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pq-Simpson's Type Inequalities Involving Generalized Convexity and Raina's Function

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Abstract: This study uses Raina's function to obtain a new coordinated pq-integral identity. Using this identity, we construct several new pq-Simpson's type inequalities for generalized convex functions on coordinates. Setting $p_1 = p_2 = 1$ in these inequalities yields well-known quantum Simpson's type inequalities for coordinated generalized convex functions. Our results have important implications for the creation of post quantum mathematical frameworks.

Keywords: quantum; post quantum; Simpson; inequalities

MSC: 05C38; 15A15; 05A15; 15A18

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1. Introduction

Simpson's rule, devised by Thomas Simpson (1710–1761), revolutionized numerical integration and definite integral estimates. This method employs a three-point Newton–Cotes quadrature rule, termed a Newton-type result, and has been a cornerstone of numerical analysis for over two centuries. Interestingly, a similar approximation technique was employed almost a century prior by German mathematician Johannes Kepler (1571–1630) and is occasionally referred to as Kepler's rule. Even today, Simpson's rule continues to hold its position as a fundamental tool in the field of mathematical analysis owing to its significant impact.

(a) Simpson's one-third rule, a variant of Simpson's quadrature formula, is a numerical integration technique that approximates definite integrals using quadratic polynomials. This method divides the interval into subintervals and employs a weighted average of function values for a more accurate estimation of the integral:

$$\int_{\pi_1}^{\pi_2} \mathcal{F}(\acute{y})\, d\acute{y} \approx \frac{\mathcal{F}(\pi_2) + 4\mathcal{F}\left(\frac{\pi_1+\pi_2}{2}\right) + \mathcal{F}(\pi_1)}{6(\pi_2-\pi_1)^{-1}}.$$

(b) Simpson's three-eighths rule, using quadratic polynomials, is a numerical integration approach that improves accuracy. This technique is particularly useful for approximating definite integrals over an interval, providing more precise results compared to simpler methods:

$$\int_{\pi_1}^{\pi_2} \mathcal{F}(\acute{y})\, d\acute{y} \approx \frac{\mathcal{F}(\pi_2)+3\mathcal{F}\left(\frac{2\pi_1+\pi_2}{3}\right)+3\mathcal{F}\left(\frac{\pi_1+2\pi_2}{3}\right)+\mathcal{F}(\pi_1)}{8(\pi_2-\pi_1)^{-1}}.$$

Numerous methods and techniques exist for estimating quantities associated with these quadrature rules. One such approach involves Simpson's inequality, which can be expressed as follows:

Theorem 1. *Suppose that $\mathcal{F} : [\pi_1, \pi_2] \to \mathbb{R}$ is a four-times continuous and differentiable mapping on (π_1, π_2), with $\left\|\mathcal{F}^{(4)}\right\|_\infty = \sup_{x\in(\pi_1,\pi_2)} \left|\mathcal{F}^{(4)}(x)\right| < \infty$. Then, the inequality holds:*

$$\left|\frac{1}{3}\left[2\mathcal{F}\left(\frac{\pi_1+\pi_2}{2}\right)+\frac{\mathcal{F}(\pi_1)+\mathcal{F}(\pi_2)}{2}\right]-\int_{\pi_1}^{\pi_2}\frac{\mathcal{F}(\acute{y})}{\pi_2-\pi_1}d(\acute{y})\right| \leq \frac{\left\|\mathcal{F}^{(4)}\right\|_\infty(\pi_2-\pi_1)^4}{2880}.$$

Since Euler's time, significant efforts have been made to gain expertise in mathematics to establish the relationship between physics and math. These breakthroughs mark significant strides in harnessing the power of quantum principles to revolutionize computational methods and mathematical frameworks. Dragomir et al. [1] not only developed unique Simpson's inequalities but also explored the extensive use of quadrature formulae for numerical integration. Simpson's inequalities have been a topic of concentration for many authors studying various functions.The work of Pečarić, Proschan, and Tong [2] clarifies the complex relationship between partial orderings, convex functions, and their statistical applications, and it serves as a thorough basis for the rest of this study. Specifically, certain mathematicians have directed their attention to the outcomes derived from Simpson's and Newton's types, aiming to formulate convex functions. Convexity theory is the subject of this discussion because of its usefulness and ability to address a wide range of complexities in the fields of practical and theoretical mathematics.

Alomari et al. [3] created unique Simpson's type inequalities for s-convex functions. Simultaneously, Sarikaya delved into investigating the variations within convexity-based Simpson-type inequalities in [4]. To gain further insights and undertake more critical studies in this field, Refs. [5,6] are suggested. These advancements find numerous applications across diverse mathematical subjects, spanning combinatorics, number theory, fundamental hypergeometric functions, and polynomial orthogonality. Furthermore, their effect extends across several scientific areas; not only can one find them in mechanics and relativity theory but also one can find their usefulness in quantum theory [7–13]. Quantum calculus integrates principles from quantum information theory within many interdisciplinary domains [14].

In recent years, the amount of research on this subject has significantly increased. For example, in 2013, Tariboon introduced the q_1-integral and ${}_{\pi_1}D_q$-difference operator [15]. Bermudo et al. [16] brought forth the ideas of the q_2-integral and the ${}^{\pi_2}D_q$-derivative in 2020. Moreover, Sadjang presented the notion of post quantum calculus, denoted as (p, q)-calculus, and expanded its application to quantum calculus [17]. A further study examined concepts related to fractional (p, q)-calculus by Soontharanon et al. [18]. Tunç [19] introduced a post quantum version of the ${}_{\pi_1}D_q$-difference operator and q_{π_1}-integral. Noteworthy contributions by Ali et al. and Noor et al. produced new estimates for convex and coordinated convex functions and supported HH-integral inequalities, employing ${}_{\pi_1}D_q, {}^{\pi_2}D_q$-derivatives and q_1q_2-integrals, as documented in [20–27].

In the study conducted by Nwaeze et al. [28], the researchers proposed parameterized quantum integral inequalities specifically designed for generalized quasi-convex functions. This novel approach contributes to the exploration of quantum integral inequalities in a broader context. Building upon this foundation, Khan et al. [29] extended the exploration by demonstrating the quantum HH-inequality. Their work involves leveraging the Green function, adding a valuable dimension to the understanding of quantum integral inequalities and expanding the scope of applications in quantum mathematics. Through their creative contributions, Budak et al. [30] have greatly enhanced the field. While Vivas-Cortez and colleagues have created quantum Newton-type inequalities, especially for coordinated convex functions, Budak obtained quantum Simpson inequalities for convex functions. Together with providing useful applications in many mathematical contexts, these developments expand the theoretical underpinnings of quantum mathematics. In their respective works cited as [31], H. Budak et al., M. A. Ali et al., and Vivas-Cortez et al. unveiled innovative quantum analogs of Simpson's and Newton's inequalities tailored for convex and coordinated convex functions. Due to its many uses, there has been a recent rise in interest in quantum versions of classical mathematical ideas. This tendency is reflected in the work of Rovelli on Quantum Gravity [32] and in the use of fractional calculus in fluid dynamics by Sengar et al. [33]. Additionally, the contributions of Ali et al. include quantum versions of well-known identities and inequalities that improve theory and practice in a variety of scientific and mathematical fields [34,35].

Kunt et al. [36] extended their work utilizing the ${}_{\pi_1}D_{(p,q)}$-difference operator and the $(p,q)^{\pi_1}$-integral. This expansion resulted in the presentation of HH-type inequalities along with corresponding left estimates, which were then elucidated by Latif et al. [37] as the accurate estimates for the HH-type inequalities previously. Introducing innovative post quantum identities, our study employs Raina's function through the utilization of a (p,q)-integral for (p,q)-differentiable generalized convex functions. Two novel identities employing Raina's function were developed by Cortez et al. [38] by utilizing the idea of right quantum derivatives.

It is worth mentioning here that there exists a deep link between the concepts of convexity and symmetry. Convexity has a strong relationship with the concept of symmetry. The literature extensively contains significant properties of symmetric convex sets. The benefit of this connection is that by focusing on one concept, we can apply it to the other. The concept of symmetry also helps in the study of convexity and inequalities. For details regarding symmetric convex sets, see [39,40].

For generalized φ-convex functions, Vivas-Cortez introduced new quantum estimates in [38]. Based on this work, we construct two new identities using Raina's function. In this work, Simpson's one-third rule's new error bounds are obtained by connecting with the post-quantum integration convexity condition of the functions over a rectangular domain. Through the effective integration of both recent and innovative findings from the literature, our study offers a fresh addition and significantly advances the field's understanding. This research contributes to the evolving landscape in this domain, offering fresh perspectives on previously established inequalities. Exploring integral inequalities, particularly through the lens of quantum and post quantum integrals across multiple function types, has been a key focus of recent studies. Exploring this area reveals the complex linkages and applications that these inherent inequalities have in a variety of scientific areas.

Section 2 of this scientific paper provides the fundamental notions required for q-calculus and related studies in this field. Section 3 focuses on (p,q)-calculus principles and highlights important research in this realm. In Section 4, we introduce midpoint-type inequalities tailored to work for the twice (p,q)-differentiable functions using (p,q)-integrals. Furthermore, we conduct a comprehensive comparison of our findings with the current body of literature. Finally, Section 5 provides recommendations for further exploration in this scientific domain. To the best of our knowledge, the results are a new addition to the literature and we hope that the ideas and techniques of this paper will stimulate further research in this area.

2. Quantum Derivatives and Integrals

Within this section, we delve into several q-calculus definitions. Throughout our investigation, we utilize real integers p and q in a manner that adheres to the condition $0 < q < p \leq 1$. The expression is established as follows:

$$\begin{aligned}[\acute{n}]_q &= \frac{1-q^{\acute{n}}}{1-q} \\ &= q^{\acute{n}-1} + q^{\acute{n}-2} + q^{\acute{n}-3} + \dots + q^3 + q^2 + 1,\end{aligned}$$

$$(1-\acute{s})_q^{\acute{n}} = (\acute{s}, q)_{\acute{n}} = \Pi_{k=0}^{\acute{n}-1}\left(1-q^k\acute{s}\right).$$

Definition 1 ([8])**.** *Let $\sigma = (\sigma(0), ..., \sigma(k), ...)$ be a bounded sequence of positive real numbers and $\nu, \mu > 0$. A non-empty set £ is said to be generalized convex if*

$$\pi_1 + t_1\check{R}_{\mu,\nu,\sigma}(\pi_2 - \pi_1) \in £, \quad \forall \pi_1, \pi_2 \in £, \quad t \in [0,1].$$

In this context, $\check{R}_{\mu,\nu,\sigma}(.)$ represents Raina's function, defined as follows:

$$\check{R}_{\mu,\nu,\sigma}(\check{z}) = \check{R}_{\mu,\nu}^{\sigma(0),\sigma(1),...}(\check{z}) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\mu k + \nu)} \check{z}^{k}, \tag{1}$$

where $\Re\mu > 0$, $|\check{z}| \leq R$, and $\Gamma(.)$ denotes the Gamma function. For more in-depth information, refer to [9].

Definition 2 ([8])**.** *Let $\sigma = (\sigma(0), ..., \sigma(k), ...)$ be a bounded sequence of positive real numbers and $\nu, \mu > 0$. Then a function $\mathcal{F} : £ \to R$ is said to be generalized convex if*

$$\mathcal{F}(\pi_1 + t\check{R}_{\mu,\nu,\sigma}(\pi_2 - \pi_1)) \leq (1-t)\mathcal{F}(\pi_1) + t\mathcal{F}(\pi_2), \quad \forall \pi_1, \pi_2 \in £, \pi_1 < \pi_2, t \in [0,1].$$

In [41], Jackson introduced the q-Jackson integral, delineating its application to mapping $\mathcal{F}$ from 0 to π_2 in the realm of mathematical analysis:

$$\int_0^{\pi_2} \mathcal{F}(\acute{y}) \; d_q \, \acute{y} \; = (1-q)\pi_2 \sum_{\acute{n}=0}^{\infty} q^{\acute{n}} \mathcal{F}\left(\pi_2 q^{\acute{n}}\right), \tag{2}$$

assuming total convergence of the sum. Furthermore, Jackson introduced the integral of mapping $\mathcal{F}$ over the interval $[\pi_1, \pi_2]$ as

$$\int_{\pi_2}^{\pi_2} \mathcal{F}(\acute{y}) \; d_q \, \acute{y} \; = \int_0^{\pi_2} \mathcal{F}(\acute{y}) \; d_q \, \acute{y} \; - \int_0^{\pi_1} \mathcal{F}(\acute{y}) \; d_q \, \acute{y} \; .$$

Definition 3 ([15])**.** *The q_{π_1} -derivative for the function $\mathcal{F} : [\pi_1, \pi_2] \to R$ is defined as follows:*

$${}_{\pi_1}D_q\mathcal{F}(\acute{y}) \;\; = \frac{1}{(1-q)(\acute{y}-\pi_1)}[\mathcal{F}(\acute{y}) - \mathcal{F}((1-q)\pi_1 + q\acute{y})], \; \acute{y} \neq \pi_1. \tag{3}$$

For $\acute{y} = \pi_1$, we imply that ${}_{\pi_1}D_q\mathcal{F}(\acute{y}) = \lim_{\acute{y}\to\pi_1} \; {}_{\pi_1}D_q\mathcal{F}(\acute{y})$ as long as it is finite and existing.

Definition 4 ([16])**.** *The q^{π_2} -derivative for the function $\mathcal{F} : [\pi_1, \pi_2] \to R$ is defined as follows:*

$${}^{\pi_2}D_q\mathcal{F}(\acute{y}) \;\; = \frac{1}{(1-q)(\pi_2-\acute{y})}[\mathcal{F}(q\acute{y} + (1-q)\pi_2) - \mathcal{F}(\acute{y})], \; \acute{y} \neq \pi_2. \tag{4}$$

For $x = \pi_2$, we imply that ${}^{\pi_2}D_q\mathcal{F}(\acute{y}) = \lim_{\acute{y}\to\pi_2} \; {}^{\pi_2}D_q\mathcal{F}(\acute{y})$ if it exists and is finite.

Definition 5 ([15])**.** *The left quantum integral $(q_{\pi_1} - definite\ integral)$ for a continuous function $\mathcal{F} : [\pi_1, \pi_2] \to R$, at $\acute{y} \in [\pi_1, \pi_2]$ is as follows:*

$$\int_{\pi_1}^{x} \mathcal{F}(\tau) \;\; {}_{\pi_1}d_q \, \tau \;\; = (\acute{y} - \pi_1)(1-q) \sum_{n=0}^{\infty} q^{\acute{n}} \mathcal{F}(q^n\acute{y} + (1-q^n)\pi_1), \tag{5}$$

where $\acute{y} \in [\pi_1, \pi_2]$.

