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Onthe Reducibility of a Class Nonlinear Almost Periodic Hamiltonian Systems

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Abstract: Inthis paper, we consider the reducibility of a class of nonlinear almost periodic Hamiltonian systems. Under suitable hypothesis of analyticity, non-resonant conditions and non-degeneracy conditions, by using KAM iteration, it is shown that the considered Hamiltonian system is reducible to an almost periodic Hamiltonian system with zero equilibrium points for most small enough parameters. As an example, we discuss the reducibility and stability of an almost periodic Hill's equation.

Keywords: almost periodic Hamiltonian systems; reducibility; KAM iteration

1. Introduction

In this paper, we are concerned with the reducibility of the almost periodic Hamiltonian system

$$\dot{x} = (A + \varepsilon Q(t))x + \varepsilon g(t) + h(x,t), \;\; x \in \mathbb{R}^{2n}, \tag{1}$$

where A has multiple possible eigenvalues, $Q(t)$, $g(t)$, and $h(x,t)$ are all analytic almost periodic with respect to t, and $\varepsilon > 0$ is a sufficiently small parameter.

First, we review some relevant definitions for almost periodic systems. If $A(t)$ is an $d \times d$ almost periodic matrix, the equation

$$\frac{dx}{dt} = A(t)x, \; x \in \mathbb{R}^d \tag{2}$$

is reducible if there exists a regular almost periodic transformation

$$x = \kappa(t)z,$$

where $\kappa(t)$ and $\kappa^{-1}(t)$ are almost periodic and bounded, which transforms Equation (2) into

$$\frac{dz}{dt} = Dz, \; y \in \mathbb{R}^d \tag{3}$$

where D is constant.

In recent years, the reducibility for linear equations has attracted the attention and been studied by many researchers. The well known Floquet theorem states that every periodic differential Equation (2) can be reduced to a constant coefficient differential Equation (3) by means of a periodic change of variables with the same period as $A(t)$. But this result no longer holds true for the quasi-periodic and almost periodic linear equation; more details can be seen in [1]. If the coefficient matrix satisfies the "full spectrum" condition, Johnson and Sell [2] proved the reducibility of the quasi-periodic linear system (2).

Later, many authors [3–12] paid attention to the reducibility for the following quasi-periodic linear system:

$$\frac{dx}{dt} = (A + \epsilon Q(t))x, \;\; x \in \mathbb{R}^d. \tag{4}$$

where ϵ is a sufficiently small parameter.

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In 1996, Xu and You [13] studied the reducibility for the almost-periodic linear system

$$\frac{dx}{dt} = (A + \varepsilon Q(t))x, \ \ x \in \mathbb{R}^n. \tag{5}$$

They proved that system (5) is reducible in the case that A has different eigenvalues, for most sufficiently small ε through KAM iteration and a "space structure". Later, ref. [14] studied the case in which system (5) is Hamiltonian and A has possible multiple eigenvalues; they obtained reducibility results similar to those in [13].

In 2017, J. Li, C. Zhu, and S. Chen [15] studied the quasi-periodic case of (1). It was shown that for most sufficiently small parameters, under some assumptions of analyticity, non-resonant and non-degeneracy conditions, through a quasi-periodic symplectic change of variables, the considered system was changed into a quasi-periodic Hamiltonian system with zero equilibrium points.

Motivated by [13–15], we will extend the reducible results of [15] to the case of almost periodic Hamiltonian systems. Under some suitable assumptions, we will obtain a similar result.

Theorem 1. *Consider the almost periodic Hamiltonian system* (1) *in which A is a $2n \times 2n$ matrix that can be diagonalized with multiple possible eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_{2n}$, $i = 1, \cdots, 2n$, and $Q(t) = \sum_{\Lambda \in \tau} Q_\Lambda(t)$, $g(t) = \sum_{\Lambda \in \tau} g_\Lambda(t)$, and $h(x,t) = \sum_{\Lambda \in \tau} h_\Lambda(x,t)$ are all analytic almost periodic functions on D_{ρ_0}; they have the same frequencies $\omega = (\omega_1, \omega_2, \cdots)$ and spatial structure $(\tau, [\cdot])$. Moreover, $h(x,t)$ is analytic with respect to x on $B_{a_0}(0)$, $h(0,t) = 0$, and $D_x h(0,t,\varepsilon) = 0$. Here, $B_{a_0}(0)$ is a ball centered on 0 with radius a_0; $\varepsilon \in (0, \varepsilon_0)$ is a sufficiently small parameter. Suppose the following:*
(1) There exists $z_0 > 0$ such that $|||Q|||_{z_0,\rho_0} < \infty$;
(2) (Non-resonant conditions) $\lambda = (\lambda_1, \cdots, \lambda_{2n})$ and $\omega = (\omega_1, \omega_2, \cdots)$ satisfy

$$|\langle k, \omega \rangle \sqrt{-1} - \lambda_i| \geq \frac{\alpha_0}{\Delta^4(|k|)\Delta^4([k])}, \tag{6}$$

$$|\langle k, \omega \rangle \sqrt{-1} - \lambda_i + \lambda_j| \geq \frac{\alpha_0}{\Delta^4(|k|)\Delta^4([k])} \tag{7}$$

for $1 \leq i, j \leq 2n, i \neq j$, and $k \in \mathbb{Z}^{\mathbb{N}} \setminus \{0\}$, where $\alpha_0 > 0$ is a small constant, and Δ is an approximation function.
(3) (Non-degeneracy conditions) Denote the solution of the equation $\dot{x} = Ax + \varepsilon g(t)$ by $\underline{x}$. Let $\widehat{Q}(t) = Q(t) + \varepsilon^{-1} D_x h(\underline{x}(t), t)$. Assume $A + \varepsilon \overline{\widehat{Q}} := A_+$ has eigenvalues $\lambda_1^+, \lambda_2^+, \cdots, \lambda_{2n}^+$ that satisfy $|\lambda_i^+ - \lambda_j^+| \geq 2\delta\varepsilon > 0$ and $|\lambda_i^+| \geq 2\delta\varepsilon > 0$, where $i \neq j$ and $i, j = 1, 2, \cdots, 2n$.
(4) $\|D_{xx} h(x,t,\varepsilon)\| \leq K_0$, $x \in B_{a_0}(0)$.

Then, there exists a positive Lebesgue measure, non-empty Cantor set $E^* \subset (0, \varepsilon_0)$, such that for $\varepsilon \in E^*$, there is an almost periodic symplectic transformation $x = \psi(t)y + \varphi(t)$ that transforms (1) *into*

$$\dot{y} = By + h_\infty(y,t), \tag{8}$$

where $\psi(t)$ and $\varphi(t)$ are almost periodic with the same frequencies and spatial structure as $Q(t)$, B is a real constant matrix, and $h_\infty(y,t) = O(y^2)$ as $y \to 0$. Moreover, meas$((0, \varepsilon_0) \setminus E^*) = o(\varepsilon_0)$ as $\varepsilon_0 \to 0$.

As an example, we will apply Theorem 1 in Section 4 to an almost-periodic Hill's equation:

$$\ddot{x} + \varepsilon a(t)x = 0. \tag{9}$$

Under some appropriate assumptions, we have that, for most small ε, Equation (9) is reducible. Furthermore, the zero equilibrium point of (9) is Lyapunov stable.

The basic framework of this paper is as follows. In Section 2, we recall some definitions and notations, present some results in the form of lemmas that will be useful in the proof of Theorem 1. The proof of Theorem 1 is presented in Section 3. In Section 4, we analyze the almost periodic Hill's equation, Equation (9).

2. Some Preliminaries

Firstly, we present some definitions.

Definition 1. *We say a function f is quasi-periodic with the basic frequencies $\omega = (\omega_1, \omega_2, \cdots, \omega_s)$ if $f(t) = F(\theta_1, \theta_2, \cdots, \theta_s)$ where F is 2π-periodic in $\theta_j = \omega_j t$ for $j = 1, 2, \cdots, s$. Moreover, if $F(\theta)$ $(\theta = (\theta_1, \theta_2, \cdots, \theta_s))$ is analytic on $D_\rho = \{\theta \in \mathbb{C}^s : |\Im\theta_j| \leq \rho, j = 1, 2, \cdots, s\}$, then $f(t)$ is analytic quasi-periodic on D_ρ.*

If $f(t)$ is analytic quasi-periodic, it can be expanded as a Fourier series

$$f(t) = \sum_{k\in\mathbb{Z}^s} f_k e^{\langle k,\omega\rangle\sqrt{-1}t}$$

with Fourier coefficients

$$f_k = \frac{1}{(2\pi)^s} \int_{\mathbb{T}^r} F(\theta) e^{-\langle k,\theta\rangle\sqrt{-1}} d\theta.$$

The norm is denoted as $||f||_\rho$:

$$||f||_\rho = \sum_{k\in\mathbb{Z}^s} |f_k| e^{|k|\rho}.$$

Assume $R(t) = (r_{ij}(t))_{1\leq i,j\leq m}$ is an $m \times m$ matrix. If all $r_{ij}(t)$ $(i, j = 1, 2, \cdots, m)$ are analytic quasi-periodic on D_ρ with frequencies $\omega = (\omega_1, \omega_2, \cdots, \omega_s)$, then matrix $R(t)$ is said to be analytic quasi-periodic on D_ρ with frequencies ω.

