the creators of this method used it and other methods in history to solve a nuclear reactor system of equations [6,7].

4. The Nuclear Reactor Equations Analytical Solution

Utilizing the Laplace residual power series method (LRPSM) to derive a series solution for the fractional neutron diffusion equations is the target in this stage.

$$\begin{gathered}\frac{1}{v_c}\mathfrak{D}_t^{\alpha}\varphi(x,t) = D\nabla^2\varphi(x,t) + \left(\gamma\sum_f - \sum_a\right)\varphi(x,t) + \frac{1}{x}\frac{\partial\varphi(x,t)}{\partial x} + \lambda\,\omega(x,t),\\ \mathfrak{D}_t^{\alpha}\omega(x,t) = \beta\gamma\sum_f\ \varphi(x,t) - \lambda\,\omega(x,t),\end{gathered} \tag{18}$$

given the specified initial conditions

$$\varphi(x,0) = \varphi_0(x),\ \ \omega(x,0) = \frac{\beta\nu\sum_f}{\lambda}\varphi_0(x), \tag{19}$$

Rewriting the coupled equations in (18) as $B = \beta\gamma\sum_f$ and $\sum = \gamma\sum_f - \sum_a$, we obtain the following:

$$\begin{gathered}\mathfrak{D}_t^{\alpha}\varphi(x,t) = v_c\ D\ \varphi_{xx}(x,t) + v_c\ \sum\varphi(x,t) + \frac{v_c}{x}\varphi_x(x,t) + v_c\lambda\,\omega(x,t),\\ \mathfrak{D}_t^{\alpha}\omega(x,t) = B\varphi(x,t) - \lambda\,\omega(x,t),\end{gathered} \tag{20}$$

The following nuclear physics initial conditions will be used

$$\varphi(x,0) = \varphi_0(x),\ \ \omega(x,0) = \frac{B}{\lambda}\varphi_0(x), \tag{21}$$

When the coupled Equation (20) are subjected to the Laplace transform, we have

$$\begin{gathered}\mathcal{L}[\mathfrak{D}_t^{\alpha}\varphi(x,t)] = v_c\ D\mathcal{L}[\varphi_{xx}(x,t)] + v_c\sum\mathcal{L}[\varphi(x,t)] + \frac{v_c}{x}\mathcal{L}[\varphi_x(x,t)] + v_c\lambda\mathcal{L}[\omega(x,t)],\\ \mathcal{L}[\mathfrak{D}_t^{\alpha}\omega(x,t)] = B[\varphi(x,t)] - \lambda\mathcal{L}[\omega(x,t)].\end{gathered} \tag{22}$$

Using the initial conditions in Equation (21) and Lemma 2, we may formulate Equation (22) in the following form

$$\begin{gathered}s^{\alpha}\Phi(x,s) - s^{\alpha-1}\varphi_0(x) = v_c\ D\ \Phi_{xx}(x,s) + v_c\sum\Phi(x,s) + \frac{v_c}{x}\Phi_x(x,s) + v_c\lambda\ \Omega(x,s),\\ s^{\alpha}\Omega(x,s) - s^{\alpha-1}\frac{B}{\lambda}\varphi_0(x) = B\Phi(x,s) - \lambda\ \Omega(x,s),\end{gathered} \tag{23}$$

where $\Phi(x,s) = \mathcal{L}[\varphi(x,t)]$, $\Omega(x,s) = \mathcal{L}[\omega(x,t)]$.