Bermudo et al. [16] expressed the idea of the q-definite integral as follows:

Definition 6 ([16])**.** *The right quantum integral* $(q^{\pi_2} - definite\ integral)$ *for a continuous function* $\mathcal{F} : [\pi_1, \pi_2] \to R$*, at* $\acute{y} \in [\pi_1, \pi_2]$ *is expressed as follows:*

$$\int_{x}^{\pi_2} \mathcal{F}(\tau)\ {}^{\pi_2}d_q\,\tau \;= (\pi_2 - \acute{y})(1-q)\sum_{n=0}^{\infty} q^n \mathcal{F}(q^n \acute{y} + (1-q^n)\pi_2) \text{ where, } \acute{y} \in [\pi_1, \pi_2]. \tag{6}$$

3. Post Quantum Derivatives and Integrals

The investigation of basic representations and ideas in (p,q)-calculus is the focus of this section's discussion. The notations used are as indicated by [17]:

$$\begin{aligned}[\acute{n}]_{(p,q)} &= \frac{q^{\acute{n}} - p^{\acute{n}}}{q-p} \\ &= p^{\acute{n}} + p^{\acute{n}-1}q + \ldots + pq^{\acute{n}-1} + q^{\acute{n}}.\end{aligned}$$

The $[\acute{n}]_{(p,q)}!$, (p,q)-factorial, $\begin{bmatrix} \acute{n} \\ i \end{bmatrix}!$, (p,q)-binomial, and $(1-s)^{\acute{n}}_{(p,q)}$, (p,q)-power are defined as follows:

$$\begin{aligned}[\acute{n}]_{(p,q)}! &= \Pi_{i=1}^{\acute{n}}[i]_{(p,q)} \quad where, \;\; \acute{n} \geq 1 \;\; and \;\; [0]_{(p,q)}! = 1, \\ \begin{bmatrix} \acute{n} \\ i \end{bmatrix}! &= \frac{[\acute{n}]_{(p,q)}!}{[\acute{n}-i]_{(p,q)}![i]_{(p,q)}!}\end{aligned}$$

and

$$(1-s)^{\acute{n}}_{(p,q)} = \Pi_{k=0}^{\acute{n}-1}\left(p^k - q^k s\right).$$

Definition 7 ([17])**.** *The post quantum derivative* $(p, q - derivative)$ *for a continuous function* $\mathcal{F} : [\pi_1, \pi_2] \to R$ *is expressed as follows:*

$$D_{(p,q)}\mathcal{F}(\acute{y}) \;= \frac{\mathcal{F}(p\acute{y}) - \mathcal{F}(q\acute{y})}{(p-q)(\acute{y})},\ x \neq 0. \tag{7}$$

Definition 8 ([19])**.** *The left derivative in terms of post quantum calculus, represented as a* $(p,q)_{\pi_1}$*-derivative, for a continuous function* $\mathcal{F} : [\pi_1, \pi_2] \to R$ *is given as*

$${}_{\pi_1}D_{(p,q)}\mathcal{F}(\acute{y}) \;= \frac{\mathcal{F}(p\acute{y} + (1-p)\pi_1) - \mathcal{F}(q\acute{y} + (1-q)\pi_2)}{(p-q)(\acute{y} - \pi_1)},\ \acute{y} \neq \pi_1. \tag{8}$$

For $\acute{y} = \pi_1$*, we assume that* ${}_{\pi_1}D_{(p,q)}\mathcal{F}(\pi_1) = \lim_{\acute{y} \to \pi_1}\ {}_{\pi_1}D_{(p,q)}\mathcal{F}(\acute{y})$ *if it exists, and the value will be finite.*

Definition 9 ([42])**.** *The right derivative in terms of post quantum calculus, represented as a* $(p,q)^{\pi_2}-$*derivative, for a continuous function* $\mathcal{F} : [\pi_1, \pi_2] \to R$ *is given as*

$${}^{\pi_2}D_{(p,q)}\mathcal{F}(\acute{y}) \;= \frac{\mathcal{F}(q\acute{y} + (1-q)\pi_2) - \mathcal{F}(p\acute{y} + (1-p)\pi_1)}{(p-q)(\pi_2 - \acute{y})},\ \acute{y} \neq \pi_2. \tag{9}$$

For $x = \pi_2$*, we assume that* ${}^{\pi_2}D_{(p,q)}\mathcal{F}(\pi_2) = \lim_{\acute{y} \to \pi_2}\ {}^{\pi_2}D_{(p,q)}\mathcal{F}(\acute{y})$ *if it exists, and it will be finite.*

Remark 1. *The substitution of* $p = 1$ *in Equations* (8) *and* (9) *evidently reduces these Equations to* (3) *and* (4)*, respectively.*

Definition 10 ([17]). *The definite integral for a continuous function $\mathcal{F}$ on $[0, \pi_2]$, in terms of $(p,q)-$calculus, is given as*

$$\int_0^{\pi_2} \mathcal{F}(\tau)\ d_{(p,q)}\,\tau\ = (\pi_2)(p-q)\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}\mathcal{F}\left(\frac{q^k}{p^{k+1}}\pi_2\right). \tag{10}$$

Moreover, the definite (p,q)-integral for the continuous function $\mathcal{F}$ on $[\pi_1, \pi_2]$ is defined as

$$\int_{\pi_1}^{\pi_2} \mathcal{F}(\tau)\ d_{(p,q)}\,\tau\ = \int_0^{\pi_2} \mathcal{F}(\tau)\ d_{(p,q)}\,\tau\ - \int_0^{\pi_1} \mathcal{F}(\tau)\ d_{(p,q)}\,\tau\ .$$

Definition 11 ([19]). *The left definite integral, $(p,q)_{\pi_1}-$integral, for a continuous function $\mathcal{F} : [\pi_1, \pi_2] \to R$, in terms of $(p,q)-$calculus, is given as*

$$\int_{\pi_1}^{x} \mathcal{F}(\tau)\ {}_{\pi_1}d_{(p,q)}\,\tau\ = \frac{(x-\pi_1)}{(p-q)^{-1}}\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}\mathcal{F}\left(\frac{\acute{y}q^k}{p^{k+1}} + \left(1-\frac{q^k}{p^{k+1}}\right)\pi_1\right). \tag{11}$$

Definition 12 ([42]). *The right definite integral, $(p,q)^{\pi_2}-$integral, for a continuous function $\mathcal{F} : [\pi_1, \pi_2] \to R$, in terms of $(p,q)-$calculus, is given as*

$$\int_{x}^{\pi_2} \mathcal{F}(\tau)\ {}^{\pi_2}d_{(p,q)}\,\tau\ = \frac{(\pi_2-\acute{y})}{(p-q)^{-1}}\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}\mathcal{F}\left(\frac{\acute{y}q^k}{p^{k+1}} + \left(1-\frac{q^k}{p^{k+1}}\right)\pi_2\right). \tag{12}$$

Remark 2. *It is obvious that with the replacement of $p = 1$ in Equations (11) and (12), these equations reduce to (6) and (5), respectively.*

Remark 3. *The substitution of $\pi_1 = 0$ and $x = \pi_2 = 1$ in (11) leads to*

$$\int_0^{1} \mathcal{F}(\tau)\ {}_0d_{(p,q)}\,\tau\ = (p-q)\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}\mathcal{F}\left(\frac{q^k}{p^{k+1}}\right). \tag{13}$$

Similarly, by substitution, $x = \pi_1 = 0$ and $\pi_2 = 1$ in (12) gives the equation of the form

$$\int_0^{1} \mathcal{F}(\tau)\ {}^1d_{(p,q)}\,\tau\ = (p-q)\sum_{k=0}^{\infty}\frac{q^{\,k}}{p^{\,k+1}}\mathcal{F}\left(1-\frac{q^{\,k}}{p^{\,k+1}}\right). \tag{14}$$

Lemma 1 ([43]). *We possess the following equality:*

$$\int_{\pi_1}^{\pi_2} (\pi_2-\acute{y})^{\beta}\ {}^{\pi_2}d_{(p,q)}\,\acute{y}\ = \frac{(\pi_2-\pi_1)^{\beta+1}}{[\beta+1]_{(p,q)}}, \tag{15}$$

$$\int_{\pi_1}^{\pi_2} (\acute{y}-\pi_1)^{\beta}\ {}_{\pi_1}d_{(p,q)}\,\acute{y}\ = \frac{(\pi_2-\pi_1)^{\beta+1}}{[\beta+1]_{(p,q)}}, \tag{16}$$

where $\beta \in R - \{-1\}$.

To enhance the readability and simplicity, we introduce some notation that will be consistently used throughout the paper.

$$
\begin{array}{rcl}
\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_2+t\eta_1(\pi_1,\pi_2),\pi_4+s\eta_2(\pi_3,\pi_4))}{{}^{\pi_2}\partial_{(p_1,q_1)}\, t\ {}^{\pi_4}\partial_{(p_2,q_2)}\, s} & = & {}^{\pi_2,\pi_4}\check{D}^{\,2}_{(p_1,q_1),(p_2,q_2)}\ \mathcal{F}, \\
{}_1\check{R}^{\sigma_1}_{\mu_1,\nu_1}(\pi_1-\pi_2) & = & \acute{O}_1, \\
{}_2\check{R}^{\sigma_2}_{\mu_2,\nu_2}(\pi_3-\pi_4) & = & \acute{O}_2, \\
{}_1\check{R}^{\sigma_1}_{\mu_1,\nu_1}(\pi_2-\pi_1) & = & \grave{O}_1, \\
{}_2\check{R}^{\sigma_2}_{\mu_2,\nu_2}(\pi_4-\pi_3) & = & \grave{O}_2, \\
[\pi_2+\acute{O}_1,\pi_2]\times[\pi_4+\acute{O}_2,\pi_4] & = & \Delta.
\end{array}
$$

4. Identities

We prove two identities using the right post quantum integrals in this part, which may be used to formulate post quantum Simpson's type inequalities.

Lemma 2. *Let $\mathcal{F}:\Delta\subseteq R^2\to R^2$ be a two-times partially $q_1^{\pi_2}q_2^{\pi_4}$-differentiable function on the interval Δ with $-\check{R}^{\sigma}_{\mu,\nu}(x-y)= {}_1\check{R}^{\sigma_1}_{\mu_1,\nu_1}(y-x)>0$ and $0<q_1,q_2<1$. The ${}^{\pi_2,\pi_4}\check{D}^{\,2}_{(p_1,q_1),(p_2,q_2)}\ \mathcal{F}$ is under the conditions of being continuous and integrable on Δ; then the following identity holds for $q_1^{\pi_2}q_2^{\pi_4}$-integrals:*

$$
q_1q_2\grave{O}_1\grave{O}_2\int_0^1\int_0^1\Lambda_{(p_1,q_1)}\ t\ \Lambda_{(p_2,q_2)}\ s\ {}^{\pi_2,\pi_4}\check{D}^{\,2}_{(p_1,q_1),(p_2,q_2)}\ \mathcal{F}\ {}^{\pi_2}d_{(p_1,q_1)}t\ {}^{\pi_4}d_{(p_2,q_2)}s \tag{17}
$$

$$
\begin{aligned}
&=\frac{(\grave{O}_1\grave{O}_2)^{-1}}{[6]_{(p_1,q_1)}[6]_{(p_2,q_2)}}\Bigg[\frac{[5]_{(p_2,q_2)}-1}{(q_1)(p_1^5)^{-1}}\mathcal{F}\left(\pi_2,\pi_4+\frac{\acute{O}_2}{[2]_{(p_2,q_2)}}\right)+\Big([5]_{(p_1,q_1)}-1\Big)\Big([5]_{(p_2,q_2)}-1\Big)\\
&\quad\times\mathcal{F}\left(\frac{\pi_2[2]_{(p_1,q_1)}+\acute{O}_1}{[2]_{(p_1,q_1)}},\pi_4+\frac{\acute{O}_2}{[2]_{(p_2,q_2)}}\right)+\frac{[5]_{(p_1,q_1)}-1}{(q_2)(p_2^5)^{-1}}\mathcal{F}\left(\frac{\pi_2[2]_{(p_1,q_1)}+\acute{O}_1}{[2]_{(p_1,q_1)}},\pi_4\right)\\
&\quad+\frac{[5]_{(p_2,q_2)}-1}{(q_1)(p_1^5)^{-1}}\mathcal{F}\left(\pi_2+\acute{O}_1,\pi_4+\frac{\acute{O}_2}{[2]_{(p_2,q_2)}}\right)+\frac{p_2^5\Big([5]_{(p_1,q_1)}-1\Big)}{q_2}\\
&\quad\times\mathcal{F}\left(\frac{\pi_2[2]_{(p_1,q_1)}+\acute{O}_1}{[2]_{(p_1,q_1)}},\pi_4+\acute{O}_2\right)\Bigg]+\frac{1}{[6]_{(p_1,q_1)}[6]_{(p_2,q_2)}}\Bigg[\frac{[5]_{(p_2,q_2)}-1}{(q_1)(p_1^5)^{-1}}\mathcal{F}(\pi_2+\acute{O}_1,\pi_4)\\
&\quad+\mathcal{F}(\pi_2,\pi_4)+\frac{[5]_{(p_1,q_1)}-1}{(q_2)(p_1^5)^{-1}}\mathcal{F}(\pi_2,\pi_4+\acute{O}_2)+\frac{p_1^5p_1^5}{q_1q_2}\mathcal{F}(\pi_2+\acute{O}_1,\pi_4+\acute{O}_2)\Bigg]\\
&\quad-\frac{1}{\grave{O}_1[6]_{(p_2,q_2)}}\Bigg[\int_{\pi_2+p_1\acute{O}_1}^{\pi_2}\frac{\mathcal{F}(x,\pi_4)}{q_1}\ {}^{\pi_2}d_{(p_1,q_1)}(x)+\frac{p_2^5}{q_1q_2}\int_{\pi_2+p_1\acute{O}_1}^{\pi_2}\mathcal{F}(x,\pi_4+\acute{O}_2)\,{}^{\pi_2}d_{(p_1,q_1)}(x)\\
&\quad+\frac{\Big([5]_{(p_2,q_2)}-1\Big)}{q_1}\int_{\pi_2+p_1\acute{O}_1}^{\pi_2}\mathcal{F}\left(x,\pi_4+\frac{\acute{O}_2}{[2]_{(p_2,q_2)}}\right)\,{}^{\pi_2}d_{(p_1,q_1)}(x)\Bigg]\\
&\quad-\frac{1}{\grave{O}_2[6]_{(p_1,q_1)}}\Bigg[\frac{1}{q_2}\int_{\pi_4+p_2\acute{O}_2}^{\pi_4}\mathcal{F}(\pi_2,y)\,{}^{\pi_4}d_{(p_2,q_2)}(y)+\frac{p_1^5}{q_1q_2}\int_{\pi_4+p_2\acute{O}_2}^{\pi_4}\mathcal{F}(\pi_2+\grave{O}_1,y)\,{}^{\pi_4}d_{(p_2,q_2)}(y)\\
&\quad+\frac{\Big([5]_{(p_1,q_1)}-1\Big)}{q_2}\int_{\pi_4+p_2\acute{O}_2}^{\pi_4}\mathcal{F}\left(\pi_2+\frac{\acute{O}_1}{[2]_{(p_1,q_1)}},y\right)\,{}^{\pi_4}d_{(p_2,q_2)}(y)\Bigg]\\
&\quad+\frac{1}{\grave{O}_1\grave{O}_2}\int_{\pi_4+p_2\acute{O}_2}^{\pi_4}\int_{\pi_4+p_2\acute{O}_2}^{\pi_4}\mathcal{F}(x,y)\ {}^{\pi_2}d_{(p_1,q_1)}(x)\ {}^{\pi_4}d_{(p_2,q_2)}(y),
\end{aligned}
$$

