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Results for Analytic Function Associated with Briot–Bouquet Differential Subordinations and Linear Fractional Integral Operators

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Abstract: In this paper, we present a new class of linear fractional differential operators that are based on classical Gaussian hypergeometric functions. Then, we utilize the new operators and the concept of differential subordination to construct a convex set of analytic functions. Moreover, through an examination of a certain operator, we establish several notable results related to differential subordination. In addition, we derive inclusion relation results by employing Briot–Bouquet differential subordinations. We also introduce a perspective study for developing subordination results using Gaussian hypergeometric functions and provide certain properties for further research in complex dynamical systems.

Keywords: univalent functions; Ruscheweh operators; Salagean operators; inclusion relation; differential operator

MSC: 05A30; 26D10; 26D15; 26A33

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1. Introduction

The geometric function theory is one of the mathematical science domains where the theory of differential and integral operators has been used. In [1], Salagean covered differential operators and devised a few enlarged classes in the open unit disk of normalized analytic functions. Since then, a number of scholars have used differential and integral operators to justify the geometric characteristics of the analytic functions [2]. In the development of several classes of univalent functions, several fractional integral operators have been used recently [3].

Let D be the unite disc $D = \{v : v \in \mathbb{C}, |v| < 1\}$ normalized by $f'(0) = 1$ and $f(0) = 0$ and $\mathcal{A} = \{f(v) : v \in D\}$. Then, every function $f \in \mathcal{A}$ has a series expansion as

$$f(v) = v + \sum_{n=2}^{\infty} a_n v^n, \quad v \in D. \tag{1}$$

In $\mathcal{A}$, the univalent function class is represented by S, the starlike function class is represented by S^*, and the convex function class is represented by K [4]. The authors of [5–7] explore the class $\mathcal{C}(\gamma)$ of convex functions of complex orders γ ($\gamma \in \mathbb{C} - \{0\}$) and analyze many extensions of the class of univalent functions.

In [8], Mocanu and Miller introduce the concept of differential subordination, which was later applied to many classes of analytic functions. If $f_1, f_2 \in \mathcal{A}$ and there is a Schwartz

function w, with $w(0) = 0$ and $|w| < 1$, such that $f_1(v) = f_2(w(v))$, for $v \in D$, then the function f_1 is subordinate to the function f_2, expressed as $f_1 \prec f_2$. If $f_2(v)$ is univalent, then $f_1(v) \prec f_2(v)$ if and only if $f_1(0) = f_2(0)$ and $f_1(D) \subseteq f_2(D)$.

The differential and integral operators have made major contributions as a result of such development in the differential subordination principle; see [9–11] for further details.

Assume $h \in S$, $\psi : \mathbb{C}^3 \times D \to \mathbb{C}$ and p is an analytic function on D that satisfies the subordination

$$\psi(p, vp', z^2p''; v) \prec h(v), \quad v \in D. \tag{2}$$

If $p \prec q$ for every p such that the connection (2) holds, then the univalent function q is said to be a dominant to the solution of the differential solution (2). If $\hat{q} \prec p$ is satisfied by any dominant p of (2), then that dominant $\hat{p}$ is the best dominant. It is evident that on the open union disk D, the best dominant is unique [8]. By employing differential subordination, the authors in [12] introduce a subclass of univalent functions and deduce numerous features of these functions. Oros uses differential subordination theory in [13] to study the Gaussian hypergeometric function.

Specific problems are connected to the differential subordination (2) and also to the differential equations associated with the subordination presented in (2). It was used in a number of physics and mathematics branches. For instance, R. W. Ibrahim uses the Brin–Bouquet differential subordination in [14] to study the class of analytic functions. By using the appropriate Schwarz function, we arrive at the Briot–Bouquet differential equation. The Briot–Bouquet differential equation is used by the authors in [15] to solve the holomorphic dynamical system

$$z_t = u(x, w), \quad w_t = v(x, w) \quad for \quad x, w \in D.$$

Rong in [16] studied holomorphic dynamical systems by using Briot–Bouquet differential equation. The authors in [17] used the Briot–Bouquet differential equation to obtain the solution of the equation of nano-shells. In this situation, the fields of transposition of the nano-shell possess a dynamic system as follows

$$z_t = u(x, w) + u_\theta(x, w), \quad w_t = v(x, w) + v_\theta(x, w) \quad for \quad x, w \in D,$$

where t is in any interval, and θ is the angle between z and w and their conjugates.

A special type of such problems is known as the Briot–Bouquet differential subordination. The Briot–Bouquet differential subordination is very useful in discussing various original results; see the following lemma.

Lemma 1 ([8]). *Assume $\beta \in \mathbb{C}$ and $h(v)$ is a function which is convex in D, where $h(0) = 1$. If p is in the form $p(v) = 1 + b_1v + b_2v^2 + ...$ and analytic in D and*

$$p(v) + \frac{1}{\beta} vp'(v) \prec h(v), \quad v \in D,$$

then

$$p(v) \prec \tilde{h}(v) = \frac{\beta}{v^\beta} \int_0^v t^{\beta-1} h(t)dt \prec h(v). \tag{3}$$

and $\tilde{h}(v)$ is the best dominant of (3)

Lemma 2 ([8,18]). *Assume $\gamma, \beta \in \mathbb{C}$ and h is a function which is convex on D, where $Re\{\beta h(v) + \gamma\} > 0$. Suppose that p is an analytic mapping on D, with $h(0) = p(0)$, and*

$$p(v) + \frac{vp'(v)}{\beta p(v) + \gamma} \prec h(v), \quad v \in D.$$

If the differential equation

$$g(v) + \frac{vg'(v)}{\beta g(v) + \gamma} = h(v)$$

has a univalent solution $g(v)$*, then*

$$p(v) \prec g(z) \prec h(v), \quad v \in D.$$

The function $g(v)$ *is the best dominant.*

Remark 1 ([8,18]). *In view of Lemma 2, we can easily show that if*

$$p(v) + \frac{vp'(v)}{\beta p(v) + \gamma} \prec g(v) + \frac{vg'(v)}{\beta g(v) + \gamma},$$

then $p(v) \prec g(v) \prec h(v), \in D$ *and* $g(z)$.

