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Optimal Choice of the Auxiliary Equation for Finding Symmetric Solutions of Reaction–Diffusion Equations

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Abstract: This paper addresses an important method for finding traveling wave solutions of nonlinear partial differential equations, solutions that correspond to a specific symmetry reduction of the equations. The method is known as the simplest equation method and it is usually applied with two a priori choices: a power series in which solutions are sought and a predefined auxiliary equation. Uninspired choices can block the solving process. We propose a procedure that allows for the establishment of their optimal forms, compatible with the nonlinear equation to be solved. The procedure will be illustrated on the rather large class of reaction–diffusion equations, with examples of two of its subclasses: those containing the Chafee–Infante and Dodd–Bullough–Mikhailov models, respectively. We will see that Riccati is the optimal auxiliary equation for solving the first model, while it cannot directly solve the second. The elliptic Jacobi equation represents the most natural and suitable choice in this second case.

Keywords: auxiliary equations; optimal choice; general reaction diffusion equation; Chafee-Infante equation; Dodd-Bullough-Mikhailov equation

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1. Introduction

Traveling waves represent a specific class of solutions for the nonlinear partial differential equations (NPDE). The existence of these solutions is closely related to the symmetry properties of the equation, more precisely to what is known as similarity reduction. Basically, the existence of this symmetry enables us, as a preliminary step of the solving procedure, to describe the evolution in space and time as a one-dimensional evolution, the NPDE becoming a nonlinear ordinary differential equation (NODE). The problem is to solve the respective NODE and a large category of methods approaching this problem consists in looking for the NODE solutions in terms of the already known solutions of a so-called "auxiliary equation" [1]. This approach, also called "simplest equations method" [2], is frequently used following a quite universal algorithm that supposes two important choices: selecting a specific auxiliary equation and determining a form in which the solutions of the equation to be solved are expressed, usually a power series. The variations refer only to the specific choices of the two [3]. A good review of the method and of its connections with other approaches for solving nonlinear differential equations is presented in [4].

Specific choices of the auxiliary equation and of the form in which the solution is sought are not always compatible. In some cases it can happen that the solving procedure is blocked and no solutions are generated for the investigated equation. How these choices can be optimized to be compatible is still an open question to which we will seek an answer in this paper.

We propose here a procedure that does not start from pre-defined choices. The procedure involves associating and evaluating the degrees of the terms that appear in the original equation. By imposing a balancing requirement, so that the terms compensate each other, we will be able to decide which auxiliary equation is optimal for solving a given model of nonlinear dynamical system and what is the form of the solution that will be generated. We will illustrate this procedure on a rather general form of second-order partial differential equation, the reaction–diffusion equation with variable coefficients, a generic equation that includes a lot of very important nonlinear dynamical models, among which Chafee–Infante (CI) [5] and Dodd–Bullough–Mikhailov (DBM) [6], equations on which we will particularly focus. We will see, for example, that the optimal auxiliary equation for solving the DBM equation is the elliptic Jacobi equation, while the Riccati equation cannot solve it directly. For solving the non-linear CI model, the Riccati equation, which is actually a sub-equation from the class of elliptic Jacobi equations, is the best choice as an auxiliary equation. No other sub-equations from this class are suitable.

The paper is structured as follows: after these introductory remarks, the auxiliary equation technique is reviewed and the general procedure we are proposing is presented in the Section 2. How the procedure for establishing the optimal auxiliary equation works for the reaction–diffusion equation is presented in the Section 3. The general procedure will be effectively applied in the Section 4 of the paper for the two important subclasses of reaction–diffusion equations that we have mentioned: the sub-class to which the CI model belongs, and the sub-class that includes the DBM Equation. Some concluding remarks, summarizing the main results reported in the paper, will be included in the last section.