where

$$\Lambda_{(p_1,q_1)}\, t = \begin{cases} t - \frac{1}{[6]_{(p_1,q_1)}}, & t \in \left[0, \frac{1}{[2]_{(p_1,q_1)}}\right), \\ t - \frac{[5]_{(p_1,q_1)}}{[6]_{(p_1,q_1)}}, & t \in \left[\frac{1}{[2]_{(p_1,q_1)}}, 1\right]; \end{cases}$$

$$\Lambda_{(p_2,q_2)}\, s = \begin{cases} s - \frac{1}{[6]_{(p_2,q_2)}}, & s \in \left[0, \frac{1}{[2]_{(p_2,q_2)}}\right), \\ s - \frac{[5]_{(p_2,q_2)}}{[6]_{(p_2,q_2)}}, & s \in \left[\frac{1}{[2]_{(p_2,q_2)}}, 1\right]. \end{cases}$$

Proof. By utilizing the fundamental properties of pq-integrals on the left-hand side of Equation (17) and subsequently applying the definitions of $\Lambda_{(p_1,q_1)}\, t$ and $\Lambda_{(p_2,q_2)}\, s$,

$$\begin{aligned} \hat{I} &= \int_0^1 \int_0^1 \Lambda_{(p_1,q_1)}\, t\, \Lambda_{(p_2,q_2)}\, s \;{}^{\pi_2,\pi_4}\check{D}^{\,2}_{(p_1,q_1),(p_2,q_2)}\, \mathcal{F}\; {}^{\pi_2}d_{(p_1,q_1)}\, t\; {}^{\pi_4}d_{(p_2,q_2)}\, s \\ &= \frac{\left([5]_{(p_2,q_2)}-1\right)\left([5]_{(p_1,q_1)}-1\right)}{[6]_{(p_2,q_2)}[6]_{(p_1,q_1)}} \int_0^{\frac{1}{[2]_{(p_1,q_1)}}} {}^{\pi_2,\pi_4}\check{D}^{\,2}_{(p_1,q_1),(p_2,q_2)}\, \mathcal{F}\, d_{(p_2,q_2)}\, s\, d_{(p_1,q_1)}\, t \\ &+ \frac{[5]_{(p_2,q_2)}-1}{[6]_{(p_2,q_2)}} \int_0^{\frac{1}{[2]_{(p_2,q_2)}}} \int_0^1 \left(t - \frac{[5]_{(p_1,q_1)}}{[6]_{(p_1,q_1)}}\right) {}^{\pi_2,\pi_4}\check{D}^{\,2}_{(p_1,q_1),(p_2,q_2)}\, \mathcal{F}\, d_{(p_2,q_2)}\, s\, d_{(p_1,q_1)}\, t \\ &+ \frac{[5]_{(p_1,q_1)}-1}{[6]_{(p_1,q_1)}} \int_0^{\frac{1}{[2]_{(p_1,q_1)}}} \int_0^1 \left(s - \frac{[5]_{(p_2,q_2)}}{[6]_{(p_2,q_2)}}\right) {}^{\pi_2,\pi_4}\check{D}^{\,2}_{(p_1,q_1),(p_2,q_2)}\, \mathcal{F}\, d_{(p_2,q_2)}\, s\, d_{(p_1,q_1)}\, t \\ &+ \int_0^1 \int_0^1 \left(s - \frac{[5]_{(p_2,q_2)}}{[6]_{(p_2,q_2)}}\right)\left(t - \frac{[5]_{(p_1,q_1)}}{[6]_{(p_1,q_1)}}\right) {}^{\pi_2,\pi_4}\check{D}^{\,2}_{(p_1,q_1),(p_2,q_2)}\, \mathcal{F}\, d_{(p_2,q_2)}\, s\, d_{(p_1,q_1)}\, t\,. \end{aligned}$$

Assuming

$$\hat{I} = I_1 + I_2 + I_3 + I_4. \tag{18}$$

Using Definition 9, evaluating integrals *in* I_1 *becomes*

$$\begin{aligned} I_1 &= \frac{\left([5]_{(p_2,q_2)}-1\right)\left([5]_{(p_1,q_1)}-1\right)(p_2-q_2)(p_1-q_1)}{(p_1-q_1)(p_2-q_2)[2]_{(p_1,q_1)}[2]_{(p_2,q_2)}[6]_{(p_1,q_1)}[6]_{(p_2,q_2)}\grave{O}_1\grave{O}_2} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}[2]_{(p_1,q_1)}[2]_{(p_2,q_2)} \\ &\times \Bigg[\mathcal{F}\left(\pi_2+\frac{q_1^{m+1}\acute{O}_1}{p_1^{m+1}[2]_{(p_1,q_1)}}, \pi_4+\frac{q_2^{n+1}\acute{O}_2}{p_2^{n+1}[2]_{(p_2,q_2)}}\right) - \mathcal{F}\left(\pi_2+\frac{q_1^{m}\acute{O}_1}{p_1^{m}[2]_{(p_1,q_1)}}, \pi_4+\frac{q_2^{n+1}\acute{O}_2}{p_2^{n+1}[2]_{(p_2,q_2)}}\right) \\ &- \mathcal{F}\left(\pi_2+\frac{q_1^{m+1}\acute{O}_1}{p_1^{m+1}[2]_{(p_1,q_1)}}, \pi_4+\frac{q_2^{n}\acute{O}_2}{p_2^{n}[2]_{(p_2,q_2)}}\right) + \mathcal{F}\left(\pi_2+\frac{q_1^{m}\acute{O}_1}{p_1^{m}[2]_{(p_1,q_1)}}, \pi_4+\frac{q_2^{n}\acute{O}_2}{p_2^{n}[2]_{(p_2,q_2)}}\right)\Bigg], \end{aligned}$$

$$\begin{aligned} I_1 = \frac{\left([5]_{(p_2,q_2)}-1\right)\left([5]_{(p_1,q_1)}-1\right)}{[6]_{(p_1,q_1)}[6]_{(p_2,q_2)}\grave{O}_1\grave{O}_2} &\Bigg[\mathcal{F}(\pi_2,\pi_4) - \mathcal{F}\left(\pi_2, \pi_4+\frac{\acute{O}_2}{[2]_{(p_2,q_2)}}\right) \\ &- \mathcal{F}\left(\frac{\pi_2[2]_{(p_1,q_1)}+\acute{O}_1}{[2]_{(p_1,q_1)}}, \pi_4\right) + \mathcal{F}\left(\frac{\pi_2[2]_{(p_1,q_1)}+\acute{O}_1}{[2]_{(p_1,q_1)}}, \pi_4+\frac{\acute{O}_2}{[2]_{(p_2,q_2)}}\right)\Bigg]. \end{aligned}$$

The integrals in I_2 can be expressed as

$$I_2 = \frac{[5]_{(p_2,q_2)}-1}{[6]_{(p_2,q_2)}} \int_0^{\frac{1}{[2]_{(p_2,q_2)}}} \int_0^1 \left(t - \frac{[5]_{(p_1,q_1)}}{[6]_{(p_1,q_1)}} \right) {}^{\pi_2,\pi_4}\check{D}^2_{(p_1,q_1),(p_2,q_2)} \mathcal{F}\, d_{(p_2,q_2)}\, s\, d_{(p_1,q_1)}\, t$$

$$= \frac{[5]_{(p_2,q_2)}-1}{\grave{O}_1\grave{O}_2[6]_{(p_2,q_2)}} \Bigg[\Bigg\{ \int_{\pi_2+p_1\acute{O}_1}^{\pi_2} \frac{\mathcal{F}(x,\pi_4)}{\grave{O}_1 q_1} d_{(p_1,q_1)}\, t - \frac{\mathcal{F}(\pi_2+\acute{O}_1,\pi_4)}{q_1} - \frac{1}{\grave{O}_1 q_1} \int_{\pi_2+p_1\acute{O}_1}^{\pi_2}$$

$$\times \mathcal{F}\left(x, \pi_4 + \frac{\acute{O}_2}{[2]_{(p_2,q_2)}} \right) d_{(p_1,q_1)}\, t + \frac{\mathcal{F}\left(\pi_2+\acute{O}_1, \pi_4 + \frac{\acute{O}_2}{[2]_{(p_2,q_2)}} \right)}{q_1} \Bigg\} - \frac{[5]_{(p_1,q_1)}}{[6]_{(p_1,q_1)}}$$

$$\times \left\{ \mathcal{F}(\pi_2,\pi_4) - \mathcal{F}\left(\pi_2, \pi_4 + \frac{\acute{O}_2}{[2]_{(p_2,q_2)}} \right) - \mathcal{F}(\pi_2+\acute{O}_1,\pi_4) + \mathcal{F}\left(\pi_2+\acute{O}_1, \pi_4 + \frac{\acute{O}_2}{[2]_{(p_2,q_2)}} \right) \right\} \Bigg].$$

Similarly, at the same steps, I_3 is evaluated as

$$I_3 = \frac{[5]_{(p_1,q_1)}-1}{[6]_{(p_1,q_1)}\grave{O}_1\grave{O}_2} \Bigg\{ \int_{\pi_4+p_2\acute{O}_2}^{\pi_4} \frac{\mathcal{F}(\pi_2,y,)}{\grave{O}_2 q_2} d_{(p_2,q_2)}(y) - \frac{1}{p_2}\mathcal{F}(\pi_2,\pi_4+\acute{O}_2)$$

$$-\frac{1}{\grave{O}_2 q_2} \int_{\pi_4+p_2\acute{O}_2}^{\pi_4} \mathcal{F}\left(\frac{\pi_2[2]_{(p_1,q_1)}+\acute{O}_1}{[2]_{(p_1,q_1)}}, y \right) d_{(p_2,q_2)}(y) + \frac{1}{q_2}\mathcal{F}\left(\frac{\pi_2[2]_{(p_1,q_1)}+\acute{O}_1}{[2]_{(p_1,q_1)}}, \pi_4+\acute{O}_2 \right) \Bigg\}$$

$$- \frac{\left([5]_{(p_1,q_1)}-1\right)[5]_{(p_2,q_2)}}{[6]_{(p_1,q_1)}[6]_{(p_2,q_2)}\grave{O}_1\grave{O}_1} \Bigg\{ \mathcal{F}(\pi_2,\pi_4) - \mathcal{F}\left(\frac{\pi_2[2]_{(p_1,q_1)}+\acute{O}_1}{[2]_{(p_1,q_1)}}, \pi_4 \right)$$

$$-\mathcal{F}(\pi_2+\acute{O}_1,\pi_4) + \mathcal{F}\left(\pi_2 + \frac{\acute{O}_2}{[2]_{(p_1,q_1)}}, \pi_4+\acute{O}_2 \right) \Bigg\}.$$

Solving (p,q)-integrals in I_4,

$$I_4 = \int_0^1 \int_0^1 st\, {}^{\pi_2,\pi_4}\check{D}^2_{(p_1,q_1),(p_2,q_2)} \mathcal{F}\, d_{(p_2,q_2)}\, s\, d_{(p_1,q_1)}\, t - \frac{[5]_{(p_2,q_2)}}{[6]_{(p_2,q_2)}} \int_0^1 {}^{\pi_2,\pi_4}\check{D}^2_{(p_1,q_1),(p_2,q_2)}$$

$$\times \mathcal{F}\, d_{(p_2,q_2)}\, s\, d_{(p_1,q_1)}(t) - \int_0^1 \int_0^1 \frac{[5]_{(p_1,q_1)}}{[6]_{(p_1,q_1)}} s\, {}^{\pi_2,\pi_4}\check{D}^2_{(p_1,q_1),(p_2,q_2)} \mathcal{F}\, d_{(p_2,q_2)}\, s\, d_{(p_1,q_1)}\, t$$

$$+ \int_0^1 \int_0^1 \left(\frac{[5]_{(p_1,q_1)}}{[6]_{(p_1,q_1)}} \right) \left(\frac{[5]_{(p_2,q_2)}}{[6]_{(p_2,q_2)}} \right) {}^{\pi_2,\pi_4}\check{D}^2_{(p_1,q_1),(p_2,q_2)} \mathcal{F}\, d_{(p_2,q_2)}\, s\, d_{(p_1,q_1)}\, t$$

$$= \frac{1}{\grave{O}_1\grave{O}_2} \left[\left\{ \mathcal{F}(\pi_2,\pi_4) - \mathcal{F}(\pi_2+\acute{O}_1,\pi_4) - \mathcal{F}(\pi_2,\pi_4+\acute{O}_2) + \mathcal{F}(\pi_2+\acute{O}_1,\pi_4+\acute{O}_2) \right\} \right.$$

$$- \frac{[5]_{(p_2,q_2)}}{[6]_{(p_2,q_2)}} \Bigg\{ \int_{\pi_2+p_1\acute{O}_1}^{\pi_2} \frac{\mathcal{F}(x,\pi_4)}{\grave{O}_1 q_1} d_{(p_1,q_1)}x - \frac{\mathcal{F}(\pi_2+\acute{O}_1,\pi_4)}{q_1} - \int_{\pi_2+p_1\acute{O}_1}^{\pi_2} \frac{\mathcal{F}(x,\pi_4+\acute{O}_2)}{\grave{O}_1 q_1} d_{(p_1,q_1)}x$$

$$+ \frac{\mathcal{F}(\pi_2+\acute{O}_1,\pi_4+\acute{O}_2)}{q_1} \Bigg\} - \frac{[5]_{(p_1,q_1)}}{[6]_{(p_1,q_1)}} \Bigg[\frac{\mathcal{F}(\pi_2+\acute{O}_1,\pi_4+\acute{O}_2)}{q_2} - \int_{\pi_4+p_2\acute{O}_2}^{\pi_4} \frac{\mathcal{F}(\pi_4+\acute{O}_2,y)}{\acute{O}_2 q_2} d_{(p_2,q_2)}y$$

$$+ \int_{\pi_4+p_2\acute{O}_2}^{\pi_4} \frac{\mathcal{F}(\pi_2,y)}{\grave{O}_2 q_2} d_{(p_2,q_2)}y - \frac{\mathcal{F}(\pi_2,\pi_4+\acute{O}_2)}{q_2} \Bigg] + \Bigg[\frac{\mathcal{F}(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{q_1 q_2}$$

$$
- \int_{\pi_2+p_1\acute{O}_1}^{\pi_2} \frac{\mathcal{F}\left(x, \pi_4 + s\acute{O}_2\right)}{q_1 q_2 \grave{O}_1} \, {}^{\pi_2}d_{(p_1,q_1)}x - \int_{\pi_4+p_2\acute{O}_2}^{\pi_4} \frac{\mathcal{F}\left(\pi_2 + t\acute{O}_1, y\right)}{q_1 q_2 \grave{O}_2} \, {}^{\pi_4}d_{(p_2,q_2)}y
$$
$$
+ \int_{\pi_2+p_1\acute{O}_1}^{\pi_2} \int_{\pi_4+p_2\acute{O}_2}^{\pi_4} \frac{\mathcal{F}(x,y)}{\grave{O}_2 \grave{O}_1 q_1 q_2} \, {}^{\pi_2}d_{(p_1,q_1)} \, t \, {}^{\pi_4}d_{(p_2,q_2)} \, s \Bigg].
$$