We can rewrite Equation (23) as

$$\begin{gathered}\Phi(x,s) - \frac{1}{s}\varphi_0(x) - \frac{v_c\ D}{s^{\alpha}}\ \Phi_{xx}(x,s) - \frac{v_c\sum}{s^{\alpha}}\ \Phi(x,s) - \frac{v_c}{xs^{\alpha}}\Phi_x(x,s) - \frac{v_c\lambda}{s^{\alpha}}\ \Omega(x,s) = 0,\\ \Omega(x,s) - \frac{B}{\lambda\ s}\varphi_0(x) - \frac{B}{s^{\alpha}}\ \Phi(x,s) + \frac{\lambda}{s^{\alpha}}\ \Omega(x,s) = 0.\end{gathered} \tag{24}$$

The coupled Equation (24) forms a system of linear partial differential equations which involve derivatives with respect to the variable x. According to the LRPS approach, the series solution for the system (24) will be as follows:

$$\begin{gathered}\Phi(x,s) = \sum_{n=0}^{\infty}\frac{y_n(x)}{s^{n\alpha+1}}, x \in I,\ s > \delta \geq 0,\\ \Omega(x,s) = \sum_{n=0}^{\infty}\frac{z_n(x)}{s^{n\alpha+1}}, x \in I,\ s > \delta \geq 0.\end{gathered} \tag{25}$$

The kth truncated series of $\Phi(x,s), \Omega(x,s)$ can be represented as follows using Lemma 2:

$$\begin{aligned} \Phi_k(x,s) &= \frac{\varphi_0(x)}{s} + \sum_{n=1}^{k} \frac{y_n(x)}{s^{n\alpha+1}}, x \in I,\ s > \delta \geq 0, \\ \Omega_k(x,s) &= \frac{B}{\lambda\, s}\varphi_0(x) + \sum_{n=1}^{k} \frac{z_n(x)}{s^{n\alpha+1}}, x \in I,\ s > \delta \geq 0. \end{aligned} \tag{26}$$

The next step is to obtain the anonymous coefficients of Equation (26) by establishing the Laplace residual functions of Equation (24)

$$\begin{aligned} LRes(\Phi(x,s)) &= \Phi(x,s) - \tfrac{1}{s}\varphi_0(x) - \tfrac{v_c\, D}{s^{\alpha}}\, \Phi_{xx}(x,s) - \tfrac{v_c \sum}{s^{\alpha}}\, \Phi(x,s) - \tfrac{v_c}{xs^{\alpha}}\Phi_x(x,s) - \tfrac{v_c\lambda}{s^{\alpha}}\, \Omega(x,s) \\ LRes(C(x,s)) &= C(x,s) - \tfrac{B}{\lambda\, s}\varphi_0(x) - \tfrac{B}{s^{\alpha}}\, \Phi(x,s) + \tfrac{\lambda}{s^{\alpha}}\, \Omega(x,s). \end{aligned} \tag{27}$$

Moreover, the kth Laplace residual serves as

$$\begin{aligned} LRes_k(\Phi(x,s)) &= \Phi_k(x,s) - \tfrac{1}{s}\varphi_0(x) - \tfrac{v_c\, D}{s^{\alpha}}\, \Phi_{kxx}(x,s) - \tfrac{v_c \sum}{s^{\alpha}}\, \Phi_k(x,s) - \tfrac{v_c}{xs^{\alpha}}\Phi_{kx}(x,s) - \tfrac{v_c\lambda}{s^{\alpha}}\, \Omega_k(x,s) \\ LRes_k(\Omega(x,s)) &= \Omega_k(x,s) - \tfrac{B}{\lambda\, s}\varphi_0(x) - \tfrac{B}{s^{\alpha}}\, \Phi_k(x,s) + \tfrac{\lambda}{s^{\alpha}}\, \Omega_k(x,s). \end{aligned} \tag{28}$$

Since $LRes(\Phi(x,s)) = 0$ and $LRes(\Omega(x,s)) = 0$, we find that $s^{k\alpha+1}LRes(\Phi(x,s)) = 0$, $s^{k\alpha+1}LRes(\Omega(x,s)) = 0$. Therefore,

$$\lim_{s\to\infty}\left(s^{k\alpha+1}LRes_k(\Phi(x,s))\right) = 0,\ \lim_{s\to\infty}\left(s^{k\alpha+1}LRes_k(\Omega(x,s))\right) = 0\ for\ k = 0,1,2,\ldots. \tag{29}$$