The norm of $R(t)$ is defined as

$$||R||_\rho = \max_{1\leq i\leq m} \sum_{j=1}^{m} ||r_{ij}||_\rho.$$

Obviously,

$$||R_1 R_2||_\rho \leq ||R_1||_\rho ||R_2||_\rho.$$

If R is a constant matrix, to simplify, we record $||R||_\rho$ as $||R||$. The average of $R(t)$ is $\overline{R} = (\overline{r}_{ij})_{1\leq i,j\leq m}$, where

$$\overline{r}_{ij} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} r_{ij}(t) dt,$$

The details can be found in [16].

Definition 2. *A function f is said to be an almost periodic function if $f(t) = \sum_{m=1}^{\infty} f_m(t)$, where $f_m(t)$ are all quasi-periodic for $m = 1, 2, \cdots$.*

In [13], we see that "spatial structure" and "approximation function" are very powerful tools to study almost periodic systems. We provide the definitions and notions from [17,18].

Definition 3 ([17]). *If τ is a set of some subsets of $\mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, then $(\tau, [\cdot])$ is said to be a finite spatial structure if τ meets the following conditions:*
1. $\emptyset \in \tau$;
2. If $\Lambda_r, \Lambda_s \in \tau$, then $\Lambda_r \cup \Lambda_s \in \tau$;
3. $\bigcup_{\Lambda\in\tau} = \mathbb{N}$, where $[\cdot]$ is a weight function defined on τ, satisfying $[\emptyset] = 0$ and $[\Lambda_r \cup \Lambda_s] \leq [\Lambda_r] + [\Lambda_s]$.

Let $l \in \mathbb{Z}^{\mathbb{N}}$. Write the support set of l as

$$suppl = \{(s_1, s_2, \cdots, s_n)|l_j \neq 0, j = s_1, s_2, \cdots, s_n, l_j = 0, as\ j = otherwise\}.$$

Write the weight value as $[l] = inf_{suppl \subset \Lambda, \Lambda \in \tau}[\Lambda]$. Denote

$$|l| = \sum_{s=1}^{\infty} |l_s|.$$

Definition 4 ([18]). *Δ is called an approximation function, if*
1. $\Delta : [0, \infty) \to [1, \infty)$ is increasing, and $\Delta(0) = 1$ is satisfied;
2. $\frac{\log \Delta(t)}{t}$ is decreasing on $[0, \infty)$;
3. $\int_0^{\infty} \frac{\log \Delta(t)}{t^2} dt < \infty$.

Remark 1. *If Δ is an approximation function, from Definition 4, it follows that Δ^4 is also an approximation function.*

Definition 5. *Let $R(t) = \sum_{\Lambda \in \tau} R_{\Lambda}(t)$. If $R_{\Lambda}(t)$ are quasi-periodic matrix functions with basic frequencies $\omega = \{\omega_s | s \in \Lambda\}$, then $R(t)$ is said to be an almost periodic matrix function with spatial structure $(\tau, [\cdot])$ and basic frequencies ω.*

We also write the average of R(t) as $\overline{R}$, where

$$\overline{R} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} R(t) dt.$$

Let $R(t) = \sum_{\Lambda \in \tau} R_{\Lambda}(t)$. For $\mu > 0$ and $\varrho > 0$,

$$||R||_{\mu,\varrho} = \sum_{\Lambda \in \tau} e^{\mu[\Lambda]} ||R_{\Lambda}||_{\varrho}$$

is the weighted norms with finite spatial structure $(\tau, [\cdot])$. From [13], we can select the weighted function

$$[\Lambda] = 1 + \sum_{s \in \Lambda} \log^q(1 + |s|), q > 2.$$

Also, we will present some lemmas in this section, which are useful for the proof of our main result.

Lemma 1 ([8]). *Let $g : B_{\sigma}(0) \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^2 function with $g(0) = 0, D_x g(0) = 0$, $||D_{xx} g(x)|| \leq M$, and $x \in B_{\sigma}(0)$. Then, $||g(x)|| \leq \frac{M}{2} ||x||^2$ and $||D_x g(x)|| \leq M||x||$.*

Lemma 2 ([8]). *Suppose that B_0 is an $m \times m$ matrix with the eigenvalues $\mu_1^0, \cdots, \mu_m^0$, which satisfy $|\mu_i^0| > \nu$, $|\mu_i^0 - \mu_j^0| > \nu, i \neq j, 1 \leq i$, and $j \leq m$. Let S_0 be a nonsingular matrix with $S_0^{-1} B_0 S_0 = diag(\mu_1^0, \cdots, \mu_m^0)$, $\beta_0 = \max\{||S_0||, ||S_0^{-1}||\}$, and ϱ is a value such that*

$$0 < \varrho < \frac{\nu}{(3m-1)\beta_0^2}.$$

If B_1 verifies $||B_1 - B_0|| \leq \varrho$, then the following results hold:
(1) B_1 has m different nonzero eigenvalues $\mu_1^1, \cdots, \mu_m^1$.
(2) There is a nonsingular matrix S_1 such that $S_1^{-1} B_1 S_1 = diag(\mu_1^1, \cdots, \mu_m^1)$, which satisfies $||S_1||, ||S_1^{-1}|| \leq \beta_1$, where $\beta_1 = 2\beta_0$.

Lemma 3. *Consider the differential equation*

$$\dot{X} = AX + \varepsilon g(t),\ X \in \mathbb{R}^{2m}, \tag{10}$$

where A is a $2m \times 2m$ constant matrix, which can be diagonalized and where the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_{2m}$ of A satisfy $|\lambda_i| \geq \xi$, and ξ is a positive constant. Also, $g(t) = \sum_{\Lambda \in \tau} g_\Lambda(t)$ is an analytic almost periodic function on D_ρ, of which its frequencies are $\omega = (\omega_1, \omega_2, \cdots)$ and spatial structure is $(\tau, [\cdot])$. If

$$|\langle k, \omega \rangle \sqrt{-1} - \lambda_i| \geq \frac{\alpha}{\Delta^4(|k|)\Delta^4([k])}$$

for all $k \in \mathbb{Z}^{\mathbb{N}} \setminus \{0\}$, $\alpha > 0$. Set $0 < \rho_1 < \rho$ and $0 < z_1 < z$. Then, for Equation (10), there is a unique analytic almost periodic solution $X(t)$ that has the same frequencies and spatial structure as $g(t)$ and satisfies

$$|||X|||_{z-z_1,\rho-\rho_1} \leq \varepsilon \frac{c\Gamma(\rho_1)\Gamma(z_1)}{\alpha} |||g|||_{z,\rho},$$

where $\Gamma(\rho) = \sup_{t \geq 0}\{\Delta^4(t)e^{-\rho t}\}$.

Proof. Make the change of variable $X = BY$, and let $h(t) = B^{-1}g$. Equation (10) becomes

$$\dot{Y} = DY + \varepsilon h(t),\ Y \in \mathbb{R}^{2m}, \tag{11}$$

where $D = B^{-1}AB = diag(\lambda_1, \cdots, \lambda_{2m})$.

Let $Y = \sum_{\Lambda \in \tau} y_\Lambda$, $h = \sum_{\Lambda \in \tau} h_\Lambda$, and

$$y_\Lambda = (\widetilde{y}^{ij}_\Lambda),\ \widetilde{y}^{ij}_\Lambda = \sum_{suppk \subset \Lambda} \widetilde{y}^{ij}_{\Lambda k} e^{\sqrt{-1}\langle k,\omega \rangle t},$$

$$h_\Lambda = (\widetilde{h}^{ij}_\Lambda),\ \widetilde{h}^{ij}_\Lambda = \sum_{suppk \subset \Lambda} \widetilde{h}^{ij}_{\Lambda k} e^{\sqrt{-1}\langle k,\omega \rangle t}.$$

By (11), we have

$$\widetilde{y}^{ij}_{\Lambda k} = \varepsilon \frac{\widetilde{h}^{ij}_{\Lambda k}}{\langle k, \omega \rangle \sqrt{-1} - \lambda_i}.$$

So,

$$\begin{aligned} ||\widetilde{y}^{ij}_\Lambda||_{\rho-\rho_1} &\leq \varepsilon \sum_{suppk \subset \Lambda} \frac{\Delta^4(|k|)\Delta^4([k])}{\alpha} |\widetilde{h}^{ij}_{\Lambda k}| e^{(\rho-\rho_1)|k|} \\ &\leq \varepsilon \frac{\Gamma(\rho_1)\Delta^4([\Lambda])}{\alpha} ||\widetilde{h}^{ij}_\Lambda||_\rho. \end{aligned}$$

Thus,

$$||y_\Lambda||_{\rho-\rho_1} \leq \varepsilon \frac{\Gamma(\rho_1)\Delta^4([\Lambda])}{\alpha} ||h_\Lambda||_\rho.$$

From Definition 5, we have

$$\begin{aligned} |||Y|||_{z-z_1,\rho-\rho_1} &= \sum_{\Lambda \in \tau} ||\widetilde{y}_\Lambda||_{\rho-\rho_1} e^{(z-z_1)[\Lambda]} \\ &\leq \varepsilon \sum_{\Lambda \in \tau} \frac{\Gamma(\rho_1)\Delta^4([\Lambda])}{\alpha} ||h_\Lambda||_\rho e^{z[\Lambda]-z_1[\Lambda]} \\ &\leq \varepsilon \frac{\Gamma(\rho_1)\Gamma(z_1)}{\alpha} |||h_\Lambda|||_{z,\rho}. \end{aligned}$$

Thus, from

$$|||h|||_{z,\rho} \leq ||B^{-1}|| \, |||g|||_{z,\rho},$$

$$|||X|||_{z-z_1,\rho-\rho_1} \le ||B||||y|||_{z-z_1,\rho-\rho_1},$$

we have

$$|||X|||_{z-z_1,\rho-\rho_1} \le \varepsilon \frac{c\Gamma(\rho_1)\Gamma(z_1)}{\alpha}|||g|||_{z,\rho}.$$

□

The following lemma is very useful in proving Theorem 1, in order to perform a step of the inductive procedure.