Now, using the values of $(I_1 - I_4)$ in (18) and multiplying $q_1 q_2 \grave{O}_1 \grave{O}_2$ on both sides gives the required result. $\square$

5. Main Results

Within this section, we utilize Lemma 2 as a foundation to derive novel Simpson's type inequalities for generalized convex functions. Before presenting our findings, we introduce pertinent concepts that will facilitate our proofs. Assuming $(p,q) = \mathring{u}$, we have

$$
A_1 \mathring{u} = \frac{2([3]_{\mathring{u}} - [2]_{\mathring{u}})}{[6]^3_{\mathring{u}} [2]_{\mathring{u}} [3]_{\mathring{u}}} - \frac{[6]_{\mathring{u}} - [3]_{\mathring{u}}}{[2]^3_{\mathring{u}} [6]_{\mathring{u}} [3]_{\mathring{u}}}, \tag{19}
$$

$$
B_1 \mathring{u} = 2 \frac{[6]_{\mathring{u}} [3]_{\mathring{u}} ([2]_{\mathring{u}} - 1) - [3]_{\mathring{u}} + [2]_{\mathring{u}}}{[6]^3_{\mathring{u}} [2]_{\mathring{u}} [3]_{\mathring{u}}} + \frac{[6]([3]_{\mathring{u}} - 1) - [3]_{\mathring{u}} \left([2]^2_{\mathring{u}} - 1\right)}{[2]^3_{\mathring{u}} [6]_{\mathring{u}} [3]_{\mathring{u}}}, \tag{20}
$$

$$
A_2 \mathring{u} = 2 \frac{[2]^3_{\mathring{u}} ([3]_{\mathring{u}} - [2]_{\mathring{u}})}{[6]^3_{\mathring{u}} [3]_{\mathring{u}} [2]_{\mathring{u}}} + \frac{[6]_{\mathring{u}} \left(1 + [2]^3_{\mathring{u}}\right) - [5]_{\mathring{u}} [3]_{\mathring{u}} \left(1 + [2]^2_{\mathring{u}}\right)}{[2]^3_{\mathring{u}} [6]_{\mathring{u}} [3]_{\mathring{u}}}, \tag{21}
$$

$$
B_2 \mathring{u} = \frac{[6]_{\mathring{u}} [2]_{\mathring{u}} 2([3]_{\mathring{u}} [6]_{\mathring{u}} + 1) - [3]_{\mathring{u}} [6]_{\mathring{u}} ([6]_{\mathring{u}} + 1)}{[2]_{\mathring{u}} [3]_{\mathring{u}} [6]^3_{\mathring{u}}} + \frac{[3]_{\mathring{u}} ([6]_{\mathring{u}} + 1) - [2]^2_{\mathring{u}} ([3]_{\mathring{u}} + [6]_{\mathring{u}})}{[2]^3_{\mathring{u}} [6]_{\mathring{u}} [3]_{\mathring{u}}}, \tag{22}
$$

$$
A_3 \mathring{u} = \frac{-2([2]_{\mathring{u}} - [3]_{\mathring{u}})}{[3]^2_{\mathring{u}} [8]^2_{\mathring{u}}} + \frac{\left([2]_{\mathring{u}} [8]_{\mathring{u}} - [3]^2_{\mathring{u}}\right)}{[3]^4_{\mathring{u}}}, \tag{23}
$$

$$
B_3 \mathring{u} = \frac{[3]_{\mathring{u}} [8]_{\mathring{u}} 2([2]_{\mathring{u}} - 1) - [2]_{\mathring{u}} + [3]_{\mathring{u}}}{[2]_{\mathring{u}} [3]_{\mathring{u}} [8]^3_{\mathring{u}}} + \frac{[3]^2_{\mathring{u}} [8]_{\mathring{u}} - [2]_{\mathring{u}} [3]^3_{\mathring{u}} - [2]_{\mathring{u}} + [3]^3_{\mathring{u}}}{[2]_{\mathring{u}} [3]^4_{\mathring{u}} [8]_{\mathring{u}}}. \tag{24}
$$

Simpson's Type Inequalities

Theorem 2. *We assume that the conditions of Lemma 2 hold. If* $\left| {}^{\pi_2,\pi_4}\check{D}^{\,2}_{(p_1,q_1),(p_2,q_2)} \mathcal{F} \right|$ *is a generalized convex function and integrable on* Δ*, then the following inequality holds for right quantum integrals:*

$$
\frac{1}{[6]_{(p_1,q_1)}[6]_{(p_2,q_2)}} \Bigg\{ q_2 p_1^5 \left([5]_{(p_2,q_2)} - 1\right) \mathcal{F}\left(\pi_2, \pi_4 + \frac{\acute{O}_2}{[2]_{(p_2,q_2)}}\right)
$$
$$
+ q_1 q_2 \left([5]_{(p_1,q_1)} - 1\right)\left([5]_{(p_2,q_2)} - 1\right) \mathcal{F}\left(\frac{\pi_2 [2]_{(p_1,q_1)} + \acute{O}_1}{[2]_{(p_1,q_1)}}, \frac{\pi_4 [2]_{(p_2,q_2)} + \acute{O}_2}{[2]_{(p_2,q_2)}}\right)
$$
$$
+ q_2 p_1^5 \left([5]_{(p_2,q_2)} - 1\right) \mathcal{F}\left(\pi_2 + \acute{O}_1, \frac{\pi_4 [2]_{(p_2,q_2)} + \acute{O}_2}{[2]_{(p_2,q_2)}}\right) + q_1 p_2^5 \left([5]_{(p_1,q_1)} - 1\right)
$$
$$
\times \mathcal{F}\left(\frac{\pi_2 [2]_{(p_1,q_1)} + \acute{O}_1}{[2]_{(p_1,q_1)}}, \pi_4 + \acute{O}_2\right) + q_1 p_2^5 \left([5]_{(p_1,q_1)} - 1\right) \mathcal{F}\left(\frac{\pi_2 [2]_{(p_1,q_1)} + \acute{O}_1}{[2]_{(p_1,q_1)}}, \pi_4\right) \Bigg\}
$$

$$+\left\{q_1p_1^5\left([5]_{(p_1,q_1)}-1\right)\mathcal{F}(\pi_2,\pi_4+Ó_2)+q_1q_2\mathcal{F}(\pi_2,\pi_4)\right.$$
$$\left.+q_2p_1^5\left([5]_{(p_2,q_2)}-1\right)\mathcal{F}(\pi_2+Ó_1,\pi_4)+p_1^5p_2^5\mathcal{F}(\pi_2+Ó_1,\pi_4+Ó_2)\right\}$$
$$+\int_{\pi_4+p_2Ó_2}^{\pi_4}\int_{\pi_4+p_2Ó_2}^{\pi_4}\frac{\mathcal{F}(x,y)}{Ò_2Ò_1}\,{}^{\pi_2}d_{(p_1,q_1)}(x)\,{}^{\pi_4}d_{(p_2,q_2)}(y)$$
$$-\int_{\pi_2+p_1Ó_1}^{\pi_2}\frac{q_2\mathcal{F}(x,\pi_4)+p_2^5\mathcal{F}(x,\pi_4+Ó_2)+q_2\left([5]_{(p_2,q_2)}-1\right)\mathcal{F}\left(x,\pi_4+\frac{Ó_2}{[2]_{(p_2,q_2)}}\right)}{Ò_1[6]_{(p_2,q_2)}}\,{}^{\pi_2}d_{(p_1,q_1)}x$$
$$-\int_{\pi_4+p_2Ó_2}^{\pi_4}\frac{q_1\left([5]_{(p_1,q_1)}-1\right)\mathcal{F}\left(\pi_2+\frac{Ó_1}{[2]_{(p_1,q_1)}},y\right)+p_1^5\mathcal{F}(\pi_2+Ó_1,y)+q_1\mathcal{F}(\pi_2,y)}{Ò_2[6]_{(p_1,q_1)}}\,{}^{\pi_4}d_{(p_2,q_2)}y$$

$$\leqslant\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,\,q_1),(p_2,\,q_2)}\mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(p_1,\,q_1)}\,t\;{}^{\pi_4}\partial_{(p_2,\,q_2)}\,s}\right|(A_1(p_1,q_1)+A_2(p_1,q_1))(A_1(p_2,q_2)+A_2(p_2,q_2))$$
$$+\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,\,q_1),(p_2,\,q_2)}\mathcal{F}(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{(p_1,\,q_1)}\,t\;{}^{\pi_4}\partial_{(p_2,\,q_2)}\,s}\right|(A_1(p_1,q_1)+A_2(p_1,q_1))(B_1(p_2,q_2)+B_2(p_2,q_2))$$
$$+\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,\,q_1),(p_2,\,q_2)}\mathcal{F}(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{(p_1,\,q_1)}\,t\;{}^{\pi_4}\partial_{(p_2,\,q_2)}\,s}\right|(A_1(p_2,q_2)+A_2(p_2,q_2))(B_1(p_1,q_1)+B_2(p_1,q_1))$$
$$+\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,\,q_1),(p_2,\,q_2)}\mathcal{F}(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{(p_1,\,q_1)}\,t\;{}^{\pi_4}\partial_{(p_2,\,q_2)}\,s}\right|(B_1(p_1,q_1)+B_2(p_1,q_1))(B_1(p_2,q_2)+B_2(p_2,q_2)).$$

Proof. On the right side of Lemma (2), using the properties of the modulus along with the definition of the *generalized convex function*, we have the inequality

$$\left|\int_0^1\int_0^1\Lambda_{(p_1,q_1)}(t)\Lambda_{(p_2,q_2)}(s)\,{}^{\pi_2,\pi_4}\check{D}^{\,2}_{(p_1,q_1),(p_2,q_2)}\,\mathcal{F}\,{}^{\pi_2}d_{(p_1,q_1)}(t)\,{}^{\pi_4}d_{(p_2,q_2)}(s)\right| \tag{25}$$
$$\leq\int_0^1\Lambda_{(p_2,q_2)}(s)\left[\int_0^{\frac{1}{[2]_{(p_1,q_1)}}}t\left|t-\frac{1}{[6]_{(p_1,q_1)}}\right|\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,\,q_1),(p_2,\,q_2)}\mathcal{F}(\pi_1,\pi_4+sÓ_2)}{{}^{\pi_2}\partial_{(p_1,\,q_1)}\,t\;{}^{\pi_4}\partial_{(p_2,\,q_2)}\,s}\right|d_{(p_1,q_1)}(t)\right.$$
$$+\int_0^{\frac{1}{[2]_{(p_1,q_1)}}}(1-t)\left|t-\frac{1}{[6]_{(p_1,q_1)}}\right|\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,\,q_1),(p_2,\,q_2)}\mathcal{F}(\pi_2,\pi_4+sÓ_2)}{{}^{\pi_2}\partial_{(p_1,\,q_1)}\,t\;{}^{\pi_4}\partial_{(p_2,\,q_2)}\,s}\right|d_{(p_1,q_1)}(t)$$

$$+\int_{\frac{1}{[2]_{(p_1,q_1)}}}^1 t\left|t-\frac{[5]_{(p_1,q_1)}}{[6]_{(p_1,q_1)}}\right|\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,\,q_1),(p_2,\,q_2)}\mathcal{F}(\pi_1,\pi_4+sÓ_2)}{{}^{\pi_2}\partial_{(p_1,\,q_1)}\,t\;{}^{\pi_4}\partial_{(p_2,\,q_2)}\,s}\right|d_{(p_1,q_1)}(t)+\int_{\frac{1}{[2]_{(p_1,q_1)}}}^1(1-t)$$
$$\left.\times\left|t-\frac{[5]_{(p_1,q_1)}}{[6]_{(p_1,q_1)}}\right|\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,\,q_1),(p_2,\,q_2)}\mathcal{F}(\pi_2,\pi_4+sÓ_2)}{{}^{\pi_2}\partial_{(p_1,\,q_1)}\,t\;{}^{\pi_4}\partial_{(p_2,\,q_2)}\,s}\right|d_{(p_1,q_1)}(t)\right\}{}^{\pi_4}d_{(p_2,q_2)}(s)\Bigg].$$

