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A Charged Particle with Anisotropic Mass in a Perpendicular Magnetic Field–Landau Gauge

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Abstract: The loss of any symmetry in a system leads to quantum problems that are typically very difficult to solve. Such a situation arises for particles with anisotropic mass, like electrons in various semiconductor host materials, where it is known that they may have an anisotropic effective mass. In this work, we consider the quantum problem of a spinless charged particle with anisotropic mass in two dimensions and study the resulting energy and eigenstate spectrum in a uniform constant perpendicular magnetic field when a Landau gauge is adopted. The exact analytic solution to the problem is obtained for arbitrary values of the anisotropic mass using a mathematical technique that relies on the scaling of the original coordinates. The characteristic features of the energy spectrum and corresponding eigenstate wave functions are analyzed. The results of this study are expected to be of interest to quantum Hall effect theory.

Keywords: charged particle; magnetic field; anisotropic mass; Landau states

1. Introduction

Great advances in the fields of nanotechnology and low-dimensional systems have enabled the precise, controlled fabrication of materials at atomic and molecular scales. The electron's quantum mechanical nature is very pronounced in this regime. Therefore, there is a great potential payoff that electronic devices built on nanoscale may manifest many desirable quantum properties. This means that current science and technology may be at the cusp of major developments that can fundamentally change our life for many decades to come. Low-dimensional systems, in particular two-dimensional (2D) systems of electrons are seen as some of the most fascinating systems for meeting the technological challenges of the future [1–5]. The great interest in 2D systems of electrons stems from the fact that the combination of low-dimensionality, confinement, discreteness of the electron's charge, electron's quantum spin and interaction/correlation effects can lead to very intriguing quantum phenomena [6,7]. The application of new and extraordinary experimental tools, in conjunction with the production of novel materials, has created an urgent need for a better understanding of the many novel unexpected physical phenomena that are observed under these conditions [8,9].

The application of a strong uniform constant magnetic field perpendicular to a 2D system of electrons dramatically changes its physics. As a matter of fact, a 2D system of electrons in a perpendicular magnetic field exhibits remarkable quantum phenomena at very low temperatures. Two novel specific physical phenomena, the integer quantum Hall effect (IQHE) [10] and fractional quantum Hall effect (FQHE) [11], stand out as two of the most important discoveries in condensed matter physics over the last decades. The appearance of plateaus in the Hall resistance plot as a function of the magnetic field was an unexpected finding. The Hall resistance on these plateaus is quantized at $R_H = h/(\nu e^2)$ values, where h is the Planck constant, e is the magnitude of electron's charge and the quantum number ν is the integer (1, 2, etc.) for the IQHE or fractional (1/3, 1/5, etc.) for the case of the FQHE.



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The IQHE has a simpler explanation that originates from the quantum physics of single-particle states in a perpendicular magnetic field. On the other hand, the FQHE represents a particular example of a novel collective quantum liquid state of matter that originates, in a unique way, from strong electronic interactions/correlations. For both cases, the first step in comprehending the phenomena is to consider the solution of the quantum problem of a spinless charged particle in a 2D system subject to a perpendicular magnetic field. This quantum problem was solved by Landau a long time ago [12]. The model typically assumes that charged particles (for instance, electrons) with a fixed given isotropic mass are confined in a 2D system that is subjected to a strong, uniform, constant and perpendicular magnetic field. The main feature of the quantum solution is that the resulting energy spectrum consists of massively degenerated discrete quantum states known as Landau levels which are separated from each other by an energy gap. For a given Landau level, the eigenstate spectrum of the many degenerated wave functions describes states with the same kinetic energy quantized by the application of the magnetic field.

Within the realm of quantum mechanics, the kinetic energy of an electron moving in a perpendicular magnetic field is quantized to values of $\hbar\omega_c/2$, $3\hbar\omega_c/2$, $5\hbar\omega_c/2$, and so on, where $\hbar = h/(2\pi)$ is the reduced Planck's constant and ω_c is the cyclotron angular frequency. The energy gap between two neighboring Landau levels is $\hbar\omega_c$ while $\hbar\omega_c/2$ is known as the lowest Landau level energy. The number of eigenstates in each Landau level represents the degeneracy of that level and is proportional to the value of the magnetic field (B) and to the area (A) of the 2D sample, $N_s = BA/\Phi_0$ where $\Phi_0 = h/e$ is the magnetic flux quantum. Note that the degeneracy, N_s , of each Landau level increases with increasing the magnetic field. A key parameter that controls the properties of the system is the filling factor, ν , which is defined as the ratio of the number of electrons, N , to the degeneracy (number of available states) of each Landau level, $\nu = N/N_s$. In fact, the filling factor represents exactly the quantum number, ν , in the expression for the Hall resistance plateaus, R_H . This means that IQHE occurs when the filling factor, ν , is an integer, while the FQHE happens when ν is fractional.

