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Formation of Singularity for Isentropic Irrotational Compressible Euler Equations

Jianli Liu ^{1,†} , Ziyi Qin ^{1,†} and Manwai Yuen ^{2,*} 
¹ Department of Mathematics, Shanghai University, Shanghai 200444, China; jlliu@shu.edu.cn (J.L.); zyqin@shu.edu.cn (Z.Q.)

² Department of Mathematics and Information Technology, The Education University of Hong Kong, Hong Kong, China

* Correspondence: yuenmw@eduhk.hk

† These authors contributed equally to this work.

Abstract: The domain of science and engineering relies heavily on an in-depth comprehension of fluid dynamics, given the prevalence of fluids such as water, air, and interstellar gas in the universe. Euler equations form the basis for the study of fluid motion. This paper is concerned with the Cauchy problem of isentropic compressible Euler equations away from the vacuum. We use the integration method with the general test function $f = f(r)$, proving that there exist the corresponding blowup results of C^1 irrotational solutions for Euler equations and Euler equations with time-dependent damping in \mathbb{R}^n ($n \geq 2$), provided the density-independent initial functional is sufficiently large. We also provide two simple and explicit test functions $f(r) = r$ and $f(r) = 1 + r$, to demonstrate the blowup phenomenon in the one-dimensional case. In particular, our results are applicable to the non-radial system.

Keywords: Euler equations; blowup; irrotational solutions; time-dependent damping; initial value problem; test function



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1. Introduction

The compressible Euler equations are used to describe the motion of an ideal fluid, incorporating the conservation laws of mass, momentum, and energy. These equations play a significant role in various applications such as analyzing aircraft engine thrust and examining fluid states at engine inlet and exhaust [1]. Readers can refer to [2–4] for a more in-depth discussion of physical background. The isentropic compressible Euler equations in \mathbb{R}^n ($n \geq 2$) are expressed as

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \rho[\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u}] + \nabla P = \mathbf{0}, \end{cases} \quad (1)$$

where unknown functions $\rho(t, \mathbf{x})$ and $\mathbf{u}(t, \mathbf{x}) = (u_1, u_2, \dots, u_n)(t, \mathbf{x})$ with $\mathbf{x} = (x_1, \dots, x_n)$ represent the density and velocity of the fluid. The pressure $P = K\rho^\gamma$ with constants $K > 0$ and $\gamma > 1$.

As the classical system in the fields of aerodynamics and mathematics, Euler equations have been the subject of extensive study. Given that Euler equations can be rephrased as first-order quasilinear hyperbolic systems, there exist local existence and uniqueness theorems for classical solutions, as can be found in [5,6]. In [7], Chen gave the local existence of smooth solutions for three-dimensional Euler equations with initial conditions away from the vacuum. The local well-posedness for Euler equations is also included in [8,9]. Particularly, the investigations of singularities and life span estimation of solutions for

Euler equations have captivated numerous mathematicians and physicists. In this paper, we focus on the blowup phenomenon of solutions for Euler Equation (1) with initial data

$$\begin{cases} (\rho(0, \mathbf{x}), \mathbf{u}(0, \mathbf{x})) = (\rho_0(\mathbf{x}) := \bar{\rho} + \rho^0 \mathbf{x}), \mathbf{u}_0(\mathbf{x}), \\ \text{supp}(\rho^0, \mathbf{u}_0) \subseteq \{\mathbf{x} : |\mathbf{x}| \leq R\}, \end{cases} \quad (2)$$

with some positive constants $\bar{\rho}$ and R . For the Cauchy problem with non-vacuum initial values and compact support, Sideris [10] initially proposed the integral functional

$$F(t) = \int_{\mathbb{R}^3} \rho \mathbf{x} \cdot \mathbf{u} dV \quad (3)$$

and established the finite-time singularity of solutions for three-dimensional non-isentropic Euler equations with sufficiently large $F(0)$. Subsequently, in [11], Zhu, Tu, and Fu further derived the corresponding blowup results with a less restraining condition for this system. Then, Yuen delved into the lifespan of two-dimensional projected C^2 solutions of Euler Equation (1) in [12] by employing a new density-independent functional

$$F(t) = \int_{\mathbb{R}^n} \mathbf{x} \cdot \mathbf{u} d\mathbf{x}. \quad (4)$$

Those interested can find more comprehensive studies on the blowup phenomena of Euler equations in [13–20].

Then, we proceed to consider irrotational solutions of Euler Equation (1), namely, the velocity satisfies

$$\nabla \times \mathbf{u} = \mathbf{0}. \quad (5)$$

By employing the extended vector analysis formula

$$(\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{1}{2} \nabla(|\mathbf{u}|^2) - \mathbf{u} \times \nabla \times \mathbf{u}, \quad (6)$$

rotational Euler Equation (1) can be written in following form.

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \rho[\mathbf{u}_t + \frac{1}{2} \nabla(|\mathbf{u}|^2)] + \nabla P = \mathbf{0}, \end{cases} \quad (7)$$

where $|\mathbf{u}| = (\sum_{i=1}^n u_i^2)^{\frac{1}{2}}$. Equation (7) denotes the irrotational Euler equations, also known as potential flows [8]. In radial symmetry, Euler Equation (1) can be written as

$$\begin{cases} \rho_t + V\rho_r + \rho V_r + \frac{n-1}{r} \rho V = 0, \\ \rho(V_t + VV_r) + P_r = 0, \end{cases} \quad (8)$$

where

$$\rho = \rho(t, r), \quad \mathbf{u} = \frac{\mathbf{x}}{r} V := \frac{\mathbf{x}}{r} V(t, r) \quad \text{and} \quad r = |\mathbf{x}| = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}. \quad (9)$$

Therefore, the solutions of Equation (8) are irrotational solutions. In [21], by concerning the preserved total mass and energy, as well as the degenerate total pressure, Suzuki investigated the non-existence of global-in-time, irrotational solutions for Euler equations in \mathbb{R}^3 with a vacuum state. In [22], Yuen improved the previous results with functional (4), providing the blowup proof of multi-dimensional irrotational solutions for Euler equations in non-radial symmetry.

Nevertheless, it is essential to acknowledge that Euler Equation (1) exhibits inherent constraints, being solely applicable to ideal fluids and disregarding the viscous effects. Consequently, it becomes imperative to incorporate more intricate and precise models,

such as adding damping terms, for practical applications. Describing the movement of compressible fluid through a porous medium involves employing the compressible Euler equations with time-dependent damping, represented by

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \rho[\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u}] + \nabla P = -\frac{\mu}{(1+t)^\lambda} \rho \mathbf{u}, \end{cases} \quad (10)$$

where $\mu > 0$ and $\lambda \in [-1, 1)$. When $\lambda \in [0, 1)$, the damping effect $-\frac{\mu}{(1+t)^\lambda} \rho \mathbf{u}$ weakens with time t , denoted as the underdamping case. Conversely, for $\lambda \in [-1, 0)$, the damping effect intensifies with time t , termed the overdamping case. When $\mu = 0$, system (10) reduces to the original Euler Equation (1). For further insights, we refer readers to [23,24] and the references therein.

The isentropic Euler equations with the friction term can also be viewed as a model of hyperbolic conservation laws with damping. In [25], Kato proved that solutions of the Cauchy problem for full quasilinear symmetric hyperbolic systems exist only for a small time interval. For the initial data around the constant states, Pan [26,27], Hou, and Yin [28] gave the global existence and blowup of smooth solutions to Equation (10) in one, two, and three dimensions, respectively. Liu's work in [29] imparted a detailed account of the boundary singular and time-asymptotic behaviors of Euler equations with linear damping near vacuum. In [30], Cheung and Wong explored the blowup of radial solutions for the initial-boundary value problem of multi-dimensional Equation (10) with $\mu > 0$, $\lambda \geq 0$. Readers can refer to [31–33] for pertinent studies of Euler equations with damping.

Similarly, articulating irrotational Euler equations with time-dependent damping is encapsulated in the following formulations.

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \rho[\mathbf{u}_t + \frac{1}{2} \nabla(|\mathbf{u}|^2)] + \nabla P = -\frac{\mu}{(1+t)^\lambda} \rho \mathbf{u}. \end{cases} \quad (11)$$

In [34], building upon the same functionals as in [19], Liu, Wang, and Yuen expanded the corresponding blowup theory to irrotational solutions for multi-dimensional Equation (11) in a vacuum setting.