In [19], Miller and colleagues examine analytical functions through the lens of differential subordination theory, yielding novel insights into the hypergeometric function. The Gaussian hypergeometric function and its use in differential subordination-based analytic function theory have been highlighted in previous studies [20–22]. More recently, refs. [23–25] have derived a number of conclusions about the differential subordination for the classical Gaussian hypergeometric function. In [26], authors explore many applications and describe innovative fractional integral operators using the Gaussian hypergeometric function.

Let a, b and c be complex numbers with $c \notin \{-1, -2, -3, ...\}$. Then, the classical Gaussian hypergeometric function ${}_2F_1(a, b, c; v)$ is defined by [27]

$${}_2F_1(a, b, c; v) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{v^k}{k!}, \tag{4}$$

where $(\alpha)_k$ represents a Pochhammer symbol that is defined in terms of the Gamma function as

$$(\alpha)_k = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)} = \begin{cases} 1, & k = 0, \\ \alpha(\alpha + 1)...(\alpha + k - 1), & k \in \mathbb{N}. \end{cases}$$

The classical Gaussian hypergeometric function ${}_2F_1(a, b, c; z)$ is an analytic function on D. If a or b is a negative integer, then the classical Gaussian hypergeometric function reduces to a polynomial.

In this study, we use classical Gaussian hypergeometric functions to construct the integral operator and introduce the linear differential operator $D_\lambda^{\mu,\eta}$. As a study of convex function subclasses, we examine this operator. Next, we establish analytic functions by following the Briot–Bouquet differential subordination.

2. Differential Subordination Results

Definition 1. *Let* f *belong to* $\mathcal{A}$, $\lambda > 0$ *and* $\mu, \eta \in \mathbb{R}$. *Then, the fractional differential operator* $D_\lambda^{\mu,\eta}$ *is defined by*

$$D_\lambda^{\mu,\eta} f(v) = \frac{1}{\Gamma(\lambda)} \frac{d}{dv} \left(v^{-(\lambda+\mu)} \int_0^v \frac{{}_2F_1(\lambda + \mu, -\eta, \lambda; 1 - t/v) f(t)}{(v - t)^{1-\lambda}} dt \right),$$

where ${}_2F_1(a, b, c; v)$ *defines a Gaussian hypergeometric Function* (4).

We define the differential operator $\mathcal{L}_{\lambda}^{\mu,\eta}$ as follows

$$\begin{aligned}\mathcal{L}_{\lambda}^{\mu,\eta}f(v) &= \frac{\Gamma(1-\mu)\Gamma(2+\lambda+\eta)}{\Gamma(2-\mu+\eta)}v^{\mu+1}D_{\lambda}^{\mu,\eta}f(v)\\ &= v+\sum_{k=2}^{\infty}\frac{(2-\mu+\eta)_{k-1}(2)_{k-1}}{(1-\mu)_{k-1}(\lambda+\eta+2)_{k-1}}a_k v^k, \end{aligned} \tag{5}$$

where $\lambda > 0$ and $\mu, \eta \in R$.

Definition 2. *Let $f \in \mathcal{A}$, $\lambda > 0$ and $\mu, \eta \in \mathbb{R}$. Then, the class $S_{\delta}(\lambda, \mu, \eta)$ consists of functions $f \in A$ satisfying the inequality*

$$Re\Big(\mathcal{L}_{\lambda}^{\mu,\eta}f(v)\Big)' > \delta,$$

where $\delta \in [0, 1]$.

Theorem 1. *Let $\lambda > 0$, $\mu, \eta \in \mathbb{R}$ and $0 \leq \delta \leq 1$. Then, the class $S_{\delta}(\lambda, \mu, \eta)$ is convex.*

Proof. Let f_1 and f_2 be arbitrary functions in $S_{\delta}(\lambda, \mu, \eta)$ such that

$$f_1(z) = z + \sum_{k=2}^{\infty} a_{1k}z^k, \quad f_2(z) = z + \sum_{k=2}^{\infty} a_{2k}z^k.$$

Then, it suffices to establish that the function $h = \alpha_1 f_1 + \alpha_2 f_2$ with $\alpha_1, \alpha_2 > 0$, $\alpha_1 + \alpha_2 = 1$ belongs to $S_{\delta}(\lambda, \mu, \eta)$. Since $h(z) = z + \sum_{k=2}^{\infty}(\alpha_1 a_{1k} + \alpha_2 a_{2k})z^k$, we get

$$\Big(\mathcal{L}_{\lambda}^{\mu,\eta}h(z)\Big)' = 1 + \sum_{k=2}^{\infty}\frac{k(2-\mu+\eta)_{k-1}(2)_{k-1}}{(1-\mu)_{k-1}(\lambda+\eta+2)_{k-1}}(\alpha_1 a_{1k} + \alpha_2 a_{2k})z^{k-1}.$$

This implies that

$$\begin{aligned} Re\left\{\Big(\mathcal{L}_{\lambda}^{\mu,\eta}h(z)\Big)'\right\} &= \alpha_1 Re\left\{1 + \sum_{k=2}^{\infty}\frac{k(2-\mu+\eta)_{k-1}(2)_{k-1}}{(1-\mu)_{k-1}(\lambda+\eta+2)_{k-1}}a_{1k}z^{k-1}\right\}\\ &+ \alpha_2 Re\left\{1 + \sum_{k=2}^{\infty}\frac{k(2-\mu+\eta)_{k-1}(2)_{k-1}}{(1-\mu)_{k-1}(\lambda+\eta+2)_{k-1}}a_{2k}z^{k-1}\right\}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} Re\Big(\mathcal{L}_{\lambda}^{\mu,\eta}h(z)\Big)' &= \alpha_1 Re\Big(\mathcal{L}_{\lambda}^{\mu,\eta}f_1(z)\Big)' + \alpha_2 Re\Big(\mathcal{L}_{\lambda}^{\mu,\eta}f_2(z)\Big)'\\ &> \alpha_1\delta + \alpha_2\delta = \delta. \end{aligned}$$

Thus, the result is obtained. □

Theorem 2. *Let $\lambda > 0$, $\mu, \eta \in \mathbb{R}$, $0 \leq \delta \leq 1$, $\beta, \gamma \in \mathbb{C}$, $\beta \neq 0$, $g(v) \in S_{\delta}(\lambda, \mu, \eta)$ with $g(0) = 1$, $Re(\beta g(v) + \gamma) > 0$ and define*

$$h(v) = g(v) + \frac{vg'(v)}{\beta g(v) + \gamma}. \tag{6}$$

If $f \in S_{\delta}(\lambda, \mu, \eta)$ and

$$F(v) = \frac{\beta+\gamma}{v^{\gamma}}\int_0^v f^{\beta}(t)t^{\gamma-1}dt, \tag{7}$$

then, the subordination

$$\frac{\left(v\mathcal{L}_{\lambda}^{\mu,\eta}f(v)\right)'}{\left(\mathcal{L}_{\lambda}^{\mu,\eta}f(v)\right)} \prec h(v)$$

reveals that

$$\frac{1}{\beta}\frac{\left(v\mathcal{L}_{\lambda}^{\mu,\eta}F(v)\right)'}{\left(\mathcal{L}_{\lambda}^{\mu,\eta}F(v)\right)} \prec g(v) \prec h(v), \tag{8}$$

and this result is sharp.