In the extreme quantum limit of a very high perpendicular magnetic field, the degeneracy of each Landau level becomes so large that all electrons may be accommodated in the lowest Landau level, with a negligible admixture of higher Landau levels. In fact, some of most important FQHE liquid states occur when the lowest Landau level is fractionally filled with electrons. Under these conditions, the kinetic energy of the electrons is essentially quenched to a constant value (that corresponds to the lowest Landau level energy per electron). The electrons also have a quantum spin that couples to the magnetic field. The energy associated with this coupling is known as Zeeman energy and this is smallest when the quantum spin of the electrons is aligned with the magnetic field. For this reason, and to simplify the treatment, one may assume that the quantum spin of electrons is "frozen" by the magnetic field and, therefore, the electrons may be seen as effectively spinless charged particles.

As the magnetic field varies, the stabilization of the novel quantum phases of electrons happens at special filling factors that generally have odd denominators. Among them, the most robust FQHE states correspond to filling factors $\nu = 1/3$ and $1/5$ and are well described by Laughlin's theory in terms of trial wave functions [13]. Differently from odd-denominator-filled states in the lowest Landau level, even-denominator-filled states with filling factors $\nu = 1/2$, $1/4$ and $1/6$ do not show typical FQHE features and behave as isotropic compressible metallic Fermi liquid states [14]. The composite fermion theory [15] for the FQHE sheds light on the Fermi-liquid nature of such even-denominator-filled states. On the other hand, Wigner crystallization occurs when the filling factor becomes around, or less than, $\nu = 1/7$, as seen in various studies [16–20].

As discussed above, the 2D model of a charged particle in a uniform constant perpendicular magnetic field has many applications in quantum mechanics ranging from theories of magnetism a century ago [21,22] to quantum Hall effect phenomena during the last few decades [23–32]. The model in which a charged particle has a given constant isotropic

mass was solved exactly by Landau in a work where the so-called Landau gauge was first introduced [12]. The exact solution of the stationary Schrödinger equation in this case is relatively straightforward for such a gauge due to the possibility of separating variables and writing the overall wave function as the product of a plane wave for one position variable and a displaced one-dimensional (1D) harmonic oscillator for the other one.

However, it is well known that any loss of symmetry in a quantum system leads to mathematical problems that generally become much more difficult to solve. A common situation of this nature arises when we deal with particles such as electrons trapped in semiconductor materials. For these conditions, the electrons may possess an anisotropic effective mass. The purpose of this work is to consider the 2D model of a spinless charged particle with anisotropic mass in a uniform constant perpendicular magnetic field and show that this quantum problem has a simple and exact analytic solution, despite the presence of mass anisotropy. The mathematical approach that we use is based on the introduction of “new” scaled distorted coordinates. The method allows one to transform the original problem of a charged particle with anisotropic mass in standard coordinates to that of a charged particle with isotropic effective mass in “new” scaled distorted coordinates.

The paper is organized as follows: In Section 2, we explain the quantum solution for the case of a charged particle with constant isotropic mass in a uniform constant perpendicular magnetic field. In Section 3, we provide the exact solution when the mass is anisotropic and point out the key details of the adopted mathematical method. In Section 4, we discuss the subtle effects that may come from the anisotropic mass of electrons in quantum Hall systems. In Section 5, we provide some concluding remarks.

2. Results—Isotropic Mass

In this section, we focus our attention on the quantum problem of a spinless particle with a constant isotropic mass, $m > 0$, and charge, q , moving in 2D space in the presence of a uniform constant perpendicular magnetic field. We clarify that by a constant quantity we mean one that does not change with time. A particle with a constant isotropic mass is a rather conventional one. For instance, it can be an electron with a bare mass, m_e , and a negative charge in studies of 2D electronic systems under ideal conditions. For more realistic experimental situations, one must take into account the fact that 2D systems of electrons are often created at the interface of a semiconductor heterojunction and/or heterostructure, such as *GaAs/AlGaAs*. Since electrons are typically confined in the conduction band of a given host semiconductor, the mass, m , for this case would represent the electron’s effective band mass for those structures that are known to have a constant isotropic effective mass (for example, it is known that the effective band mass of electrons is isotropic and has the value, $m = 0.067 m_e$, in a *GaAs* host semiconductor).