Then, to obtain $y_1(x)$ and $z_1(x)$ in Equation (26), we substitute in the first Laplace residual functions as $\Phi_1(x,s) = \frac{\varphi_0(x)}{s} + \frac{y_1(x)}{s^{\alpha+1}}$ and $\Omega_1(x,s) = \frac{B}{\lambda s}\varphi_0(x) + \frac{z_1(x)}{s^{\alpha+1}}$, and we obtain

$$\begin{aligned} LRes_1(\Phi(x,s)) &= \tfrac{y_1(x)}{s^{\alpha+1}} - \tfrac{v_c\, D}{s^{\alpha}}\left[\tfrac{\varphi_0''(x)}{s} + \tfrac{y_1''(x)}{s^{\alpha+1}}\right] - \tfrac{v_c \sum}{s^{\alpha}}\left[\tfrac{\varphi_0(x)}{s} + \tfrac{y_1(x)}{s^{\alpha+1}}\right] - \tfrac{v_c}{xs^{\alpha}}\left[\tfrac{\varphi_0'(x)}{s} + \tfrac{y_1'(x)}{s^{\alpha+1}}\right] - \tfrac{v_c\lambda}{s^{\alpha}}\left[\tfrac{B}{\lambda\, s}\varphi_0(x) + \tfrac{z_1(x)}{s^{\alpha+1}},\right] \\ LRes_1(\Omega(x,s)) &= \tfrac{z_1(x)}{s^{\alpha+1}} - \tfrac{B}{s^{\alpha}}\left[\tfrac{\varphi_0(x)}{s} + \tfrac{y_1(x)}{s^{\alpha+1}}\right] + \tfrac{\lambda}{s^{\alpha}}\left[\tfrac{B}{\lambda\, s}\varphi_0(x) + \tfrac{z_1(x)}{s^{\alpha+1}}\right]. \end{aligned} \tag{30}$$

Now, by solving $\lim_{s\to\infty} s^{\alpha+1}LRes_1(\Phi(x,s)) = 0$, $\lim_{s\to\infty} s^{\alpha+1}LRes_1(C(x,s)) = 0$, we find that

$$y_1(x) = v_c\ D\ \varphi_0''(x) + v_c\ \sum \varphi_0(x) + \frac{v_c}{x}\varphi_0'(x) + v_c B\omega_0(x), \tag{31}$$

$$z_1(x) = 0. \tag{32}$$

To obtain $y_2(x)$ and $z_2(x)$ in Equation (26), we substitute in the second Laplace residual functions $\Phi_2(x,s) = \frac{\varphi_0(x)}{s} + \frac{y_1(x)}{s^{\alpha+1}} + \frac{y_2(x)}{s^{2\alpha+1}}$ and $\Omega_2(x,s) = \frac{B}{\lambda s}\varphi_0(x) + \frac{z_2(x)}{s^{2\alpha+1}}$, and we obtain

$$\begin{aligned} LRes_2(\Phi(x,s)) &= \tfrac{y_1(x)}{s^{\alpha+1}} + \tfrac{y_2(x)}{s^{2\alpha+1}} - \tfrac{v_c\, D}{s^{\alpha}}\left[\tfrac{\varphi_0''(x)}{s} + \tfrac{y_1''(x)}{s^{\alpha+1}} + \tfrac{y_2''(x)}{s^{2\alpha+1}}\right] - \tfrac{v_c \sum}{s^{\alpha}}\left[\tfrac{\varphi_0(x)}{s} + \tfrac{y_1(x)}{s^{\alpha+1}} + \tfrac{y_2(x)}{s^{2\alpha+1}}\right] - \\ &\tfrac{v_c}{xs^{\alpha}}\left[\tfrac{\varphi_0'(x)}{s} + \tfrac{y_1'(x)}{s^{\alpha+1}} + \tfrac{y_2'(x)}{s^{2\alpha+1}}\right] - \tfrac{v_c\lambda}{s^{\alpha}}\left[\tfrac{B}{\lambda\, s}\varphi_0(x) + \tfrac{z_2(x)}{s^{2\alpha+1}}\right], \\ LRes_2(\Omega(x,s)) &= \tfrac{z_2(x)}{s^{2\alpha+1}} - \tfrac{B}{s^{\alpha}}\left[\tfrac{\varphi_0(x)}{s} + \tfrac{y_1(x)}{s^{\alpha+1}} + \tfrac{y_2(x)}{s^{2\alpha+1}}\right] + \tfrac{\lambda}{s^{\alpha}}\left[\tfrac{B}{\lambda\, s}\varphi_0(x) + \tfrac{z_2(x)}{s^{2\alpha+1}}\right]. \end{aligned} \tag{33}$$