2. Materials and Methods

In the study of the blowup for solutions, the integration functional method is frequently employed, with the objective of demonstrating that the singularity of solutions will inevitably develop within a finite time if the initial data of the functional are sufficiently large. In the following research, we shall use the integration method with the test function to illustrate the blowup conditions of solutions for Equations (7) and (11).

There are certain blowup findings regarding Euler equations obtained through the utilization of the integral functional with test functions. In [19], Lei, Du, and Zhang utilized test functions $\frac{1}{re^r}$ and the modified Bessel function associated with the radius r , establishing that solutions for Euler equations in \mathbb{R}^2 and \mathbb{R}^3 must undergo blowup in finite time when initial values exhibit radial symmetry and involve vacuum conditions. Then, Wong and Yuen [35] discovered a non-negative and strictly increasing C^1 test function $f(r)$ and applied the functional

$$F(t) = \int_0^\infty f(r) V(t, r) dr, \quad (12)$$

to derive new blowup conditions of solutions for multi-dimensional Euler Equation (8). Their findings indicated that singular solutions with radial symmetry must occur in finite time when $F(0)$ reaches a sufficient value. Additionally, Cheung, Wong and Yuen con-

structed a test function $f = f(r)$ that represents an increasing C^1 property on $[0, +\infty)$ and vanishes at $r = 1$ in [36]. They used the functional

$$F(t) = \int_{\Omega} f \rho \mathbf{x} \cdot \mathbf{u} d\mathbf{x}, \quad (13)$$

to address the initial-boundary value problem with $\mathbf{u} \cdot \mathbf{n}|_{|\mathbf{x}|=1} = 0$ and $\Omega := \{\mathbf{x} : |\mathbf{x}| > 1\}$ of three-dimensional non-isentropic Euler equations. Recently, Wu and Wang formulated the functional

$$F(t) = \int_0^\infty e^{-r} \rho V(t, r) dr \quad (14)$$

with an exact test function $f(r) = e^{-r}$ in [37], to demonstrate the blowup phenomena of spherically symmetric solutions for non-isentropic Euler equations, without requiring the initial velocity to have a compact support and the initial density and entropy to be equal to a constant outside the support of the initial velocity.

In this article, we discover two general test functions $f_1 = f_1(r)$ for $\gamma > 1$ and $f_2 = f_2(r)$ for $\gamma \geq 2$ with $r = |\mathbf{x}|$, where $f_1(r)$ is a strictly increasing C^1 function on $[0, +\infty)$ satisfying $f_1(0) = 0$ and $f_2(r)$ is a strictly increasing C^1 function on $[0, +\infty)$ satisfying $f_2(0) \geq \frac{1}{n}$. By utilizing density-independent functionals

$$F_1(t) = \int_{\mathbb{R}^n} f_1 \mathbf{x} \cdot \mathbf{u} d\mathbf{x} \quad (15)$$

and

$$F_2(t) = \int_{\mathbb{R}^n} f_2 \mathbf{x} \cdot \mathbf{u} d\mathbf{x}, \quad (16)$$

we examine the blowup behavior of irrotational C^1 solutions for n -dimensional compressible Euler Equation (7) and Euler equations with time-dependent damping (11) within the setting of initial conditions (2). For enhanced comprehension, the corresponding blowup criteria of the one-dimensional case with two simple test functions $f_1(r) = r$ and $f_2(r) = 1 + r$ are also included. Our results further expand the conclusions presented in [22] and remain applicable for the non-radial system.

3. Results

In this section, we shall present our research results and furnish detailed proofs.

3.1. Main Theorems

In this part, we introduce the main theorems unveiled through our investigation. For Euler Equation (7), the theorems are as follows.

Theorem 1. Fix $T \in [0, +\infty)$ and $a \in (0, n)$. Consider the C^1 solutions of system (7) and (2) with $\gamma > 1$ in \mathbb{R}^n . If $F_1(0)$ is sufficient such that

$$F_1(0) \geq \sqrt{\frac{2f_1(R + \sigma T)K_1(T)}{a}}(R + \sigma T)V(T) \quad (17)$$

and

$$F_1(0) \geq \frac{2}{n-a} \left(\int_0^T \frac{dt}{(R + \sigma t)^2 f_1(R + \sigma t) V(t)} \right)^{-1}, \quad (18)$$

where

$$K_1(t) := \frac{K\gamma}{\gamma-1} \bar{\rho}^{\gamma-1} \left[n f_1(R + \sigma t) + (R + \sigma t) \max_{r \leq R + \sigma t} f_1'(r) \right] \quad (19)$$

and

$$V(t) := \frac{\pi^{\frac{n}{2}} (R + \sigma t)^n}{\Gamma(\frac{n}{2} + 1)}, \quad (20)$$

with Gamma function $\Gamma(\eta) = \int_0^{+\infty} \frac{s^{\eta-1}}{e^s} ds$ ($\eta > 0$) and $\sigma = \sqrt{K\gamma\bar{\rho}^{\gamma-1}}$, then solutions will blow up on or before time T .

Theorem 2. Fix $T \in [0, +\infty)$. Consider the C^1 solutions of system (7) and (2) with $\gamma = 2$ in \mathbb{R}^n . Let $V(t)$ be defined by (20),

$$M(0) := \int_{\mathbb{R}^n} (\rho_0 - \bar{\rho}) dx \quad (21)$$

and

$$K_2(t) := 2K\bar{\rho} \left[n f_2(R + \sigma t) + (R + \sigma t) \max_{r \leq R + \sigma t} f_2'(r) \right]. \quad (22)$$

1. For $M(0) \geq \frac{K_2(T)V(T)}{2K}$: If $F_2(0)$ is sufficient such that

$$F_2(0) \geq \frac{2}{n} \left(\int_0^T \frac{dt}{(R + \sigma t)^2 f_2(R + \sigma t) V(t)} \right)^{-1}, \quad (23)$$

then solutions will blow up on or before time T .

2. For $M(0) < \frac{K_2(T)V(T)}{2K}$: Fix $b \in (0, n)$. If $F_2(0)$ is sufficient such that

$$F_2(0) \geq \sqrt{\frac{2f_2(R + \sigma T)C_1(T)V(T)}{b}} (R + \sigma T) \quad (24)$$

and

$$F_2(0) \geq \frac{2}{n-b} \left(\int_0^T \frac{dt}{(R + \sigma t)^2 f_2(R + \sigma t) V(t)} \right)^{-1}, \quad (25)$$

where

$$C_1(t) := -(2KM(0) - K_2(t)V(t)), \quad (26)$$

then solutions will blow up on or before time T .

Theorem 3. Fix $T \in [0, +\infty)$ and $c \in (0, n)$. Consider the C^1 solutions of system (7) and (2) with $\gamma > 2$ in \mathbb{R}^n . If $F_2(0)$ is sufficient such that

$$F_2^2(0) \geq \frac{2C_2(R + \sigma T)^2 f_2(R + \sigma T) V(T)}{c} \quad (27)$$

and

$$F_2(0) \geq \frac{2}{n-c} \left(\int_0^T \frac{dt}{(R + \sigma t)^2 f_2(R + \sigma t) V(t)} \right)^{-1}, \quad (28)$$

where

$$C_2 := - \left[\frac{K\gamma}{\gamma-1} \min_{t \in [0, T]} (M(0) + \bar{\rho}V(t))^{\gamma-1} V^{2-\gamma}(t) - K_3(T)V(T) \right], \quad (29)$$

$$K_3(t) := \frac{K\gamma}{\gamma-1} \bar{\rho}^{\gamma-1} \left[n f_2(R + \sigma t) + (R + \sigma t) \max_{r \leq R + \sigma t} f_2'(r) \right], \quad (30)$$

$V(t)$ is defined by (20) and $M(0)$ is defined by (21), then solutions will blow up on or before time T .