Lemma 4. *Consider the equation*

$$\dot{P}(t) = AP(t) - P(t)A + M(t), \tag{12}$$

where A is a $2m \times 2m$ Hamiltonian matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_{2m}$. Suppose $|\lambda_i| \ge \sigma$ and $|\lambda_i - \lambda_j| \ge \sigma$ for $i \ne j$, and $i, j = 1, \cdots, 2m$. Furthermore, $M(t) = \sum_{\Lambda \in \tau} M_\Lambda(t)$ is analytic almost periodic on D_ρ with frequencies $\omega = (\omega_1, \omega_2, \cdots)$ and has finite spatial structure $(\tau, [\cdot])$. Then, $\overline{M} = 0$, and

$$|\langle k, \omega\rangle\sqrt{-1} - \lambda_i + \lambda_j| \ge \frac{\alpha}{\Delta^4(|k|)\Delta^4([k])}$$

for all $k \in \mathbb{Z}^{\mathbb{N}} \setminus \{0\}$. Set $0 < \rho_1 < \rho$ and $0 < z_1 < z$. Then equation (12) has a unique analytic almost periodic Hamiltonian solution $P(t)$ with $\overline{P} = 0$, where $P(t)$ has the same frequencies and spatial structure as $M(t)$, and satisfies

$$|||P|||_{z-z_1,\rho-\rho_1} \le c\frac{\Gamma(\bar{z})\Gamma(\bar{\rho})}{\alpha}|||M|||_{z,\rho}.$$

Proof. Choose the matrix S such that $S^{-1}AS = D = diag(\lambda_1, \cdots, \lambda_{2n})$, make the change of variable $P(t) = SW(t)S^{-1}$, and define $R(t) = S^{-1}M(t)S$. Equation (12) becomes

$$\dot{W}(t) = DW(t) - W(t)D + R(t). \tag{13}$$

Let $W = \sum_{\Lambda\in\tau} W_\Lambda$ and $R = \sum_{\Lambda\in\tau} R_\Lambda$, where

$$R_\Lambda = (r^{ij}_\Lambda),\ \ r^{ij}_\Lambda = \sum_{suppk\subset\Lambda} r^{ij}_{\Lambda k} e^{\sqrt{-1}\langle k,\omega\rangle t},$$

$$W_\Lambda = (w^{ij}_\Lambda),\ \ w^{ij}_\Lambda = \sum_{suppk\subset\Lambda} w^{ij}_{\Lambda k} e^{\sqrt{-1}\langle k,\omega\rangle t}.$$

Substitute these into (13). We have $w^{ij}_{\Lambda 0} = 0$ and

$$w^{ij}_{\Lambda k} = \frac{r^{ij}_{\Lambda k}}{\langle k,\omega\rangle\sqrt{-1} - \lambda_i + \lambda_j},\ \ k \ne 0.$$

Since $M(t)$ and $R(t)$ are analytic on D_ρ, we have

$$\begin{aligned} ||w^{ij}_\Lambda||_{\rho-\rho_1} &\le \sum_{suppk\subset\Lambda} \frac{\Delta^4(|k|)\Delta^4([k])}{\alpha}|r^{ij}_{\Lambda k}|e^{(\rho-\rho_1)|k|} \\ &\le \frac{\Gamma(\rho_1)\Delta^4([\Lambda])}{\alpha}||r^{ij}_\Lambda||_\rho. \end{aligned}$$

Thus,

$$||W_\Lambda||_{\rho-\rho_1} \le \frac{\Gamma(\rho_1)\Delta^4([\Lambda])}{\alpha}||R_\Lambda||_\rho.$$

By Definition 5, we have

$$\begin{aligned} |||W|||_{z-z_1,\rho-\rho_1} &= \sum_{\Lambda\in\tau} ||W_\Lambda||_{\rho-\rho_1} e^{(z-z_1)[\Lambda]} \\ &\leq \sum_{\Lambda\in\tau} \frac{\Gamma(\rho_1)\Delta^4([\Lambda])}{\alpha} ||R_\Lambda||_\rho e^{z[\Lambda]-z_1[\Lambda]} \\ &\leq \frac{\Gamma(\rho_1)\Gamma(z_1)}{\alpha} |||R|||_{z,\rho}. \end{aligned}$$

Since,

$$|||P|||_{z-z_1,\rho-\rho_1} \leq ||S||\,|||W|||_{z-z_1,\rho-\rho_1}||S^{-1}||,$$

$$|||R|||_{z,\rho} \leq ||S^{-1}||\,|||M|||_{z,\rho}||S||.$$

Hence,

$$|||P|||_{z-z_1,\rho-\rho_1} \leq c\frac{\Gamma(z_1)\Gamma(\rho_1)}{\alpha}|||M|||_{z,\rho}.$$

From now on, the symbol c is used to denote different constants.

Now, we verify that $P = \sum_{\Lambda\in\tau} P_\Lambda$ is Hamiltonian. Since A and $M = \sum_{\Lambda\in\tau} M_\Lambda$ are Hamiltonian, we have

$$A = JA_J,\ M = JM_J$$

where A_J and M_J are symmetric. Let $P_J = J^{-1}P$. If P_J is symmetric, then P is Hamiltonian. Now, we demonstrate that P_J is symmetric. Substitute $P = JP_I$ into Equation (12). We have

$$\dot{P}_J = A_J JP_J - P_J JA_J + M_J, \tag{14}$$

Transposing Equation (14), we obtain

$$\dot{P_J}^T = A_J JP_J^T - P_J^T JA_J + M_J.$$

Obviously, JP_J and JP_J^T are all solutions of (12). Furthermore, $\overline{JP_J} = \overline{JP_J^T} = 0$. From the uniqueness of solution of (12) with $\overline{P} = 0$, it follows that $JP_J = JP_J^T$; hence, P is Hamiltonian. □

Lemma 5. *Consider the following Hamiltonian system:*

$$\dot{x} = (A + \varepsilon Q(t))x + \varepsilon g(t) + h(x,t),\ x \in \mathbb{R}^{2n}, \tag{15}$$

where A is a $2n \times 2n$ matrix that can be diagonalized with the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_{2n}$, and $|\lambda_i| \geq \sigma$, $\sigma > 0$ is a constant. Assume that $Q(t) = \sum_{\Lambda\in\tau} Q_\Lambda(t)$, $g(t) = \sum_{\Lambda\in\tau} g_\Lambda(t)$, and $h(x,t) = \sum_{\Lambda\in\tau} h_\Lambda(x,t)$ are analytic almost periodic on D_ρ. Their frequencies are $\omega = (\omega_1, \omega_2, \cdots)$, and they have the spatial structure $(\tau, [\cdot])$. Suppose that $h(x,t)$ is analytic about x on $B_a(0)$, where $||D_{xx}h(x)|| \leq M, \forall x \in B_a(0)$. Furthermore,

$$|\langle k,\omega\rangle\sqrt{-1} - \lambda_i| \geq \frac{\alpha}{\Delta^4(|k|)\Delta^4([k])}$$

holds for all $k \in \mathbb{Z}^{\mathbb{N}} \setminus \{0\}$, and the constant $\alpha > 0$. Let $0 < \rho_1 < \rho$ and $0 < z_1 < z$. Then, there exists a symplectic transformation $x = y + \underline{x}$ that transforms (15) into

$$\dot{y} = (A + \varepsilon\overline{\widehat{Q}}(t))y + \varepsilon^2\widehat{g}(t) + \widehat{h}(y,t),$$

satisfing

$$|||\widehat{Q}|||_{z-z_1,\rho-\rho_1} \leq |||Q|||_{z,\rho} + M\frac{c\Gamma(\rho_1)\Gamma(z_1)}{\alpha}|||g|||_{z,\rho}$$

and

$$|||\widehat{g}|||_{z-z_1,\rho-\rho_1} \le \frac{c\Gamma(\rho_1)\Gamma(z_1)}{\alpha}|||Q|||_{z,\rho}\ |||g|||_{z,\rho} + Mc\left(\frac{\Gamma(\rho_1)\Gamma(z_1)}{\alpha}\right)^2|||g|||^2_{z,\rho},$$

where $a_1 = a_0 - |||\underline{x}|||_{z-z_1,\rho-\rho_1}$, $y \in B_{a_1}(0)$.