Using the definitions of the generalized convex function and $\Lambda_{(p_2,q_2)}\,s$,

$$\left|\int_0^1\int_0^1\Lambda_{(p_1,q_1)}(t)\Lambda_{(p_2,q_2)}(s)\,{}^{\pi_2,\pi_4}\check{D}^{\,2}_{(p_1,q_1),(p_2,q_2)}\,\mathcal{F}\,{}^{\pi_2}d_{(p_1,q_1)}(t)\,{}^{\pi_4}d_{(p_2,q_2)}(s)\right|$$
$$\leqslant\int_0^{\frac{1}{[2]_{(p_2,q_2)}}}\left\{s\left|s-\frac{1}{[6]_{(p_2,q_2)}}\right|\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,\,q_1),(p_2,\,q_2)}\mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(p_1,\,q_1)}\,t\;{}^{\pi_4}\partial_{(p_2,\,q_2)}\,s}\right|(A_1(p_1,q_1)+A_2(p_1,q_1))\right.$$

$$+(1-s)\left|s-\frac{1}{[6]_{(p_2,q_2)}}\right|\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,\,q_1),(p_2,\,q_2)}\mathcal{F}(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{(p_1,\,q_1)}\,t\;{}^{\pi_4}\partial_{(p_2,\,q_2)}\,s}\right|(A_1(p_1,q_1)+A_2(p_1,q_1))\Bigg\}\,d_{(p_2q_2)}(s)$$
$$+\int_0^{\frac{1}{[2]_{(p_2,q_2)}}}\Bigg\{s\left|s-\frac{1}{[6]_{(p_2,q_2)}}\right|\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,\,q_1),(p_2,\,q_2)}\mathcal{F}(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{(p_1,\,q_1)}\,t\;{}^{\pi_4}\partial_{(p_2,\,q_2)}\,s}\right|(B_1(p_1,q_1)+B_2(p_1,q_1))$$

$$+(1-s)\left|s-\frac{1}{[6]_{(p_2,q_2)}}\right|\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,\,q_1),(p_2,\,q_2)}\mathcal{F}(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{(p_1,\,q_1)}\,t\;{}^{\pi_4}\partial_{(p_2,\,q_2)}\,s}\right|(B_1(p_1,q_1)+B_2(p_1,q_1))\Bigg\}\,d_{(p_2q_2)}(s)$$
$$+\int_{\frac{1}{[2]_{(p_2,q_2)}}}^{1}\Bigg\{s\left|s-\frac{[5]_{(p_2,q_2)}}{[6]_{(p_2,q_2)}}\right|\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,\,q_1),(p_2,\,q_2)}\mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(p_1,\,q_1)}\,t\;{}^{\pi_4}\partial_{(p_2,\,q_2)}\,s}\right|(A_1(p_1,q_1)+A_2(p_1,q_1))$$

$$+(1-s)\left|s-\frac{[5]_{(p_2,q_2)}}{[6]_{(p_2,q_2)}}\right|\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,\,q_1),(p_2,\,q_2)}\mathcal{F}(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{(p_1,\,q_1)}\,t\;{}^{\pi_4}\partial_{(p_2,\,q_2)}\,s}\right|(A_1(p_1,q_1)+A_2(p_1,q_1))\Bigg\}\,d_{(p_2q_2)}(s)$$
$$+\int_0^{\frac{1}{[2](p_2,q_2)}}\Bigg\{s\left|s-\frac{[5]_{(p_2,q_2)}}{[6]_{(p_2,q_2)}}\right|\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,\,q_1),(p_2,\,q_2)}\mathcal{F}(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{(p_1,\,q_1)}\,t\;{}^{\pi_4}\partial_{(p_2,\,q_2)}\,s}\right|(B_1(p_1,q_1)+B_2(p_1,q_1))$$
$$+(1-s)\left|s-\frac{[5]_{(p_2,q_2)}}{[6]_{(p_2,q_2)}}\right|\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,\,q_1),(p_2,\,q_2)}\mathcal{F}(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{(p_1,\,q_1)}\,t\;{}^{\pi_4}\partial_{(p_2,\,q_2)}\,s}\right|(B_1(p_1,q_1)+B_2(p_1,q_1))\Bigg\}\,d_{(p_2q_2)}(s).$$

Taking the values of $A_1(p_i,q_i)$, $A_2(p_i,q_i)$, $B_1(p_i,q_i)$, and $B_2(p_i,q_i)$, where $i\in\{1,2\}$, from (19)–(22), then after simplification, we have the required result. □

Remark 4. *By replacing $p_1=1=p_2$ in Theorem 2, we have the identity*

$$\frac{1}{[6]_{q_1}[6]_{q_2}}\Bigg[q_1^2q_2^2[4]_{q_1}[4]_{q_2}\mathcal{F}\left(\frac{\pi_2[2]_{q_1}+Ó_1}{[2]_{q_1}},\frac{\pi_4[2]_{q_2}+Ó_2}{[2]_{q_2}}\right)+q_2^2[4]_{q_2}\mathcal{F}\left(\pi_2,\pi_4+\frac{Ó_2}{[2]_{q_2}}\right)$$
$$+q_1^2[4]_{q_1}\mathcal{F}\left(\pi_2+\frac{Ó_1}{[2]_{q_1}},\pi_4\right)+q_2^2[4]_{q_2}\mathcal{F}\left(\pi_2+Ó_1,\frac{\pi_4[2]_{q_2}+Ó_2}{[2]_{q_2}}\right)+q_1^2[4]_{q_1}$$
$$\times\mathcal{F}\left(\pi_2+\frac{Ó_1}{[2]_{q_1}},\pi_4+Ó_2\right)\Bigg]+\frac{1}{[6]_{q_1}[6]_{q_2}}\Big[q_1q_2\mathcal{F}(\pi_2,\pi_4)+q_2^2[4]_{q_2}\mathcal{F}(\pi_2+Ó_1,\pi_4)$$

$$+q_1^2[4]_{(q_1)}\mathcal{F}(\pi_2,\pi_4+Ó_2)+\mathcal{F}(\pi_2+Ó_1,\pi_4+Ó_2)\Big]-\frac{1}{Ò_1[6]_{(q_2)}}\int_{\pi_2+Ó_1}^{\pi_2}\Big[q_2\mathcal{F}(x,\pi_4)$$
$$+\mathcal{F}(x,\pi_4+Ó_2)+q_2^2[4]_{(q_2)}\mathcal{F}\left(x,\pi_4+\frac{Ó_2}{[2]_{(q_2)}}\right)\Bigg]\,{}^{\pi_2}d_{(q_1)}x-\frac{1}{Ò_2[6]_{(q_1)}}\int_{\pi_4+Ó_2}^{\pi_4}$$
$$\left[q_1\mathcal{F}(\pi_2,y)+\mathcal{F}(\pi_2+Ó_1,y)+q_1^2[4]_{(q_1)}\mathcal{F}\left(\pi_2+\frac{Ó_1}{[2]_{(q_1)}},y\right)\right]\,{}^{\pi_4}d_{(q_2)}y$$
$$+\frac{1}{Ò_2Ò_1}\int_{\pi_4+Ó_2}^{\pi_4}\int_{\pi_4+Ó_2}^{\pi_4}\mathcal{F}(x,y)\;{}^{\pi_2}d_{(q_1)}x\,{}^{\pi_4}d_{(q_2)}y$$

$$
\begin{aligned}
&\leqslant \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{(q_1),(q_2)} \mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(q_1)}\, t\; {}^{\pi_4}\partial_{(q_2)}\, s} \right| (A_1(q_1)+A_2(q_1))(A_1(q_2)+A_2(q_2)) \\
&+ \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{(q_1),(q_2)} \mathcal{F}(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{(q_1)}\, t\; {}^{\pi_4}\partial_{(q_2)}\, s} \right| (A_1(q_1)+A_2(q_1))(B_1(q_2)+B_2(q_2)) \\
&+ \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{(q_1),(q_2)} \mathcal{F}(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{(q_1)}\, t\; {}^{\pi_4}\partial_{(q_2)}\, s} \right| (B_1(q_1)+B_2(q_1))(A_1(q_2)+A_2(q_2)) \\
&+ \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{(q_1),(q_2)} \mathcal{F}(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{(q_1)}\, t\; {}^{\pi_4}\partial_{(q_2)}\, s} \right| (B_1(q_1)+B_2(q_1))(B_1(q_2)+B_2(q_2)).
\end{aligned}
$$

Theorem 3. *We assume that conditions of Lemma 2 hold. If* ${}^{\pi_2,\pi_4}\check{D}^{\,2}_{(p_1,q_1),(p_2,q_2)} \mathcal{F}$ *is a generalized convex function and integrable on* I*, where* $t_1 > 1$ *with* $\frac{1}{r_1} + \frac{1}{t_1} = 1$*, then we have the following inequality:*

$$
\begin{aligned}
&\Bigg| \frac{(\grave{O}_1\grave{O}_2)^{-1}}{[6]_{(p_1,q_1)}[6]_{(p_2,q_2)}} \Bigg[\Big([5]_{(p_1,q_1)}-1\Big)\Big([5]_{(p_2,q_2)}-1\Big) \mathcal{F}\left(\frac{\pi_2[2]_{(p_1,q_1)}+\acute{O}_1}{[2]_{(p_1,q_1)}}, \pi_4 + \frac{\acute{O}_2}{[2]_{(p_2,q_2)}} \right) \\
&+ \frac{p_1^5 \mathcal{F}\left(\pi_2, \pi_4 + \frac{\acute{O}_2}{[2]_{(p_2,q_2)}}\right)}{q_1\left([5]_{(p_2,q_2)}-1\right)^{-1}} + \frac{p_2^5 \mathcal{F}\left(\frac{\pi_2[2]_{(p_1,q_1)}+\acute{O}_1}{[2]_{(p_1,q_1)}}, \pi_4\right)}{q_2\left([5]_{(p_1,q_1)}-1\right)^{-1}} + \frac{\mathcal{F}\left(\pi_2+\acute{O}_1, \pi_4 + \frac{\acute{O}_2}{[2]_{(p_2,q_2)}}\right)}{(p_1^5)^{-1} q_1\left([5]_{(p_2,q_2)}-1\right)^{-1}} \\
&+ \frac{\mathcal{F}\left(\frac{\pi_2[2]_{(p_1,q_1)}+\acute{O}_1}{[2]_{(p_1,q_1)}}, \pi_4+\acute{O}_2\right)}{(p_2^5)^{-1} q_2\left([5]_{(p_1,q_1)}-1\right)^{-1}} \Bigg] + \frac{1}{[6]_{(p_1,q_1)}[6]_{(p_2,q_2)}} \Bigg[\frac{p_1^5\mathcal{F}(\pi_2+\acute{O}_1,\pi_4)}{q_1\left([5]_{(p_2,q_2)}-1\right)^{-1}} + \mathcal{F}(\pi_2,\pi_4) \\
&+ \frac{p_1^5\mathcal{F}(\pi_2,\pi_4+\acute{O}_2)}{q_2\left([5]_{(p_1,q_1)}-1\right)^{-1}} + \frac{\mathcal{F}(\pi_2+\acute{O}_1,\pi_4+\acute{O}_2)}{(p_1^5p_1^5)^{-1}q_1q_2} \Bigg] - \frac{1}{\grave{O}_1[6]_{(p_2,q_2)}} \Bigg[\int\limits_{\pi_2+p_1\acute{O}_1}^{\pi_2} \frac{\mathcal{F}(x,\pi_4)}{q_1}\, {}^{\pi_2}d_{(p_1,q_1)}x \\
&+ \int\limits_{\pi_2+p_1\acute{O}_1}^{\pi_2} \frac{p_2^5\, \mathcal{F}(x,\pi_4+\acute{O}_2)}{q_1q_2}\, {}^{\pi_2}d_{(p_1,q_1)}x + \int\limits_{\pi_2+p_1\acute{O}_1}^{\pi_2} \frac{\mathcal{F}\left(x, \pi_4+\frac{\acute{O}_2}{[2]_{(p_2,q_2)}}\right)}{q_1\left([5]_{(p_2,q_2)}-1\right)^{-1}}\, {}^{\pi_2}d_{(p_1,q_1)}x \Bigg] \\
&- \frac{1}{\grave{O}_2[6]_{(p_1,q_1)}} \Bigg[\int\limits_{\pi_4+p_2\acute{O}_2}^{\pi_4} \frac{\mathcal{F}(\pi_2,y)}{q_2}\, {}^{\pi_4}d_{(p_2,q_2)}y + \int\limits_{\pi_4+p_2\acute{O}_2}^{\pi_4} \frac{p_1^5\mathcal{F}(\pi_2+\acute{O}_1,y)}{q_1q_2}\, {}^{\pi_4}d_{(p_2,q_2)}y \\
&+ \int\limits_{\pi_4+p_2\acute{O}_2}^{\pi_4} \frac{\mathcal{F}\left(\pi_2+\frac{\acute{O}_1}{[2]_{(p_1,q_1)}}, y\right)}{q_2\left([5]_{(p_1,q_1)}-1\right)^{-1}}\, {}^{\pi_4}d_{(p_2,q_2)}y \Bigg] + \int\limits_{\pi_4+p_2\acute{O}_2}^{\pi_4} \int\limits_{\pi_4+p_2\acute{O}_2}^{\pi_4} \frac{\mathcal{F}(x,y)}{\grave{O}_2\grave{O}_1}\, {}^{\pi_2}d_{(p_1,q_1)}x\; {}^{\pi_4}d_{(p_2,q_2)}y \Bigg|
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
&\leq q_1q_2\grave{O}_1\grave{O}_2 \left(\int_0^1 \int_0^1 \left| \Lambda_{(p_1,q_1)}\, t\; \Lambda_{(p_2,q_2)}\, s \right|^{r_1} d_{(p_1,q_1)}\, t\; d_{(p_2,q_2)}\, s \right)^{\frac{1}{r_1}} \\
&\times \Bigg[\frac{1}{[2]_{(p_1,q_1)}[2]_{(p_2,q_2)}} \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)} \mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}\, t\; {}^{\pi_4}\partial_{(p_2,q_2)}\, s} \right|^{t_1} + \frac{[2]_{(p_1,q_1)}-1}{[2]_{(p_1,q_1)}[2]_{(p_2,q_2)}} \\
&\times \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)} \mathcal{F}(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{(p_1,q_1)}\, t\; {}^{\pi_4}\partial_{(p_2,q_2)}\, s} \right|^{t_1} + \frac{[2]_{(p_2,q_2)}-1}{[2]_{(p_1,q_1)}[2]_{(p_2,q_2)}} \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)} \mathcal{F}(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}\, t\; {}^{\pi_4}\partial_{(p_2,q_2)}\, s} \right|^{t_1}
\end{aligned}
$$

$$+\frac{\left([2]_{(p_1,q_1)}-1\right)\left([2]_{(p_2,q_2)}-1\right)}{[2]_{(p_1,q_1)}[2]_{(p_2,q_2)}}\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{(p_1,q_1)}\,t\,{}^{\pi_4}\partial_{(p_2,q_2)}\,s}\right|^{t_1}\Bigg]^{\frac{1}{t_1}}.$$

where $0<q_1<1$ *and* $0<q_2<1$.