Proof. By following Equation (7) and employing simple computations, we derive

$$f(v) = \left[\frac{F(v)}{\beta+\gamma}\left(\frac{vF'(v)}{F(v)}+\gamma\right)\right]^{\frac{1}{\beta}}.$$

Hence, by (5) we obtain

$$\mathcal{L}_{\lambda}^{\mu,\eta}f(v) = \left[\frac{\mathcal{L}_{\lambda}^{\mu,\eta}F(v)}{\beta+\gamma}\left(\frac{v\mathcal{L}_{\lambda}^{\mu,\eta}(F(v))'}{\mathcal{L}_{\lambda}^{\mu,\eta}F(v)}+\gamma\right)\right]^{\frac{1}{\beta}}.$$

Computing (twice) the logarithmic derivative of the above relation reveals

$$\frac{v(\mathcal{L}_{\lambda}^{\mu,\eta}f(v))'}{\mathcal{L}_{\lambda}^{\mu,\eta}f(v)} = \frac{1}{\beta}\frac{v(\mathcal{L}_{\lambda}^{\mu,\eta}F(v))'}{\mathcal{L}_{\lambda}^{\mu,\eta}F(v)} + \frac{v\left(\frac{1}{\beta}\frac{v(\mathcal{L}_{\lambda}^{\mu,\eta}F(v))'}{\mathcal{L}_{\lambda}^{\mu,\eta}F(v)}\right)'}{\frac{v(\mathcal{L}_{\lambda}^{\mu,\eta}F(v))'}{\mathcal{L}_{\lambda}^{\mu,\eta}F(v)}+\gamma}. \tag{9}$$

Now, define the following notation

$$p(v) = \frac{1}{\beta}\frac{z(\mathcal{L}_{\lambda}^{\mu,\eta}F(v))'}{\mathcal{L}_{\lambda}^{\mu,\eta}F(v)}, \quad v \in D. \tag{10}$$

Therefore, the subordination (8) can be given as

$$p(v) + \frac{vp'(v)}{\beta p(v)+\gamma} \prec g(v) + \frac{vg'(v)}{\beta g(v)+\gamma}, \quad v \in D.$$

From Lemma 2, we get

$$p(v) \prec g(v) \prec h(v), \quad v \in D.$$

Thus, the differential subordination (8) is established and h is the best dominant. This ends the proof of our result. □

Proposition 1. *Let $\lambda > 0$, $\mu, \eta \in \mathbb{R}$, $0 \leq \delta \leq 1$, $\beta, \gamma \in \mathbb{C}$ and $h(z) \in S_{\delta}(\lambda, \mu, \eta)$ be a convex function. We define*

$$\mathcal{L}_{\lambda}^{\mu,\eta}F(z) = \frac{\beta+\gamma}{z^{\gamma}}\int_0^z H^{\beta}(t)t^{\gamma-1}dt, \; z \in D, \tag{11}$$

where

$$H(z) = z.\exp\left(\int_0^z \frac{h(t)-1}{t}dt\right),\ z \in D. \tag{12}$$

Then the differential equation

$$g(z) = \frac{\beta+\gamma}{\beta}\frac{H^\beta(z)}{\mathcal{L}_\lambda^{\mu,\eta}F(z)} - \frac{\gamma}{\beta} \tag{13}$$

is a formal solution to (6).

Proof. Consider the function $g(z)$ in (13), we get

$$\beta g(z) + \gamma = (\beta+\gamma)\frac{H^\beta(z)}{\mathcal{L}_\lambda^{\mu,\eta}F(z)}, \qquad z \in D.$$

By computing the logarithmic derivative, we derive

$$\frac{zg'(z)}{\beta g(z)+\gamma} = \frac{z\left(H^\beta(z)\right)'}{\beta H^\beta(z)} - \frac{z\left(\mathcal{L}_\lambda^{\mu,\eta}F(z)\right)'}{\beta\mathcal{L}_\lambda^{\mu,\eta}F(z)}, \qquad z \in D. \tag{14}$$

The relations (11) and (13) becomes

$$g(z) = \frac{1}{\beta}\frac{z\left(\mathcal{L}_\lambda^{\mu,\eta}F(z)\right)'}{\mathcal{L}_\lambda^{\mu,\eta}F(z)}, \quad z \in D. \tag{15}$$

Hence, from (14) and (15), we obtain

$$g(z) + \frac{zg'(z)}{\beta g(z)+\gamma} = \frac{z\left(H^\beta(z)\right)'}{\beta H^\beta(z)}, \qquad z \in D. \tag{16}$$

Now, from the Equation (12), we have

$$H^\beta(z) = z^\beta.\exp\beta\left(\int_0^z \frac{h(t)-1}{t}dt\right), \qquad z \in D.$$

Computing (twice) the logarithmic derivative of the above equation reveals

$$h(z) = \frac{z\left(H^\beta(z)\right)'}{\beta H^\beta(z)}, \qquad z \in D. \tag{17}$$

Thus, by aid of (16) and (17), the relation (6) is established. The proof of Proposition 1 is completed. □

Consequently, utilizing Theorem 2, we derive the following result.