2. The Auxiliary Equation Method

2.1. General Facts on the Auxiliary Equations

As already mentioned, the *auxiliary equation method* or the *simplest equation method* is an indirect approach to finding solutions to more complicated equations, $\Delta([u(\overrightarrow{r}, t)]) = 0$, that cannot be solved directly. If we are interested in traveling wave solutions, we can first proceed to a dimensional reduction, introducing the wave variable ξ. The initial NPDE becomes a NODE in the variable $u(\xi)$ and its derivatives, $u' \equiv du/d\xi, u'' \equiv d^2u/d\xi^2, \ldots$:

$$\Delta(u, u', u'', \ldots) = 0. \tag{1}$$

Solving (1) may not be trivial either and one of the proposed approaches was to express its solutions as functions of a predefined expression $G(\xi)$:

$$u(\xi) = u([G(\xi)]). \tag{2}$$

In the simplest approach, $u(G)$ is considered as a polynomial of G, usually represented as a power series of the form:

$$u(\xi) = u(G(\xi)) \equiv \sum_{j=0}^{N} a_j G^j(\xi);\ a_j = const. \tag{3}$$

The limit N where the expansion is stopping can be fixed by a balancing between the terms containing the higher-order derivative and the higher nonlinearity [7].

Depending on the choice of $G(\xi)$, various solving methods were developed: the tanh method [8], the F-expansion method [9], the exp-function method [10], and many others. It is clear that more choices of $G(\xi)$ lead to more possible solutions of (1). From this remark to the auxiliary equation method, there was only one more step: the idea of considering $G(\xi)$ as the whole set of known solutions of "simplest equations" appeared [2]. Each solution $G(\xi)$ of the set will generate one or more solutions $u(\xi)$. More complex auxiliary equations will automatically generate more sophisticated solutions of the investigated NODEs.

Speaking of auxiliary equations, let us mention that practically all integrable equations can be chosen for this position. However, there are some equations traditionally chosen to play the role of simplest equation. The most frequent choice is the Riccati equation with constant coefficients [11]:

$$G'(\xi) = b_0 + b_1 G + b_2 G^2,\; b_i = const. \tag{4}$$

Another example of an auxiliary equation that is used quite frequently is the second-order linear equation of the form:

$$G'' + \lambda G' + \mu G = 0. \tag{5}$$

It is important to note that when a second-order differential equation such as (5) is considered as an auxiliary equation, the solutions $u(\xi)$ of (1) could also depend on the first derivative $G' \equiv dG/d\xi$. The relation (2) takes the following form:

$$u(\xi) = u(G,\; G'). \tag{6}$$

In this case, the expansion (3) is carried out in terms of both G and G', following specific techniques, such as the G'/G-method [12,13], or the more general *functional expansion* [14].

Let us now conclude the list of commonly used auxiliary equations with a very general equation that will be intensively evoked in this paper, *the general elliptic equation* [15]. Its version with 5 parameters, b_i, $i = \overline{0,4}$, has the following form:

$$\left(G'\right)^2 = \left(\frac{dG(\xi)}{d\xi}\right)^2 = b_0 + b_1 G(\xi) + b_2 G^2(\xi) + b_3 G^3(\xi) + b_4 G^4(\xi). \tag{7}$$

It is a quite general equation whose solutions can be expressed in terms of the Jacobi special functions. For specific values of the parameters, we can generate sub-equations of (7) with fewer parameters and with mathematical expressions similar to other well-known equations [16]. Among the reduced equation, we can recover the Riccati Equation (4) or other equations, such as [17]:

$$\left(G'\right)^2 = b_0 + b_2 G^2(\xi) + b_4 G^4(\xi). \tag{8}$$

$$\left(G'\right)^2 = b_0 + b_1 G(\xi) + b_3 G^3(\xi). \tag{9}$$

$$\left(G'\right)^2 = b_2 G^2(\xi) + b_3 G^3(\xi) + b_4 G^4(\xi). \tag{10}$$

All of them can be solved in special cases and are often used as auxiliary equations.

2.2. General Procedure for Establishing the Optimal Auxiliary Equation

We look now for the simplest way to choose, in a compatible manner, both the auxiliary equation and the form (2) in which we want the solution of $\Delta([u(\xi)]) = 0$ to be expressed. We have to note that in almost all the published papers, the authors used *pre-defined and fixed choices*. These choices do not allow an *a priori* decision if they work together and if they will really lead to solutions for the considered equation. As we will see, there are situations when it is not possible to find the coefficients $\{a_j,\; j = 1, \ldots, N\}$ from (3); thus, the procedure must be restarted with new choices. How to avoid entering in such a circle is what we will try to propose here.