Providing certain details to the solution of this known problem is beneficial for understanding how the emerging new problem of a particle with anisotropic mass can be mapped back to the known results. To begin with, the magnetic field perpendicular to the 2D plane is written as follows:

$$\vec{B} = (0, 0, B_z) . \quad (1)$$

Any magnetic field is given in terms of a vector potential, so that

$$\vec{B} = \vec{\nabla} \times \vec{A}(x, y) , \quad (2)$$

where $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$ is the nabla or del operator, $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are partial derivatives and $\vec{A}(x, y)$ is the vector potential for the given magnetic field. The choice of $\vec{A}(x, y)$ is not unique. The simplest choice is the so-called Landau gauge, which may take the following two flavours:

$$\vec{A}(x, y) = B_z (0, x, 0) , \quad (3)$$

or

$$\vec{A}(x, y) = B_z (-y, 0, 0) . \quad (4)$$

In this work, we choose the Landau gauge in Equation (3).
The general quantum Hamiltonian is

$$\hat{H} = \frac{1}{2m} \left[\hat{\vec{p}} - q \vec{A}(x, y) \right]^2, \quad (5)$$

where $\hat{\vec{p}} = (\hat{p}_x, \hat{p}_y)$ is the 2D linear momentum operator. The x and y components of the 2D linear momentum operator may be explicitly written as follows:

$$\hat{p}_x = -i\hbar \frac{\partial}{\partial x} \quad ; \quad \hat{p}_y = -i\hbar \frac{\partial}{\partial y}, \quad (6)$$

where $i = \sqrt{-1}$ is the imaginary unit and \hbar is the reduced Planck's constant. One must notice that the interaction of the particle's quantum spin with the magnetic field (the Zeeman effect) is not included in the Hamiltonian of Equation (5), since, for simplicity, we are assuming a spinless charged particle.

One can write the quantum Hamiltonian as follows:

$$\hat{H} = \frac{1}{2m} \left[\hat{p}_x - q A_x(x, y) \right]^2 + \frac{1}{2m} \left[\hat{p}_y - q A_y(x, y) \right]^2. \quad (7)$$

For the Landau gauge in Equation (3), the quantum Hamiltonian in Equation (7) becomes

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{(\hat{p}_y - q B_z x)^2}{2m} = \frac{1}{2m} \left[\hat{p}_x^2 + (\hat{p}_y - q B_z x)^2 \right]. \quad (8)$$

The stationary Schrödinger equation to solve is

$$\hat{H} \Psi(x, y) = E \Psi(x, y), \quad (9)$$

where E is the energy and $\Psi(x, y)$ is the wave function. In order to solve this equation, one envisions the particle as being constrained in a 2D area, $L_x L_y$, where $-L_x/2 \leq x \leq +L_x/2$, $0 \leq y \leq L_y$ and $L_x \rightarrow \infty$ and $L_y \rightarrow \infty$. Hence, in the x -direction, one has $-\infty < x < +\infty$. On the other hand, periodic boundary conditions (PBC) for the wave function are imposed in the y -direction:

$$\Psi(x, y) = \Psi(x, y + L_y). \quad (10)$$

Given the form of the Hamiltonian in Equation (8) and the PBC choice in Equation (10), one searches for a wave function that solves Equation (9) as a product of a plane wave state in the y -direction and a function that depends on coordinate x in the other direction:

$$\Psi(x, y) = \frac{e^{ik_y y}}{\sqrt{L_y}} \Phi(x). \quad (11)$$

The overall normalization of the wave function must be such that

$$\int_{-\infty}^{+\infty} dx \int_0^{L_y} dy |\Psi(x, y)|^2 = 1. \quad (12)$$

A substitution of the expression from Equation (11) into Equation (9) gives

$$\left[\frac{\hat{p}_x^2}{2m} + \frac{(\hbar k_y - q B_z x)^2}{2m} \right] \Phi(x) = E \Phi(x). \quad (13)$$

One can rewrite Equation (13) as

$$\left[\frac{\hat{p}_x^2}{2m} + \frac{m}{2} \left(\frac{q B_z}{m} \right)^2 \left(x - \frac{\hbar k_y}{q B_z} \right)^2 \right] \Phi(x) = E \Phi(x). \quad (14)$$

At this juncture, we introduce the explicit definition of the cyclotron angular frequency:

$$\omega_c = \frac{|q| |B_z|}{m} > 0, \quad (15)$$

where $|q| > 0$ is the magnitude of the charge of the particle, $|B_z| > 0$ is the magnitude of the magnetic field perpendicular to the 2D plane and $m > 0$ is the constant isotropic mass of the charged particle. Classically speaking, a charged particle, such as the one described above, moving perpendicular to the direction of a uniform constant magnetic field, will undergo a uniform circular motion with a given rotational cyclotron frequency, $f_c = \omega_c / (2\pi)$. The cyclotron frequency is independent of the radius of the circle of rotation and velocity. All charged particles with the same charge-to-mass ratio will undergo a circular motion with the same frequency.