Theorem 3. *Let* $n \in \mathbb{N}, \lambda > 0, \mu, \eta \in \mathbb{R}, 0 \le \delta \le 1, \beta, \gamma \in \mathbb{C}, \beta \neq 0, g(v) \in S_\delta(\lambda,\mu,\eta), v \in D$ *with* $g(0) = 1$, $Re(\beta g(v)+\gamma) > 0$ *and define*

$$h(v) = g(v) + \frac{vg'(v)}{\beta g(v)+\gamma}. \tag{18}$$

If $f \in S_\delta(\lambda,\mu,\eta)$ *and*

$$F_n(v) = \frac{\beta+\gamma}{z^{\gamma+n}}\int_0^v f^\beta(t)t^{\gamma-1}dt,$$

then the following differential subordination

$$\frac{\left(v\mathcal{L}_{\lambda}^{\mu,\eta}f(v)\right)'}{\left(\mathcal{L}_{\lambda}^{\mu,\eta}f(v)\right)} \prec h(v)$$

implies that

$$\frac{1}{\beta}\left(n+\frac{\left(v\mathcal{L}_{\lambda}^{\mu,\eta}F(z)\right)'}{\left(\mathcal{L}_{\lambda}^{\mu,\eta}F(v)\right)}\right) \prec g(v) \prec h(v),$$

and this result is sharp.

Proposition 2. *Let $n \in \mathbb{N}$, $\lambda > 0$, $\mu, \eta \in \mathbb{R}$, $0 \leq \delta \leq 1$, $\beta, \gamma \in \mathbb{C}$ and $h \in S_{\delta}(\lambda, \mu, \eta)$ be a convex function. We define*

$$\mathcal{L}_{\lambda}^{\mu,\eta}F_n(z) = \frac{\beta+\gamma}{z^{\gamma+n}}\int_0^z H^{\beta}(t)t^{\gamma-1}dt,\ z \in D, \tag{19}$$

where

$$H_1(z) = z.\exp\left(\int_0^z \frac{\beta h(t)-1}{t}dt\right),\ z \in D. \tag{20}$$

Then, the differential equation

$$g(z) = \frac{\beta+\gamma}{\beta}\frac{H_1^{\beta}(z)}{z^n\mathcal{L}_{\lambda}^{\mu,\eta}F(z)} - \frac{\gamma}{\beta} \tag{21}$$

is a formal solution of (18).

Proof. By considering the function $g(z)$ in (21), we get

$$\beta g(z) + \gamma = (\beta+\gamma)\frac{H_1^{\beta}(z)}{z^n\mathcal{L}_{\lambda}^{\mu,\eta}F(z)}, \qquad z \in D.$$

By computing the logarithmic derivative, we derive

$$\frac{zg'(z)}{\beta g(z)+\gamma} = \frac{z\left(H_1^{\beta}(z)\right)'}{\beta H_1^{\beta}(z)} - \frac{1}{\beta}\left(\frac{z\left(\mathcal{L}_{\lambda}^{\mu,\eta}F(z)\right)'}{\mathcal{L}_{\lambda}^{\mu,\eta}F(z)} + n\right), \qquad z \in D. \tag{22}$$

Therefore, the relations (20) and (21) become

$$g(z) = \frac{1}{\beta}\left(\frac{z\left(\mathcal{L}_{\lambda}^{\mu,\eta}F_n(z)\right)'}{\mathcal{L}_{\lambda}^{\mu,\eta}F(z)} + n\right), \quad z \in D. \tag{23}$$

Hence, by utilizing (22) and (23), we obtain

$$g(z) + \frac{zg'(z)}{\beta g(z)+\gamma} = \frac{z\left(H_1^{\beta}(z)\right)'}{\beta H_1^{\beta}(z)}, \qquad z \in D. \tag{24}$$

Now, by using the equation included in (20), we have

$$H_1^{\beta}(z) = z^{\beta}.\exp\beta\left(\int_0^z \frac{\beta h(t)-1}{t}dt\right), \qquad z \in D.$$

Once again, computing twice the logarithmic derivative of the above equation reveals

$$h(z) = \frac{z\left(H_1^{\beta}(z)\right)'}{\beta H_1^{\beta}(z)}, \qquad z \in D. \tag{25}$$

Thus, taking into account (24) and (25), the relation (18) follows. The proof of Proposition 2 is completed. □

In the next result, we derive the fascinating conclusion for the class $S_{\delta}(\lambda,\mu,\eta)$.

Theorem 4. *Let* $\lambda > 0$, $\mu,\eta \in \mathbb{R}$, $0 \le \delta \le 1$, $\beta,\gamma \in \mathbb{C}$, $\gamma,\beta \neq 0$, $h(z) = \frac{1}{\beta}\left(\frac{2(\delta-1)z}{(1+(2\delta-1)z)(1+z)}\right)$, $\delta \in [0,1)$ *and* $I_{\beta}^{\gamma}f(z) = \frac{\beta+\gamma}{z^{\gamma}}\int_0^z f^{\beta}(t)t^{\gamma-1}dt$, $z \in D$. *Then,*

$$I_{\beta}^{\gamma}[S_{\delta}(\lambda,\mu,\eta)] \subseteq S_{\delta^*}(\lambda,\mu,\eta),$$

where

$$\delta^* = (\beta+\gamma)(1-\delta)\left[1-\frac{2}{\gamma}\int_0^1 \frac{t^{\gamma-1}}{1+t}dt\right].$$

Proof. By virtue of Theorem 2, we reach the following differential subordination

$$p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \prec h(z),\ z \in D,$$

where p is given by (10). Also, by applying Theorem 2 and Remark 1, we derive

$$\frac{1}{\beta}\frac{z(\mathcal{L}_{\lambda}^{\mu,\eta}F(z))'}{\mathcal{L}_{\lambda}^{\mu,\eta}F(z)} \prec g(z) \prec h(z), \quad z \in D.$$

Now, by using Proposition 1, we have

$$H(z) = z\exp\int_0^z \frac{h(t)-1}{t}dt = z\exp\int_0^z \frac{\frac{1}{\beta}\left(\frac{2(\delta-1)t}{(1+(2\delta-1)z)(1+t)}\right)-1}{t}dt = \left[\frac{1+(2\delta-1)z}{1+z}\right]^{\frac{1}{\beta}}.$$