Proof. Applying the well-known quantum Hölders integral inequality on the integrals on the right side of (25),

$$\left|q_1q_2\grave{O}_1\grave{O}_2\int_0^1\int_0^1\Lambda_{(p_1,q_1)}\,t\,\Lambda_{(p_2,q_2)}\,s\ {}^{\pi_2,\pi_4}\check{D}^{\,2}_{(p_1,q_1),(p_2,q_2)}\ \mathcal{F}\ d_{(p_1,q_1)}\,t\,d_{(p_2,q_2)}\,s\right| \tag{27}$$

$$\leq q_1q_2\grave{O}_1\grave{O}_2\left(\int_0^1\int_0^1\left|\Lambda_{(p_1,q_1)}\,t\,\Lambda_{(p_2,q_2)}\,s\right|^{r_1}d_{(p_1,q_1)}\,t\,d_{(p_2,q_2)}\,s\right)^{\frac{1}{r_1}}$$
$$\times\left(\int_0^1\int_0^1\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_2+t\acute{O}_1,\pi_4+s\acute{O}_2)}{{}^{\pi_2}\partial_{(p_1,q_1)}\,t\,{}^{\pi_4}\partial_{(p_2,q_2)}\,s}\right|^{t_1}d_{(p_1,q_1)}\,t\,d_{(p_2,q_2)}\,s\right).$$

By using the definition of generalized convex function, we have

$$\left|q_1q_2\grave{O}_1\grave{O}_2\int_0^1\int_0^1\Lambda_{(p_1,q_1)}\,t\,\Lambda_{(p_2,q_2)}\,s\ {}^{\pi_2,\pi_4}\check{D}^{\,2}_{(p_1,q_1),(p_2,q_2)}\ \mathcal{F}d_{(p_1,q_1)}\,t\ d_{(p_2,q_2)}\,s\right| \tag{28}$$

$$\leq q_1q_2\grave{O}_1\grave{O}_2\Bigg[\left(\int_0^1\int_0^1\left|\Lambda_{(p_1,q_1)}\,t\,\Lambda_{(p_2,q_2)}\,s\right|^{r_1}d_{(p_1,q_1)}\,t\,d_{(p_2,q_2)}\,s\right)^{\frac{1}{r_1}}$$
$$\times\left(\int_0^1\int_0^1 ts\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}\,t\,{}^{\pi_4}\partial_{(p_2,q_2)}\,s}\right|^{t_1}+\frac{t}{(1-s)^{-1}}\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{(p_1,q_1)}\,t\,{}^{\pi_4}\partial_{(p_2,q_2)}\,s}\right|^{t_1}+\frac{(1-t)}{s^{-1}}\right.$$
$$\left.\times\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}\,t\,{}^{\pi_4}\partial_{(p_2,q_2)}\,s}\right|^{t_1}+\frac{(1-t)}{(1-s)^{-1}}\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{(p_1,q_1)}\,t\,{}^{\pi_4}\partial_{(p_2,q_2)}\,s}\right|^{t_1}d_{(p_1,q_1)}\,t\,d_{(p_2,q_2)}\,s\right)^{\frac{1}{t_1}}\Bigg]$$

Now, if we apply the concept of Lemma [43] for $\alpha=0$ to the above quantum integrals, we obtain

$$\int_0^1\int_0^1 tsd_{(p_1,q_1)}\,t\,d_{(p_2,q_2)}\,s\ =\frac{1}{[2]_{(p_1,q_1)}[2]_{(p_2,q_2)}}, \tag{29}$$

$$\int_0^1\int_0^1 t(1-s)d_{(p_1,q_1)}\,t\,d_{(p_2,q_2)}\,s\ =\frac{[2]_{(p_2-q_2)}-1}{[2]_{(p_1,q_1)}[2]_{(p_2-q_2)}}, \tag{30}$$

$$\int_0^1\int_0^1 s(1-t)d_{(p_1,q_1)}\,t\,d_{(p_2,q_2)}\,s\ =\frac{[2]_{(p_1,q_1)}-1}{[2]_{(p_1,q_1)}[2]_{(p_2,q_2)}}, \tag{31}$$

$$\int_0^1\int_0^1(1-t)(1-s)d_{(p_1,q_1)}\,t\,d_{(p_2,q_2)}\,s\ =\frac{\left([2]_{(p_1,q_1)}-1\right)\left([2]_{(p_2,q_2)}-1\right)}{[2]_{(p_1,q_1)}[2]_{(p_2,q_2)}}. \tag{32}$$

By substituting the calculated integrals (29)–(32) in (28), then we obtain the desired inequality (26), which finishes the proof. □

Remark 5. *By substitution of $p_1 = p_2 = 1$ in Theorem 3, then we have the following inequality:*

$$\begin{aligned}
&\left| \frac{(\grave{O}_1\grave{O}_2)^{-1}}{[6]_{q_1}[6]_{q_2}} \left[\frac{q_1[4]_{q_1}}{q_2} \mathcal{F}\left(\pi_2 + \frac{\acute{O}_1}{[2]_{q_1}}, \pi_4\right) + \frac{q_2[4]_{q_2}}{q_1} \mathcal{F}\left(\pi_2, \frac{\pi_4[2]_{q_2} + \acute{O}_2}{[2]_{q_2}}\right) + \frac{q_2[4]_{q_2}}{q_1} \right.\right. \\
&\times \mathcal{F}\left(\pi_2 + \acute{O}_1, \frac{\pi_4[2]_{q_2} + \acute{O}_2}{[2]_{q_2}}\right) + q_1 q_2 [4]_{q_1}[4]_{q_2} \mathcal{F}\left(\pi_2 + \frac{\acute{O}_1}{[2]_{q_1}}, \frac{\pi_4[2]_{q_2} + \acute{O}_2}{[2]_{q_2}}\right) + \frac{q_1[4]_{q_1}}{q_2} \\
&\times \mathcal{F}\left(\pi_2 + \frac{\acute{O}_1}{[2]_{q_1}}, \pi_4 + \acute{O}_2\right)\Big] + \frac{1}{[6]_{q_1}[6]_{q_2}} \Big[\frac{q_2[4]_{q_2}}{q_1} \mathcal{F}(\pi_2 + \acute{O}_1, \pi_4) + \mathcal{F}(\pi_2, \pi_4) \\
&+ \frac{q_1[4]_{q_1}}{q_2} \mathcal{F}(\pi_2, \pi_4 + \acute{O}_2) + \frac{\mathcal{F}(\pi_2 + \acute{O}_1, \pi_4 + \acute{O}_2)}{q_1 q_2} \Big] - \frac{1}{\grave{O}_1 [6]_{q_2}} \Big[\int_{\pi_2 + \acute{O}_1}^{\pi_2 \pi_2} \frac{\mathcal{F}(x, \pi_4)}{q_1} d_{q_1}(x) \\
&+ \int_{\pi_2 + \acute{O}_1}^{\pi_2} \frac{\mathcal{F}(x, \pi_4 + \acute{O}_2)}{q_1 q_2} {}^{\pi_2}d_{q_1}(x) + \int_{\pi_2 + \acute{O}k_1}^{\pi_2} \frac{q_2[4]_{q_2}}{q_1} \mathcal{F}\left(x, \pi_4 + \frac{\acute{O}_2}{[2]_{q_2}}\right) {}^{\pi_2}d_{q_1}(x) \Big] \\
&- \frac{1}{\grave{O}_2 [6]_{q_1}} \Big[\int_{\pi_4 + \acute{O}_2}^{\pi_4} \frac{\mathcal{F}(\pi_2, y)}{q_2} {}^{\pi_4}d_{q_2} y + \int_{\pi_4 + \acute{O}_2}^{\pi_4} \frac{\mathcal{F}(\pi_2 + \acute{O}_1, y)}{q_1 q_2} {}^{\pi_4}d_{q_2} y + \frac{q_1[4]_{q_1}}{q_2} \int_{\pi_4 + \acute{O}_2}^{\pi_4} \\
&\left.\times \mathcal{F}\left(\pi_2 + \frac{\acute{O}_1}{[2]_{(q_1)}}, y\right) {}^{\pi_4}d_{q_2} y \Big] + \int_{\pi_4 + \acute{O}_2}^{\pi_4} \int_{\pi_4 + \acute{O}_2}^{\pi_4} \frac{\mathcal{F}(x, y)}{\grave{O}_2 \grave{O}_1} {}^{\pi_2}d_{(q_1)}(x)^{\pi_4} d_{q_2} y \right| \\
&\leq q_1 q_2 \grave{O}_1 \grave{O}_2 \left(\int_0^1 \int_0^1 \left| \Lambda_{(q_1)}\ t\ \Lambda_{(q_2)}\ s \right|^{r_1} d_{q_1}\ t\ d_{q_2}\ s \right)^{\frac{1}{r_1}} \\
&\times \left[\frac{1}{[2]_{q_1}[2]_{q_2}} \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)} \mathcal{F}(\pi_1, \pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}\ t\ {}^{\pi_4}\partial_{(p_2,q_2)}\ s} \right|^{r_1} + \frac{q_1}{[2]_{q_1}[2]_{q_2}} \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{(q_1),(q_2)} \mathcal{F}(\pi_1, \pi_4)}{{}^{\pi_2}\partial_{q_1}\ t\ {}^{\pi_4}\partial_{q_2}\ s} \right|^{r_1} \right. \\
&\left. + \frac{q_2}{[2]_{q_1}[2]_{q_2}} \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2} \mathcal{F}(\pi_2, \pi_3)}{{}^{\pi_2}\partial_{q_1}\ t\ {}^{\pi_4}\partial_{q_2}\ s} \right|^{r_1} + \frac{q_1 q_2}{[2]_{q_1}[2]_{q_2}} \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2} \mathcal{F}(\pi_2, \pi_4)}{{}^{\pi_2}\partial_{q_1}\ t\ {}^{\pi_4}\partial_{q_2}\ s} \right|^{r_1} \right]^{\frac{1}{r_1}}.
\end{aligned}$$

Theorem 4. *Let $\left| {}^{\pi_2,\pi_4}\check{D}^{\,2}_{(p_1,q_1),(p_2,q_2)} \mathcal{F} \right|$ be a generalized convex function and integrable on Δ, where $t_1 > 1$, and assume that Lemma 2 holds. Then the following inequality holds:*

$$\begin{aligned}
&\left| \frac{1}{[6]_{(p_1,q_1)}[6]_{(p_2,q_2)}} \left[\left([5]_{(p_1,q_1)} - 1\right)\left([5]_{(p_2,q_2)} - 1\right) \mathcal{F}\left(\frac{\pi_2[2]_{(p_1,q_1)} + \acute{O}_1}{[2]_{(p_1,q_1)}}, \pi_4 + \frac{\acute{O}_2}{[2]_{(p_2,q_2)}}\right) \right.\right. \\
&+ \frac{p_1^5\left([5]_{(p_2,q_2)} - 1\right)}{q_1} \mathcal{F}\left(\pi_2, \pi_4 + \frac{\acute{O}_2}{[2]_{(p_2,q_2)}}\right) + \frac{p_2^5\left([5]_{(p_1,q_1)} - 1\right)}{q_2} \mathcal{F}\left(\frac{\pi_2[2]_{(p_1,q_1)} + \acute{O}_1}{[2]_{(p_1,q_1)}}, \pi_4\right) \\
&+ \frac{p_1^5\left([5]_{(p_2,q_2)} - 1\right)}{q_1} \mathcal{F}\left(\pi_2 + \acute{O}_1, \pi_4 + \frac{\acute{O}_2}{[2]_{(p_2,q_2)}}\right) + \frac{p_2^5\left([5]_{(p_1,q_1)} - 1\right)}{q_2} \\
&\times \mathcal{F}\left(\frac{\pi_2[2]_{(p_1,q_1)} + \acute{O}_1}{[2]_{(p_1,q_1)}}, \pi_4 + \acute{O}_2\right) \Big] + \frac{1}{[6]_{(p_1,q_1)}[6]_{(p_2,q_2)}} \left[\left(\frac{p_1^5 \mathcal{F}(\pi_2 + \acute{O}_1, \pi_4)}{q_1\left([5]_{(p_2,q_2)} - 1\right)^{-1}} \right) \right. \\
&\left. + \mathcal{F}(\pi_2, \pi_4) + \frac{p_1^5 \mathcal{F}(\pi_2, \pi_4 + \acute{O}_2)}{q_2\left([5]_{(p_1,q_1)} - 1\right)^{-1}} + \frac{p_1^5 p_1^5}{q_1 q_2} \mathcal{F}(\pi_2 + \acute{O}_1, \pi_4 + \acute{O}_2) \right] - \frac{1}{\grave{O}_1 [6]_{(p_2,q_2)}} \\
&\times \left[\frac{1}{q_1} \int_{\pi_2 + p_1 \acute{O}_1}^{\pi_2} \mathcal{F}(x, \pi_4)^{\pi_2} d_{(p_1,q_1)} x + \frac{p_2^5}{q_1 q_2} \int_{\pi_2 + p_1 \acute{O}_1}^{\pi_2} \mathcal{F}(x, \pi_4 + \acute{O}_2)^{\pi_2} d_{(p_1,q_1)} x \right.
\end{aligned}$$

$$+\frac{\left([5]_{(p_2,q_2)}-1\right)}{q_1}\int_{\pi_2+p_1\acute{O}_1}^{\pi_2}\mathcal{F}\left(x,\pi_4+\frac{\acute{O}_2}{[2]_{(p_2,q_2)}}\right){}^{\pi_2}d_{(p_1,q_1)}x\Bigg]-\frac{1}{\grave{O}_2[6]_{(p_1,q_1)}}$$

$$\times\Bigg[\frac{1}{q_2}\int_{\pi_4+p_2\acute{O}_2}^{\pi_4}\mathcal{F}(\pi_2,y)^{\pi_4}d_{(p_2,q_2)}y+\frac{p_1^5}{q_1q_2}\int_{\pi_4+p_2\acute{O}_2}^{\pi_4}\mathcal{F}\left(\pi_2+\acute{O}_1,y\right)^{\pi_4}d_{(p_2,q_2)}y$$

$$+\frac{\left([5]_{(p_1,q_1)}-1\right)}{q_2}\int_{\pi_4+p_2\acute{O}_2}^{\pi_4}\mathcal{F}\left(\pi_2+\frac{\acute{O}_1}{[2]_{(p_1,q_1)}},y\right){}^{\pi_4}d_{(p_2,q_2)}y\Bigg]$$