Remark 1. If $M(0) \geq \left(\frac{\gamma-1}{K\gamma} K_3(T) \right)^{\frac{1}{\gamma-1}} V(T)$, we have $M(0) > 0$ and

$$\frac{K\gamma}{\gamma-1} \min_{t \in [0, T]} (M(0) + \bar{\rho}V(t))^{\gamma-1} V^{2-\gamma}(t) > \frac{K\gamma}{\gamma-1} \min_{t \in [0, T]} M^{\gamma-1}(0) V^{2-\gamma}(t) \quad (31)$$

$$\geq \frac{K\gamma}{\gamma-1} M^{\gamma-1}(0) V^{2-\gamma}(T) \quad (32)$$

$$\geq K_3(T)V(T). \quad (33)$$

Then, $C_2 < 0$. Therefore, the condition (27) in Theorem 3 can be removed.

For Euler equations with time-dependent damping (11), the theorem is outlined as follows.

Theorem 4. Fix $T \in [0, +\infty)$ and $d \in (0, n)$. Consider the C^1 solutions of system (11) and (2) with $\gamma > 1$ in \mathbb{R}^n . Let $K_1(t)$ and $V(t)$ be defined by (19) and (20),

$$A =: \frac{d}{2(R + \sigma T)^2 f_1(R + \sigma T) V(T)} \quad (34)$$

and

$$C =: K_1(T) V(T). \quad (35)$$

1. For the underdamping case: If $F_1(0)$ is sufficient such that

$$F_1(0) \geq \frac{\mu + \sqrt{\mu^2 + 4AC}}{2A} \quad (36)$$

and

$$F_1(0) \geq \frac{2}{n-d} \left(\int_0^T \frac{dt}{(R + \sigma t)^2 f_1(R + \sigma t) V(t)} \right)^{-1}, \quad (37)$$

then solutions will blow up at or before time T .

2. For the overdamping case: If $F_1(0)$ is sufficient such that

$$F_1(0) \geq \frac{B + \sqrt{B^2 + 4AC}}{2A} \quad (38)$$

and inequality (37) is satisfied, where

$$B =: \frac{\mu}{(1+T)^\lambda}, \quad (39)$$

then solutions will blow up at or before time T .

3.2. Preliminaries

In this part, we give several important conclusions, and the subsequent research is based on them. The first two lemmas both imply that solutions of Euler equations will always be in the non-vacuum over time in the support of velocity if initial values are in the non-vacuum.

Lemma 1 (Proposition 1.1 in [10]). Let (ρ, \mathbf{u}) be the C^1 solutions of the n -dimensional Euler Equation (7) with initial conditions (2) and a life span $T > 0$. Then, we have

$$(\rho, \mathbf{u}) = (\bar{\rho}, \mathbf{0}) \quad (40)$$

for $t \in [0, T]$ and $|\mathbf{x}| \geq R + \sigma t$, where $\sigma = \sqrt{K\gamma\bar{\rho}^{\gamma-1}} > 0$.

Lemma 2. If (ρ, \mathbf{u}) are the C^1 solutions of Euler Equation (7) and $\rho(0, \mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^n$, then $\rho(t, \mathbf{x}) > 0$ for all $t \geq 0$ and $\mathbf{x} \in \mathbb{R}^n$.

Proof. Along the characteristic curve $\mathbf{x} = \mathbf{x}(t; \mathbf{x}_0)$ that passes through any fixed point $(0, \mathbf{x}_0)$ on the initial axis $t = 0$, the first one of Equation (7) becomes

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \quad (41)$$

with the material derivative $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$.

Through this integration, we obtain

$$\rho(t, \mathbf{x}) = \rho(0, \mathbf{x}_0) \exp\left(-\int_0^t \nabla \cdot \mathbf{u}(s, \mathbf{x}) ds\right) > 0. \quad (42)$$

The proof is completed. \square

In order to further advance the process of proof for Theorems 2 and 3, we need the following lemmas.

Lemma 3. For the C^1 solutions of Euler Equation (7) with initial data (2), the total mass function is conserved, namely,

$$M(0) = M(t) = \int_{\mathbb{R}^n} (\rho - \bar{\rho}) d\mathbf{x}. \quad (43)$$

Proof. Combining the first one of Equation (7) and Green's formula, we have

$$\frac{d}{dt} M(t) = \int_{\mathbb{R}^n} \rho_t d\mathbf{x} = - \int_{\mathbb{R}^n} \nabla(\rho \mathbf{u}) d\mathbf{x} = - \int_{|\mathbf{x}| \leq R+\sigma t} \nabla(\rho \mathbf{u}) d\mathbf{x} = - \int_{|\mathbf{x}|=R+\sigma t} \rho \mathbf{u} \cdot \mathbf{n} dS = 0 \quad (44)$$

with Lemma 1, where \mathbf{n} is the unit outward normal to $S := \{\mathbf{x} : |\mathbf{x}| = R + \sigma t\}$. This means that $M(t) = M(0)$. \square

Then, we obtain the following corollaries immediately.

Corollary 1. For the C^1 solutions of Euler Equation (7) with initial data (2), we have

$$\int_{|\mathbf{x}| \leq R+\sigma t} \rho d\mathbf{x} = M(0) + \bar{\rho} V(t), \quad (45)$$

where $V(t) := \frac{\pi^{\frac{n}{2}} (R+\sigma t)^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ is the volume of an n -dimensional sphere of radius $R + \sigma t$ and $\Gamma(\eta) = \int_0^{+\infty} \frac{s^{\eta-1}}{e^s} ds$ ($\eta > 0$) is the Gamma function.

Proof. According to Lemma 3, it follows that

$$\int_{|\mathbf{x}| \leq R+\sigma t} (\rho - \bar{\rho}) d\mathbf{x} = \int_{\mathbb{R}^n} (\rho - \bar{\rho}) d\mathbf{x} = \int_{\mathbb{R}^n} (\rho_0 - \bar{\rho}) d\mathbf{x} = M(0). \quad (46)$$

Therefore, we have

$$\int_{|\mathbf{x}| \leq R+\sigma t} \rho d\mathbf{x} = M(0) + \int_{|\mathbf{x}| \leq R+\sigma t} \bar{\rho} d\mathbf{x} = M(0) + \bar{\rho} V(t). \quad (47)$$

The proof is completed. \square

Corollary 2. For the C^1 solutions of Euler Equation (7) with initial data (2) and $\gamma \geq 2$, we have

$$\int_{|\mathbf{x}| \leq R+\sigma t} \rho^{\gamma-1} d\mathbf{x} \geq (M(0) + \bar{\rho} V(t))^{\gamma-1} V^{2-\gamma}(t). \quad (48)$$

Proof. In fact, the Hölder inequality can be applied to confirm that

$$\left(\int_{|\mathbf{x}| \leq R+\sigma t} \rho d\mathbf{x} \right)^{\gamma-1} \leq \left(\int_{|\mathbf{x}| \leq R+\sigma t} \rho^{\gamma-1} d\mathbf{x} \right) \left(\int_{|\mathbf{x}| \leq R+\sigma t} 1 d\mathbf{x} \right)^{\gamma-2} \quad (49)$$

$$= V^{\gamma-2}(t) \int_{|\mathbf{x}| \leq R+\sigma t} \rho^{\gamma-1} d\mathbf{x} \quad (50)$$

for $\gamma \geq 2$. This means that

$$\int_{|\mathbf{x}| \leq R+\sigma t} \rho^{\gamma-1} d\mathbf{x} \geq \left(\int_{|\mathbf{x}| \leq R+\sigma t} \rho d\mathbf{x} \right)^{\gamma-1} V^{2-\gamma}(t).$$

The conclusion is obviously true with Corollary 1. \square

3.3. Integration Methods with Test Functions

In this part, we shall give the proof of our main results to show the formation of singular solutions by using the integration method. We first demonstrate the blowup phenomenon of irrotational solutions of ideal compressible Euler Equation (7).