By using Equation (11), we obtain

$$\begin{aligned}
\mathcal{L}_{\lambda}^{\mu,\eta}F(z) &= \frac{\beta+\gamma}{z^{\gamma}}\int_0^z H^{\beta}(t)t^{\gamma-1}dt = \frac{\beta+\gamma}{z^{\gamma}}\int_0^z \frac{1+(2\delta-1)t}{1+t}t^{\gamma-1}dt \\
&= \frac{(\beta+\gamma)(2\delta-1)}{z^{\gamma}}\int_0^z t^{\gamma-1}dt + \frac{2(1-\delta)(\beta+\gamma)}{z^{\gamma}}\int_0^z \frac{t^{\gamma-1}}{1+t}dt \\
&= \frac{(\beta+\gamma)(2\delta-1)}{\gamma} + \frac{2(1-\delta)(\beta+\gamma)}{z^{\gamma}}\int_0^z \frac{t^{\gamma-1}}{1+t}dt.
\end{aligned}$$

Equation (13) hence implies that

$$z(\mathcal{L}_{\lambda}^{\mu,\eta}F(z))' = (\beta+\gamma)H^{\beta}(z) - \gamma\mathcal{L}_{\lambda}^{\mu,\eta}F(z)), \quad z \in D.$$

Therefore, we derive

$$(\mathcal{L}_{\lambda}^{\mu,\eta}F(z))' = (\beta+\gamma)\left(\frac{1+(2\delta-1)z}{z(1+z)}\right) - \frac{(\beta+\gamma)(2\delta-1)}{z} - \frac{2(1-\delta)(\beta+\gamma)}{\gamma z^{\gamma+1}}\int_0^z \frac{t^{\gamma-1}}{1+t}dt.$$

In view of Equation (13) and the fact that g is convex, $g(D)$ is symmetric with respect to the real axis. Therefore, we derive

$$\begin{aligned} & Re\{\left(\mathcal{L}_{\lambda}^{\mu,\eta}F(z)\right)'\} \geq \\ & min_{|z|=1}Re\left\{(\beta+\gamma)\left(\frac{1+(2\delta-1)z}{z(1+z)}\right) - \frac{(\beta+\gamma)(2\delta-1)}{z} - \frac{2(1-\delta)(\beta+\gamma)}{\gamma z^{\gamma+1}}\int_0^z \frac{t^{\gamma-1}}{1+t}dt\right\} \\ = \quad & (\beta+\gamma)(1-\delta)\left[1-\frac{2}{\gamma}\int_0^1 \frac{t^{\gamma-1}}{1+t}dt\right] = \delta^*. \end{aligned}$$

This ends the proof. □

Following the proof of Theorem 5, we then prove an inclusion for the class $S_{\delta}(\lambda,\mu,\eta)$ by utilizing Proposition 2 and Theorem 3.

Theorem 5. *Let $\lambda > 0$, $\mu,\eta \in \mathbb{R}$, $0 \leq \delta \leq 1$, $\beta,\gamma \in \mathbb{C}$, $\gamma,\beta \neq 0$, $h(z) = \frac{1}{\beta^2}\frac{(\delta-2)z}{(1+(\delta-1)z)(1+z)}$, $\delta \in [0,1)$ and $I_{n,\beta}^{\gamma}f(z) = \frac{\beta+\gamma}{z^{\gamma+n}}\int_0^z f^{\beta}(t)t^{\gamma-1}dt$, $z \in D$. Then, we have the following inclusion*

$$I_{n,\beta}^{\gamma}[S_{\delta}(\lambda,\mu,\eta)] \subseteq S_{\delta^*}(\lambda,\mu,\eta),$$

where

$$\delta^* = (\beta+\gamma)\left[\frac{\delta}{2} + \frac{(\gamma+n)(1-\delta)}{\gamma} - (\gamma+n)(\delta-2)\int_0^1 \frac{t^{\gamma-1}}{1+t}dt\right].$$

Theorem 6. *Let $\lambda > 0$, $\mu,\eta \in \mathbb{R}$, $0 \leq \delta \leq 1$, $\beta,\gamma \in \mathbb{C}$, $\beta \neq 0$ and $h(v) \in S_{\delta}(\lambda,\mu,\eta)$. If the mapping $f \in \mathcal{A}$ satisfies the following differential subordination*

$$\frac{1}{\beta}\left[\frac{v(1-(\mathcal{L}_{\lambda}^{\mu,\eta}f(v))')}{\mathcal{L}_{\lambda}^{\mu,\eta}f(v)} + 1 - \gamma\right] \prec h(v),\ v \in D,$$

then

$$\frac{1}{\beta}\left[\frac{v}{\mathcal{L}_{\lambda}^{\mu,\eta}f(v)} - \gamma\right] \prec g(v) \prec h(v), v \in D. \tag{26}$$

The result is sharp.

Proof. Let $p(v) = \frac{1}{\beta}\left[\frac{v}{\mathcal{L}_{\lambda}^{\mu,\eta}f(v)} - \gamma\right]$. Then, we have $\beta p(v) + \gamma = \frac{v}{\mathcal{L}_{\lambda}^{\mu,\eta}f(v)}$. By computing the logarithmic derivative, it can be read as

$$\frac{vp'(v)}{\beta p(v)+\gamma} = \frac{1}{\beta} - \frac{1}{\beta}\frac{v(\mathcal{L}_{\lambda}^{\mu,\eta}f(v))'}{\mathcal{L}_{\lambda}^{\mu,\eta}f(v)},\ v \in D \tag{27}$$

The relation (27) becomes

$$p(v) + \frac{vp'(v)}{\beta p(v)+\gamma} = \frac{1}{\beta}\left[\frac{v(1-(\mathcal{L}_{\lambda}^{\mu,\eta}f(v))')}{\mathcal{L}_{\lambda}^{\mu,\eta}f(v)} + 1 - \gamma\right],\ v \in D.$$

By applying Lemma 2 and Remark 1, we establish that $p(v) \prec g(v) \prec h(v)$. Thus, the differential subordination in (30) is established. The proof is finished. □

Theorem 7. *Let $\lambda > 0$, $\mu, \eta \in \mathbb{R}$, $0 \leq \delta \leq 1$, $\beta, \gamma \in \mathbb{C}$, $\beta \neq 0$ and $h(v) \in S_\delta(\lambda, \mu, \eta)$. If $f \in \mathcal{A}$ satisfies the following differential subordination*

$$\frac{1}{\beta}\left[\frac{v\left(\mathcal{L}_\lambda^{\mu,\eta} f(v)\right)'}{\mathcal{L}_\lambda^{\mu,\eta} f(v)} - \left(1 + \frac{v\left(\mathcal{L}_\lambda^{\mu,\eta} f(v)\right)''}{\left(\mathcal{L}_\lambda^{\mu,\eta} f(v)\right)'}\right) + \frac{\mathcal{L}_\lambda^{\mu,\eta} f(v)}{v\left(\mathcal{L}_\lambda^{\mu,\eta} f(v)\right)'} - \gamma\right] \prec h(v),\ v \in D,$$

which implies that

$$\frac{1}{\beta}\left[\frac{\mathcal{L}_\lambda^{\mu,\eta} f(v)}{v\left(\mathcal{L}_\lambda^{\mu,\eta} f(z)\right)'} - \gamma\right] \prec g(v) \prec h(v), v \in D, \tag{28}$$

and this result is sharp.