Proof. The solution of Equation $\dot{x} = Ax + \varepsilon g(t)$ is denoted by $\underline{x}$. From Lemma 3, it follows that

$$|||\underline{x}|||_{z-z_1,\rho-\rho_1} \le \varepsilon\frac{c\Gamma(\rho_1)\Gamma(z_1)}{\alpha}|||g|||_{z,\rho}.$$

By the symplectic transformation $x = y + \underline{x}$, Equation (15) is transformed into

$$\dot{y} = (A + \varepsilon\overline{\widehat{Q}}(t))y + \varepsilon^2\widehat{g}(t) + \widehat{h}(y,t),$$

where

$$\widehat{Q} = Q(t) + \frac{1}{\varepsilon}D_x h(\underline{x}, t),$$

$$\widehat{g} = \frac{1}{\varepsilon^2}h(\underline{x}, t) + \frac{1}{\varepsilon}Q(t)\underline{x}$$

and

$$\widehat{h} = h(\underline{x} + y, t) - h(\underline{x}, t) - D_x h(\underline{x}, t)y.$$

From Lemmas 1 and 3, it follows that

$$\begin{aligned} |||\widehat{Q}|||_{z-z_1,\rho-\rho_1} &\le |||Q|||_{z,\rho} + \frac{M}{\varepsilon}|||\underline{x}|||_{z-z_1,\rho-\rho_1} \\ &\le |||Q|||_{z,\rho} + M\frac{c\Gamma(\rho_1)\Gamma(z_1)}{\alpha}|||g|||_{z,\rho} \end{aligned}$$

and

$$\begin{aligned} |||\widehat{g}|||_{z-z_1,\rho-\rho_1} &\le \frac{1}{\varepsilon^2}\frac{M}{2}|||\underline{x}|||^2_{z-z_1,\rho-\rho_1} + \frac{1}{\varepsilon}||Q|||_{z-z_1,\rho-\rho_1}\ |||\underline{x}|||_{z-z_1,\rho-\rho_1} \\ &\le \frac{c\Gamma(\rho_1)\Gamma(z_1)}{\alpha}|||Q|||_{z,\rho}\ |||g|||_{z,\rho} + Mc\left(\frac{\Gamma(\rho_1)\Gamma(z_1)}{\alpha}\right)^2|||g|||^2_{z,\rho}. \end{aligned}$$

The results are obtained. □

3. Proof of Theorem 1

3.1. *The First KAM Step*

In the first step, we will change A in the Equation (1) from the case with multiple eigenvalues into the case with different eigenvalues, and the ε of $\varepsilon Q(t)$ and $\varepsilon g(t)$ become ε^2.

First of all, for Equation (1), by the symplectic transformation $x = \underline{x}_0 + y$, where $\underline{x}_0$ is the solution of

$$\dot{\underline{x}}_0 = A\underline{x}_0 + \varepsilon g(t).$$

Hamiltonian system (1) is changed into

$$\dot{y} = (A + \varepsilon\widehat{Q}(t))y + \varepsilon^2\widehat{g}(t) + \widehat{h}(y,t). \tag{16}$$

Here,

$$\widehat{Q} = Q(t) + \frac{1}{\varepsilon}D_x h(\underline{x}_0, t),$$

$$\widehat{g} = \frac{1}{\varepsilon^2}h(\underline{x}_0, t) + \frac{1}{\varepsilon}Q(t)\underline{x}_0(t),$$

$$\widehat{h} = h(\underline{x}_0 + y, t) - h(\underline{x}_0, t) - D_x h(\underline{x}_0, t)y.$$

By the assumptions of Theorem 1 and Lemma 3, we have

$$|||\underline{x}_0|||_{z_0-\bar{z}_1,\rho_0-\bar{\rho}_1} \leq \varepsilon\frac{c\Gamma(\bar{\rho}_1)\Gamma(\bar{z}_1)}{\alpha_0}|||g|||_{z,\rho}, \tag{17}$$

where $0 < \bar{z}_1 < \frac{1}{2}z_0$ and $0 < \bar{\rho}_1 < \frac{1}{2}\rho_0$. Define the average of $\widehat{Q}$ by $\overline{\widehat{Q}}$. Equation (16) is changed into

$$\dot{y} = (A_1 + \varepsilon\widetilde{Q}(t))y + \varepsilon^2\widetilde{g}(t) + \widetilde{h}(y,t), \tag{18}$$

where

$$A_1 = A + \varepsilon\overline{\widehat{Q}},\ \widehat{Q} - \overline{\widehat{Q}} = \widetilde{Q},\ \widehat{g} = \widetilde{g},\ \widehat{h} = \widetilde{h}.$$

From the assumptions of Theorem 1, we see that the eigenvalues of A_1 are $\lambda_1^+, \lambda_2^+, \cdots, \lambda_{2n}^+$, which satisfy $|\lambda_i^+ - \lambda_j^+| \geq 2\delta\varepsilon > 0$ and $|\lambda_i^+| \geq 2\delta\varepsilon > 0,\ i,j = 1,\cdots,2n,\ i \neq j$.

Introduce the transformation $y = e^{\varepsilon P_0(t)}x_1$. By this symplectic transformation, system (18) is changed into

$$\begin{aligned}\dot{x}_1 &= (e^{-\varepsilon P_0(t)}(A_1 + \varepsilon\widetilde{Q} - \varepsilon\dot{P}_0)e^{\varepsilon P_0(t)} \\ &\quad + e^{-\varepsilon P_0(t)}(\varepsilon\dot{P}_0 e^{\varepsilon P_0(t)} - \frac{d}{dt}(e^{\varepsilon P_0(t)}))x_1 \\ &\quad + e^{-\varepsilon P_0(t)}\varepsilon^2\widetilde{g}(t) + e^{-\varepsilon P_0(t)}\widetilde{h}(e^{\varepsilon P_0(t)}x_1, t),\end{aligned} \tag{19}$$

where $x_1 \in B_{a_1}(0)$.

Expand $e^{\varepsilon P_0}$ and $e^{-\varepsilon P_0}$ into

$$e^{\varepsilon P_0} = I + \varepsilon P_0 + B,\ \ e^{-\varepsilon P_0} = I - \varepsilon P_0 + \widetilde{B},$$

where

$$B = \frac{(\varepsilon P_0)^2}{2!} + \frac{(\varepsilon P_0)^3}{3!} + \cdots,\ \ \widetilde{B} = \frac{(\varepsilon P_0)^2}{2!} - \frac{(\varepsilon P_0)^3}{3!} + \cdots.$$

System (19) is rewritten as follows:

$$\begin{aligned}\dot{x}_1 &= ((I - \varepsilon P_0 + \widetilde{B})(A_1 + \varepsilon\widetilde{Q} - \varepsilon\dot{P}_0)(I + \varepsilon P_0 + B) \\ &\quad + e^{-\varepsilon P_0(t)}(\varepsilon\dot{P}_0 e^{\varepsilon P_0(t)} - \frac{d}{dt}(e^{\varepsilon P_0(t)})))x_1 \\ &\quad + e^{-\varepsilon P_0(t)}\varepsilon^2\widetilde{g}(t) + e^{-\varepsilon P_0(t)}\widetilde{h}(e^{\varepsilon P_0(t)}x_1, t) \\ &= (A_1 + \varepsilon\widetilde{Q} - \varepsilon\dot{P}_0 + \varepsilon A_1 P_0 - \varepsilon P_0 A_1 + Q^{(1)})x_1 \\ &\quad + e^{-\varepsilon P_0(t)}\varepsilon^2\widetilde{g}(t) + e^{-\varepsilon P_0(t)}\widetilde{h}(e^{\varepsilon P_0(t)}x_1, t),\end{aligned} \tag{20}$$

where

$$\begin{aligned}Q^{(1)} &= -\varepsilon^2 P_0(\widetilde{Q} - \dot{P}_0) + \varepsilon^2(\widetilde{Q} - \dot{P}_0)P_0 - \varepsilon^2 P_0(A_1 + \varepsilon\widetilde{Q} - \varepsilon\dot{P}_0)P_0 \\ &\quad - \varepsilon P_0(A_1 + \varepsilon\widetilde{Q} - \varepsilon\dot{P}_0)B + (A_1 + \varepsilon\widetilde{Q} - \varepsilon\dot{P}_0)B \\ &\quad + \widetilde{B}(A_1 + \varepsilon\widetilde{Q} - \varepsilon\dot{P}_0)e^{\varepsilon P_0} + e^{-\varepsilon P_0}\left(\varepsilon\dot{P}_0 e^{\varepsilon P_0} - \frac{d}{dt}e^{\varepsilon P_0}\right).\end{aligned}$$

We would like to have

$$\widetilde{Q} - \dot{P}_0 + A_1 P_0 - P_0 A_1 = 0,$$

which is equivalent to

$$\dot{P}_0 = A_1 P_0 - P_0 A_1 + \widehat{Q} - \overline{\widehat{Q}}. \tag{21}$$

According to Lemma 4, if

$$|\langle k,\omega\rangle\sqrt{-1}-\lambda_i^+ + \lambda_j^+| \geq \frac{\alpha_1}{\Delta^4(|k|)\Delta^4([k])}$$

for all $k \in \mathbb{Z}^{\mathbb{N}} \setminus \{0\}$, where $\alpha_1 = \frac{\alpha}{4}$, then Equation (21) has a unique analytic almost periodic solution $P(t)$, its frequencies are ω, and it has a spatial structure $(\tau, [\cdot])$, which satisfies

$$|||P_0|||_{z_0-2\bar{z}_1,\rho_0-2\bar{\rho}_1} \leq c\frac{\Gamma(\bar{z}_1)\Gamma(\bar{\rho}_1)}{\alpha_1}|||\widehat{Q}|||_{z_0-\bar{z}_1,\rho_0-\bar{\rho}_1}. \tag{22}$$

System (20) becomes

$$\dot{x}_1 = (A_1 + \varepsilon^2 Q_1(t))x_1 + \varepsilon^2 g_1(t) + h_1(x_1,t), \tag{23}$$

where

$$\varepsilon^2 Q_1 = Q^{(1)},\ g_1(t) = e^{-\varepsilon P_0}\widetilde{g},\ h_1(x_1,t) = e^{-\varepsilon P_0}\widetilde{h}(e^{\varepsilon P_0}x_1,t).$$