Proof of Theorem 1. For $\gamma > 1$, we consider the second one of Equation (7), which can be written as

$$\mathbf{u}_t + \frac{1}{2} \nabla(|\mathbf{u}|^2) + \frac{K\gamma}{\gamma-1} \nabla(\rho^{\gamma-1} - \bar{\rho}^{\gamma-1}) = 0. \quad (51)$$

Multiplying the above equation by \mathbf{x} and f_1 and integrating over \mathbb{R}^n , by the solutions (ρ, \mathbf{u}) for $|\mathbf{x}| \geq R + \sigma t$ in Lemma 1, we obtain

$$\int_{\mathbb{R}^n} f_1 \mathbf{x} \cdot \mathbf{u}_t d\mathbf{x} = - \int_{\mathbb{R}^n} f_1 \mathbf{x} \cdot \left[\frac{1}{2} \nabla(|\mathbf{u}|^2) + \frac{K\gamma}{\gamma-1} \nabla(\rho^{\gamma-1} - \bar{\rho}^{\gamma-1}) \right] d\mathbf{x} \quad (52)$$

$$= - \int_{|\mathbf{x}| \leq R+\sigma t} f_1 \mathbf{x} \cdot \left[\frac{1}{2} \nabla(|\mathbf{u}|^2) + \frac{K\gamma}{\gamma-1} \nabla(\rho^{\gamma-1} - \bar{\rho}^{\gamma-1}) \right] d\mathbf{x}. \quad (53)$$

Combined with Green's formula, we have

$$\frac{d}{dt} \int_{\mathbb{R}^n} f_1 \mathbf{x} \cdot \mathbf{u} d\mathbf{x} = \int_{|\mathbf{x}| \leq R+\sigma t} \left[\frac{1}{2} |\mathbf{u}|^2 + \frac{K\gamma}{\gamma-1} (\rho^{\gamma-1} - \bar{\rho}^{\gamma-1}) \right] \nabla \cdot (f_1 \mathbf{x}) d\mathbf{x} \quad (54)$$

$$= \int_{|\mathbf{x}| \leq R+\sigma t} \left[\frac{1}{2} |\mathbf{u}|^2 + \frac{K\gamma}{\gamma-1} (\rho^{\gamma-1} - \bar{\rho}^{\gamma-1}) \right] \left(n f_1 + \sum_{i=1}^n x_i \frac{\partial f_1}{\partial x_i} \right) d\mathbf{x} \quad (55)$$

$$= \int_{|\mathbf{x}| \leq R+\sigma t} \left[\frac{1}{2} |\mathbf{u}|^2 + \frac{K\gamma}{\gamma-1} (\rho^{\gamma-1} - \bar{\rho}^{\gamma-1}) \right] (n f_1 + f_1' |\mathbf{x}|) d\mathbf{x}. \quad (56)$$

Because f_1 is strictly increasing and $f_1(0) = 0$, it is easily known that $f_1 \geq 0$ and $f_1' > 0$. Then, we have

$$\frac{d}{dt} \int_{\mathbb{R}^n} f_1 \mathbf{x} \cdot \mathbf{u} d\mathbf{x} = \frac{1}{2} \int_{|\mathbf{x}| \leq R+\sigma t} |\mathbf{u}|^2 (n f_1 + f_1' |\mathbf{x}|) d\mathbf{x} + \frac{K\gamma}{\gamma-1} \int_{|\mathbf{x}| \leq R+\sigma t} (\rho^{\gamma-1} - \bar{\rho}^{\gamma-1}) (n f_1 + f_1' |\mathbf{x}|) d\mathbf{x} \quad (57)$$

$$\geq \frac{n}{2} \int_{|\mathbf{x}| \leq R+\sigma t} f_1 |\mathbf{u}|^2 d\mathbf{x} + \frac{K\gamma}{\gamma-1} \int_{|\mathbf{x}| \leq R+\sigma t} (\rho^{\gamma-1} - \bar{\rho}^{\gamma-1}) (n f_1 + f_1' |\mathbf{x}|) d\mathbf{x}. \quad (58)$$

By Lemma 2 and the continuity of the first derivative of $f_1(r)$, it follows that

$$\frac{K\gamma}{\gamma-1} \int_{|\mathbf{x}| \leq R+\sigma t} (\rho^{\gamma-1} - \bar{\rho}^{\gamma-1}) (n f_1 + f_1' |\mathbf{x}|) d\mathbf{x} \quad (59)$$

$$> - \frac{K\gamma}{\gamma-1} \int_{|\mathbf{x}| \leq R+\sigma t} \bar{\rho}^{\gamma-1} (n f_1 + f_1' |\mathbf{x}|) d\mathbf{x} \quad (60)$$

$$\geq - \frac{K\gamma \bar{\rho}^{\gamma-1}}{\gamma-1} \left[n f_1(R + \sigma t) + (R + \sigma t) \max_{r \leq R+\sigma t} f_1'(r) \right] V(t) \quad (61)$$

$$\geq - \frac{K\gamma \bar{\rho}^{\gamma-1}}{\gamma-1} \left[n f_1(R + \sigma T) + (R + \sigma T) \max_{r \leq R+\sigma T} f_1'(r) \right] V(T) \quad (62)$$

$$=: -K_1(T) V(T), \quad (63)$$

where $V(t) := \frac{\pi^{\frac{n}{2}}(R+\sigma t)^n}{\Gamma(\frac{n}{2}+1)}$ is the volume of an n -dimensional sphere with the radius of $R + \sigma t$ and $\Gamma(\eta)$ is the Gamma function with $\eta > 0$.

Moreover, by the Hölder inequality, we hold

$$F_1^2(t) = \left(\int_{\mathbb{R}^n} f_1 \mathbf{x} \cdot \mathbf{u} d\mathbf{x} \right)^2 \quad (64)$$

$$= \left(\int_{|\mathbf{x}| \leq R+\sigma t} f_1 \mathbf{x} \cdot \mathbf{u} d\mathbf{x} \right)^2 \quad (65)$$

$$\leq \left(\int_{|\mathbf{x}| \leq R+\sigma t} f_1 |\mathbf{x}|^2 d\mathbf{x} \right) \left(\int_{|\mathbf{x}| \leq R+\sigma t} f_1 |\mathbf{u}|^2 d\mathbf{x} \right) \quad (66)$$

$$\leq f_1(R+\sigma t)(R+\sigma t)^2 \left(\int_{|\mathbf{x}| \leq R+\sigma t} 1 d\mathbf{x} \right) \left(\int_{|\mathbf{x}| \leq R+\sigma t} f_1 |\mathbf{u}|^2 d\mathbf{x} \right) \quad (67)$$

$$= \frac{\pi^{\frac{n}{2}}(R+\sigma t)^{n+2}}{\Gamma(\frac{n}{2}+1)} f_1(R+\sigma t) \int_{|\mathbf{x}| \leq R+\sigma t} f_1 |\mathbf{u}|^2 d\mathbf{x}. \quad (68)$$

Thus, we have

$$\int_{|\mathbf{x}| \leq R+\sigma t} f_1 |\mathbf{u}|^2 d\mathbf{x} \geq \frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}}(R+\sigma t)^{n+2} f_1(R+\sigma t)} F_1^2(t). \quad (69)$$

Fix $T > 0$. Then, for any $0 \leq t \leq T$, inequality (58) can be converted to

$$\dot{F}_1(t) > \frac{n\Gamma(\frac{n}{2}+1)}{2\pi^{\frac{n}{2}}(R+\sigma t)^{n+2} f_1(R+\sigma t)} F_1^2(t) - K_1(T)V(T) \quad (70)$$

$$= \frac{(n-a)\Gamma(\frac{n}{2}+1)}{2\pi^{\frac{n}{2}}(R+\sigma t)^{n+2} f_1(R+\sigma t)} F_1^2(t) + \frac{a\Gamma(\frac{n}{2}+1)}{2\pi^{\frac{n}{2}}(R+\sigma t)^{n+2} f_1(R+\sigma t)} F_1^2(t) - K_1(T)V(T) \quad (71)$$

$$\geq \frac{(n-a)\Gamma(\frac{n}{2}+1)}{2\pi^{\frac{n}{2}}(R+\sigma t)^{n+2} f_1(R+\sigma t)} F_1^2(t) + \left[\frac{a\Gamma(\frac{n}{2}+1)}{2\pi^{\frac{n}{2}}(R+\sigma T)^{n+2} f_1(R+\sigma T)} F_1^2(t) - K_1(T)V(T) \right] \quad (72)$$

$$=: \frac{(n-a)\Gamma(\frac{n}{2}+1)}{2\pi^{\frac{n}{2}}(R+\sigma t)^{n+2} f_1(R+\sigma t)} F_1^2(t) + G_1(t), \quad (73)$$

where a is a positive constant, such that $0 < a < n$.