Proof. $p(v) = \frac{1}{\beta}\left[\frac{\mathcal{L}_\lambda^{\mu,\eta} f(v)}{v\left(\mathcal{L}_\lambda^{\mu,\eta} f(v)\right)'} - \gamma\right]$ leads to $\beta p(v) + \gamma = \frac{\mathcal{L}_\lambda^{\mu,\eta} f(v)}{v\left(\mathcal{L}_\lambda^{\mu,\eta} f(v)\right)}, v \in D$. By computing the logarithmic derivative, we get

$$\frac{v p'(v)}{\beta p(v) + \gamma} = \frac{1}{\beta}\left[\frac{v(\mathcal{L}_\lambda^{\mu,\eta} f(v))'}{\mathcal{L}_\lambda^{\mu,\eta} f(v)} - 1 - \frac{v(\mathcal{L}_\lambda^{\mu,\eta} f(v))''}{\left(\mathcal{L}_\lambda^{\mu,\eta} f(v)\right)'}\right],\ v \in D \tag{29}$$

The relation (29), for $z \in D$, becomes

$$p(v) + \frac{v p'(v)}{\beta p(v) + \gamma} = \frac{1}{\beta}\left[\frac{v\left(\mathcal{L}_\lambda^{\mu,\eta} f(v)\right)'}{\mathcal{L}_\lambda^{\mu,\eta} f(v)} - \left(1 + \frac{v\left(\mathcal{L}_\lambda^{\mu,\eta} f(v)\right)''}{\left(\mathcal{L}_\lambda^{\mu,\eta} f(v)\right)'}\right) + \frac{\mathcal{L}_\lambda^{\mu,\eta} f(v)}{v\left(\mathcal{L}_\lambda^{\mu,\eta} f(v)\right)'} - \gamma\right].$$

By employing Lemma 2 and Remark 1, we infer that $p \prec g \prec h$. Thus, the differential subordination in (28) is established. The proof of Theorem 7 is completed. □

Similarly, from Theorems 6 and 7, we establish the following results.

Theorem 8. *Let $\lambda > 0$, $\mu, \eta \in \mathbb{R}$, $0 \leq \delta \leq 1$, $\beta, \gamma \in \mathbb{C}$, $\beta \neq 0$ and $h(v) \in S_\delta(\lambda, \mu, \eta)$. If $f \in \mathcal{A}$ satisfies the following differential subordination*

$$\frac{1}{\beta}\left[\frac{v(\mathcal{L}_\lambda^{\mu,\eta} f(v))'}{\mathcal{L}_\lambda^{\mu,\eta} f(v)} + \mathcal{L}_\lambda^{\mu,\eta} f(v) - \gamma\right] \prec h(v), \qquad v \in D,$$

then we have

$$\frac{1}{\beta}\left[\mathcal{L}_\lambda^{\mu,\eta} f(v) - \gamma\right] \prec g(v) \prec h(v),\ \ v \in D.$$

The result is sharp.

Theorem 9. *Let $\lambda > 0$, $\mu, \eta \in \mathbb{R}$, $0 \leq \delta \leq 1$, $\beta, \gamma \in \mathbb{C}$, $\beta \neq 0$ and $h(v) \in S_\delta(\lambda, \mu, \eta)$. If $f \in \mathcal{A}$ satisfies the following differential subordination*

$$\frac{1}{\beta}\left[1 - \frac{v(\mathcal{L}_\lambda^{\mu,\eta} f(v))'}{\mathcal{L}_\lambda^{\mu,\eta} f(v)} + \left(\frac{v}{\mathcal{L}_\lambda^{\mu,\eta} f(v)}\right)^\delta - \gamma\right] \prec h(v),\ \ v \in D,$$

then we have

$$\frac{1}{\beta}\left[\left(\frac{v}{\mathcal{L}_{\lambda}^{\mu,\eta}f(v)}\right)^{\delta}-\gamma\right]\prec g(v)\prec h(v),\ v\in D. \tag{30}$$

The result is sharp.

3. Application

As an application of this theory, we introduce the following results.

Theorem 10. *Let $\lambda>0$, $\mu,\eta\in\mathbb{R}$, $0\leq\delta\leq 1$, $0\leq\zeta<1$, $f\in\mathcal{A}$ and $h(v)\in S_{\delta}(\lambda,\mu,\eta)$. If $Re((1-\zeta)p(v)+\zeta)>0$, then the following differential subordination*

$$\frac{1}{1-\zeta}\left(1+\frac{v(\mathcal{L}_{\lambda}^{\mu,\eta}f(v))''}{(\mathcal{L}_{\lambda}^{\mu,\eta}f(v))'}-\zeta\right)\prec h(v), \tag{31}$$

implies that

$$\frac{1}{1-\zeta}\left(\frac{v(\mathcal{L}_{\lambda}^{\mu,\eta}f(v))'}{\mathcal{L}_{\lambda}^{\mu,\eta}f(v)}-\zeta\right)\prec g(v)\prec h(v). \tag{32}$$

The result is sharp.