Now, by solving $\lim_{s\to\infty} s^{2\alpha+1}LRes_2(\Phi(x,s)) = 0, \lim_{s\to\infty} s^{2\alpha+1}LRes_2(\Omega(x,s)) = 0$, we find that

$$y_2(x) = v_c\ D\ y_1''(x) + v_c \sum y_1(x) + \frac{v_c}{x}y_1'(x), \tag{34}$$

$$z_2(x) = By_1(x). \tag{35}$$

Then, we have the following:

$$LRes_3(\Phi(x,s)) = \frac{y_1(x)}{s^{\alpha+1}} + \frac{y_2(x)}{s^{2\alpha+1}} + \frac{y_3(x)}{s^{3\alpha+1}} - \frac{v_c\ D}{s^{\alpha}}\left[\frac{\varphi_0''(x)}{s} + \frac{y_1''(x)}{s^{\alpha+1}} + \frac{y_2''(x)}{s^{2\alpha+1}} + \frac{y_3''(x)}{s^{3\alpha+1}}\right] -$$

$$\frac{v_c \sum}{s^{\alpha}}\left[\frac{\varphi_0'(x)}{s} + \frac{y_1'(x)}{s^{\alpha+1}} + \frac{y_2'(x)}{s^{2\alpha+1}} + \frac{y_3'(x)}{s^{3\alpha+1}}\right] - \frac{v_c\lambda}{s^{\alpha}}\left[\frac{B}{\lambda\ s}\varphi_0(x) + \frac{z_2(x)}{s^{2\alpha+1}} + \frac{z_3(x)}{s^{3\alpha+1}}\right], \tag{36}$$

$$LRes_3(\Omega(x,s)) = \frac{z_2(x)}{s^{2\alpha+1}} + \frac{z_3(x)}{s^{3\alpha+1}} - \frac{B}{s^{\alpha}}\left[\frac{\varphi_0(x)}{s} + \frac{y_1(x)}{s^{\alpha+1}} + \frac{y_2(x)}{s^{2\alpha+1}} + \frac{y_3(x)}{s^{3\alpha+1}}\right] + \frac{\lambda}{s^{\alpha}}\left[\frac{B}{\lambda\ s}\varphi_0(x) + \frac{z_2(x)}{s^{2\alpha+1}} + \frac{z_3(x)}{s^{3\alpha+1}}\right].$$

Now, by solving $\lim_{s\to\infty} s^{3\alpha+1} LRes_3(\Phi(x,s)) = 0, \lim_{s\to\infty} s^{3\alpha+1} LRes_3(\Omega(x,s)) = 0$, we find that

$$y_3(x) = v_c\ D\ y_2''(x) + v_c\ \sum y_2(x) + \frac{v_c}{x} y_2'(x) + v_c\lambda z_2(x), \tag{37}$$

$$z_3(x) = By_2(x) - \lambda z_2(x). \tag{38}$$

Proceeding similarly, we replace the kth truncated series $\Phi_k(x,s)$, $\Omega_k(x,s)$ with the kth Laplace residual function $LRes_k(\Phi(x,s)), LRes_k(\Omega(x,s))$. Subsequently, we multiply the resulting equations by $s^{k\alpha+1}$ and consider the limit as $s \to \infty$. The recurrence relation shown below can be used to obtain $y_{k+1}(x)$, $z_{k+1}(x)$ for $k \geq 2$.

$$y_n(x) = v_c\ D\ y_{n-1}''(x) + v_c\ \sum y_{n-1}(x) + \frac{v_c}{x} y_{n-1}'(x) + v_c\lambda z_{n-1}(x), \tag{39}$$

$$z_n(x) = By_{n-1}(x) - \lambda z_{n-1}(x),\ n = 1,2,3,\ldots \tag{40}$$

where $z_0(x) = \frac{B}{\lambda}\varphi_0(x)$.