Obviously, we can obtain $G_1(0) \geq 0$ from inequality (17), which implies $G_1(t) \geq 0$ for $t \in [0, T]$ and

$$\dot{F}_1(t) > \frac{(n-a)\Gamma(\frac{n}{2}+1)}{2\pi^{\frac{n}{2}}(R+\sigma t)^{n+2} f_1(R+\sigma t)} F_1^2(t) \geq 0 \quad \text{for } t \in [0, T]. \quad (74)$$

When condition (18) is satisfied, $F_1(t)$ is a strictly increasing function such that

$$F_1(t) > F_1(0) \geq \frac{2\pi^{\frac{n}{2}}}{(n-a)\Gamma(\frac{n}{2}+1)} \left(\int_0^T \frac{dt}{(R+\sigma t)^{n+2} f_1(R+\sigma t)} \right)^{-1} > 0 \quad \text{for } t \in (0, T]. \quad (75)$$

Then, for $0 < t \leq T$, we take integration of inequality (74) with respect to time over $(0, t)$, yielding

$$\frac{1}{F_1(0)} - \frac{1}{F_1(t)} > \frac{(n-a)\Gamma(\frac{n}{2}+1)}{2\pi^{\frac{n}{2}}} \int_0^t \frac{1}{(R+\sigma s)^{n+2} f_1(R+\sigma s)} ds. \quad (76)$$

That is,

$$0 < \frac{1}{F_1(t)} < \frac{1}{F_1(0)} - \frac{(n-a)\Gamma(\frac{n}{2}+1)}{2\pi^{\frac{n}{2}}} \int_0^t \frac{1}{(R+\sigma s)^{n+2} f_1(R+\sigma s)} ds. \quad (77)$$

The right term of the inequality (77) is less than or equal to 0 on $t = T$ under the condition of (18) in Theorem 1, which leads to a contradiction. Therefore, solutions will blow up on or before T .

The proof is completed. \square

Therefore, we have the following corollary for one-dimensional Euler Equation (7) with an exact test function $f_1(r) = r = |x|$.

Corollary 3. Fix $T \in [0, +\infty)$ and $e \in (0, 1)$. Consider the C^1 solutions of system (7) and (2) with $n = 1$ and $\gamma > 1$. If $F_1(0)$ is sufficient such that

$$F_1(0) \geq 2\sqrt{\frac{2K\gamma\bar{\rho}^{\gamma-1}}{e(\gamma-1)}}(R + \sigma T)^3 \quad (78)$$

and

$$F_1(0) \geq \frac{12\sigma R^3(R + \sigma T)^3}{(1-e)(\sigma^3 T^3 + 3R\sigma^2 T^2 + 3R^2\sigma T)}, \quad (79)$$

then solutions will blow up on or before time T .

Proof of Corollary 3. The 1-dimensional Euler Equation (7) can be written as

$$\begin{cases} \rho_t + u_x = 0, \\ u_t + \frac{1}{2}\partial_x(u^2) + \frac{K\gamma}{\gamma-1}\partial_x(\rho^{\gamma-1} - \bar{\rho}^{\gamma-1}) = 0. \end{cases} \quad (80)$$

As before, multiplying Equation (80) by $|x|$ and x on both sides and taking the integration with respect to x , we have

$$\int_{-\infty}^{+\infty} |x|xu_t dx = - \int_{-\infty}^{+\infty} |x|x \left[\frac{1}{2}\partial_x(u^2) + \frac{K\gamma}{\gamma-1}\partial_x(\rho^{\gamma-1} - \bar{\rho}^{\gamma-1}) \right] dx \quad (81)$$

$$= \int_{-R-\sigma t}^{R+\sigma t} 2|x| \left[\frac{1}{2}u^2 + \frac{K\gamma}{\gamma-1}(\rho^{\gamma-1} - \bar{\rho}^{\gamma-1}) \right] dx \quad (82)$$

$$> \frac{1}{2} \int_{-R-\sigma t}^{R+\sigma t} |x|u^2 dx - \frac{2K\gamma\bar{\rho}^{\gamma-1}}{\gamma-1} \int_{-R-\sigma t}^{R+\sigma t} |x| dx \quad (83)$$

$$= \frac{1}{2} \int_{-R-\sigma t}^{R+\sigma t} |x|u^2 dx - \frac{2K\gamma\bar{\rho}^{\gamma-1}}{\gamma-1} (R + \sigma t)^2. \quad (84)$$

On the other hand, by Lemma 1, we have

$$F_1^2(t) = \left(\int_{-R-\sigma t}^{R+\sigma t} |x|xudx \right)^2 \quad (85)$$

$$\leq \left(\int_{-R-\sigma t}^{R+\sigma t} |x|u^2 dx \right) \left(\int_{-R-\sigma t}^{R+\sigma t} |x|x^2 dx \right) \quad (86)$$

$$\leq 2(R + \sigma t)^4 \int_{-R-\sigma t}^{R+\sigma t} |x|u^2 dx. \quad (87)$$

Thus, we have

$$\dot{F}_1(t) > \frac{1}{4(R + \sigma t)^4} F_1^2(t) - \frac{2K\gamma\bar{\rho}^{\gamma-1}}{\gamma-1} (R + \sigma t)^2 \quad (88)$$

$$\geq \frac{1-e}{4(R + \sigma t)^4} F_1^2(t) + \left[\frac{e}{4(R + \sigma T)^4} F_1^2(t) - \frac{2K\gamma\bar{\rho}^{\gamma-1}}{\gamma-1} (R + \sigma T)^2 \right] \quad (89)$$

$$=: \frac{1-e}{4(R + \sigma t)^4} F_1^2(t) + \bar{G}_1(t), \quad (90)$$

where $0 < e < 1$.

Based on our previous analysis, we have $\bar{G}_1(t) \geq 0$ with Equation (78) for $t \in [0, T]$. Therefore,

$$\dot{F}_1(t) > \frac{1-e}{4(R+\sigma t)^4} F_1^2(t). \quad (91)$$

The conclusion can be obtained accordingly with Equation (79). \square

Then, we exploit Corollary 1 to certify Theorem 2.

Proof of Theorem 2. Because f_2 is strictly increasing and $f_2(0) \geq \frac{1}{n}$, we have $nf_2 \geq 1$ for $r \in [0, +\infty)$. From (57) and (61) in the proof of Theorem 1, there are the same conclusions that

$$\dot{F}_2(t) = \frac{d}{dt} \int_{\mathbb{R}^n} f_2 \mathbf{x} \cdot \mathbf{u} d\mathbf{x} \quad (92)$$

$$= \frac{1}{2} \int_{|\mathbf{x}| \leq R+\sigma t} |\mathbf{u}|^2 (nf_2 + f_2'|\mathbf{x}|) d\mathbf{x} + \frac{K\gamma}{\gamma-1} \int_{|\mathbf{x}| \leq R+\sigma t} (\rho^{\gamma-1} - \bar{\rho}^{\gamma-1}) (nf_2 + f_2'|\mathbf{x}|) d\mathbf{x} \quad (93)$$

$$> \frac{n}{2} \int_{|\mathbf{x}| \leq R+\sigma t} f_2 |\mathbf{u}|^2 d\mathbf{x} + \frac{K\gamma}{\gamma-1} \int_{|\mathbf{x}| \leq R+\sigma t} \rho^{\gamma-1} d\mathbf{x} - \frac{K\gamma}{\gamma-1} \int_{|\mathbf{x}| \leq R+\sigma t} \bar{\rho}^{\gamma-1} (nf_2 + f_2'|\mathbf{x}|) d\mathbf{x} \quad (94)$$

and

$$- \frac{K\gamma}{\gamma-1} \int_{|\mathbf{x}| \leq R+\sigma t} \bar{\rho}^{\gamma-1} (nf_2 + f_2'|\mathbf{x}|) d\mathbf{x} \quad (95)$$

$$\geq - \frac{K\gamma \bar{\rho}^{\gamma-1}}{\gamma-1} \left[nf_2(R+\sigma t) + (R+\sigma t) \max_{r \leq R+\sigma t} f_2'(r) \right] V(t) \quad (96)$$

$$\geq - \frac{K\gamma \bar{\rho}^{\gamma-1}}{\gamma-1} \left[nf_2(R+\sigma T) + (R+\sigma T) \max_{r \leq R+\sigma T} f_2'(r) \right] V(T) \quad (97)$$

$$=: -K_3(T)V(T). \quad (98)$$

Similarly, by the Hölder inequality, we obtain

$$\int_{|\mathbf{x}| \leq R+\sigma t} f_2 |\mathbf{u}|^2 d\mathbf{x} \geq \frac{F_2^2(t)}{(R+\sigma t)^2 f_2(R+\sigma t) V(t)}. \quad (99)$$