Proof. First, we define

$$p(v)=\frac{1}{1-\zeta}\left(\frac{v(\mathcal{L}_{\lambda}^{\mu,\eta}f(v))'}{\mathcal{L}_{\lambda}^{\mu,\eta}f(v)}-\zeta\right), \tag{33}$$

where $p\in\mathcal{P}$. We compute the logarithmic derivative of relation (33) to have

$$\frac{1}{1-\zeta}\left(1+\frac{v(\mathcal{L}_{\lambda}^{\mu,\eta}f(v))''}{(\mathcal{L}_{\lambda}^{\mu,\eta}f(v))'}-\zeta\right)=p(v)+\frac{vp'(v)}{(1-\zeta)p(v)+\zeta},\ v\in D.$$

Therefore, we can rewrite the differential subordination (31) in the form

$$p(v)+\frac{vp'(v)}{(1-\zeta)p(v)+\zeta}\prec h(v),\ \ v\in D.$$

Now, by taking $\beta=1-\zeta$ and $\gamma=\zeta$, we apply Lemma 2 to obtain the differential subordination (32) and announce h as the best dominant. This ends the proof. □

Corollary 1. *Let $\lambda>0$, $\mu,\eta\in\mathbb{R}$, $0\leq\delta\leq 1$, $\zeta\in\mathbb{C}$, $\zeta\neq 0$, $h(v)=\frac{1+(2\delta-1)v}{1+v}$ and $F(v)=\frac{\beta+\gamma}{v^{\gamma}}\int_0^v f^{1-\zeta}(t)t^{\zeta-1}dt$, $v\in D$. Then, the following differential subordination*

$$\frac{1}{1-\zeta}\left(1+\frac{v(\mathcal{L}_{\lambda}^{\mu,\eta}f(v))''}{(\mathcal{L}_{\lambda}^{\mu,\eta}f(v))'}-\zeta\right)\prec\frac{1+(2\delta-1)v}{1+v},$$

implies that

$$\frac{1}{1-\zeta}\left(\frac{v(\mathcal{L}_{\lambda}^{\mu,\eta}f(v))'}{\mathcal{L}_{\lambda}^{\mu,\eta}f(v)}-\zeta\right)\prec\frac{1}{1-\zeta}\frac{(2\delta-1)(1-\zeta)v-\zeta}{1+v}. \tag{34}$$

Proof. In view of Theorem 2, there can be found a convex function $g(v)$ with $g(0) = 1$, satisfying

$$h(v) = g(v) + \frac{vg'(v)}{(1-\zeta)g(v)+\zeta}, \quad v \in D. \tag{35}$$

Now, by using Equation (11), we derive

$$H(v) = v\exp\int_0^v \frac{h(t)-1}{t}dt = v\exp\int_0^v \frac{\frac{1+(2\delta-1)t}{1+t}-1}{t}dt = v(1+v)^{2(\delta-1)}.$$

By using Equation (38), we obtain

$$\mathcal{L}_\lambda^{\mu,\eta}F(v) = \frac{1}{v^\zeta}\int_0^v H^{1-\zeta}(t)t^{\zeta-1}dt = \frac{1}{v^\zeta}\int_0^v (1+v)^{2(1-\zeta)(\delta-1)}dv = \frac{(1+v)^{2(1-\zeta)(\delta-1)+1}}{[2(1-\zeta)(\delta-1)+1]v^\zeta}.$$

Since $\beta = 1-\zeta$ and $\gamma = \zeta$, Equation (13) implies that

$$g(v) = \frac{1}{1-\zeta}\left(\frac{H^{1-\zeta}(v)}{\mathcal{L}_\lambda^{\mu,\eta}F(v)} - \zeta\right) = \frac{1}{1-\zeta}\frac{(2\delta-1)(1-\zeta)v-\zeta}{1+v}.$$

Thus, the differential subordination (34) is derived. This ends the proof of our corollary. □

Corollary 2. *Let $\lambda > 0$, $\mu, \eta \in \mathbb{R}$, $A, B, \zeta \in \mathbb{C}$, $A, B, \zeta \neq 0$, $A \neq B$,$h(z) = \frac{1+Az}{1+Bz}$ and $F(z) = \frac{\beta+\gamma}{z^\gamma}\int_0^z f^{1-\zeta}(t)t^{\zeta-1}dt$, $z \in D$. Then, the following differential subordination*

$$\frac{1}{1-\zeta}\left(1 + \frac{z(\mathcal{L}_\lambda^{\mu,\eta}f(z))''}{(\mathcal{L}_\lambda^{\mu,\eta}f(z))'} - \zeta\right) \prec \frac{1+Az}{1+Bz},$$

which implies that

$$\frac{1}{1-\zeta}\left(\frac{z(\mathcal{L}_\lambda^{\mu,\eta}f(z))'}{\mathcal{L}_\lambda^{\mu,\eta}f(z)} - \zeta\right) \prec \frac{1}{1-\zeta}\frac{A(1-\zeta)-\zeta}{1+Bz}. \tag{36}$$

Proof. Following similar proof to the proof of Corollary 1, we write

$$H(z) = z\exp\int_0^z \frac{h(t)-1}{t}dt = z\exp\int_0^z \frac{\frac{1+At}{1+Bt}-1}{t}dt = z(1+Bz)^{\frac{A-B}{B}}.$$

Hence, by using Equation (38), we establish

$$\mathcal{L}_\lambda^{\mu,\eta}F(z) = \frac{1}{z^\zeta}\int_0^z H^{1-\zeta}(t)t^{\zeta-1}dt = \frac{1}{z^\zeta}\int_0^z (1+Bt)^{\frac{A-B}{B}(1-\zeta)}dt = \frac{(1+Bz)^{\frac{A-B}{B}(1-\zeta)+1}}{((A-B)(1-\zeta)+B)z^\zeta}.$$

Equation (13) then reveals

$$g(z) = \frac{1}{1-\zeta}\left(\frac{H^{1-\zeta}(z)}{\mathcal{L}_\lambda^{\mu,\eta}F(z)} - \zeta\right) = \frac{1}{1-\zeta}\frac{A(1-\zeta)-\zeta}{1+Bz}.$$

Hence, the differential subordination (36) is derived. This ends the proof of Corollary 2. □

Corollary 3. *Let $\lambda > 0$, $\mu, \eta \in \mathbb{R}$, $0 \leq \alpha < 1$, $\zeta \neq 0$, $h(v) = \frac{1}{1-\zeta}\frac{1+2(\alpha-1)v}{1+v}$ and $F_n(v) = \frac{\beta+\gamma}{v^{\gamma+n}}\int_0^v f^{1-\zeta}(t)t^{\zeta-1}dt$, $v \in D$. Then, the following differential subordination*

$$\frac{1}{1-\zeta}\left(1+\frac{v(\mathcal{L}_{\lambda}^{\mu,\eta}f(v))''}{(\mathcal{L}_{\lambda}^{\mu,\eta}f(v))'}-\zeta\right)\prec\frac{1}{1-\zeta}(\frac{1+2(\alpha-1)v}{1+v}),$$

implies that

$$\frac{1}{1-\zeta}\left(\frac{v(\mathcal{L}_{\lambda}^{\mu,\eta}f(v))'}{\mathcal{L}_{\lambda}^{\mu,\eta}f(v)}-\zeta\right)\prec\frac{1}{1-\zeta}\frac{(2\alpha-1)(1-\zeta)v-\zeta}{1+v}. \tag{37}$$