With help from the definition in Equation (15), one can write Equation (14) as follows:

$$\left[\frac{\hat{p}_x^2}{2m} + \frac{m}{2} \omega_c^2 \left(x - \frac{\hbar k_y}{q B_z} \right)^2 \right] \Phi(x) = E \Phi(x). \quad (16)$$

The PBC in the y -direction, as specified by Equation (10), fixes the allowed values of the k_y wave vector:

$$k_y = \frac{2\pi}{L_y} j ; \quad j = 0, \pm 1, \pm 2 \dots \quad (17)$$

Note that Equation (16) represents a displaced 1D quantum oscillator centered at $\hbar k_y / (q B_z)$, which has a known solution. The resulting discrete energy eigenvalues are

$$E_n = \hbar \omega_c \left(n + \frac{1}{2} \right) ; \quad n = 0, 1, \dots \quad (18)$$

These are the Landau levels. Note that the discrete Landau level energies are highly degenerate since the quantum number, k_y , does not enter the energy expression. The normalized eigenfunctions corresponding to the above energy levels may be written as follows:

$$\Psi_{n k_y}(X, y) = \frac{e^{i k_y y}}{\sqrt{L_y}} \Phi_n(X), \quad (19)$$

where

$$X = x - \frac{\hbar k_y}{q B_z}, \quad (20)$$

and $\Phi_n(X)$ is the normalized eigenfunction of a 1D harmonic oscillator of mass m and frequency ω_c . Such a wave function is given by

$$\Phi_n(X) = N_n \exp\left(-\frac{\alpha^2 X^2}{2}\right) H_n(\alpha X), \quad (21)$$

where

$$N_n = \sqrt{\frac{\alpha}{\sqrt{\pi} 2^n n!}}, \quad (22)$$

is the normalization constant,

$$\alpha = \sqrt{\frac{m \omega_c}{\hbar}}, \quad (23)$$

is a parameter with the dimensionality of an inverse length and $H_n(x)$ are the Hermite polynomials. By using Equation (15), one sees that

$$\alpha = \sqrt{\frac{m \omega_c}{\hbar}} = \sqrt{\frac{|q| |B_z|}{\hbar}} = \frac{1}{l_B}, \quad (24)$$

where

$$l_B = \sqrt{\frac{\hbar}{|q| |B_z|}}, \quad (25)$$

is known as the magnetic length.

By looking at the expression obtained in Equation (19), one concludes that the normalized eigenfunctions for a charged particle studied in a Landau gauge look like stripes [33]. It is straightforward to note that the probability density for such eigenfunctions, $|\Psi_{nk_y}(X, y)|^2$, depends only on the variable X (thus, x), but not y . This means that one can view such states as extended in one direction (in this case, along the y -direction), but exponentially localized around a given set of centers in the other perpendicular direction (in this case, the x -direction).

3. Results—Anisotropic Mass

Let us now consider the same quantum problem but with the assumption that the charged particle has a constant anisotropic (effective) mass of the following form:

$$m_x > 0 \ ; \ m_y > 0, \quad (26)$$

along the respective x and y directions. This situation would apply to electrons hosted in semiconductors in which the lowest energies of the conduction band are locally approximated in parabolic form by an anisotropic dispersion relation:

$$E(\vec{k}) = E_0 + \frac{\hbar^2}{2m_x}(k_x - k_{0x})^2 + \frac{\hbar^2}{2m_y}(k_y - k_{0y})^2 + \frac{\hbar^2}{2m_z}(k_z - k_{0z})^2, \quad (27)$$

where $E(\vec{k})$ is the energy of an electron at wavevector \vec{k} in that band, E_0 is the minimum energy of that band, $m_{x,y,z}$ are the effective masses along the different axes and $\vec{k}_0 = (k_{0x}, k_{0y}, k_{0z})$ represents the wavevector of the conduction band minimum (that, in principle, may be offset from the zero value). The effective (isotropic or anisotropic) masses of conduction band electrons in common host semiconductor materials (*GaAs*, *GaP*, *InAs*, *AlAs*, *Si*, *Ge*, etc.) are all positive. However, if semiconductor band structures of certain exotic materials exhibit saddle points (e.g., in heterostructures, curved 2D materials, or topological insulator systems), there could be situations where the effective mass is positive in one direction ($m_x > 0$) and negative in another ($m_y < 0$). Dealing with the concept of a particle with a negative mass, either classically or quantum mechanically, is beyond the scope of this work.

The scenario envisioned by Equation (26) would lead to a starting quantum Hamiltonian

$$\hat{H} = \frac{1}{2m_x} [\hat{p}_x - q A_x(x, y)]^2 + \frac{1}{2m_y} [\hat{p}_y - q A_y(x, y)]^2. \quad (28)$$

For the assumption of a Landau gauge, as in Equation (3), one has

$$\hat{H} = \frac{1}{2m_x} \left(-i\hbar \frac{\partial}{\partial x} \right)^2 + \frac{1}{2m_y} \left(-i\hbar \frac{\partial}{\partial y} - q B_z x \right)^2. \quad (29)$$

The Hamiltonian in Equation (29) is the anisotropic mass counterpart to that in Equation (8), with the 2D linear momentum operators written in explicit form.