The system of Equation (24) in accordance with what was introduced is

$$\begin{aligned} \Phi(x,s) &= \frac{\varphi_0(x)}{s} + \sum_{n=1}^{\infty} \frac{y_n(x)}{s^{n\alpha+1}},\ x \in I,\ s > \delta \geq 0. \\ \Omega(x,s) &= \frac{B}{\lambda\ s}\varphi_0(x) + \sum_{n=1}^{\infty} \frac{z_n(x)}{s^{n\alpha+1}},\ x \in I,\ s > \delta \geq 0. \end{aligned} \tag{41}$$

Here, the inverse Laplace transform is applied in Equation (41) enables us to derive the series solution for the system outlined in Equation (18) as follows:

$$\begin{aligned} \varphi(x,t) &= \varphi_0(x) + \sum_{n=1}^{\infty} \frac{y_n(x)\ t^{n\alpha}}{\Gamma(n\alpha+1)}, t \geq 0, x \in I. \\ \omega(x,t) &= \frac{B}{\lambda}\varphi_0(x) + \sum_{n=2}^{\infty} \frac{z_n(x)\ t^{n\alpha}}{\Gamma(n\alpha+1)}, t \geq 0, x \in I, \end{aligned} \tag{42}$$

where

$$\begin{aligned} y_n(x) &= v_c\ D\ y_{n-1}''(x) + v_c\ \sum y_{n-1}(x) + \tfrac{v_c}{x}\ y_{n-1}'(x) + v_c\lambda z_{n-1}(x), \\ z_n(x) &= By_{n-1}(x) - \lambda z_{n-1}(x),\ n = 1,2,3,\ldots \end{aligned} \tag{43}$$

where $z_0(x) = \frac{B}{\lambda}\varphi_0(x)$.

5. Numerical Results and Discussion

The proposed theoretical framework is verified by tackling the fractional time-dependent nuclear reactor equations and neutron diffusion equation and incorporating delayed neutrons through LRPSM. The results obtained from this approach will be compared with numerical results derived from the neutron diffusion cross-section data as referenced, as in [10]. The necessary nuclear reactor constants are provided in Table 1.

Table 1. The neutron diffusion cross-section data.

v_c	B	D	λ	Σ
220,000 cm/s	0.000735 cm^{-1}	0.356	0.08 s^{-1}	0.005 cm^{-2}.

It is important to emphasize that these data are specified in Section 1.

The functions $y_n(x)$ and $z_n(x)$ were calculated using codes implemented in MATHEMATICA software 12. Once these values are determined, the data necessary for constructing the tables and figures are obtained.

In this study, we will examine two numerical cases depending on the initial condition.

The first example considers the initial condition $\varphi(x,0) = 1$. The distribution of the neutron in the cylindrical reactor, represented by the neutron flux, is tabulated in Table 1 and illustrated in Figure 1.