For $\gamma = 2$, according to Corollary 1, we have

$$\frac{K\gamma}{\gamma-1} \int_{|\mathbf{x}| \leq R+\sigma t} \rho^{\gamma-1} d\mathbf{x} = 2K \int_{|\mathbf{x}| \leq R+\sigma t} \rho d\mathbf{x} = 2K(M(0) + \bar{\rho}V(t)) > 2KM(0) \quad (100)$$

and

$$- \frac{K\gamma}{\gamma-1} \int_{|\mathbf{x}| \leq R+\sigma t} \bar{\rho}^{\gamma-1} (nf_2 + f_2'|\mathbf{x}|) d\mathbf{x} \quad (101)$$

$$\geq -2K\bar{\rho} \left[nf_2(R+\sigma t) + (R+\sigma t) \max_{r \leq R+\sigma t} f_2'(r) \right] V(t) \quad (102)$$

$$\geq -2K\bar{\rho} \left[nf_2(R+\sigma T) + (R+\sigma T) \max_{r \leq R+\sigma T} f_2'(r) \right] V(T) \quad (103)$$

$$=: -K_2(T)V(T). \quad (104)$$

Therefore, inequality (94) can be estimated by

$$\dot{F}_2(t) > \frac{nF_2^2(t)}{2(R+\sigma t)^2 f_2(R+\sigma t) V(t)} + 2KM(0) - K_2(T)V(T). \quad (105)$$

If $M(0) \geq \frac{K_2(T)V(T)}{2K} > 0$, we have

$$2KM(0) - K_2(T)V(T) \geq 0. \quad (106)$$

Thus, inequality (105) becomes

$$\dot{F}_2(t) > \frac{nF_2^2(t)}{2(R + \sigma t)^2 f_2(R + \sigma t)V(t)} \geq 0, \quad (107)$$

which means $F_2(t)$ is a strictly increasing function. If inequality (23) is satisfied, we have

$$F_2(t) > F_2(0) > 0 \quad \text{for } t \in (0, T]. \quad (108)$$

Then, for $0 < t \leq T$, we have

$$0 < \frac{1}{F_2(t)} < \frac{1}{F_2(0)} - \frac{n}{2} \int_0^t \frac{1}{(R + \sigma s)^2 f_2(R + \sigma s)V(s)} ds. \quad (109)$$

More precisely, we have

$$F_2(0) < \frac{2}{n} \left(\int_0^T \frac{1}{(R + \sigma t)^2 f_2(R + \sigma t)V(t)} dt \right)^{-1}, \quad (110)$$

which contradicts the condition given by inequality (23). Hence, solutions will blow up before or on T .

If $M(0) < \frac{K_2(T)V(T)}{2K}$, we have

$$2KM(0) - K_2(T)V(T) < 0. \quad (111)$$

Thus, inequality (105) becomes

$$\dot{F}_2(t) > \frac{(n-b)F_2^2(t)}{2(R + \sigma t)^2 f_2(R + \sigma t)V(t)} + \left[\frac{bF_2^2(t)}{2(R + \sigma t)^2 f_2(R + \sigma t)V(t)} + 2KM(0) - K_2(T)V(T) \right] \quad (112)$$

$$\geq \frac{(n-b)F_2^2(t)}{2(R + \sigma t)^2 f_2(R + \sigma t)V(t)} + \left[\frac{bF_2^2(t)}{2(R + \sigma T)^2 f_2(R + \sigma T)V(T)} + 2KM(0) - K_2(T)V(T) \right] \quad (113)$$

$$=: \frac{(n-b)F_2^2(t)}{2(R + \sigma t)^2 f_2(R + \sigma t)V(t)} + G_2(t), \quad (114)$$

where b is a constant, such that $0 < b < n$.

From inequality (24), we have $G_2(t) \geq 0$ for $t \in [0, T]$, which means

$$\dot{F}_2(t) > \frac{(1-b)F_2^2(t)}{2(R + \sigma t)^2 f_2^2(R + \sigma t)V(t)} \geq 0. \quad (115)$$

Then, for $0 < t \leq T$, if we have inequality (25), it follows that

$$0 < \frac{1}{F_2(t)} < \frac{1}{F_2(0)} - \frac{(n-b)}{2} \int_0^t \frac{1}{(R + \sigma s)^2 f_2(R + \sigma s)V(s)} ds \quad (116)$$

and

$$F_2(0) < \frac{2}{n-b} \left(\int_0^T \frac{1}{(R + \sigma t)^2 f_2(R + \sigma t)V(t)} dt \right)^{-1}. \quad (117)$$

An argument arises and the solutions must blow up in finite time.

The proof is completed. \square

Then, we give a corresponding corollary for one-dimensional Euler Equation (7) with a explicit text function $f_2(r) = 1 + r = 1 + |x|$.

Corollary 4. Fix $T \in [0, +\infty)$. Consider the C^1 solutions of system (7) and (2) with $n = 1$ and $\gamma = 2$. Let

$$M(0) = \int_{-\infty}^{+\infty} (\rho_0 - \bar{\rho}) dx. \quad (118)$$

1. For $M(0) \geq 2\bar{\rho}(R + \sigma T)(1 + R + \sigma T)$: If $F_2(0)$ is sufficient such that

$$F_2(0) \geq 4 \left(\int_0^T \frac{dt}{(R + \sigma t)^3 (1 + R + \sigma t)} \right)^{-1}, \quad (119)$$

then solutions will blow up on or before time T .

2. For $M(0) < 2\bar{\rho}(R + \sigma T)(1 + R + \sigma T)$: Fix $f \in (0, 1)$. If $F_2(0)$ is sufficient such that

$$F_2(0) \geq 2(R + \sigma T) \sqrt{\frac{2K(R + \sigma T)(1 + R + \sigma T)[2\bar{\rho}(R + \sigma T)(1 + R + \sigma T) - M(0)]}{f}} \quad (120)$$

and

$$F_2(0) \geq \frac{4}{1-f} \left(\int_0^T \frac{dt}{(R + \sigma t)^3 (1 + R + \sigma t)} \right)^{-1}, \quad (121)$$

then solutions will blow up on or before time T .

Proof of Corollary 4. For $\gamma = 2$, the second one of Equation (80) becomes

$$u_t + \frac{1}{2} \partial_x (u^2) + 2K \partial_x (\rho - \bar{\rho}) = 0. \quad (122)$$

As before, we multiply the equation above by $1 + |x|$ and x on both sides and take the integration with respect to x , yielding

$$\int_{-\infty}^{+\infty} (1 + |x|) x u_t dx = - \int_{-\infty}^{+\infty} (1 + |x|) x \left[\frac{1}{2} \partial_x (u^2) + 2K \partial_x (\rho - \bar{\rho}) \right] dx \quad (123)$$

$$= \int_{-R-\sigma t}^{R+\sigma t} (1 + 2|x|) \left[\frac{1}{2} u^2 + 2K(\rho - \bar{\rho}) \right] dx \quad (124)$$

$$> \frac{1}{2} \int_{-R-\sigma t}^{R+\sigma t} (1 + |x|) u^2 dx + 2K \int_{-R-\sigma t}^{R+\sigma t} \rho dx - 2K\bar{\rho} \int_{-R-\sigma t}^{R+\sigma t} (1 + 2|x|) dx \quad (125)$$

$$= \frac{1}{2} \int_{-R-\sigma t}^{R+\sigma t} (1 + |x|) u^2 dx + 2K \int_{-R-\sigma t}^{R+\sigma t} \rho dx - 4K\bar{\rho}(1 + R + \sigma t)(R + \sigma t). \quad (126)$$

Moreover, we have

$$F_2^2(t) = \left(\int_{-R-\sigma t}^{R+\sigma t} (1 + |x|) x u dx \right)^2 \quad (127)$$

$$\leq \left(\int_{-R-\sigma t}^{R+\sigma t} (1 + |x|) u^2 dx \right) \left(\int_{-R-\sigma t}^{R+\sigma t} (1 + |x|) x^2 dx \right) \quad (128)$$

$$\leq 2(R + \sigma t)^3 (1 + R + \sigma t) \int_{-R-\sigma t}^{R+\sigma t} (1 + |x|) u^2 dx. \quad (129)$$

By Lemma 3, we obtain

$$\int_{-R-\sigma t}^{R+\sigma t} \rho dx = M(0) + 2\bar{\rho}(R + \sigma t), \quad (130)$$

where

$$M(0) = \int_{-\infty}^{+\infty} (\rho_0 - \bar{\rho}) dx. \quad (131)$$

Therefore,

$$\dot{F}_2(t) \geq \frac{1}{4(R + \sigma t)^3(1 + R + \sigma t)} F_2^2(t) + 2KM(0) - 4K\bar{\rho}(R + \sigma t)(1 + R + \sigma t). \quad (132)$$

If $M(0) \geq 2\bar{\rho}(R + \sigma T)(1 + R + \sigma T)$, it is evident that

$$\dot{F}_2(t) > \frac{1}{4(R + \sigma t)^3(1 + R + \sigma t)} F_2^2(t). \quad (133)$$

Then, we can derive a contradiction from Equation (119).