The idea behind the solution of this quantum problem is to try to identify some “new” coordinates that will allow us to see the problem of a particle with anisotropic mass in “old” coordinates as that of a “new” particle with constant “isotropic” mass in these “new” coordinates. This means that the solution of the problem will be achieved elegantly if this process comes to fruition, given that, at this juncture, one can rely on already known results.

Being inspired by this idea, the mathematical approach that we follow is centered on scaling the original coordinates, x and y . We start by scaling the variable x to γx , where γ is seen as a real positive scaling parameter whose precise value would be determined at a later stage. Let us write Equation (29) as

$$\hat{H} = \frac{1}{2m_x} \left[-i\hbar \gamma \frac{\partial}{\partial(\gamma x)} \right]^2 + \frac{1}{2m_y} \left[-i\hbar \frac{\partial}{\partial y} - \frac{qB_z}{\gamma} (\gamma x) \right]^2. \quad (30)$$

Since γx is going to be one of the “new” scaled coordinate variables, it immediately transpires that the scaling parameter, γ , can be factorized out of the second term in the right-hand side expression of Equation (30) if one rescales the other coordinate variable y to y/γ :

$$\hat{H} = \frac{1}{2m_x} \left[-i\hbar \gamma \frac{\partial}{\partial(\gamma x)} \right]^2 + \frac{1}{2m_y} \left[-\frac{i\hbar}{\gamma} \frac{\partial}{\partial\left(\frac{y}{\gamma}\right)} - \frac{qB_z}{\gamma} (\gamma x) \right]^2. \quad (31)$$

At this junction, one can check that the quantum Hamiltonian in Equation (31) can be rewritten as follows:

$$\hat{H} = \frac{\gamma^2}{2m_x} \left[-i\hbar \frac{\partial}{\partial(\gamma x)} \right]^2 + \frac{1}{2m_y \gamma^2} \left[-i\hbar \frac{\partial}{\partial\left(\frac{y}{\gamma}\right)} - qB_z (\gamma x) \right]^2. \quad (32)$$

Let us choose the value of γ , such that

$$\frac{\gamma^2}{2m_x} = \frac{1}{2m_y \gamma^2}. \quad (33)$$

This choice leads to

$$\gamma^2 = \sqrt{\frac{m_x}{m_y}}. \quad (34)$$

For the choice of γ^2 , as in Equation (34), one has

$$\frac{\gamma^2}{m_x} = \frac{1}{m_y \gamma^2} = \frac{1}{\sqrt{m_x m_y}}; \quad \gamma^2 = \sqrt{\frac{m_x}{m_y}}. \quad (35)$$

This means that one can use the result from Equation (35) to write the Hamiltonian in Equation (32) as

$$\hat{H} = \frac{1}{2m_c} \left(-i\hbar \frac{\partial}{\partial x'} \right)^2 + \frac{1}{2m_c} \left(-i\hbar \frac{\partial}{\partial y'} - qB_z x' \right)^2, \quad (36)$$

where the two “new” scaled coordinate variables are

$$x' = \gamma x; \quad y' = \frac{y}{\gamma}, \quad (37)$$

and

$$m_c = \sqrt{m_x m_y}, \quad (38)$$

represents the effective cyclotron mass of a particle with anisotropic mass [34].

One can rewrite the quantum Hamiltonian in Equation (36) in a more compact form as follows:

$$\hat{H} = \frac{1}{2m_c} \left[\hat{p}_{x'}^2 + \left(\hat{p}_{y'} - qB_z x' \right)^2 \right], \quad (39)$$

where $\hat{p}' = (\hat{p}_{x'}, \hat{p}_{y'})$ is the 2D linear momentum operator with respect to the “new” primed coordinates.

A comparison of the result from Equation (39) with the original Hamiltonian for a particle with constant isotropic mass, as seen in Equation (8), indicates that the problem of a particle with anisotropic mass in variables x and y has been transformed to that of a particle with isotropic mass, $m_c = \sqrt{m_x m_y}$ in terms of “new” scaled variables x' and y' . Note that $dx dy = dx' dy'$. However, one must be careful to write

$$\int_{-\infty}^{+\infty} dx \int_0^{L_y} dy = \int_{-\infty}^{+\infty} dx' \int_0^{L_y/\gamma} dy', \quad (40)$$

when the overall normalization condition of the wave function is applied. The solution of the quantum problem for a particle with constant isotropic mass is well known. Thus, one can immediately use such known results with the only consideration taken that all expressions must be written in terms of the “new” primed variables. The energy eigenvalues are

$$E_n = \hbar \omega'_c \left(n + \frac{1}{2} \right) ; \quad n = 0, 1, \dots \quad (41)$$

where

$$\omega'_c = \frac{|q| |B_z|}{m_c}. \quad (42)$$

Note that ω'_c takes the place of the cyclotron angular frequency, ω_c , for the case of a particle with constant isotropic mass, m . Likewise, $m_c = \sqrt{m_x m_y}$ takes the place of mass, m , for the case of a particle with constant isotropic mass.