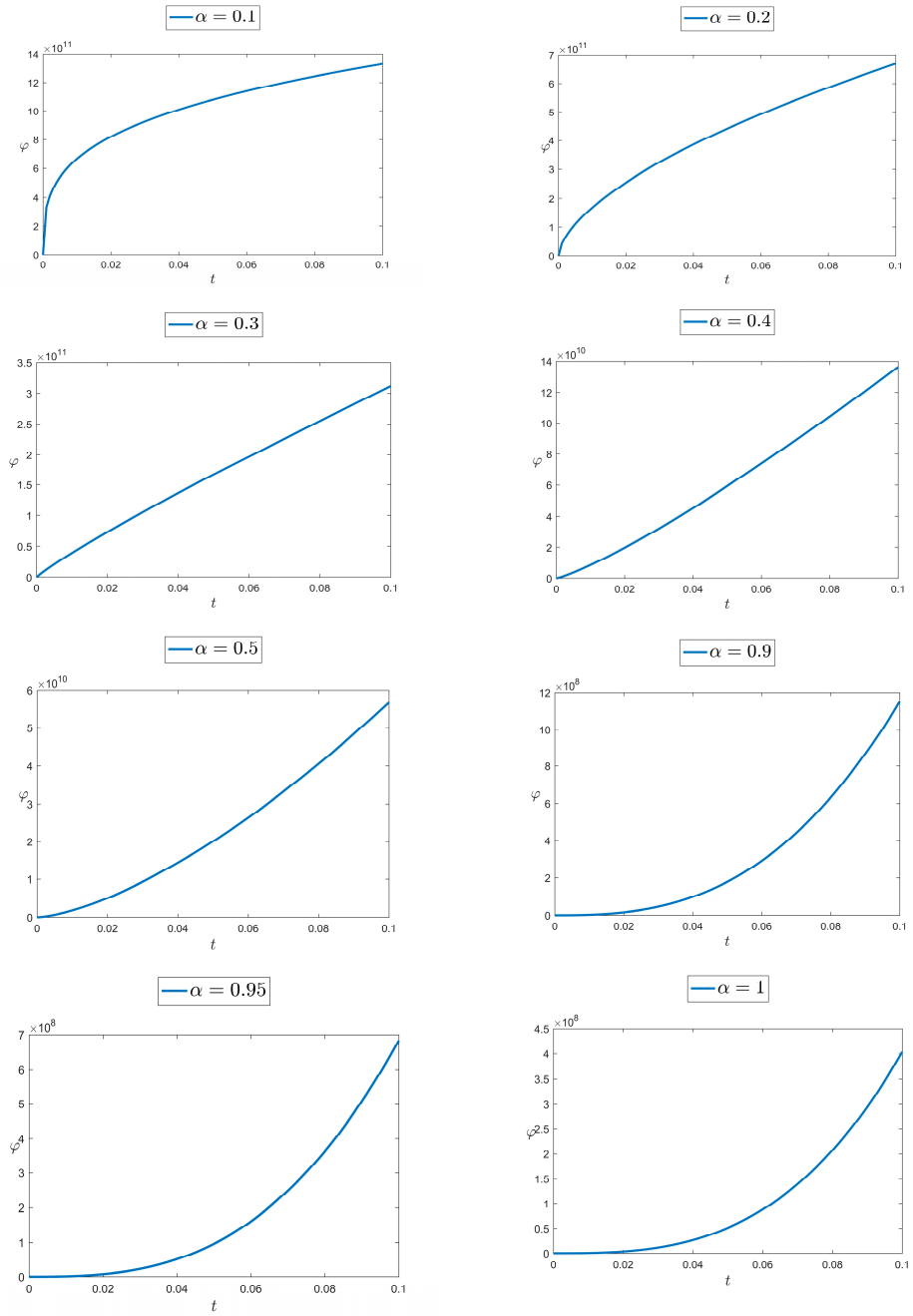


Figure 1. Graphs of neutron flux $\varphi(x,t)$ in two dimensions for different values of α.

The flux was computed at various values of time, considering both a non-Gaussian distribution, ($\alpha < 1$, sub-diffusion), and Gaussian distribution, ($\alpha = 1$, integer order case). In addition to that, Figure 1 and Table 2 show that the neutron flux increases with time.

The second case study considers the initial condition $\varphi(x, 0) = x^2$ where the neutron flux for different values of time under this condition is illustrated in Figure 2.