We will not initially choose specific forms for $\Psi([G]) = 0$ or for $u(\xi) = u[G(\xi)]$ in an attempt to see what are the most appropriate choices that. These choices must be, on the one hand, simple enough and, on the other hand, must be compatible in the sense that they can lead to nontrivial solutions $u(\xi)$ for each solution $G(\xi)$ of the auxiliary equation.

The following suppositions are at the base of our procedure:

Conjecture 1: We limit ourselves exclusively to second-order NPDEs generating, by passing to the wave variable, NODEs of the form (1). We assume that all coefficients of the terms involving derivatives in (1) are polynomials in u. This assumption holds for almost all equations of interest, and even if it does not, there are transformations available to make it so [4].

Conjecture 2: The solutions $u(\xi)$ of (1) and their derivatives are expressed as functions of $G(\xi)$:

$$u(\xi) = u(G(\xi)),\ u' = u_G G',\ u'' = u_G G'' + u_{2G} G'^2. \tag{11}$$

Conjecture 3: Without considering $u = u(G)$ as an expansion of the form (3), we still keep it in mind, so are able to talk about minimum and maximum degrees in G appearing in $u(G)$.

Conjecture 4: The auxiliary equation is not predefined; it is established from the wide class of equations of the form

$$G' = f^k(G(\xi)),\ k\ rational. \tag{12}$$

Establishing the equation mean in this case establishing $f(G)$ and k.

Conjecture 5: With (11) and (12), the Equation (1) can be written as a sum of terms with different powers of G. We impose a *compatibility condition* asking for the cancellation of the coefficients of all these powers of G. It leads to an algebraic system with the unknowns $f(G), k$ and $n_G^{\min}, n_G^{\max}$, the minimal and maximal degrees in G of $u(G)$.

Conjecture 6: Solving the algebraic system, we can establish all the forms of $u = u(G)$ and of the auxiliary equations from (12) that are compatible. The *principle of the simplest choices* shows us the *optimal auxiliary equation* and the form of the solution it generates.

Remark 1. *The previously mentioned algebraic system could be difficult to solve and could require the use of specific approaches, like expanding the number of parameters with additional ghost variables. We fall within the BRST theory* [18,19]*.*

Remark 2. *The self-imposed restriction to limit the analysis to second-order differential equations leads to the idea of looking for auxiliary equations of the first order, with $f(G)$ depending on $G(\xi)$ only, not on its derivative. The principle of the simplest choices enabled us to choose it in the canonical form* (12)*. This choice includes two important categories of equations that will be considered here:*

- *The case $k = 1$, corresponding to an auxiliary equation with the generic form:*

$$G' = f(G), \tag{13}$$

- *The case $k = 1/2$, equivalent with the choice of the auxiliary equation as follows:*

$$G'^2 = f(G). \tag{14}$$

How this procedure actually works is illustrated in the next section by applying it to solving a rather general example of equation, the reaction–diffusion equation.

3. Optimal Auxiliary Equations for the Reaction–Diffusion Equations

The Reaction–Diffusion Equation

As we already mentioned, we investigate the problem of choosing in an optimal way the auxiliary equation that has to be used for obtaining traveling wave solutions of the reaction–diffusion equations. The typical form of these equations in a 2D space-time described by the coordinates $\{x, t\}$ is [20]:

$$u_t = \frac{\partial}{\partial x}\left(A(u)\frac{\partial u}{\partial x}\right) + C(u)u_x + E(u). \tag{15}$$

Here, $A(u)$ is the diffusion function (diffusivity), $C(u)$ is a velocity function, while $E(u)$ represents the reaction function. More generally, the class of the nonlinear reaction–diffusion equations contains any second-order NPDE that, by passing to the wave variable, can be reduced to a NODE of the form

$$A(u)u'' + B(u)u'^2 + C(u)u' + E(u) = 0. \tag{16}$$

The Equation (16) includes many nonlinear equations of interest in physics, engineering and biomathematics. In fact, the large majority of the equations describing evolutionary phenomena belongs to this class of equations, as long as it contains the main evolution paths: reaction, diffusion and convection. Of practical importance are the various sub-classes of the equations arising for specific values of the parametric functions. For example,

- If $B(u) = 0$ we get:
$$A(u)u'' + C(u)u' + E(u) = 0.$$
In particular, this sub-class of equation contains the Chafee–Infante and Fisher's equations.
- When $C(u) = 0$ we have:
$$A(u)u'' + B(u)u'^2 + E(u) = 0. \tag{17}$$
To this category belongs to the Dodd–Bullough–Mikhailov equation, describing fluid flows or systems from Quantum Fields.