Proof. In view of Theorem 3, there can be formed a convex function g such that

$$h(v)=g(v)+\frac{vg'(v)}{(1-\zeta)g(v)+\zeta},\ \ v\in D. \tag{38}$$

Now, by applying Equation (20) and the fact that $\beta=1-\zeta$, we write

$$H_1(v)=v.\exp\int_0^v\frac{(1-\zeta)h(t)-1}{t}dt=v.\exp\int_0^v\frac{2(\alpha-1)}{1+t}dt=v(1+v)^{2(\alpha-1)}.$$

Thus, by invoking Equation (19) and the fact that $\gamma=\zeta$, we establish that

$$\begin{aligned}\mathcal{L}_{\lambda}^{\mu,\eta}F_n(v) &= \frac{1}{v^{\zeta+n}}\int_0^v H_1^{1-\zeta}(t)t^{\zeta-1}dt\\ &= \frac{1}{v^{\zeta+n}}\int_0^v(1+t)^{2(\alpha-1)(1-\zeta)}dt\\ &= \frac{1}{v^{\zeta+n}}\frac{(1+v)^{2(\alpha-1)(1-\zeta)+1}}{2(\alpha-1)(1-\zeta)+1}.\end{aligned}$$

Equation (21), therefore, reveals

$$g(v)=\frac{1}{1-\zeta}\left(n+\frac{v\left(\mathcal{L}_{\lambda}^{\mu,\eta}F_n(v)\right)'}{\mathcal{L}_{\lambda}^{\mu,\eta}F(v)}\right)=\frac{1}{1-\zeta}\frac{(2\alpha-1)(1-\zeta)v-\zeta}{1+v}.$$

Thus, the subordination (37) is derived. This finishes the proof of Corollary 3. □

Theorem 11. *Let $\lambda>0$, $\mu,\eta\in\mathbb{R}$, $\zeta\geq1$, $0\leq\delta\leq1$ and $f\in\mathcal{A}$. If then the following differential subordination*

$$\left(\mathcal{L}_{\lambda}^{\mu,\eta}f(v)\right)'\prec\frac{1+(1-2\delta)v}{1-v},\quad(v\in D)$$

then we have

$$Re\left\{\left(\frac{\mathcal{L}_{\lambda}^{\mu,\eta}f(v)}{v}\right)^{\frac{1}{\zeta}}\right\}>\left(2\delta-1+2(1-\delta)\frac{ln(1-r)}{r}\right)^{\frac{1}{\zeta}}, \tag{39}$$

where $|v|=r$.

Proof. First, we define

$$p(v)=\frac{\mathcal{L}_{\lambda}^{\mu,\eta}f(v)}{v}.$$

For this function, we find that

$$p(v)+vp'(v)=\left(\mathcal{L}_{\lambda}^{\mu,\eta}f(v)\right)'$$

Thus, by using Lemma (1), we derive

$$p(v) \prec \frac{1}{v}\int_0^v \frac{1+(1-2\delta)t}{1-t}dt$$

or

$$\frac{\mathcal{L}_\lambda^{\mu,\eta} f(v)}{v} = \int_0^1 \frac{1+(1-2\delta)uw(v)}{1-uw(v)}du$$

where $w(v)$ is a Schwarz function.
Since

$$\int_0^1 \frac{1+(1-2\delta)uw(v)}{1-uw(v)}du = 2\delta - 1 - 2(1-\delta)\int_0^1 \frac{du}{1+uw(v)}$$

we obtain that

$$\begin{aligned} Re\left\{\frac{\mathcal{L}_\lambda^{\mu,\eta} f(v)}{v}\right\} &= Re\left\{2\delta - 1 - 2(1-\delta)\int_0^1 \frac{du}{1+uw(v)}\right\} \\ &\geq 2\delta - 1 - 2(1-\delta)\left|\int_0^1 \frac{du}{1+uw(v)}\right| \\ &\geq 2\delta - 1 - 2(1-\delta)\int_0^1 \frac{du}{|1+uw(v)|}. \end{aligned} \tag{40}$$

Now, we set $|v| = r$ and we have the following inequalities

$$\begin{aligned} 1-ur &= 1-u|v| \leq 1-u|w(v)| \\ &= |1-u|w(v)|| \leq |1-uw(v)| \leq |1+uw(v)|. \end{aligned} \tag{41}$$

It follows from the inequality (40) and (41) that

$$\begin{aligned} Re\left\{\frac{\mathcal{L}_\lambda^{\mu,\eta} f(v)}{v}\right\} &\geq 2\delta - 1 - 2(1-\delta)\int_0^1 \frac{du}{1-ur} \\ &= 2\delta - 1 + 2(1-\delta)\frac{ln(1-r)}{r}, \qquad (v \in D, |v| = r). \end{aligned} \tag{42}$$

Since $Re\left(v^{\frac{1}{\zeta}}\right) \geq (Re(v))^{\frac{1}{\zeta}}$ for $Re(v) > 0$ and $\zeta \geq 1$, from inequality (42), the inequality (39) is derived. This end the proof. □

4. Conclusions

This article uses the inspirational classical Gaussian hypergeometric functions to define a linear fractional differential operator $D_\lambda^{\mu,\eta}$. The new differential operator is then used to introduce a number of analytic function subclasses. Moreover, the convex function class $S_\delta(\lambda,\mu,\eta)$ is provided, and its characteristics are described. Furthermore, a number of conclusions involving starlike and convex functions of complex order are achieved by using the Briot–Bouquet differential subordination. Briot–Bouquet differential subordination was used to construct a variety of inclusion relationships. Following the establishment of this idea, scholars have considered a number of applications for this theory. As a result, the findings of this research might be expanded to examine Gaussian hypergeometric functions and significant properties of complex dynamical systems.

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