With some care, one can write the normalized eigenfunctions corresponding to the above energy levels as follows:

$$\Psi_{nk'_y}(X', y') = \frac{e^{ik'_y y'}}{\sqrt{L_y/\gamma}} \Phi_n(X'), \quad (43)$$

where

$$X' = x' - \frac{\hbar k'_y}{q B_z}, \quad (44)$$

and $\Phi_n(X')$ is the normalized eigenfunction of a displaced 1D harmonic oscillator of mass, m_c , and frequency, ω'_c , in “new” primed coordinates. In order to have the PBC in Equation (10) still be valid, one has

$$k'_y = \gamma k_y, \quad (45)$$

where k_y is given from Equation (17).

The displaced 1D quantum oscillator wave function in the “new” primed coordinates is written as follows:

$$\Phi_n(X') = N'_n \exp\left(-\frac{\alpha'^2 X'^2}{2}\right) H_n(\alpha' X'), \quad (46)$$

where

$$N'_n = \sqrt{\frac{\alpha'}{\sqrt{\pi} 2^n n!}}, \quad (47)$$

and

$$\alpha' = \sqrt{\frac{m_c \omega'_c}{\hbar}}. \quad (48)$$

4. Discussion

Classically speaking, a charged particle experiences a magnetic force when moving through a magnetic field. The fundamental question that one must answer is what happens to the particle if this magnetic field is uniform over the motion of the charged particle. The simplest case occurs when a charged particle with fixed isotropic mass moves perpendicular to a uniform constant magnetic field. Since the magnetic force is perpendicular to the direction of travel, a charged particle follows a circular path in a magnetic field. Another way to look at this is that the magnetic force is always perpendicular to velocity, so that it does no work on the charged particle. As a result, the particle's kinetic energy and speed (magnitude of velocity) remain constant. In a nutshell, the direction of motion is affected but not the speed. The classical description above becomes more nuanced when the charged particle has a constant anisotropic mass with values $m_x \neq m_y$ along the respective x and y directions. We have investigated the classical 2D motion of a charged particle with such an anisotropic mass in the presence of a uniform constant magnetic field that is perpendicular to the plane of motion and have found that the trajectory of the particle for such a case is elliptical [34]. We also have verified that, as expected, such a trajectory becomes circular when the mass becomes isotropic ($m_x = m_y$). Overall, it was found that the resulting classical motion and trajectory of such a particle is very sensitive to the direction of the initial velocity.

The solution of the quantum counterpart to this problem is much more complicated. The main reason is that, unlike the classical scenario, the quantum Hamiltonian is given in terms of the vector potential and not the magnetic field. There are different vector potentials that can generate the same magnetic field. The freedom to choose various vector potentials, $\vec{A}(x, y)$, that lead to the same magnetic field is known as the choice of the gauge. The most common gauges used are the symmetric gauge and Landau gauge. A step-by-step solution to the quantum problem of a charged particle with constant isotropic mass in a perpendicular uniform constant magnetic field for the case of a symmetric gauge is readily available in the literature [35]. The quantum problem of a charged particle with isotropic mass is easier to solve for a Landau gauge. By providing the full details of such a solution, we reminded the reader of the peculiarities of the Landau gauge and also prepared the ground for tackling the much more difficult quantum problem for the counterpart case of a charged particle with anisotropic mass. It is shown in this work that the quantum problem of a charged particle with anisotropic mass in a perpendicular uniform constant magnetic field can be solved rather elegantly by adopting a mathematical method that rescales the original coordinates to new distorted ones. This procedure allows one to restore the mass symmetry of the stationary Schrödinger differential equation, albeit in "new" distorted coordinates.

The quantum problem of a charged particle (with or without an isotropic mass) undergoing 2D motion in a perpendicular uniform constant magnetic field leads to the physics of Landau states. The properties and the nature of Landau states is fundamental to explain a plethora of important phenomena in physics, such as the IQHE and FQHE. The unique nature of the IQHE/FQHE phases has always been fertile ground for paradigm-shifting ideas in theoretical condensed matter physics and materials science. Novel theories, phases of matter and concepts such as topological states, incompressible quantum Hall liquids or composite fermions are now well-known in the literature and all originate from studies of these two phenomena. For all these cases, the starting model assumes a standard Coulomb interaction potential between the charged particles. Obviously, a Coulomb interaction potential is isotropic, in the sense that the interaction energy of any pair of charged particles depends only on their separation distance. The same presumption is valid for many inherently anisotropic phases, such as charge density waves, liquid crystalline phases, Wigner solid phases, etc. The assumption made is that the interaction potential is isotropic (for instance, a Coulomb interaction potential) and there is no intrinsic anisotropy.