The procedure of establishing the most suitable choice of the auxiliary equation presented above will be illustrated in the sections ahead on these two sub-classes of equations, more exactly on the CI equation, as an equation with $B(u) = 0$, and on the DBM equation, as an equation with $C(u) = 0$.

Remark 3. *With (11), the general reaction–diffusion Equation (16) will take the form:*

$$A(u)u_G G'' + [A(u)u_{2G} + B(u)u_G^2]G'^2 + C(u)u_G G' + E(u) = 0. \tag{18}$$

It is the most general form of the equation that does not yet take into account a choice for the auxiliary equation.

Remark 4. *The form (12) of the auxiliary equation can be extended to higher-order auxiliary equations, as in [4], where Ψ is considered as*

$$\Psi \equiv \frac{d^l G}{d\xi^l} - \left(\sum_{j=0}^{N} b_j G^j\right)^k = 0, \textit{ with } j, l \text{ integer}, \ k \textit{ rational}. \tag{19}$$

This extension is makes no sense in the case when the equation to be solved is the reaction–diffusion equation, a second-order equation. We will consider exclusively the case $l = 1$. Contrary to the choice (19), we will not depart from the assumption that $f(G)$ is a series expansion. We will keep it as an arbitrary function whose polynomial dependence on G will be determined.

For an auxiliary equation of the form $G' = f^k(G)$, the reaction–diffusion Equation (18) becomes

$$kA(u)u_G f_G f^{2k-1} + [A(u)u_{2G} + B(u)u_G^2]f^{2k} + C(u)u_G f^k + E(u) = 0. \tag{20}$$

From this point on, we have to find possible expressions for $f(G)$, k and $u(G)$ that are compatible with (20). Finding them is equivalent with finding what auxiliary equations can be used for solving (16) and how the corresponding solution looks in terms of $G(\xi)$. The procedure we proposed in the section above suggests considering $u(G)$ and $f(G)$, given by

(11) and (12), respectively, as polynomials in G with constant coefficients. In the first step, we will evaluate the maximum and minimum degrees in G that the terms in Equation (20) could have to compensate each other. In the second step, we will determine the exact forms of $f(G)$ and $u(G)$, using their previously established generic form and finding the polynomial coefficients that transform (20) into the identity.

To effectively apply the procedure, let us first fix some notations: let n_u denote the polynomial degrees in G of the terms appearing in $u(G)$ given by (11), and n_f—the degrees in G of the terms appearing in $f(G)$ from (12). As both $f(G)$ and $u(G)$ could contain many terms, n_u and n_f will be in fact sets of values that have to be determined from the compatibility requirements imposed on (20):

$$n_u = \{n_u^{\min}, \dots, n_u^{\max}\}, \tag{21}$$

$$n_f = \{n_f^{\min}, \dots, n_f^{\max}\}. \tag{22}$$

Polynomial degrees in G will also be attached to $A(u(G))$, $B(u(G))$, $C(u(G))$ and $E(u(G))$ from (16), seen in the most general case as polynomials in $u(G)$. We can consider, for example, that

$$\begin{gathered} A(u) = \sum_{n(A)=n^{\min}(A)}^{n^{\max}(A)} a_n u^{n(A)};\; B(u) = \sum_{n(B)=n^{\min}(B)}^{n^{\max}(B)} b_n u^{n(B)}; \\ C(u) = \sum_{n(C)=n^{\min}(C)}^{n^{\max}(C)} c_n u^{n(C)};\; E(u) = \sum_{n(E)=n^{\min}(E)}^{n^{\max}(E)} e_n u^{n(E)}. \end{gathered} \tag{23}$$

We denoted by $n(A)$, $n(B)$, $n(C)$, $n(E)$ the degrees of the respective functions seen as polynomials in u. Considering for them the summation values from (23), we can conclude that the set of degrees in G for the same functions are

$$n_A = \{n^{\max}(A)n_u,\ (n^{\max}(A)-1)n_u, \dots, n^{\min}(A)n_u\}\ , \tag{24}$$
$$n_B = \{n^{\max}(B)n_u,\ (n^{\max}(B)-1)n_u, \dots, n^{\min}(B)n_u\}, \tag{25}$$
$$n_C = \{n^{\max}(C)n_u,\ (n^{\max}(C)-1)n_u, \dots, n^{\min}(C)n_u\}, \tag{26}$$
$$n_E = \{n^{\max}(E)n_u,\ (n^{\max}(E)-1)n_u, \dots, n^{\min}(E)n_u\}. \tag{27}$$