Since $\widetilde{Q} - \dot{P}_0 = P_0A_1 - A_1P_0$,

$$\begin{aligned}\varepsilon^2 Q_1 \quad = \quad & Q^{(1)} = -\varepsilon^2 P_0(P_0A_1 - A_1P_0) + \varepsilon^2(P_0A_1 - A_1P_0)P_0 \\ & -\varepsilon^2 P_0(A_1 + \varepsilon P_0A_1 - \varepsilon A_1P_0)P_0 \\ & -\varepsilon P_0(A_1 + \varepsilon P_0A_1 - \varepsilon A_1P_0)B + (A_1 + \varepsilon P_0A_1 - \varepsilon A_1P_0)B \\ & +\widetilde{B}(A_1 + \varepsilon P_0A_1 - \varepsilon A_1P_0)e^{\varepsilon P_0} + e^{-\varepsilon P_0}\left(\varepsilon\dot{P}_0e^{\varepsilon P_0} - \frac{d}{dt}e^{\varepsilon P_0}\right).\end{aligned}$$

Thus, the symplectic transformation is $T_0x_1 = \underline{x}_0 + e^{\varepsilon P}x_1 = \phi_0(t) + \psi_0(t)x_1$. If

$$|||P_0|||_{z_0-2\bar{z}_1,\rho_0-2\bar{\rho}_1} \leq \frac{1}{2},$$

then by (17) and (22), we have

$$|||\phi_0|||_{z_0-2\bar{z}_1,\rho_0-2\bar{\rho}_1} \leq \varepsilon\frac{c\Gamma(\bar{\rho}_1)\Gamma(\bar{z})_1}{\alpha_0}|||g|||_{z,\rho}.$$

$$|||\psi_0 - I|||_{z_0-2\bar{z}_1,\rho_0-2\bar{\rho}_1} \leq \varepsilon\frac{c\Gamma(\bar{z}_1)\Gamma(\bar{\rho}_1)}{\alpha_1}|||\widehat{Q}|||_{z_0-\bar{z}_1,\rho_0-\bar{\rho}_1}.$$

Hence, under the symplectic transformation $x = T_0x_1$, system (1) becomes Hamiltonian system (23).

3.2. The mth KAM Step

The first step has been completed. That is, A_1 has $2n$ different eigenvalues, and $\varepsilon^2 Q_1(t)$ and $\varepsilon^2 g_1(t)$ are smaller perturbations. In the mth step, consider the Hamiltonian system

$$\dot{x}_m = (A_m + \varepsilon^{2^m}Q_m(t))x_m + \varepsilon^{2^m}g_m(t) + h_m(x_m,t),\ m \geq 1, \tag{24}$$

where $x_m \in B_{a_m}(0), Q_m, g_m, h_m$ are analytic almost periodic on D_{ρ_m}, with frequencies ω and the same spatial structure $(\tau, [\cdot])$. A_m has $2n$ different eigenvalues $\lambda_1^m, \cdots, \lambda_{2n}^m$ with

$$|\lambda_i^m| \geq \delta\varepsilon,\ \ |\lambda_i^m - \lambda_j^m| \geq \delta\varepsilon,\ \ i \neq j,\ 1 \leq i,j \leq 2n,$$

where we denote $\lambda_i^1 = \lambda_i^+, i = 1, \cdots, 2n$.

By the symplectic transformation $x_m = \underline{x}_m + y$, where $\underline{x}_m$ is solution of $\dot{\underline{x}}_m = A_m\underline{x}_m + \varepsilon^{2^m}g(t)$ on $D_{\rho_m-\bar{\rho}_{m+1}}$, Hamiltonian system (24) becomes

$$\dot{y} = (A_m + \varepsilon^{2^m}\widehat{Q}_m(t))y + \varepsilon^{2^{m+1}}\widehat{g}(t) + \widehat{h}(y,t), \tag{25}$$

where

$$\widehat{Q}_m = Q_m(t) + \frac{1}{\varepsilon^{2^m}} D_x h_m(\underline{x}_m, t),$$

$$\widehat{g}_m = \frac{1}{\varepsilon^{2^{m+1}}} h_m(\underline{x}_m, t) + \frac{1}{\varepsilon^{2^m}} Q_m(t)\underline{x}_m(t),$$

$$\widehat{h}_m = h_m(\underline{x}_m + y, t) - h_m(\underline{x}_m, t) - D_x h_m(\underline{x}_m, t)y.$$

By Lemma 3, if

$$|\langle k, \omega\rangle\sqrt{-1} - \lambda_i^m| \geq \frac{\alpha_m}{\Delta^4(|k|)\Delta^4([k])}, k \in \mathbb{Z}^{\mathbb{N}},$$

we have

$$|||\underline{x}_m|||_{z_m - \bar{z}_{m+1}, \rho_m - \bar{\rho}_{m+1}} \leq \varepsilon^{2^m} \frac{c\Gamma(\bar{\rho}_{m+1})\Gamma(\bar{z}_{m+1})}{\alpha_m} |||g_m|||_{z_m, \rho_m}. \tag{26}$$

Define the average of $\widehat{Q}_m$ by $\overline{\widehat{Q}}_m$. Equation (25) is changed into

$$\dot{y} = (A_{m+1} + \varepsilon^{2^m} \widetilde{Q}_m(t))y + \varepsilon^{2^{m+1}} \widetilde{g}_m(t) + \widetilde{h}_m(y, t), \tag{27}$$

where

$$A_{m+1} = A_m + \varepsilon^{2^m} \overline{\widehat{Q}}_m, \widehat{Q}_m - \overline{\widehat{Q}}_m = \widetilde{Q}_m, \widehat{g}_m = \widetilde{g}_m, \widehat{h}_m = \widetilde{h}_m.$$

Denote the eigenvalues of A_{m+1} by $\lambda_1^{m+1}, \lambda_2^{m+1}, \cdots, \lambda_{2n}^{m+1}$.

In making the change of variables $y = e^{\varepsilon^{2^m} P_m(t)} x_{m+1}$, where $P_m(t)$ is to be determined later, by the symplectic transformation, Hamiltonian system (27) becomes the new system

$$\begin{aligned} \dot{x}_{m+1} \quad = \quad & (e^{-\varepsilon^{2^m} P_m(t)} (A_{m+1} + \varepsilon^{2^m} \widetilde{Q}_m - \varepsilon^{2^m} \dot{P}_m) e^{\varepsilon^{2^m} P_m(t)} \\ & + e^{-\varepsilon^{2^m} P_m(t)} (\varepsilon^{2^m} \dot{P}_m e^{\varepsilon^{2^m} P_m(t)} - \frac{d}{dt}(e^{\varepsilon^{2^m} P_m(t)}))) x_{m+1} \\ & + e^{-\varepsilon^{2^m} P_m(t)} \varepsilon^{2^{m+1}} \widetilde{g}_m(t) \\ & + e^{-\varepsilon^{2^m} P_m(t)} \widetilde{h}_m(e^{\varepsilon^{2^m} P_m(t)} x_{m+1}, t), \end{aligned} \tag{28}$$

where $x_{m+1} \in B_{a_{m+1}}(0)$.

Expand $e^{\varepsilon^{2^m} P_m}$ and $e^{-\varepsilon^{2^m} P_m}$ into

$$e^{\varepsilon^{2^m} P_m} = I + \varepsilon^{2^m} P_m + B_m, \;\; e^{-\varepsilon^{2^m} P_m} = I - \varepsilon^{2^m} P_m + \widetilde{B}_m,$$

where

$$B_m = \frac{(\varepsilon^{2^m} P_m)^2}{2!} + \frac{(\varepsilon^{2^m} P_m)^3}{3!} + \cdots, \;\; \widetilde{B}_m = \frac{(\varepsilon^{2^m} P_m)^2}{2!} - \frac{(\varepsilon^{2^m} P_m)^3}{3!} + \cdots.$$

Then, system (28) can be rewritten as follows:

$$\begin{aligned} \dot{x}_{m+1} \quad = \quad & ((I - \varepsilon^{2^m} P_m + \widetilde{B}_m)(A_{m+1} + \varepsilon^{2^m} \widetilde{Q} - \varepsilon^{2^m} \dot{P}_m)(I + \varepsilon^{2^m} P_m + B_m) \\ & + e^{-\varepsilon^{2^m} P_m(t)} (\varepsilon^{2^m} \dot{P}_m e^{\varepsilon^{2^m} P_m(t)} - \frac{d}{dt}(e^{\varepsilon^{2^m} P_m(t)}))) x_{m+1} \\ & + e^{-\varepsilon^{2^m} P_m(t)} \varepsilon^{2^{m+1}} \widetilde{g}_m(t) + e^{-\varepsilon^{2^m} P_m(t)} \widetilde{h}_m(e^{\varepsilon^{2^m} P_m(t)} x_{m+1}, t) \\ = \quad & (A_{m+1} + \varepsilon^{2^m} \widetilde{Q}_m - \varepsilon^{2^m} \dot{P}_m + \varepsilon^{2^m} A_{m+1} P_m - \varepsilon^{2^m} P_m A_{m+1} + Q_m^{(1)}) x_{m+1} \\ & + e^{-\varepsilon^{2^m} P_m(t)} \varepsilon^{2^{m+1}} \widetilde{g}_m(t) + e^{-\varepsilon^{2^m} P_m(t)} \widetilde{h}_m(e^{\varepsilon^{2^m} P_m(t)} x_{m+1}, t), \end{aligned} \tag{29}$$

where

$$
\begin{aligned}
Q_m^{(1)} \quad = \quad & -\varepsilon^{2^{m+1}} P_m(\widetilde{Q}_m - \dot{P}_m) + \varepsilon^{2^{m+1}}(\widetilde{Q}_m - \dot{P}_m)P_m \\
& -\varepsilon^{2^{m+1}} P_m(A_{m+1} + \varepsilon^{2^m}\widetilde{Q}_m - \varepsilon\dot{P}_m)P_m \\
& -\varepsilon^{2^m} P_m(A_{m+1} + \varepsilon^{2^m}\widetilde{Q}_m - \varepsilon^{2^m}\dot{P}_m)B_m \\
& +(A_{m+1} + \varepsilon^{2^m}\widetilde{Q}_m - \varepsilon^{2^m}\dot{P}_m)B_m \\
& +\widetilde{B}_m(A_{m+1} + \varepsilon^{2^m}\widetilde{Q}_m - \varepsilon^{2^m}\dot{P}_m)e^{\varepsilon^{2^m}P_m} \\
& +e^{-\varepsilon^{2^m}P_m}\left(\varepsilon^{2^m}\dot{P}_m e^{\varepsilon^{2^m}P_m} - \frac{d}{dt}e^{\varepsilon^{2^m}P_m}\right).
\end{aligned}
$$