$$+\frac{1}{\grave{O}_2\grave{O}_1}\int_{\pi_4+p_2\acute{O}_2}^{\pi_4}\int_{\pi_4+p_2\acute{O}_2}^{\pi_4}\mathcal{F}(x,y)\,{}^{\pi_2}d_{(p_1,q_1)}x^{\pi_4}d_{(p_2,q_2)}y\Bigg|$$

$$\leq q_1q_2\grave{O}_1\grave{O}_2[K(p_1,q_1)K(p_2,q_2)]^{1-\frac{1}{t_1}}$$

$$\times\Bigg[A_1(p_1,q_1)\Bigg(A_1(p_2,q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}t\,{}^{\pi_4}\partial_{(p_2,q_2)}s}\right|^{t_1}+B_1(p_2,q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{(p_1,q_1)}t\,{}^{\pi_4}\partial_{(p_2,q_2)}s}\right|^{t_1}\Bigg)$$

$$+B_1(p_1,q_1)\Bigg(A_1(p_2,q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}t\,{}^{\pi_4}\partial_{(p_2,q_2)}s}\right|^{t_1}+B_1(p_2,q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{(p_1,q_1)}t\,{}^{\pi_4}\partial_{(p_2,q_2)}s}\right|^{t_1}\Bigg)\Bigg]^{\frac{1}{t_1}}$$

$$+q_1q_2\grave{O}_1\grave{O}_2[K(p_1,q_1)J(p_2,q_2)]^{1-\frac{1}{t_1}}$$

$$\times\Bigg[A_1(p_1,q_1)\Bigg(A_2(p_2,q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}t\,{}^{\pi_4}\partial_{(p_2,q_2)}s}\right|^{t_1}+B_2(p_2,q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{(p_1,q_1)}t\,{}^{\pi_4}\partial_{(p_2,q_2)}s}\right|^{t_1}\Bigg)$$

$$+B_1(p_1,q_1)\Bigg(A_2(p_2,q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}t\,{}^{\pi_4}\partial_{(p_2,q_2)}s}\right|^{t_1}+B_2(p_2,q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{(p_1,q_1)}t\,{}^{\pi_4}\partial_{(p_2,q_2)}s}\right|^{t_1}\Bigg)\Bigg]^{\frac{1}{t_1}}$$

$$+q_1q_2\grave{O}_1\grave{O}_2[J(p_1,q_1)K(p_2,q_2)]^{1-\frac{1}{t_1}}$$

$$\times\Bigg[A_2(p_1,q_1)\Bigg(A_1(p_2,q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}t\,{}^{\pi_4}\partial_{(p_2,q_2)}s}\right|^{t_1}+B_1(p_2,q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{(p_1,q_1)}t\,{}^{\pi_4}\partial_{(p_2,q_2)}s}\right|^{t_1}\Bigg)$$

$$+B_2(p_1,q_1)\Bigg(+A_1(p_2,q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}t\,{}^{\pi_4}\partial_{(p_2,q_2)}s}\right|^{t_1}+B_1(p_2,q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{(p_1,q_1)}t\,{}^{\pi_4}\partial_{(p_2,q_2)}s}\right|^{t_1}\Bigg)\Bigg]^{\frac{1}{t_1}}$$

$$+q_1q_2\grave{O}_1\grave{O}_2[J(p_1,q_1)J(p_2,q_2)]^{1-\frac{1}{t_1}}$$

$$\times\Bigg[A_2(p_1,q_1)\Bigg(A_2(p_2,q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}t\,{}^{\pi_4}\partial_{(p_2,q_2)}s}\right|^{t_1}+B_2(p_2,q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{(p_1,q_1)}t\,{}^{\pi_4}\partial_{(p_2,q_2)}s}\right|^{t_1}\Bigg)$$

$$B_2(p_1,q_1)\Bigg(A_2(p_2,q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}t\,{}^{\pi_4}\partial_{(p_2,q_2)}s}\right|^{t_1}+B_2(p_2,q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{(p_1,q_1)}t\,{}^{\pi_4}\partial_{(p_2,q_2)}s}\right|^{t_1}\Bigg)\Bigg]^{\frac{1}{t_1}}$$

Proof. By using the power mean inequality on the right side of (17) with definitions of $\Lambda_{(p_1,q_1)}\, t$ and $\Lambda_{(p_2,q_2)}\, s$, owing to the generalized convexity of $\left| {}^{\pi_2,\pi_4}\check{D}^{\,2}_{(p_1,q_1),(p_2,q_2)}\, \mathcal{F} \right|^{t_1}$, we find that

$$\left| q_1 q_2 \grave{O}_1 \grave{O}_2 \int_0^1 \int_0^1 \Lambda_{(p_1,q_1)}\, t\, \Lambda_{(p_2,q_2)}\, s\; {}^{\pi_2,\pi_4}\check{D}^{\,2}_{(p_1,q_1),(p_2,q_2)}\, \mathcal{F}_{(p_1,q_1)}\, t\, d_{(p_2,q_2)}\, s \right|$$

$$\leq q_1 q_2 \grave{O}_1 \grave{O}_2 \Bigg[\left(\int_0^{\frac{1}{[2]_{(p_1,q_1)}}} \int_0^{\frac{1}{[2]_{(p_2,q_2)}}} \left| t - \frac{1}{[6]_{(p_1,q_1)}} \right| \left| s - \frac{1}{[6]_{(p_2,q_2)}} \right| d_{(p_1,q_1)}\, t\, d_{(p_2,q_2)}\, s \right)^{1-\frac{1}{t_1}}$$

$$\times \left(\int_0^{\frac{1}{[2]_{(p_1,q_1)}}} \int_0^{\frac{1}{[2]_{(p_2,q_2)}}} \left| t - \frac{1}{[6]_{(p_1,q_1)}} \right| \left| s - \frac{1}{[6]_{(p_2,q_2)}} \right| d_{(p_1,q_1)}\, t\, d_{(p_2,q_2)}\, s \right.$$

$$\left. \times \left| {}^{\pi_2,\pi_4}\check{D}^{\,2}_{(p_1,q_1),(p_2,q_2)}\, \mathcal{F} \right|^{t_1} d_{(p_1,q_1)}\, t\, d_{(p_2,q_2)}\, s \right)^{\frac{1}{t_1}}$$

$$+ \left(\int_0^{\frac{1}{[2]_{(p_1,q_1)}}} \int_{\frac{1}{[2]_{(p_2,q_2)}}}^1 \left| t - \frac{1}{[6]_{(p_1,q_1)}} \right| \left| s - \frac{[5]_{(p_2,q_2)}}{[6]_{(p_2,q_2)}} \right| d_{(p_1,q_1)}\, t\, d_{(p_2,q_2)}\, s \right)^{1-\frac{1}{t_1}}$$

$$\times \left(\int_0^{\frac{1}{[2]_{(p_1,q_1)}}} \int_{\frac{1}{[2]_{(p_2,q_2)}}}^1 \left| t - \frac{1}{[6]_{(p_1,q_1)}} \right| \left| s - \frac{[5]_{(p_2,q_2)}}{[6]_{(p_2,q_2)}} \right| d_{(p_1,q_1)}\, t\, d_{(p_2,q_2)}\, s \right.$$

$$\left. \times \left| {}^{\pi_2,\pi_4}\check{D}^{\,2}_{(p_1,q_1),(p_2,q_2)}\, \mathcal{F} \right|^{t_1} d_{(p_1,q_1)}\, t\, d_{(p_2,q_2)}\, s \right)^{\frac{1}{t_1}}$$

$$+ \left(\int_{\frac{1}{[2]_{(p_1,q_1)}}}^1 \int_0^{\frac{1}{[2]_{(p_2,q_2)}}} \left| t - \frac{[5]_{(p_1,q_1)}}{[6]_{(p_1,q_1)}} \right| \left| s - \frac{1}{[6]_{(p_2,q_2)}} \right| d_{(p_1,q_1)}\, t\, d_{(p_2,q_2)}\, s \right)^{1-\frac{1}{t_1}}$$

$$\times \left(\int_{\frac{1}{[2]_{(p_1,q_1)}}}^1 \int_0^{\frac{1}{[2]_{(p_2,q_2)}}} \left| t - \frac{[5]_{(p_1,q_1)}}{[6]_{(p_1,q_1)}} \right| \left| s - \frac{1}{[6]_{(p_2,q_2)}} \right| d_{(p_1,q_1)}\, t\, d_{(p_2,q_2)}\, s \right.$$

$$\left. \times \left| {}^{\pi_2,\pi_4}\check{D}^{\,2}_{(p_1,q_1),(p_2,q_2)}\, \mathcal{F} \right|^{t_1} d_{(p_1,q_1)}\, t\, d_{(p_2,q_2)}\, s \right)^{\frac{1}{t_1}}$$

$$+ \left(\int_{\frac{1}{[2]_{(p_1,q_1)}}}^1 \int_{\frac{1}{[2]_{(p_2,q_2)}}}^1 \left| t - \frac{[5]_{(p_1,q_1)}}{[6]_{(p_1,q_1)}} \right| \left| s - \frac{[5]_{(p_2,q_2)}}{[6]_{(p_2,q_2)}} \right| d_{(p_1,q_1)}\, t\, d_{(p_2,q_2)}\, s \right)^{1-\frac{1}{t_1}}$$

$$\times \left(\int_{\frac{1}{[2]_{(p_1,q_1)}}}^1 \int_0^{\frac{1}{[2]_{(p_2,q_2)}}} \left| t - \frac{[5]_{(p_1,q_1)}}{[6]_{(p_1,q_1)}} \right| \left| s - \frac{[5]_{(p_2,q_2)}}{[6]_{(p_2,q_2)}} \right| d_{(p_1,q_1)}\, t\, d_{(p_2,q_2)}\, s \right.$$

$$\left. \times \left| {}^{\pi_2,\pi_4}\check{D}^{\,2}_{(p_1,q_1),(p_2,q_2)}\, \mathcal{F} \right|^{t_1} d_{(p_1,q_1)}\, t\, d_{(p_2,q_2)}\, s \right)^{\frac{1}{t_1}} \Bigg].$$

Using the values from Equations (19)–(22) and again using the generalized convexity of the function on the second coordinate,

$$\left| q_1 q_2 \grave{O}_1 \grave{O}_2 \int_0^1 \int_0^1 \Lambda_{(p_1,q_1)}\, t\, \Lambda_{(p_2,q_2)}\, s \; {}^{\pi_2,\pi_4}\check{D}^{2}_{(p_1,q_1),(p_2,q_2)}\; \mathcal{F}_{(p_1,q_1)}\, t\, d_{(p_2,q_2)}\, s \right|$$

$$\le q_1 q_2 \grave{O}_1 \grave{O}_2 [K(p_1,q_1)K(p_2,q_2)]^{1-\frac{1}{t_1}}$$
$$\times \left[A_1(p_1,q_1) \left(\begin{array}{c} \int_0^{\frac{1}{[2](p_2,\,q_2)}} (s) \left| s - \frac{1}{[6]_{(p_2,q_2)}} \right| d_{(p_2,q_2)}\, s \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)} \mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}\, t\; {}^{\pi_4}\partial_{(p_2,q_2)}\, s} \right|^{t_1} \\ + \int_0^{\frac{1}{[2](p_2,\,q_2)}} (1-s) \left| s - \frac{1}{[6]_{(p_2,q_2)}} \right| d_{(p_2,q_2)}\, s \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)} \mathcal{F}(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{(p_1,q_1)}\, t\; {}^{\pi_4}\partial_{(p_2,q_2)}\, s} \right|^{t_1} \end{array} \right) \right.$$
$$\left. + B_1(p_1,q_1) \left(\begin{array}{c} \int_0^{\frac{1}{[2](p_2,\,q_2)}} s \left| s - \frac{1}{[6]_{(p_2,q_2)}} \right| d_{(p_2,q_2)}\, s \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)} \mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}\, t\; {}^{\pi_4}\partial_{(p_2,q_2)}\, s} \right|^{t_1} \\ + \int_0^{\frac{1}{[2](p_2,\,q_2)}} (1-s) \left| s - \frac{1}{[6]_{(p_2,q_2)}} \right| d_{(p_2,q_2)}\, s \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)} \mathcal{F}(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{(p_1,q_1)}\, t\; {}^{\pi_4}\partial_{(p_2,q_2)}\, s} \right|^{t_1} \end{array} \right) \right]^{\frac{1}{t_1}}$$

$$+ q_1 q_2 \grave{O}_1 \grave{O}_2 [K(p_1,q_1)J(p_2,q_2)]^{1-\frac{1}{t_1}}$$
$$\times \left[A_1(p_1,q_1) \left(\begin{array}{c} \int_{\frac{1}{[2](p_2,\,q_2)}}^1 \left| s - \frac{[5](p_2,\,q_2)}{[6]_{(p_2,q_2)}} \right| d_{(p_2,q_2)}\, s \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)} \mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}\, t\; {}^{\pi_4}\partial_{(p_2,q_2)}\, s} \right|^{t_1} \\ + \int_{\frac{1}{[2](p_2,\,q_2)}}^1 (1-s) \left| s - \frac{[5](p_2,\,q_2)}{[6]_{(p_2,q_2)}} \right| d_{(p_2,q_2)}\, s \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)} \mathcal{F}(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{(p_1,q_1)}\, t\; {}^{\pi_4}\partial_{(p_2,q_2)}\, s} \right|^{t_1} \end{array} \right) \right.$$
$$\left. + B_1(p_1,q_1) \left(\begin{array}{c} \int_{\frac{1}{[2](p_2,\,q_2)}}^1 \left| s - \frac{[5](p_2,\,q_2)}{[6]_{(p_2,q_2)}} \right| d_{(p_2,q_2)}\, s \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)} \mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}\, t\; {}^{\pi_4}\partial_{(p_2,q_2)}\, s} \right|^{t_1} \\ + \int_{\frac{1}{[2](p_2,\,q_2)}}^1 (1-s) \left| s - \frac{[5](p_2,\,q_2)}{[6]_{(p_2,q_2)}} \right| d_{(p_2,q_2)}\, s \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)} \mathcal{F}(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{(p_1,q_1)}\, t\; {}^{\pi_4}\partial_{(p_2,q_2)}\, s} \right|^{t_1} \end{array} \right) \right]^{\frac{1}{t_1}}$$