A simple evaluation shows now that the degrees in G of the terms appearing in (20) are respectively

$$n_A + n_u + 2n_f - 2,\ n_A + n_u + 2n_f - 2,\ n_B + 2n_u + 2n_f - 2,\ n_C + n_f, n_E. \tag{28}$$

The basic requirement to be imposed is that all these terms can compensate each other. We must take into account that the degrees involved in (28) actually represent sets of possible values, and the compensation must be performed for all the terms included in these sets. As for a given model of reaction–diffusion equation $n(A), n(B), n(C)$ and $n(E)$ are known and they refer to at most 1–2 terms, the detailed analysis will be reduced to the possible values for n_u and n_f. This analysis will allow us to determine the maximum and minimum values, and, as a result, to actually write the series expansions for the two functions:

$$u(G) = a_{\min} G^{n_u^{\min}} + a_{\min+1} G^{n_u^{\min+1}} + \ldots + a_{\max} G^{n_u^{\max}}, \tag{29}$$
$$f(G) = b_{\min} G^{n_f^{\min}} + b_{\min+1} G^{n_f^{\min+1}} + \ldots + b_{\max} G^{n_f^{\max}}. \tag{30}$$

The next stage of the procedure we are proposing consists in the effective determination of the coefficients $\{a_i,\ i = n_u^{\min}, \dots, n_u^{\max}\}$ and $\{b_j, j = n_f^{\min}, \dots, n_f^{\max}\}$, respecrively, that appear in (29) and (30). We can do this by solving the algebraic system generated when we insert these relations in (20) and ask for the cancellation of the coefficients of different powers of G.

4. Optimal Choices for Specific Models

In this section, we will demonstrate the functionality of the proposed procedure using specific models belonging to the class of reaction-diffusion equations, and we will show how the solutions for those models are effectively obtained. Two important models will be used to illustrate the procedure: the Chafee–Infante equation belonging to the sub-class of equations with $B(u) = 0$, and the Dodd–Bulough–Mikhailov equation, which belong to the sub-equations with $C(u) = 0$.

4.1. The Case $B = 0$, the Chafee–Infante Equation

Let us start the application of our procedure on specific models of the general reaction–diffusion Equation (16) with the study of a model that belongs to the case $B(u) = 0$. We chose the interesting model of the Chafee–Infante (CI) equation, for which $A(u) = 1$, $C(u) = V = const$ (*the wave velocity*), and $E(u) = -\alpha u(u+1)(u-1)$. The equation will be considered of the following form:

$$u'' + Vu' = \alpha u(u+1)(u-1). \tag{31}$$

When the generic auxiliary Equation (12) is considered, the CI equation will be expressed as

$$ku_G f_G f^{2k-1} + u_{2G} f^{2k} + Vu_G f^k = \alpha u^3 - \alpha u. \tag{32}$$

The degrees in G of the terms from this equation are

$$n_u + 2(kn_f - 1), n_u + (kn_f - 1), 3n_u, n_u. \tag{33}$$

We are looking for the simplest possible solutions of the CI equation, corresponding to $n_u^{\min} = 0$ and $n_u^{\max} = 1$, that is to solutions of the form

$$u(G) = a_0 + a_1 G. \tag{34}$$

We note that the choice of these extreme values of n_u will not increase the nonlinearity in G more than necessary, in the sense that the power in G of the first term from (32) is the smallest that we can consider. In this case, the degrees from (33) that have to be compensated become

$$2kn_f - 2, kn_f - 1, 2kn_f - 1, kn_f, 3, 1, 0, 0.$$

The compensation is possible is we choose $kn_f^{\min} = 1$ and $kn_f^{\max} = 2$.