We would like to have

$$
\widetilde{Q}_m - \dot{P}_m + A_{m+1}P_m - P_m A_{m+1} = 0,
$$

which is equivalent to

$$
\dot{P}_m = A_{m+1}P_m - P_m A_{m+1} + \widehat{Q}_m - \overline{\widehat{Q}}_m. \tag{30}
$$

By Lemma 4, if

$$
|\lambda_i^{m+1}| \geq \delta\varepsilon,\ \ |\lambda_i^{m+1} - \lambda_j^{m+1}| \geq \delta\varepsilon,\ \ i \neq j,\ 1 \leq i,j \leq 2n
$$

and

$$
|\langle k,\omega\rangle\sqrt{-1} - \lambda_i^{m+1} + \lambda_j^{m+1}| \geq \frac{\alpha_{m+1}}{\Delta^4(|k|)\Delta^4([k])}
$$

for $k \in \mathbb{Z}^{\mathbb{N}} \setminus \{0\}$, then Equation (30) has a unique almost periodic Hamiltonian solution $P_m(t)$. Furthermore,

$$
|||P_m|||_{z_m - 2\overline{z}_{m+1}, \rho_m - 2\overline{\rho}_{m+1}} \leq c\frac{\Gamma(\overline{z}_{m+1})\Gamma(\overline{\rho}_{m+1})}{\alpha_{m+1}}|||\widehat{Q}_m|||_{z_m - \overline{z}_{m+1}, \rho_m - \overline{\rho}_{m+1}}. \tag{31}
$$

System (29) becomes

$$
\dot{x}_{m+1} = (A_{m+1} + \varepsilon^{2^{m+1}}Q_{m+1}(t))x_{m+1} + \varepsilon^{2^{m+1}}g_{m+1}(t) + h_{m+1}(x_{m+1}, t), \tag{32}
$$

where

$$
\varepsilon^{2^{m+1}}Q_1 = Q_m^{(1)},\ g_{m+1}(t) = e^{-\varepsilon^{2^m}P_m}\widetilde{g}_m,
$$

$$
h_{m+1}(x_{m+1}, t) = e^{-\varepsilon^{2^m}P_m}\widetilde{h}_m(e^{\varepsilon^{2^m}P_m}x_{m+1}, t).
$$

Since $\widetilde{Q}_m - \dot{P}_m = P_m A_{m+1} - A_{m+1}P_m$,

$$
\begin{aligned}
\varepsilon^{2^{m+1}}Q_{m+1} \quad = \quad & Q_m^{(1)} = -\varepsilon^{2^{m+1}}P_m(P_m A_{m+1} - A_{m+1}P_m) \\
& +\varepsilon^{2^{m+1}}(P_m A_{m+1} - A_{m+1}P_m)P_m \\
& -\varepsilon^{2^{m+1}}P_m(A_{m+1} + \varepsilon^{2^m}P_m A_{m+1} - \varepsilon^{2^m}A_{m+1}P_m)P_m \\
& -\varepsilon^{2^m}P_m(A_{m+1} + \varepsilon^{2^m}P_m A_{m+1} - \varepsilon^{2^m}A_{m+1}P_m)B_m \\
& +(A_{m+1} + \varepsilon^{2^m}P_m A_{m+1} - \varepsilon^{2^m}A_{m+1}P_m)B_m \\
& +\widetilde{B}_m(A_{m+1} + \varepsilon^{2^m}P_m A_{m+1} - \varepsilon^{2^m}A_{m+1}P_m)e^{\varepsilon^{2^m}P_m} \\
& +e^{-\varepsilon^{2^m}P_m}\left(\varepsilon^{2^m}\dot{P}_m e^{\varepsilon^{2^m}P_m} - \frac{d}{dt}e^{\varepsilon^{2^m}P_m}\right).
\end{aligned} \tag{33}
$$

Hence, the symplectic changes of variables are

$$T_m x_{m+1} = \underline{x}_m + e^{\varepsilon^{2^m} P_m} x_{m+1} = \phi_m(t) + \psi_m(t) x_{m+1}.$$

If $|||P_m|||_{z_m - 2\bar{z}_{m+1}, \rho_m - 2\bar{\rho}_{m+1}} \leq \frac{1}{2}$, by (26) and (31), we have

$$|||\phi_m|||_{z_m - 2\bar{z}_{m+1}, \rho_m - 2\bar{\rho}_{m+1}} \leq \varepsilon^{2^m} \frac{c\Gamma(\bar{\rho}_{m+1})\Gamma(\bar{z}_{m+1})}{\alpha_m} |||g_m|||_{z_m, \rho_m}.$$

$$|||\psi_m - I|||_{z_m - 2\bar{z}_{m+1}, \rho_m - 2\bar{\rho}_{m+1}} \leq \varepsilon^{2^m} \frac{c\Gamma(\bar{z}_{m+1})\Gamma(\bar{\rho}_{m+1})}{\alpha_{m+1}} |||\widehat{Q}_m|||_{z_m - \bar{z}_{m+1}, \rho_m - \bar{\rho}_{m+1}}.$$

Under the symplectic transformation $x_m = T_m x_{m+1}$, system (24) becomes system (32).

3.3. Iteration

In this section, we prove the convergence of the iteration as $m \to \infty$.

From the arbitrariness of $\bar{z}$ and $\bar{\rho}$, we set z_m, ρ_m as follows:

$$z_m = z - \sum_{\nu=1}^{m} 2\bar{z}_\nu, \ \rho_m = \rho - \sum_{\nu=1}^{m} 2\bar{\rho}_\nu,$$

where $\bar{z}_\nu \downarrow 0$ and $\bar{\rho}_\nu \downarrow 0$ satisfy

$$\sum_{\nu=1}^{\infty} \bar{z}_\nu = \frac{1}{4} z, \ \sum_{\nu=1}^{\infty} \bar{\rho}_\nu = \frac{1}{4} \rho.$$

Moreover, we choose

$$\alpha_m = \frac{\alpha_0}{(m+1)^2}, \ \ a_{m+1} = \frac{a_m - |||\underline{x}_m|||_{z_{m+1}, \rho_{m+1}}}{e^{\varepsilon^{2^m}} |||P_m|||_{z_{m+1}, \rho_{m+1}}}.$$

If $|||\varepsilon^{2^m} P_m|||_{z_{m+1}, \rho_{m+1}} \leq \frac{1}{2}$, we have

$$a_{m+1} \geq \frac{a_m - |||\underline{x}_m|||_{z_{m+1}, \rho_{m+1}}}{1 + 2\varepsilon^{2^m} |||P_m|||_{z_{m+1}, \rho_{m+1}}}. \tag{34}$$

If ε is small enough, from [8], it follows that

$$a_\infty = \lim_{m \to \infty} a_m \geq \sigma > 0.$$

By Lemma 5, we have

$$|||\widehat{Q}_m|||_{z_m - \bar{z}_{m+1}, \rho_m - \bar{\rho}_{m+1}} \leq |||Q_m|||_{z_m, \rho_m} + K_m \frac{c\Gamma(\bar{\rho}_{m+1})\Gamma(\bar{z}_{m+1})}{\alpha_m} |||g_m|||_{z_m, \rho_m}. \tag{35}$$

Thus, by (31) and (35), we have

$$\begin{aligned} & |||P_m|||_{z_m - 2\bar{z}_{m+1}, \rho_m - 2\bar{\rho}_{m+1}} \\ \leq \ & \frac{c\Gamma(\bar{\rho}_{m+1})\Gamma(\bar{z}_{m+1})}{\alpha_{m+1}} \left(|||Q_m|||_{z_m, \rho_m} + K_m \frac{c\Gamma(\bar{\rho}_{m+1})\Gamma(\bar{z}_{m+1})}{\alpha_m} |||g_m|||_{z_m, \rho_m} \right). \end{aligned}$$

From K_m being convergent (see below), it follows that there exists $c_0 > 1$ such that $K_m \le c_0$. Thus, we have

$$\begin{aligned} & |||P_m|||_{z_m-2\bar{z}_{m+1},\rho_m-2\bar{\rho}_{m+1}} \\ \le\ & c_0 c\left(\frac{\Gamma(\bar{\rho}_{m+1})\Gamma(\bar{z}_{m+1})}{\alpha_{m+1}}\right)^2(|||Q_m|||_{z_m,\rho_m}+|||g_m|||_{z_m,\rho_m}) \\ \le\ & c\left(\frac{\Gamma(\bar{\rho}_{m+1})\Gamma(\bar{z}_{m+1})}{\alpha_{m+1}}\right)^2(|||Q_m|||_{z_m,\rho_m}+|||g_m|||_{z_m,\rho_m}). \end{aligned} \tag{36}$$