If $M(0) < 2\bar{\rho}(R + \sigma T)(1 + R + \sigma T)$, we have

$$\begin{aligned} \dot{F}_2(t) &> \frac{1-f}{4(R + \sigma t)^3(1 + R + \sigma t)} F_2^2(t) \\ &+ \left[\frac{f}{4(R + \sigma T)^3(1 + R + \sigma T)} F_2^2(t) + 2KM(0) - 4K\bar{\rho}(R + \sigma T)(1 + R + \sigma T) \right] \end{aligned} \quad (134)$$

$$=: \frac{1-f}{4(R + \sigma t)^3(1 + R + \sigma t)} F_2^2(t) + \bar{G}_2(t), \quad (135)$$

where $0 < f < 1$.

Thus, $\bar{G}_2(t) \geq 0$ with Equation (120) for $t \in [0, T]$. Then,

$$\dot{F}_2(t) > \frac{1-f}{4(R + \sigma t)^3(1 + R + \sigma t)} F_2^2(t). \quad (136)$$

A contradiction arises from Equation (121). \square

Next, we prove Theorem 3 by using Corollary 2.

Proof of Theorem 3. For $\gamma > 2$, by Corollary 2, inequality (94) becomes

$$\begin{aligned} \dot{F}_2(t) &> \frac{(n-c)}{2(R + \sigma t)^2 f_2(R + \sigma t) V(t)} F_2^2(t) + \left[\frac{c}{2(R + \sigma T)^2 f_2(R + \sigma T) V(T)} F_2^2(t) \right. \\ &\quad \left. + \frac{K\gamma}{\gamma-1} \min_{t \in [0, T]} (M(0) + \bar{\rho} V(t))^{\gamma-1} V^{2-\gamma}(t) - K_3(T) V(T) \right] \end{aligned} \quad (137)$$

$$=: \frac{(n-c)}{2(R + \sigma t)^2 f_2(R + \sigma t) V(t)} F_2(t) + G_3(t), \quad (138)$$

where c is a constant, such that $0 < c < n$.

It is easy to obtain $G_3(t) \geq 0$ on $[0, T]$ with condition (27). Therefore, we have

$$\dot{F}_2(t) > \frac{(n-c)F_2^2(t)}{2(R + \sigma t)^2 f_2(R + \sigma t) V(t)}. \quad (139)$$

As before, for $0 < t \leq T$, we have

$$0 < \frac{1}{F_2(t)} < \frac{1}{F_2(0)} - \frac{n-c}{2} \int_0^T \frac{1}{(R + \sigma t)^2 f_2(R + \sigma t) V(t)} dt, \quad (140)$$

provided that condition (28) is satisfied. It follows that the solutions blow up before or on T .

The proof is completed. \square

Therefore, for $\gamma > 2$, we also have the corresponding corollary for one-dimensional Euler Equation (7) with a explicit text function $f_2(r) = 1 + r = 1 + |x|$.

Corollary 5. Fix $T \in [0, +\infty)$ and $g \in (0, 1)$. Consider the C^1 solutions of system (7) and (2) with $n = 1$ and $\gamma > 2$. If $F_2(0)$ is sufficient such that

$$F_2^2(0) \geq \frac{16K\bar{C}_2(R + \sigma T)^3(1 + R + \sigma T)}{g} \quad (141)$$

and

$$F_2(0) \geq \frac{4}{1-g} \left(\int_0^T \frac{dt}{(R + \sigma t)^3(1 + R + \sigma t)} \right)^{-1}, \quad (142)$$

where

$$\bar{C}_2 := \bar{\rho}(R + \sigma T)(1 + R + \sigma T) - \frac{\gamma}{(\gamma - 1)2^\gamma} \min_{t \in [0, T]} (R + \sigma t)^{2-\gamma} [M(0) + 2\bar{\rho}(R + \sigma t)]^{\gamma-1}, \quad (143)$$

then solutions will blow up on or before time T .

Proof of Corollary 5. Multiplying the second one of Equation (80) by $1 + |x|$ and x on both sides and taking the integration with respect to x , we have

$$\int_{-\infty}^{+\infty} (1 + |x|)xu_t dx = - \int_{-\infty}^{+\infty} (1 + |x|)x \left[\frac{1}{2} \partial_x(u^2) + \frac{K\gamma}{\gamma - 1} \partial_x(\rho^{\gamma-1} - \bar{\rho}^{\gamma-1}) \right] dx \quad (144)$$

$$= \int_{-R-\sigma t}^{R+\sigma t} (1 + 2|x|) \left[\frac{1}{2} u^2 + \frac{K\gamma}{\gamma - 1} (\rho^{\gamma-1} - \bar{\rho}^{\gamma-1}) \right] dx \quad (145)$$

$$\begin{aligned} &> \frac{1}{2} \int_{-R-\sigma t}^{R+\sigma t} (1 + |x|)u^2 dx \\ &\quad + \frac{K\gamma}{\gamma - 1} \int_{-R-\sigma t}^{R+\sigma t} \rho^{\gamma-1} dx - \frac{K\gamma\bar{\rho}^{\gamma-1}}{\gamma - 1} \int_{-R-\sigma t}^{R+\sigma t} (1 + 2|x|) dx \end{aligned} \quad (146)$$

$$\begin{aligned} &= \frac{1}{2} \int_{-R-\sigma t}^{R+\sigma t} (1 + |x|)u^2 dx \\ &\quad + \frac{K\gamma}{\gamma - 1} \int_{-R-\sigma t}^{R+\sigma t} \rho^{\gamma-1} dx - \frac{2K\gamma\bar{\rho}^{\gamma-1}}{\gamma - 1} (R + \sigma t)(1 + R + \sigma t). \end{aligned} \quad (147)$$

For $n = 1$, from Corollary 2, we have

$$\int_{-R-\sigma t}^{R+\sigma t} \rho^{\gamma-1} dx \geq 2^{2-\gamma} (R + \sigma t)^{2-\gamma} [M(0) + 2\bar{\rho}(R + \sigma t)]^{\gamma-1}. \quad (148)$$

Therefore, combined with inequality (129),

$$\begin{aligned} \dot{F}_2(t) &> \frac{1-g}{4(R + \sigma t)^3(1 + R + \sigma t)} F_2^2(t) + \left[\frac{g}{4(R + \sigma T)^3(1 + R + \sigma T)} F_2^2(t) \right. \\ &\quad \left. + \frac{2^{2-\gamma}K\gamma}{\gamma - 1} \min_{t \in [0, T]} (R + \sigma t)^{2-\gamma} [M(0) + 2\bar{\rho}(R + \sigma t)]^{\gamma-1} - 4K\bar{\rho}(R + \sigma T)(1 + R + \sigma T) \right] \end{aligned} \quad (149)$$

$$= : \frac{1-g}{4(R + \sigma t)^3(1 + R + \sigma t)} F_2^2(t) + \bar{G}_3(t), \quad (150)$$

where $0 < g < 1$.