However, in a real quantum Hall sample, electrons may possess an anisotropic effective mass tensor or may interact via an effective anisotropic interaction potential (mediated from

the substrate). This situation calls for a re-examination of the role played by anisotropic factors such as an anisotropic effective mass [36]. To be more specific, we consider a 2D system of charged particles with anisotropic band mass values, $m_x > 0$ and $m_y > 0$, along two respective directions labelled x and y . The charged particles interact with the usual (isotropic) Coulomb interaction potential:

$$v_C(\vec{r}_i - \vec{r}_j) = \frac{k q^2}{\sqrt{|x_i - x_j|^2 + |y_i - y_j|^2}}, \quad (49)$$

where k is Coulomb's electric constant, q is the charge of the particles and $\vec{r}_i - \vec{r}_j = (x_i - x_j, y_i - y_j)$ is the 2D vector that separates the positions of particles i and j . The anisotropic mass of the charged particles breaks the rotational symmetry of the quantum kinetic energy operator (with or without a magnetic field). The transformation of variables in Equation (37) allows one to restore the rotational symmetry of the quantum kinetic energy in the "new" scaled coordinates. The same transformation of coordinates, when applied to the Coulomb interaction potential, would transform it into the following anisotropic Coulomb interaction potential:

$$v_\gamma(\vec{r}'_i - \vec{r}'_j) = \frac{k q^2}{\sqrt{\frac{|x'_i - x'_j|^2}{\gamma^2} + \gamma^2 |y'_i - y'_j|^2}}, \quad (50)$$

where $\gamma > 0$ is an interaction anisotropy parameter that leads to anisotropy when different from 1. This parameter (γ) is the same as the scaling parameter of the coordinates that was discussed earlier. From the perspective of Equation (50), one can view the directions of x and y (primed or unprimed) as corresponding to the two principal axes of the dielectric tensor. The potential becomes the standard isotropic Coulomb interaction potential for $\gamma = 1$. In primed variables, this anisotropic Coulomb interaction potential (for $\gamma \neq 1$) breaks the usual assumption of isotropic pair interaction potentials. It is expected that an anisotropic interaction potential of this nature can steer us towards novel conceptual frameworks [37]. The idea is to deal with the rarely tackled, but considerably more difficult, problem of understanding how anisotropic order arises in a quantum system in which the constituent particles interact with an anisotropic interaction potential. An anisotropic interaction potential, alone or in conjunction with other intrinsic degrees of anisotropy in the system, may be seen as a game changer that can lead to novel physics in the field [38,39].

The interaction/correlation effects in systems of electrons may lead to the formation of novel quantum phases of matter. Under certain conditions, one can describe various properties in terms of the underlying topology of the system. This is the case for topological insulators in general and FQHE systems in particular. In fact, FQHE liquids are the ultimate examples of a phenomenon with topological features. As already noticed, the FQHE is observed in certain 2D materials (in the presence of a large perpendicular magnetic field near absolute zero temperature). The quantum effects related to the magnetic field cause a gap to open up between energy bands in the bulk material. As a result, the electrons in the bulk become localized (they cannot move freely). This leads to bulk states of electrons that represent an insulator. However, the electrons at the edge can still move and, thus, they can conduct (this is the physics of "edge states"), while the bulk phase remains insulating. Overall, such phases are characterized by the presence of an energy band-gap within the bulk of the material, while the material's edge/boundary or surface hosts topologically protected gapless conducting modes. The non-trivial topology of the FQHE gives rise to fractionally charged elementary excitations which, in some cases, may even possess non-Abelian braiding statistics (for instance, the Pfaffian state at filling factor $\nu = 5/2$). The interaction/correlation effects between electrons are the key ingredients that lead to this sort of physics (in fact, there is no FQHE without interactions between electrons). The predominant view since the time of Laughlin's theory [13] has been that FQHE states

represent isotropic quantum liquid phases that have rotational symmetry. However, recent work [40] in quantum Hall fluids has revealed the importance of a novel internal geometric degree of freedom (or metric) that has been previously overlooked. It has been pointed out that topological liquid phases arising in the FQHE regime are not required to be rotationally invariant. This means that the presence of an internal anisotropy (such as anisotropic mass and/or anisotropic interaction potential) may drastically modify our view on the remarkable topological properties of such systems. Therefore, the quantum solution of this problem for the case of a charged particle with an anisotropic mass ($m_x \neq m_y$) is very useful for analyzing situations in which the charged particles (for instance, electrons and/or holes) possess a pronounced anisotropic (effective) mass.