We first estimate $|||g_{m+1}|||_{z_{m+1},\rho_{m+1}}$. By Lemma 5, we have

$$\begin{aligned} & |||g_{m+1}|||_{z_{m+1},\rho_{m+1}} \\ \le\ & \frac{c\Gamma(\bar{\rho}_{m+1})\Gamma(\bar{z}_{m+1})}{\alpha_m}|||Q_m|||_{z_m,\rho_m}|||g_m|||_{z_m,\rho_m} \\ & +c\left(\frac{\Gamma(\bar{\rho}_{m+1})\Gamma(\bar{z}_{m+1})}{\alpha_m}\right)^2|||g_m|||^2_{z_m,\rho_m} \\ \le\ & c\left(\frac{\Gamma(\bar{\rho}_{m+1})\Gamma(\bar{z}_{m+1})}{\alpha_m}\right)^2\Big(|||Q_m|||_{z_m,\rho_m}|||g_m|||_{z_m,\rho_m}+|||g_m|||^2_{z_m,\rho_m}\Big). \end{aligned} \tag{37}$$

Now, we estimate $|||Q_{m+1}|||_{z_{m+1},\rho_{m+1}}$. If $|||\varepsilon^{2^m}P_m|||_{z_{m+1},\rho_{m+1}} \le \frac{1}{2}$, it follows that

$$\begin{aligned} & |||e^{\pm\varepsilon^{2^m}P_m}|||_{z_m-2\bar{z}_{m+1},\rho_m-2\bar{\rho}_{m+1}} \\ \le\ & 1+|||\varepsilon^{2^m}P_m|||_{z_m-2\bar{z}_{m+1},\rho_m-2\bar{\rho}_{m+1}}+\frac{|||\varepsilon^{2^m}P_m|||^2_{z_m-2\bar{z}_{m+1},\rho_m-2\bar{\rho}_{m+1}}}{2!}+\cdots \\ \le\ & 2. \end{aligned}$$

Moreover, if $|||\varepsilon^{2^m}P_m|||_{z_m-\bar{z}_{m+1},\rho_m-\bar{\rho}_{m+1}} \le \frac{1}{2}$, by

$$\begin{aligned} & |||\frac{d}{dt}(\varepsilon^{2^m}P_m)^n|||_{z_m-2\bar{z}_{m+1},\rho_m-2\bar{\rho}_{m+1}} \\ \le\ & n|||\varepsilon^{2^m}\dot{P}_m|||_{z_m-2\bar{z}_{m+1},\rho_m-2\bar{\rho}_{m+1}}(|||\varepsilon^{2^m}P_m|||_{z_m-2\bar{z}_{m+1},\rho_m-2\bar{\rho}_{m+1}})^{n-1} \end{aligned}$$

for $n \in \mathbb{Z}^+$, we have

$$\begin{aligned} & |||\varepsilon^{2^m}\dot{P}_m e^{\varepsilon^{2^m}P_m}-\frac{d}{dt}e^{\varepsilon^{2^m}P_m}|||_{z_m-2\bar{z}_{m+1},\rho_m-2\bar{\rho}_{m+1}} \\ \le\ & 4|||\varepsilon^{2^m}\dot{P}_m|||_{z_m-2\bar{z}_{m+1},\rho_m-2\bar{\rho}_{m+1}}|||\varepsilon^{2^m}P_m|||_{z_m-2\bar{z}_{m+1},\rho_m-2\bar{\rho}_{m+1}}. \end{aligned}$$

So,

$$\begin{aligned} & |||e^{-\varepsilon^{2^m}P_m}(\varepsilon^{2^m}\dot{P}_m e^{\varepsilon^{2^m}P_m}-\frac{d}{dt}e^{\varepsilon^{2^m}P_m})|||_{z_{m+1},\rho_{m+1}} \\ \le\ & c\varepsilon^{2^{m+1}}(|||P_m|||^2_{z_{m+1},\rho_{m+1}}+|||P_m|||_{z_{m+1},\rho_{m+1}}|||Q_m|||_{z_m,\rho_m} \\ & +||P_m|||_{z_{m+1},\rho_{m+1}}K_m\frac{c\Gamma(\bar{\rho}_{m+1})\Gamma(\bar{z}_{m+1})}{\alpha_m}|||g_m|||_{z_m,\rho_m}). \end{aligned} \tag{38}$$

From the representations of B_m and $\widetilde{B}_m$, we have

$$\begin{aligned} & |||B_m|||_{z_m-2\bar{z}_{m+1},\rho_m-2\bar{\rho}_{m+1}},\ |||\widetilde{B}_m|||_{z_m-2\bar{z}_{m+1},\rho_m-2\bar{\rho}_{m+1}} \\ \leq\ & 2|||\varepsilon^{2^m}P_m|||_{z_m-2\bar{z}_{m+1},\rho_m-2\bar{\rho}_{m+1}}. \end{aligned} \tag{39}$$

Then from (33) and (38), it follows that

$$\begin{aligned} & |||Q_{m+1}|||_{z_m-2\bar{z}_{m+1},\rho_m-2\bar{\rho}_{m+1}} \\ \leq\ & \frac{c\Gamma(\bar{\rho}_{m+1})\Gamma(\bar{z}_{m+1})}{\alpha_m}(|||P_m|||^2_{z_{m+1},\rho_{m+1}} \\ & +|||P_m|||_{z_{m+1},\rho_{m+1}}|||Q_m|||_{z_m,\rho_m}+|||P_m|||_{z_{m+1},\rho_{m+1}}|||g_m|||_{z_m,\rho_m}). \end{aligned}$$

Then by (36), we have

$$\begin{aligned} & |||Q_{m+1}|||_{z_{m+1},\rho_{m+1}} \\ \leq\ & c\left(\frac{\Gamma(\bar{\rho}_{m+1})\Gamma(\bar{z}_{m+1})}{\alpha_{m+1}}\right)^5(|||Q_m|||^2_{z_m,\rho_m}+|||g_m|||^2_{z_m,\rho_m} \\ & +|||Q_m|||_{z_m,\rho_m}|||g_m|||_{z_m,\rho_m}). \end{aligned}$$

Set

$$C_m=\left[(m+2)^{2^{-m+2}}(m+1)^{2^{-m+1}}m^{2^{-m}}\cdots 2^{-2}\right]^5$$

and

$$\Phi_m(z)=\Pi_{\nu=1}^{m+1}[\Gamma(\bar{z}_\nu)]^{5-\nu},\quad \Phi_m(\rho)=\Pi_{\nu=1}^{m+1}[\Gamma(\bar{\rho}_\nu)]^{5-\nu}.$$

From [18], $C_m, \Phi_m(z)$, and $\Phi_m(\rho)$ are all convergent as $m\to\infty$.

Let

$$C_1=\max\{\frac{c}{\alpha_0^5},1\},\ \ \gamma=\max\{|||Q|||_{z_0,\rho_0},|||g|||_{z_0,\rho_0}\}$$

and

$$M=\max\{1,\sup_m(C_1C_m\Phi_m(z)\Phi_m(\rho)\}\gamma.$$

Then, we have

$$|||Q_{m+1}|||_{z_{m+1},\rho_{m+1}}<M^{2^{m+2}},\ |||g_{m+1}|||_{z_{m+1},\rho_{m+1}}<M^{2^{m+2}}.$$

If $0<\varepsilon M^2<1$, then

$$\lim_{m\to\infty}\varepsilon^{2^m}Q_m=0,\ \lim_{m\to\infty}\varepsilon^{2^m}g_m=0.$$

Moreover, by (35), we obtain

$$\begin{aligned} ||A_{m+1}-A_m|| &\leq \varepsilon^{2^m}||\widehat{\bar{Q}}_m|| \\ &\leq \varepsilon^{2^m}(|||Q_m|||_{z_m,\rho_m}+\frac{cK_m\Gamma(\bar{\rho}_{m+1})\Gamma(\bar{z}_{m+1})}{\alpha_m}|||Q_m|||_{z_m,\rho_m}) \\ &\leq (\varepsilon M^2)^{2^m}. \end{aligned}$$

Thus, $\lim_{m\to\infty}||A_{m+1}-A_m||=0$. That is, A_m is convergent when $m\to\infty$. Let

$$A_m\to B\ (m\to\infty).$$

Furthermore, if $(\varepsilon M^2)^{2^m} < \frac{\delta\varepsilon}{(6n-1)\beta_m^2}$, we have

$$||A_{m+1} - A_m|| \le \frac{\delta\varepsilon}{(6n-1)\beta_m^2}, \text{ for any } m \ge 1,$$

where $\beta_m = \max\{||S_m||, ||S_m^{-1}||\}$, and S_m is the nonsingular matrix in Lemma 2 satisfing

$$S_m^{-1} A_m S_m = diag(\lambda_1^m, \cdots, \lambda_{2n}^m).$$

Therefore, from Lemma 2, it follows that the eigenvalues $\lambda_1^{m+1}, \cdots, \lambda_{2n}^{m+1}$ of A_{m+1} are different. Moreover,

$$|\lambda_i^{m+1} - \lambda_j^{m+1}| \ge \varepsilon\delta, \ 1 \le i, j \le 2n, \ i \ne j,$$

and

$$|\lambda_j^{m+1}| \ge \varepsilon\delta, \ j = 1, \cdots, 2n.$$

Next we present the proof of the above inequalities:

$$\begin{aligned} \left|\lambda_i^{m+1} - \lambda_j^{m+1}\right| &\ge \left|\lambda_i^1 - \lambda_j^1\right| - \sum_{l=1}^{m} \left(|\lambda_i^{l+1} - \lambda_i^l| + |\lambda_j^{l+1} - \lambda_j^l|\right) \\ &\ge \left|\lambda_i^1 - \lambda_j^1\right| - 2\sum_{l=1}^{m} ||A_{l+1} - A_l|| \\ &\ge \left|\lambda_i^1 - \lambda_j^1\right| - 2\sum_{l=1}^{m} (\varepsilon M^2)^{2^l} \\ &\ge 2\delta\varepsilon - 2\sum_{l=1}^{m} (\varepsilon M^2)^{2^l} \\ &\ge 2\delta\varepsilon - 4(\varepsilon M^2)^2. \end{aligned}$$