Conclusion: For the simplest choice (34) of the CI solution, $f(G)$ from the auxiliary Equation (12) will be optimally chosen if it has

$$n_f^{\min} = \frac{1}{k},\ n_f^{\max} = \frac{2}{k}. \tag{35}$$

The optimal auxiliary equation for the Chafee–Infante model will have the most general form:

$$G' = \left(b_{\min} G^{\frac{1}{k}} + \ldots + b_{\max} G^{\frac{2}{k}}\right)^k,\ \ k \text{ rational}. \tag{36}$$

As the usual auxiliary equations are polynomials of positive and integer degrees in G, only rational and sub-unitary values for k have to be considered in (36). For $k = 1$, the relation (36) corresponds to a Riccati type equation. We will write it as

$$G' = b_{11} G + b_{21} G^2. \tag{37}$$

For $k = 1/2$, (36) is an elliptic equation of the form

$$G' = (b_{22}G^2 + b_{32}G^3 + b_{42}G^4)^{1/2}. \tag{38}$$

The choice of other values for k is also possible, but it leads to auxiliary equations that have not obvious solutions to be used in (34).

Let us verify now if and when exactly the choices (37) and (38) lead to CI solutions. In the first case, when we look for solutions of the form (34), the compatibility conditions will impose some constraints among the possible values of the parameters α and V from (31), $a_0,\ a_1$ from (34), and $b_{11},\ b_{21}$ from (37):

$$a_1 = b_{21}\sqrt{\frac{2}{\alpha}};\ b_{11} = \sqrt{2\alpha} - \frac{V}{3}.$$

Considering $a_0,\ \alpha,\ V,\ b_{21}$ as free parameters, we get that the simplest CI solution of the form (34) will be

$$u(G) = a_0 + b_{21}\sqrt{\frac{2}{\alpha}}G. \tag{39}$$

It is generated by the auxiliary Equation (37) with the specific form

$$G' = \left(\sqrt{2\alpha} - \frac{V}{3}\right)G + b_{21}G^2. \tag{40}$$

In the case when (38) will be considered as an auxiliary equation, the compatibility conditions will restrict $a_0 = 0$ or $a_0 = \pm 1$. These two possibilities lead to two possible sets of constraints between the parameters, but both lead to the same, unique form for the auxiliary equation. Considering now b_{22} and b_{42} as free parameters, it can be written as

$$G'^2 = b_{22}G^2 \pm 2\sqrt{b_{22}b_{42}}G^3 + b_{42}G^4 = (\sqrt{b_{22}}G \pm \sqrt{b_{42}}G^2)^2. \tag{41}$$

It is simple to see that, by extracting the square root and with the identification $\sqrt{2\alpha} - \frac{V}{3} = \sqrt{b_{22}};\ b_{21} = \pm\sqrt{b_{42}}$, the Jacobi equation from above reduces in fact to the Riccati Equation (37), and we recover (39) as the corresponding CI solution.

As a conclusion on the Chafee–Infante equation, we can generate solutions of the form (34) if auxiliary equations of the generic form (36) are considered. Both Riccati and Jacobi equations are included here, so both of them seem to be suitable. Our explicit check shows that, in fact, Riccati is the only suitable auxiliary equation to be used, regardless of whether it is considered itself, as in (37), or is viewed as a special sub-class of the Jacobi Equation (38). No other Jacobi sub-equations are suitable.

4.2. The Case $C = 0$. The Dodd–Bullough–Mikhailov Equation

Let us consider now the DBM model that corresponds to the reaction–diffusion Equation (16) with the following coefficient functions: $A(u) = -Vu,\ B(u) = V,\ C(u) = 0,\ E(u) = u^3 + 1$. Its specific form is

$$-Vu\ u'' + Vu'^2 + u^3 + 1 = 0. \tag{42}$$

The equation describes various nonlinear phenomena appearing in optical fibers [21], plasma physics [22–24], or in quantum fields. In Quantum Field Theory, the DBM equation can be assimilated with the mechanical equation attached to Yang-Mills fields, transforming the field dynamics into a particle dynamics. This is an example of reducing a theory with an infinite number of degrees of freedom to a theory with a finite number of degrees of freedom. It is one of the possible approaches for gauge theories, the other alternative being offered by the BRST theory [25], which assumes the introduction of additional ghost variables.