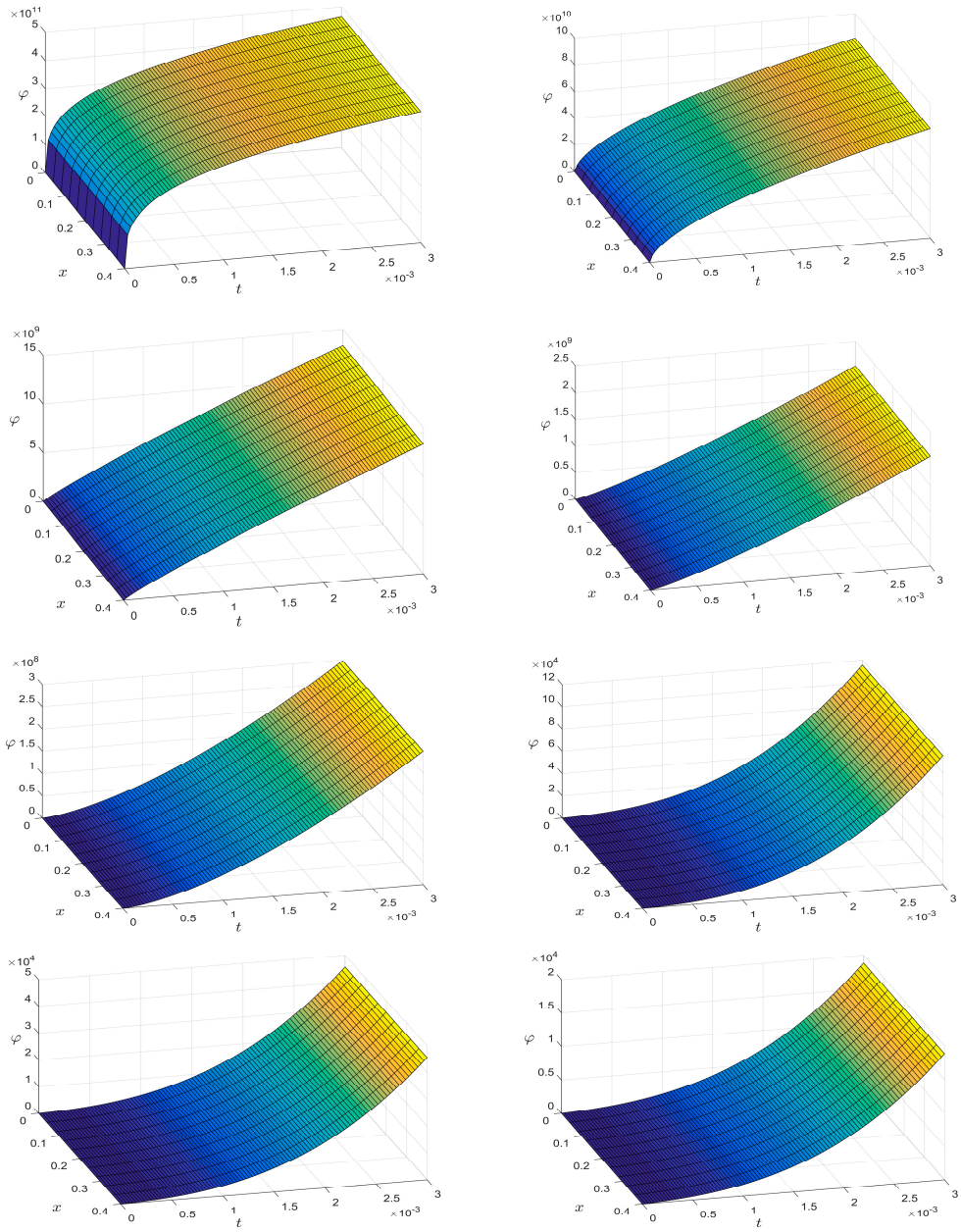


Figure 2. Graphs of neutron flux $\varphi(x, t)$ in three dimensions with $\varphi_0(x) = x^2$ at α = 0.1, 0.2, 0.3, 0.4, 0.5, 0.90, 0.95, and 1.

Presently, a varied initial value problem $\varphi(x, 0) = x^2$ is employed in examining the following scenario, and the outcomes are elucidated in Figure 2.

It is evident that the behavior of the flux becomes closer to the non- fractional case as α approaches to unity. This indicates that the anomalous behavior decreases which means the effect of the delayed neutrons diminishes and eventually vanishes.