$$+ q_1 q_2 \grave{O}_1 \grave{O}_2 [J(p_1,q_1)K(p_2,q_2)]^{1-\frac{1}{t_1}}$$
$$\times \left[B_2(p_1,q_1) \left(\begin{array}{c} \int_{\frac{1}{[2](p_2,\,q_2)}}^1 (s) \left| s - \frac{[5](p_2,\,q_2)}{[6]_{(p_2,q_2)}} \right| d_{(p_2,q_2)}\, s \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)} \mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}\, t\; {}^{\pi_4}\partial_{(p_2,q_2)}\, s} \right|^{t_1} \\ + \int_0^{\frac{1}{[2](p_2,\,q_2)}} (1-s) \left| s - \frac{1}{[6]_{(p_2,q_2)}} \right| d_{(p_2,q_2)}\, s \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)} \mathcal{F}(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{(p_1,q_1)}\, t\; {}^{\pi_4}\partial_{(p_2,q_2)}\, s} \right|^{t_1} \end{array} \right) \right.$$
$$\left. + B_2(p_1,q_1) \left(\begin{array}{c} + \int_{\frac{1}{[2](p_2,\,q_2)}}^1 (s) \left| s - \frac{[5](p_2,\,q_2)}{[6]_{(p_2,q_2)}} \right| d_{(p_2,q_2)}\, s \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)} \mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}\, t\; {}^{\pi_4}\partial_{(p_2,q_2)}\, s} \right|^{t_1} \\ + \int_0^{\frac{1}{[2](p_2,\,q_2)}} (1-s) \left| s - \frac{1}{[6]_{(p_2,q_2)}} \right| d_{(p_2,q_2)}\, s \left| \frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)} \mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}\, t\; {}^{\pi_4}\partial_{(p_2,q_2)}\, s} \right|^{t_1} \end{array} \right) \right]^{\frac{1}{t_1}}$$

$$+q_1q_2\grave{O}_1\grave{O}_2[J(p_1,q_1)J(p_2,q_2)]^{1-\frac{1}{t_1}}$$

$$\times\left[B_2(p_1,q_1)\left(\begin{array}{c}\int_{\frac{1}{[2]_{(p_2,q_2)}}}^{1}(s)\left|s-\frac{[5]_{(p_2,q_2)}}{[6]_{(p_2,q_2)}}\right|d_{(p_2,q_2)}\,s\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}\,t\;{}^{\pi_4}\partial_{(p_2,q_2)}\,s}\right|^{t_1}\\ \int_{\frac{1}{[2]_{(p_2,q_2)}}}^{1}(1-s)\left|s-\frac{[5]_{(p_2,q_2)}}{[6]_{(p_2,q_2)}}\right|d_{(p_2,q_2)}\,s\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}\,t\;{}^{\pi_4}\partial_{(p_2,q_2)}\,s}\right|^{t_1}\end{array}\right)\right.$$

$$\left.B_2(p_1,q_1)\left(\begin{array}{c}\int_{\frac{1}{[2]_{(p_2,q_2)}}}^{1}(s)\left|s-\frac{[5]_{(p_2,q_2)}}{[6]_{(p_2,q_2)}}\right|d_{(p_2,q_2)}\,s\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}\,t\;{}^{\pi_4}\partial_{(p_2,q_2)}\,s}\right|^{t_1}\\ \int_{\frac{1}{[2]_{(p_2,q_2)}}}^{1}(1-s)\left|s-\frac{[5]_{(p_2,q_2)}}{[6]_{(p_2,q_2)}}\right|d_{(p_2,q_2)}\,s\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{(p_1,q_1),(p_2,q_2)}\mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{(p_1,q_1)}\,t\;{}^{\pi_4}\partial_{(p_2,q_2)}\,s}\right|^{t_1}\end{array}\right)\right]^{\frac{1}{t_1}}.$$

Here,

$$J(p_i,q_i)=\int_0^{\frac{1}{[2]_{(p_i,q_i)}}}\left|t-\frac{1}{[6]_{(p_i,q_i)}}\right|d_{(p_i,q_i)}\;t\;=\frac{2\left([2]_{(p_i,q_i)}-1\right)}{[2]_{(p_i,q_i)}[6]^2_{(p_i,q_i)}}+\frac{[6]_{(p_i,q_i)}-[2]^2_{(p_i,q_i)}}{[2]^3_{(p_i,q_i)}[6]_{(p_i,q_i)}},$$

$$\begin{aligned}K(p_i,q_i)&=\int_{\frac{1}{[2]_{(p_i,q_i)}}}^{1}\left|s-\frac{[5]_{(p_i,q_i)}}{[6]_{(p_i,q_i)}}\right|d_{(p_i,q_i)}\;s\\&=\frac{2[5]^2_{(p_i,q_i)}\left([2]_{(p_i,q_i)}-1\right)}{[2]_{(p_i,q_i)}[6]^2_{(p_i,q_i)}}+\frac{[6]_{(p_i,q_i)}-[2]^2_{(p_i,q_i)}[5]_{(p_i,q_i)}}{[2]^3_{(p_i,q_i)}[6]_{(p_i,q_i)}}+\frac{1}{[2]_{(p_i,q_i)}}-\frac{[5]_{(p_i,q_i)}}{[6]_{(p_i,q_i)}},\end{aligned}$$

where $i\in\{1,2\}$. Taking the values of $A_1(p_i,q_i)$, $A_2(p_i,q_i)$, $B_1(p_i,q_i)$, and $B_2(p_i,q_i)$, where $i\in\{1,2\}$, from (19)–(22) gives us the required result. □

Remark 6. *By substituting $p_1=p_2=1$ in Theorem 4, we have the identity*

$$\Bigg|\tfrac{1}{[6]_{q_1}[6]_{q_2}}\Big[q_1^2[4]_{q_1}\mathcal{F}\Big(\pi_2+\tfrac{\pi_1-\pi_2}{[2]_{q_1}},\pi_3\Big)+q_1^2q_2^2[4]_{q_1}[4]_{q_2}\mathcal{F}\Big(\pi_2+\tfrac{\pi_1-\pi_2}{[2]_{q_1}},\pi_4+\tfrac{\pi_3-\pi_4}{[2]_{q_2}}\Big)$$
$$+q_1^2q_2[4]_{q_1}\mathcal{F}\Big(\pi_2+\tfrac{\pi_1-\pi_2}{[2]_{q_1}},\pi_4\Big)+q_1q_2^2[4]_{q_2}\mathcal{F}\Big(\pi_2,\pi_4+\tfrac{\pi_3-\pi_4}{[2]_{q_2}}\Big)+q_2^2[4]_{q_2}\mathcal{F}\Big(\pi_1,\pi_4+\tfrac{\pi_3-\pi_4}{[2]_{q2}}\Big)\Big]$$

$$+\tfrac{q_1\mathcal{F}(\pi_2,\pi_3)+\mathcal{F}(\pi_1,\pi_3)+q_2\mathcal{F}(\pi_1,\pi_4)+q_1q_2\mathcal{F}(\pi_2,\pi_4)}{[6]_{q_1}[6]_{q_2}}+\int_{\pi_1}^{\pi_2}\int_{\pi_3}^{\pi_4}\tfrac{\mathcal{F}(x,y)\;{}^{\pi_2}d_{q_1}(x)\;{}^{\pi_4}d_{q_2}(y)}{(\pi_2-\pi_1)(\pi_4-\pi_3)}$$
$$-\tfrac{1}{[6]_{q_2}(\pi_2-\pi_1)}\int_{\pi_1}^{\pi_2}\Big[\mathcal{F}(x,\pi_3)+q_2\mathcal{F}(x,\pi_4)+q_2^2[4]_{q_2}\mathcal{F}\Big(x,\pi_4+\tfrac{\pi_3-\pi_4}{[2]_{q_2}}\Big)\Big]\;{}^{\pi_2}d_{q_1}(x)$$

$$-\tfrac{1}{[6]_{q_1}(\pi_4-\pi_3)}\int_{\pi_3}^{\pi_4}\Big[q_1\mathcal{F}(\pi_2,y)+q_1^2[4]_{q_1}\mathcal{F}\Big(\pi_2+\tfrac{(\pi_1-\pi_2)}{[2]_{q1}},y\Big)+\mathcal{F}(\pi_1,y)\Big]\;{}^{\pi_4}d_{q_2}y\Bigg|$$
$$\leq q_1q_2(\pi_2-\pi_1)(\pi_4-\pi_3)\Bigg[\Bigg\{\Bigg(\tfrac{2q_1}{[2]_{q_1}[6]^2_{q_1}}+\tfrac{q_1^3[3]_{q_1}-q_1}{[6]_{q_1}[2]^3_{q_1}}\Bigg)\Bigg(\tfrac{2q_2}{[2]_{q_2}[6]^2_{q_2}}+\frac{q_2^3[3]_{q_2}-q_2}{[6]_{q_2}[2]^3_{q_2}}\Bigg)\Bigg\}^{1-\frac{1}{p_1}}$$

$$\times\Bigg\{A_1(q_1)\Bigg(A_1(q_2)\left|\tfrac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}\mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}\,t\;{}^{\pi_4}\partial_{q_2}\,s}\right|^{p_1}+B_1(q_2)\left|\tfrac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}\mathcal{F}(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}\,t\;{}^{\pi_4}\partial_{q_2}\,s}\right|^{p_1}\Bigg)+B_1(q_1)$$

$$\Bigg(A_1(q_2)\left|\tfrac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}\mathcal{F}(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}\,t\;{}^{\pi_4}\partial_{q_2}\,s}\right|^{p_1}+B_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}\mathcal{F}(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}\;t\;{}^{\pi_4}\partial_{q_2}\;s}\right|^{p_1}\Bigg)\Bigg\}^{\frac{1}{p_1}}$$

$$+\left\{\left(\frac{2q_2[5]_{q_2}^2}{[2]_{q_2}[6]_{q_2}^2}+\frac{1}{[2]_{q_2}}-\frac{[5]_{q_2}}{[6]_{q_2}}-\frac{[2]_{q_2}^2[5]_{q_2}-[6]_{q_2}}{[6]_{q_2}[2]_{q_2}^3}\right)\left(\frac{2q_1}{[2]_{q_1}[6]_{q_1}^2}+\frac{q_1^3[3]_{q_1}-q_1}{[6]_{q_1}[2]_{q_1}^3}\right)\right\}^{1-\frac{1}{p_1}}$$

$$\times\left\{A_1(q_1)\left(A_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}\mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}t\,{}^{\pi_4}\partial_{q_2}s}\right|^{p_1}+B_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}\mathcal{F}(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}t\,{}^{\pi_4}\partial_{q_2}s}\right|^{p_1}\right)\right.$$

$$\left.+B_1(q_1)\left(A_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}\mathcal{F}(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}t\,{}^{\pi_4}\partial_{q_2}s}\right|^{p_1}+B_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}\mathcal{F}(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}t\,{}^{\pi_4}\partial_{q_2}s}\right|^{p_1}\right)\right\}^{\frac{1}{p_1}}$$

$$+\left\{\left(\frac{2q_1[5]_{q_1}^2}{[2]_{q_1}[6]_{q_1}^2}+\frac{1}{[2]_{q_1}}-\frac{[5]_{q_1}}{[6]_{q_1}}-\frac{[2]_{q_1}^2[5]_{q_1}-[6]_{q_1}}{[6]_{q_1}[2]_{q_1}^3}\right)\left(\frac{2q_2}{[2]_{q_2}[6]_{q_2}^2}+\frac{q_1^3[3]_{q_2}-q_2}{[6]_{q_2}[2]_{q_2}^3}\right)\right\}^{1-\frac{1}{p_1}}$$

$$\times\left\{A_2(q_1)\left(A_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}\mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}t\,{}^{\pi_4}\partial_{q_2}s}\right|^{p_1}+B_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}\mathcal{F}(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}t\,{}^{\pi_4}\partial_{q_2}s}\right|^{p_1}\right)\right.$$

$$\left.+B_2(q_1)\left(A_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}\mathcal{F}(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}t\,{}^{\pi_4}\partial_{q_2}s}\right|^{p_1}+B_1(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}\mathcal{F}(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}t\,{}^{\pi_4}\partial_{q_2}s}\right|^{p_1}\right)\right\}^{\frac{1}{p_1}}$$

$$+\left\{\left(2q_1\frac{[5]_{q_1}^2}{[2]_{q_1}[6]_{q_1}^2}+\frac{1}{[2]_{q_1}}-\frac{[5]_{q_1}}{[6]_{q_1}}-\frac{[2]_{q_1}^2[5]_{q_1}-[6]_{q_1}}{[6]_{q_1}[2]_{q_1}^3}\right)\right.$$

$$\left.\times\left(2q_2\frac{[5]_{q_2}^2}{[2]_{q_2}[6]_{q_2}^2}+\frac{1}{[2]_{q_2}}-\frac{[5]_{q_2}}{[6]_{q_2}}-\frac{[2]_{q_2}^2[5]_{q_2}-[6]_{q_2}}{[6]_{q_2}[2]_{q_2}^3}\right)\right\}^{1-\frac{1}{p_1}}$$

$$\times\left\{A_2(q_1)\left(A_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}\mathcal{F}(\pi_1,\pi_3)}{{}^{\pi_2}\partial_{q_1}t\,{}^{\pi_4}\partial_{q_2}s}\right|^{p_1}+B_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}\mathcal{F}(\pi_1,\pi_4)}{{}^{\pi_2}\partial_{q_1}t\,{}^{\pi_4}\partial_{q_2}s}\right|^{p_1}\right)\right.$$

$$\left.\left.+B_2(q_1)\left(A_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}\mathcal{F}(\pi_2,\pi_3)}{{}^{\pi_2}\partial_{q_1}t\,{}^{\pi_4}\partial_{q_2}s}\right|^{p_1}+B_2(q_2)\left|\frac{{}^{\pi_2,\pi_4}\partial^2_{q_1,q_2}\mathcal{F}(\pi_2,\pi_4)}{{}^{\pi_2}\partial_{q_1}t\,{}^{\pi_4}\partial_{q_2}s}\right|^{p_1}\right)\right\}\right]^{\frac{1}{p_1}}.$$

6. Conclusions

Finally, this work proposes novel post quantum integral identities for coordinated generalized convex functions in the context of the right post quantum integrals, utilizing Raina's function. These findings are used to develop the pq-version of a Simpson's type inequality for coordinated generalized convex functions. Furthermore, setting $p_1 = p_2 = 1$ yields previously known results from the literature. Future research can build upon these findings to investigate new inequalities for different types of generalized convex functions, making use of this coordinated version to explore novel and exciting problems in the field of post quantum calculus. Overall, this study contributes to the understanding of symmetry in post quantum calculus and provides new avenues for further exploration in the future. It is worth mentioning that using the post-quantum calculus approach and symmetric functions one can establish Fejér type of inequalities on coordinates. This will be an interesting future research topic.

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