We note that for $C(u) = 0$, (20) takes the form

$$\frac{1}{2}A(u)u_G(f^{2k})_G + [A(u)u_{2G} + B(u)u_G^2]f^{2k} + E(u) = 0. \quad (43)$$

A first remark is that this equation becomes linear if we use the notation $f^{2k}(G) = h(G)$. It is equivalent to consider that the auxiliary Equation (12) is chosen as

$$G' = h^{1/2}(G) \text{ or } G'^2 = h(G). \quad (44)$$

The equation that has to be solved can be written in the variable $f(G)$ or using $h(G)$:

$$-kVu\, u_G f^{2k-1} f_G + V\left(u_G^2 - uu_{2G}\right)f^{2k} + u^3 + 1 = 0, \quad (45)$$

$$-\frac{V}{2}u\, u_G h_G + V\left(u_G^2 - u\, u_{2G}\right)h + u^3 + 1 = 0. \quad (46)$$

Considering u and f as polynomials in G, we can evaluate the degrees of the terms from the previous equations. Let us first consider the degrees of the terms appearing in the Equation (45). At the level of the full equations, the following degrees have to compensate each other: $2\left(n_u + kn_f - 1\right)$, and $3n_u, 0$. The first degree is always an even number, regardless of the values of n_u, k and n_f, while the degree $3n_u$ can be odd for n_u odd, a situation in which the term $3n_u$ cannot be compensated by any other terms. In conclusion, the degree n_u cannot be odd. Therefore, we have to consider that

$$n_u \in \{0, 2, 4, \ldots\}. \quad (47)$$

Using the principle of the simplest choice, we could consider that

$$u(G) = a_2G^2 + a_0. \quad (48)$$

We look now for a form of $f(G)$ that could be compatible with the form of $u(G)$ established above. In other words, we are looking for the optimal choice for the auxiliary equation, which can be equivalently found by using (48) directly in (46) for finding the function $h(u)$. From (48) and (46), we get

$$(-Va_2^2G^3 - Va_0a_2G)h_G + (2Va_2^2G^2 - Va_0a_2)h + \left(a_2G^2 + a_0\right)^3 + 1 = 0. \quad (49)$$

This equation contains terms whose degrees in G are: $n_h + 2$, $n_h, 6, 4, 2, 0$. All the terms could compensate if $n_h^{\max} = 4$ and $n_h^{\min} = 0$. So, the most general form to consider for $h(G)$ would be

$$h(G) = b_4G^4 + b_3G^3 + b_2G^2 + b_1G + b_0. \quad (50)$$

Transposed for the variable $f(u)$, this dependence can be written as

$$f(G) = \left(b_4G^4 + b_3G^3 + b_2G^2 + b_1G + b_0\right)^{\frac{1}{2k}}. \quad (51)$$

We go now to the second stage of the general procedure, the stage that involves the effective determination of the optimal auxiliary equation and of the DBM solutions, by determining the coefficients $\{a_i, b_i\}$. We introduce (50) into the Equation (49) and group the terms according to the powers of G. Canceling the coefficients will lead us, as a general compatibility requirement, to $b_1 = b_3 = 0$, and to two constraints between the remaining coefficients:

$$3a_0^2 + 4V^2b_4b_0 - 4Va_0b_2 = 0,$$
$$a_0^3 - 4V^2b_4b_0a_0 + 1 = 0.$$

We can choose, for example, b_2 and b_4 as independent parameters. The optimal auxiliary equation will now be the two–parametric Jacobi elliptic sub-equation:

$$G'^2 = b_4 G^4 + b_2 G^2 - \frac{b_2^2}{b_4}. \tag{52}$$

Any solution $G(\xi)$ of (52) generates a DBM solution of the form

$$u(G) = -\frac{1}{\sqrt[3]{2}}\left(\frac{b_4}{b_2}G^2 + 1\right). \tag{53}$$

Conclusion: Regardless of the values that k from relation (51) take, the appropriate auxiliary equation to search for DBM solutions in the form (48) is of Jacobi sub-equation $G'^2 = b_4 G^4 + b_2 G^2 + b_0$. This conclusion remains valid for any reaction–diffusion equation with $C = 0$.