The nonfractional case (α = 0.1) for the cylindrical reactor is presented in the literature [5].

It is important to note that, in both examined numerical cases [9,10,12], the time-dependent flux exhibits an increase over time.

Table 2. The flux simulations for various values of α.

t	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
0.00010	1.0001×10^{14}	2.3998×10^{12}	5.1358×10^{10}	1.0172×10^{9}	1.9570×10^{7}	409,582	12,938.5	1069.01	218.945	67.1205
0.00039	1.7233×10^{14}	7.1214×10^{12}	2.6187×10^{11}	8.8447×10^{9}	2.8255×10^{8}	8.93672×10^{6}	311,117	15,528	1642	366.761
0.00068	2.1524×10^{14}	1.1106×10^{13}	5.0966×10^{11}	2.1440×10^{10}	8.4808×10^{8}	3.26117×10^{7}	1.30286×10^{6}	63,492.3	5186.97	887.538
0.00097	2.4809×10^{14}	1.4753×10^{13}	7.7999×10^{11}	3.7766×10^{10}	1.7145×10^{9}	7.50461×10^{7}	3.32891×10^{6}	167,752	12,286.8	1733.47
0.00126	2.7544×10^{14}	1.8184×10^{13}	1.0671×10^{11}	5.7313×10^{10}	2.8813×10^{9}	1.38975×10^{8}	6.69906×10^{6}	352,590	24,648.1	3033.51
0.00155	2.9923×10^{14}	2.1459×10^{13}	1.3677×10^{12}	7.9754×10^{10}	4.3478×10^{9}	2.26674×10^{8}	1.17040×10^{7}	643,464	44,222.3	4941.57
0.00184	3.2047×10^{14}	2.4613×10^{13}	1.6798×10^{12}	1.0485×10^{11}	6.1139×10^{9}	3.40119×10^{8}	1.86209×10^{7}	1.0668×10^{6}	73,184.9	7636.5
0.00213	3.3979×10^{14}	2.7669×10^{13}	2.0019×10^{12}	1.3244×10^{11}	8.1791×10^{9}	4.81079×10^{8}	2.77163×10^{7}	1.64988×10^{6}	113,920	11,322.1
0.00242	3.5758×10^{14}	3.0642×10^{13}	2.3329×10^{12}	1.6236×10^{11}	1.0543×10^{10}	6.51161×10^{8}	3.92481×10^{7}	2.42072×10^{6}	169,007	16,227.1
0.00271	3.7414×10^{14}	3.3544×10^{13}	2.6718×10^{12}	1.9451×10^{11}	1.3206×10^{10}	8.51847×10^{8}	5.34675×10^{7}	3.40801×10^{6}	241,213	22,605.1
0.00300	3.8966×10^{14}	3.6384×10^{13}	3.0181×10^{12}	2.2879×10^{11}	1.6168×10^{10}	1.08452×10^{9}	7.06193×10^{7}	4.64106×10^{6}	333,482	30,734.9

The significance of this work lies in the investigation of the behavior of delayed neutron flux, which affects irregular diffusion. The solution of the time-dependent fractional neutron diffusion equations, as shown in Equation (42), provides a mathematical expression for this flux behavior.

6. Conclusions

The fractional neutron diffusion equations, including belated neutrons, were effectively studied utilizing the Laplace residual power series approach. We were able to produce an analytical fractional series solution using this method.

The soundness of the method's theoretical framework is verified through two distinct numerical case studies in nuclear physics. Flux calculations for both cases are derived, shedding light on the phenomenon of anomalous diffusion, $\alpha < 1$ (sub-diffusion). The investigation involves the consideration of neutron fluxes that exhibit an increase over time. Using LRPSM in fractional calculus, the novel notion of irregular diffusion, which characterizes diffusion patterns that are not Gaussian, has been thoroughly described. The application of LRPSM in cylindrical nuclear reactors can improve the study more complicated nuclear reactor geometries and the investigation of different scientific problems.

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