Then, we have $\bar{G}_3(t) \geq 0$ for $t \in [0, T]$ with (141). Thus,

$$\dot{F}_2(t) > \frac{1-g}{4(R + \sigma t)^3(1 + R + \sigma t)} F_2^2(t). \quad (151)$$

We can obtain a contradiction by Equation (142). \square

Finally, we use the same idea to consider the solutions of Euler equations with time-dependent damping.

Proof of Theorem 4. For $\gamma > 1$, the second one of Equation (11) can be written as

$$\mathbf{u} + \frac{1}{2} \nabla(|\mathbf{u}|^2) + \frac{K\gamma}{\gamma-1} \nabla(\rho^{\gamma-1} - \bar{\rho}^{\gamma-1}) = -\frac{\mu}{(1+t)^\lambda} \mathbf{u}. \quad (152)$$

Multiplying the above equation by \mathbf{x} and f_1 and integrating over \mathbb{R}^n , we have

$$\int_{\mathbb{R}^n} f_1 \mathbf{x} \cdot \mathbf{u}_t d\mathbf{x} = - \int_{\mathbb{R}^n} f_1 \mathbf{x} \cdot \left[\frac{1}{2} \nabla(|\mathbf{u}|^2) + \frac{K\gamma}{\gamma-1} \nabla(\rho^{\gamma-1} - \bar{\rho}^{\gamma-1}) \right] d\mathbf{x} - \frac{\mu}{(1+t)^\lambda} \int_{\mathbb{R}^n} f_1 \mathbf{u} \cdot \mathbf{x} d\mathbf{x}. \quad (153)$$

Combining Equations (58), (63) and (69), we have

$$\dot{F}_1(t) = \int_{|\mathbf{x}| \leq R+\sigma t} \left[\frac{1}{2} |\mathbf{u}|^2 + \frac{K\gamma}{\gamma-1} (\rho^{\gamma-1} - \bar{\rho}^{\gamma-1}) \right] \nabla \cdot (f_1 \mathbf{x}) d\mathbf{x} - \frac{\mu}{(1+t)^\lambda} \int_{|\mathbf{x}| \leq R+\sigma t} f_1 \mathbf{u} \cdot \mathbf{x} d\mathbf{x} \quad (154)$$

$$> \frac{n\Gamma(\frac{n}{2}+1)}{2\pi^{\frac{n}{2}}(R+\sigma t)^{n+2}f_1(R+\sigma t)} F_1^2(t) - K_1(T)V(T) - \frac{\mu}{(1+t)^\lambda} F_1(t) \quad (155)$$

$$\geq \frac{(n-d)\Gamma(\frac{n}{2}+1)}{2\pi^{\frac{n}{2}}(R+\sigma t)^{n+2}f_1(R+\sigma t)} F_1^2(t) + \left[\frac{d\Gamma(\frac{n}{2}+1)}{2\pi^{\frac{n}{2}}(R+\sigma T)^{n+2}f_1(R+\sigma T)} F_1^2(t) - \frac{\mu}{(1+t)^\lambda} F_1(t) - K_1(T)V(T) \right] \quad (156)$$

$$=: \frac{(n-d)\Gamma(\frac{n}{2}+1)}{2\pi^{\frac{n}{2}}(R+\sigma t)^{n+2}f_1(R+\sigma t)} F_1^2(t) + G_4(t), \quad (157)$$

where d is a constant and satisfies $0 < d < n$.

In fact, if $F_1(0) \geq 0$ and $G_4(t) \geq 0$ for $t \in [0, T]$, we have

$$\dot{F}_1(t) > \frac{(n-d)\Gamma(\frac{n}{2}+1)}{2\pi^{\frac{n}{2}}(R+\sigma t)^{n+2}f_1(R+\sigma t)} F_1^2(t) \quad \text{for } t \in [0, T]. \quad (158)$$

Hence, $F_1(t)$ is an increasing function and $F_1(t) > F_1(0) \geq 0$.

For the underdamping case: Because $\mu > 0$ and $\lambda \in [0, 1)$, it is clear that $-\frac{\mu}{(1+t)^\lambda} F_1(t) \geq -\mu F_1(t)$. It follows that

$$G_4(t) \geq \frac{d\Gamma(\frac{n}{2}+1)}{2\pi^{\frac{n}{2}}(R+\sigma T)^{n+2}f_1(R+\sigma T)} F_1^2(t) - \mu F_1(t) - K_1(T)V(T) \quad (159)$$

$$:= AF_1^2(t) - \mu F_1(t) - C \quad (160)$$

$$:= G_5(F_1(t)). \quad (161)$$

Apparently, $G_5(F_1(t))$ is a quadratic equation about $F_1(t)$ and $F_1(t) \in [F_1(0), F_1(T)]$ for $t \in [0, T]$. We have $G_5(F_1(t)) \geq 0$ if $F_1(0) \geq \frac{\mu}{2A}$ and $G_5(F_1(0)) \geq 0$. Thus, by (36), we obtain $G_4(t) \geq G_5(F_1(t)) \geq 0$ for $t \in [0, T]$. From condition (37) and inequality (158), we have

$$0 < \frac{1}{F_1(t)} < \frac{1}{F_1(0)} - \frac{(n-d)\Gamma(\frac{n}{2}+1)}{2\pi^{\frac{n}{2}}} \int_0^T \frac{1}{(R+\sigma t)^{n+2}f_1(R+\sigma t)} dt \quad (162)$$

and

$$F(0) < \frac{2\pi^{\frac{n}{2}}}{(n-d)\Gamma(\frac{n}{2}+1)} \left(\int_0^T \frac{1}{(R+\sigma t)^{n+2}f_1(R+\sigma t)} dt \right)^{-1}. \quad (163)$$

There is a contradiction between inequality (37) and the inequality above.

For the overdamping case: As before, if $F_1(0) \geq 0$ and $G_4(t) \geq 0$ for $t \in [0, T]$, we have $-\frac{\mu}{(1+t)^\lambda} F_1(t) \geq -\frac{\mu}{(1+T)^\lambda} F_1(t)$ with $\mu > 0$ and $\lambda \in [-1, 0)$. Therefore,

$$G_4(t) \geq \frac{d\Gamma(\frac{n}{2} + 1)}{2\pi^{\frac{n}{2}}(R + \sigma T)^{n+2}f_1(R + \sigma T)} F_1^2(t) - \frac{\mu}{(1+T)^\lambda} F_1(t) - K_1(T)V(T) \quad (164)$$

$$:= AF_1^2(t) - BF_1(t) - C \quad (165)$$

$$:= G_6(F_1(t)). \quad (166)$$

We have $G_6(F_1(t)) \geq 0$ if $F_1(0) \geq \frac{B}{2A}$ and $G_6(F_1(0)) \geq 0$. When (38) is satisfied, $G_4(t) \geq G_6(F_1(t)) \geq 0$ is true for $t \in [0, T]$. Hence, the process of obtaining inequalities (158) and (163) is smooth with (37), which is the desired contradiction.

The proof is completed. \square

4. Discussion

As observed in [36], the functional (15) has potential applications in exploring three-dimensional non-isentropic rotational solutions of Euler equations. The formation of singularity of irrotational solutions for compressible Euler equations with general time-dependent damping in \mathbb{R}^n

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \rho[\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u}] + \nabla P = -a(t)\rho \mathbf{u}, \end{cases} \quad (167)$$

could be analyzed using a similar approach, where $a(t) > 0$. If $a(t)$ is a constant, Equation (167) represents the Euler equations with linear damping. Then, $a(t) \int_{\mathbb{R}^n} f \mathbf{x} \cdot \mathbf{u} d\mathbf{x}$ must be estimated. There may hold a form similar to

$$\dot{F}(t) \geq A(t)F^2(t) + C(t)F(t) + D(t) \quad (168)$$

with $A(t) > 0$. Furthermore, the conditions that $A(t)$, $C(t)$, and $D(t)$ should satisfy have to be considered.

5. Conclusions

This paper primarily discusses the singularity formation of irrotational solutions for n -dimensional compressible Euler equations with non-vacuum initial data. We find the general test function $f_1(r)$ to consider Euler equations and Euler equations with underdamping and overdamping for $\gamma > 1$ and the general test function $f_2(r)$ for $\gamma \geq 2$. By constructing novel functionals related to the test functions, we demonstrate that solutions always blow up in finite time. Moreover, we utilize the integration method with two exact test functions to yield the homologous blowup results of irrotational solutions for Euler equations in one dimension.

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