Remark 5. *Let us see what would happen if, without any preliminary analysis, we decide to solve the DBM equation using as auxiliary equation a Riccati equation of the form (4), $G'(\xi) = b_0 + b_1 G + b_2 G^2$. Looking again for DBM solutions of the simplest form (48), with $n_u^{max} = 2$, $n_u^{min} = 0, k = 1$, the degree analysis will lead to request for compensation of the following degrees: $2\left(n_f + 1\right), 2\left(n_f - 1\right), 6, 0, 0$. To compensate the terms of the form G^6, we should consider that $n_f^{max} = 2$ and to compensate terms with G^0, we should have $n_f^{min} = 0$, so $n_f = \{0, 1, 2\}$. A simple check shows that it is not possible to compensate all the terms appearing in the DBM Equation (45). In other words, the choice of the Riccati as auxiliary equation cannot directly give DBM solutions of the form (48).*

Remark 6. *The optimal Equation (52) and the corresponding DBM solution (53) were obtained for the simplest choice (48) of $u(\xi)$. What would happen if we decided to look for more complex solutions for $u(\xi)$? According to (47), we can choose, for example, a more complex dependency of $u = u(G)$, with maximal degree in G given by the next possible value, $n_u^{max} = 4$, in which case we would have the following:*

$$u = a_4 G^4 + a_2 G^2 + a_0. \tag{54}$$

A careful analysis of this possibility, similar to the previous one, will lead to DBM solutions and optimal auxiliary equations expressed through three parameters. By choosing b_4, b_6 and V as free parameters, we will get

$$\begin{aligned} u(G) &= 8Vb_6 G^4 + 4Vb_4 G^2 - 1, \\ G'^2 &= b_6 G^6 + b_4 G^4 - 3G^2/4V. \end{aligned} \tag{55}$$

In conclusion, the optimal auxiliary equation depends both on the type of equation to be solved and on the choice of the form in which the solution $u(x,t)$ is sought. If we look for solutions of the DBM equation of the form (48), the optimal auxiliary equation will be of the form (52), and for the choice (54), the optimal auxiliary equation has the form (55).

5. Conclusions

This paper considered key issues regarding the integrability of nonlinear differential equations [26] and their complex solutions [27]. As we mentioned, we investigated a direct method for finding traveling wave solutions, a specific class of solutions generated by the Lie symmetry technique and the similarity reduction procedure [28,29]. More specifically, we investigated how the auxiliary equation method can be optimally applied to solve a second-order reaction–diffusion equation. In its general form, the method assumes two important choices: selecting the auxiliary equation and determining the expression (2) in which the solution of the equation to be solved is expressed. The two choices cannot be made arbitrarily; they can generate incompatibilities and can generate deadlocks of the solving algorithm. How to avoid such a mismatch represented the main aim of the paper. We have proposed a procedure that enables us to choose the optimal auxiliary equation that leads to predefined forms of solutions for different equations. The procedure is based on an analysis of polynomial degrees, similar to the one performed in the homogeneous balance method, but without departing from the assumption of predefined polynomial forms. To be very concrete, we have illustrated the procedure for solving the reaction–diffusion Equation (16), looking for optimal auxiliary equations of the type (12). Two important examples of auxiliary equations from this class, the Riccati Equation (4) and the generalized elliptic Equation (7), represented the main focus. The proposed procedure was effectively considered for two sub-cases of (16): $C(u) = 0$ and $B(u) = 0$. General rules for the choice of the auxiliary equation in the case $B(u) = 0$ are presented in Section 4.1 for the Chafee–Infante equation, while the Dodd–Bullough–Mikhailov equation is considered in Section 4.2 for the case $C(u) = 0$.

The main results of our investigation were that not all the auxiliary equations are suitable for solving a given nonlinear differential equation; however, an a priori choice of them is possible. We found that the reaction–diffusion equations with $B(u) = 0$ can be apparently solved both through Riccati and Jacobi equations. In reality, from the class of Jacobi equations, only the equations that reduce to Riccati can be used, so Riccati is the optimal auxiliary equation to choose for solving reaction–diffusion equations with $B = 0$. For the reaction–diffusion equations with $C(u) = 0$ the most suitable auxiliary equation is of Jacobi type, while an auxiliary equation of Riccati type is not compatible and cannot directly solve the DBM Equation. The form of the optimal auxiliary equation equally depends on the form in which we are looking for solutions. For the simplest choice (48) of the DBM solution, we can use the Jacobi sub-equation of order 4 (52), while for finding more complex solutions of the form (54), we have to use the Jacobi sub-equation (55) of order 6, as the most suitable auxiliary equation.

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