5. Conclusions

The quantum problem of a charged particle confined in 2D space in a uniform constant perpendicular magnetic field is the foundation of many important phenomena in physics, where the IQHE/FQHE stand out as two major discoveries from the last few decades. The basic features of all quantum Hall phenomena were initially understood by using the standard model of charged particles (electrons) with constant isotropic mass in a uniform constant perpendicular magnetic field. However, experimental sample refinements have led to an increased interest in grasping more subtle systems involving electrons that possess an effective anisotropic band mass.

Breaking any symmetry in a quantum system leads to problems that are not easy to solve in analytical form. The case of an (effective) particle with anisotropic mass is one such scenario that is important not only from a mathematical perspective, but also for real experimental applications, for instance when dealing with electrons in a semiconductor material. As a matter of fact, the concept of an effective anisotropic mass tensor is routine when it comes to studying the properties of electrons in periodic potentials, such as the ones created by the crystal structure of many semiconductor materials. The standard studies of 2D systems of electrons in a perpendicular magnetic field originally involved GaAs/AlGaAs heterostructures. In these systems, the electrons typically manifest an (effective) isotropic mass. However, the improvements in experimental samples and materials now allow one to study 2D systems of electrons for regimes that were not accessible before, including those in which the electrons manifest an (effective) anisotropic mass. Any source of internal anisotropy in a quantum system may lead to novel subtle quantum phenomena, involving scenarios that may have not been observed before. This means that the system's symmetry (or lack of it) influences the patterns of various observed quantum phenomena. In particular, this work further emphasizes the role that symmetry (or lack of it) plays in quantum problems that involve 2D systems of charged particles with anisotropic mass subject to a perpendicular magnetic field.

For a 2D system of electrons with an applied uniform constant magnetic field in the z -direction and a homogeneous electric field, E_x , in the x -direction, all states drift in the transverse y -direction (where the plane waves are). As a result, the current density in the y -direction, j_y , will be given by

$$j_y = \nu \frac{e^2}{h} E_x, \quad (51)$$

where, for simplicity, we consider IQHE states with filling factor, $\nu = 1, 2, \dots$. Sample details, including effects from the presence of an anisotropic (effective) mass of the electrons, are not expected to play any role in the measured Hall resistance plateaus, $R_H = |V_y/I_x| = h/(\nu e^2)$, where V_y is the Hall (transverse) voltage and I_x is the longitudinal current. However, the ranges of the magnetic field and/or the electron density where the plateaus appear will be affected by an anisotropic mass. The quantized Hall resistance shows a universal behavior, but it is known that the current distribution in real quantum Hall devices is quite complicated [41]. Therefore, an anisotropic mass of the charge carriers is expected to affect the patterns of the current distribution, too. The occurrence of such phenomena may be detected from the experimental observation of

the unexpected magneto-transport anisotropy features of a 2D system of electrons in the quantum Hall regime.

Another interesting scenario where an anisotropic mass may have a profound impact arises when one discusses the effects of Earth's gravity on the quantum Hall behavior of 2D systems of electrons. To this effect, one may consider a 2D quantum Hall sample of electrons oriented in such a way that the gravitational field of the Earth acts on the 2D plane of the sample perpendicular to the magnetic field. The effect of gravity for such an orientation is to act as an effective in-plane constant homogeneous electric field due to the same linear nature of the gravitational potential. For the geometry adopted in this work, one may choose the gravitational field to act in the x -direction. The new twist that comes from the (effective) mass anisotropy of the electrons is that the resulting gravitational potential energy has the gravitational field coupled to the mass of the electrons in the x -direction, resulting in an energy term of the form, $m_x g x$, where g is the acceleration due to gravity on the Earth's surface. If the quantum Hall effect is affected by the gravitational field, the effects of the mass anisotropy should show up as a modification to the current density expression when one uses the quantum Hall effect to probe the inverse-square law of gravity, as recently suggested [42]:

$$j_y(m_x) = \left(1 + \frac{m_x g}{e E_x}\right) \nu \frac{e^2}{h} E_x. \quad (52)$$

This additional energy term correction due to Earth's gravity ($\propto m_x/E_x$) may lead to subtler effects than the case study of a constant isotropic mass, m , previously considered in a recent work, which takes a fresh look at the influence of gravity on the quantum Hall effect states (more precisely, on the IQHE states) of electrons for a variety of conditions [42].

Based on these considerations, one can promptly recognize the reasons why it is important to consider the 2D quantum problem of a charged particle with an anisotropic mass subject to a uniform constant perpendicular magnetic field when a Landau gauge is adopted. This problem is important to the physics of the quantum Hall effect for those situations in which the charged particles (electrons) have an effective band mass anisotropy [43–47]. It is shown that this model, despite exhibiting no axial symmetry, allows an exact analytic calculation of the energy and eigenfunctions for any value of anisotropic mass and magnetic field. The solution to the quantum problem is obtained elegantly by a scaling transformation of the original coordinates. The results of this study would be of interest to a broad audience of individuals working in quantum mechanics, as well as researchers that study the applications of quantum theory in materials science.

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