Thus, if $\varepsilon < \frac{\delta}{4M^4}$, we have

$$|\lambda_i^{m+1} - \lambda_j^{m+1}| \ge \varepsilon\delta, \ 1 \le i, j \le 2n, \ i \ne j.$$

Similarly, we obtain

$$|\lambda_i^{m+1}| \ge \varepsilon\delta, \ i = 1, \cdots, 2n.$$

Then,

$$\lim_{m\to\infty} |||\phi_m|||_{z_{m+1},\rho_{m+1}} = 0, \ \lim_{m\to\infty} |||\psi_m - I|||_{z_{m+1},\rho_{m+1}} = 0.$$

Let $T^m = T_0 \circ T_1 \circ \cdots \circ T_{m-1}$. Thus, T^m is convergent on $D_{\frac{z_0}{2},\frac{\rho_0}{2}}$. Assume that $T^m \to T$ as $m \to \infty$. By (32), we have

$$|||D_{x_{m+1}x_{m+1}} h_{m+1}|||_{z_{m+1},\rho_{m+1}} \le \frac{(1 + 2|||\varepsilon^{2^m}|||_{z_{m+1},\rho_{m+1}})^2}{1 - 2|||\varepsilon^{2^m} P_m|||_{z_{m+1},\rho_{m+1}}} K_m. \tag{40}$$

If $|||\varepsilon^{2^m} P_m|||_{z_{m+1},\rho_{m+1}} \le \frac{1}{4}$, by (40), we have $K_m \le (\frac{9}{2})^m K_0$. Since, $\frac{1}{1-x} \le 1 + 2x$, if $0 \le x \le \frac{1}{2}$,

$$K_{m+1} \le (1 + 4\varepsilon^{2^m} |||P_m|||_{z_{m+1},\rho_{m+1}})^3 K_m. \tag{41}$$

Since $K_m \le (\frac{9}{2})^m K_0$, then by (41), we obtain the convergence of K_m. Let $K_m \to K_\infty$ $(m \to \infty)$. Hence,

$$\lim_{m\to\infty} h_m(x_m, t) = h_\infty(y, t) = O(y^2),$$

as $y \to 0$.

Thus, under the transformation

$$x = Ty = \psi(t)y + \varphi(t),$$

Hamiltonian system (1) is changed into system (8).

3.4. Measure Estimate

Firstly, we prove that the following non-resonant conditions

$$|\langle k,\omega\rangle\sqrt{-1} - \lambda_i^m| \geq \frac{\alpha_m}{\Delta^4(|k|)\Delta^4([k])} \quad (42)$$

and

$$|\langle k,\omega\rangle\sqrt{-1} - \lambda_i^{m+1} + \lambda_j^{m+1}| \geq \frac{\alpha_{m+1}}{\Delta^4(|k|)\Delta^4([k])} \quad (43)$$

hold for most small $\varepsilon \in (0,\varepsilon_0)$, where $k \in \mathbb{Z}^{\mathbb{N}} \setminus \{0\}$.

By Theorem B in [13], there exist ε_0 and non-empty set $E^* \subset (0,\varepsilon_0)$ such that for $\varepsilon \in E^*$, we have

$$|\langle k,\omega\rangle\sqrt{-1} - \lambda_i^{m+1} + \lambda_j^{m+1}| \geq \frac{\alpha_m}{2\Delta^4(|k|)\Delta^4([k])},$$

$$\lim_{\varepsilon_0\to 0}\frac{meas(E^*)}{\varepsilon_0} = 1.$$

Obviously, (43) holds.

In the same way as above, we can obtain

$$|\langle k,\omega\rangle\sqrt{-1} - \lambda_i^m| \geq \frac{\alpha_m}{\Delta^4(|k|)\Delta^4([k])},$$

$$\lim_{\varepsilon_0\to 0}\frac{meas(E^*)}{\varepsilon_0} = 1.$$

That is, (42) holds.

Thus, there exists a non-empty Cantor subset $E^* \subset (0,\varepsilon_0)$ such that for $\varepsilon \in E^*$, there exists an almost periodic symplectic transformation

$$x = \psi(t,\varepsilon)y + \varphi(t,\varepsilon)$$

that changes (1) into

$$\dot{y} = By + h_\infty(y,t).$$

where E^* has a positive Lebesgue measure, and $\psi(t)$ and $\varphi(t)$ have the same basic frequencies and spatial structure as $Q(t)$. The matrix B is a real constant, and $h_\infty(y,t) = O(y^2)$ as $y \to 0$. Moreover, $meas((0,\varepsilon_0) \setminus E^*) = o(\varepsilon_0)$ as $\varepsilon_0 \to 0$. Therefore, we have completed the proof of Theorem 1.

4. Application

Now, we apply Theorem 1 to the almost periodic Hill's equation

$$\ddot{x} + \varepsilon a(t)x = 0, \quad (44)$$

where $a(t)$ is an analytic almost periodic function on D_ρ with frequencies $\omega = (\omega_1,\omega_2,\cdots)$ and has spatial structure $(\tau,[\cdot])$. Denote the average of $a(t)$ by $\bar{a}$.

Let $\dot{x} = y$, Equation (44) equivalently becomes of the form

$$\dot{x} = y, \ \ \dot{y} = -\varepsilon a(t)x. \quad (45)$$

In order to apply Theorem 1, we rewrite (45) as follows:

$$\dot{Z} = (A + \varepsilon Q(t))Z, \tag{46}$$

where

$$Z = \begin{pmatrix} x \\ y \end{pmatrix}, \ A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ Q = \begin{pmatrix} 0 & 0 \\ -a(t) & 0 \end{pmatrix}$$

and $g(t) \equiv 0$, and $h(x,t) \equiv 0$. It is easy to see that A has multiple eigenvalues $\lambda_1 = \lambda_2 = 0$ and that $A + \varepsilon\overline{Q}$ has two different eigenvalues $\mu_1 = i\sqrt{\bar{a}\varepsilon}$ and $\mu_2 = -i\sqrt{\bar{a}\varepsilon}$, where $i = \sqrt{-1}$. Obviously,

$$|\mu_i| = \sqrt{\bar{a}}\sqrt{\varepsilon} \geq 2\delta\varepsilon, \ i = 1, 2,$$

and

$$|\mu_1 - \mu_2| = 2\sqrt{\bar{a}}\sqrt{\varepsilon} \geq 2\delta\varepsilon$$

hold, where $\delta = \frac{1}{2}\sqrt{\bar{a}}$. Thus, by Theorem 1, we have the result as follows.

Theorem 2. *Assume that $a(t) = \sum_{\Lambda\in\tau} a_\Lambda(t)$ is analytic almost periodic on D_ρ with frequencies $\omega = (\omega_1, \omega_2, \cdots)$ and has spatial structure $(\tau, [\cdot])$. If $\bar{a} > 0$ and*

$$|\langle k, \omega\rangle\sqrt{-1}| \geq \frac{\alpha_0}{\Delta^4(|k|)\Delta^4([k])} \tag{47}$$

holds for all $k \in \mathbb{Z}^{\mathbb{N}} \setminus \{0\}$, where the constant $\alpha_0 > 0$, and Δ is an approximation function.

Then, there exist small enough $\varepsilon_0 > 0$ and the non-empty Cantor subset $E_{\varepsilon_0} \subset (0, \varepsilon_0)$ such that for $\varepsilon \in E_{\varepsilon_0}$, there exists an almost periodic symplectic transformation that changes (46) into a constant coefficient linear system. In addition, $\frac{meas(E_{\varepsilon_0})}{\varepsilon_0} \to 1$ as $\varepsilon_0 \to 0$.

From Theorem 2, we see that, for most small $\varepsilon > 0$, Equation (44) is changed into a constant coefficient system. Hence, similar to Xue [12], by an analytic almost periodic transformation, Equation (44) is transformed into

$$\ddot{x}_\infty + bx_\infty = 0, \tag{48}$$

where $b = \bar{a}\varepsilon + O(\varepsilon^2)$, which depends on $\bar{a}$ and ε only. Obviously, Equation (48) is elliptic, so the equilibrium is Lyapunov stable for most small enough ε.

5. Conclusions

In this paper, we considered the reducibility of almost-periodic nonlinear Hamiltonian systems and proved that, for most small enough ε, system (1) was reduced to a Hamiltonian system with an equilibrium. The result was proved by using some non-resonant conditions, non-degeneracy conditions and the KAM iterations. Application to the almost periodic Hill's equation was also presented.

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