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On the Sums over Inverse Powers of Zeros of the Hurwitz Zeta Function and Some Related Properties of These Zeros

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Abstract: Recently, we have applied the generalized Littlewood theorem concerning contour integrals of the logarithm of the analytical function to find the sums over inverse powers of zeros for the incomplete gamma and Riemann zeta functions, polygamma functions, and elliptical functions. Here, the same theorem is applied to study such sums for the zeros of the Hurwitz zeta function $\zeta(s,z)$, including the sum over the inverse first power of its appropriately defined non-trivial zeros. We also study some related properties of the Hurwitz zeta function zeros. In particular, we show that, for any natural N and small real ε, when z tends to $n = 0, -1, -2\ldots$ we can find at least N zeros of $\zeta(s,z)$ in the ε neighborhood of 0 for sufficiently small $|z+n|$, as well as one simple zero tending to 1, etc.

Keywords: logarithm of an analytical function; generalized Littlewood theorem; Hurwitz zeta function; zeros and poles of analytical function

MSC: 30E20; 30C15; 33B20; 33B99

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1. Introduction

For any fixed complex $z \neq 0, -1, -2\ldots$ and Re $s > 1$, the Hurwitz zeta function is defined as

$$\zeta(s,z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^s} \tag{1}$$

and with analytic continuation for Re$s \leq 1$. The only singularity of the function is the simple pole at $s = 1$ with the residue one; see, e.g., references [1–5] for the discussion of the main properties of this function. They are well known, so below, we give only a rather short account.

Many integral representations of the Hurwitz zeta function can be established. These, of course, include the "classic" representation $\zeta(s,z) = \frac{1}{\Gamma(s)}\int\limits_0^{\infty} \frac{x^{s-1}e^{-ax}}{1-e^{-x}}dx$, where Re $s > 1$ and Re $a > 0$, but the most used for analytical continuation is Hermite's integral representation, valid for Re $z > 0$ and $s \neq 1$:

$$\zeta(s,z) = \frac{1}{2}z^{-s} + \frac{z^{1-s}}{s-1} + 2\int\limits_0^{\infty} \frac{\sin(s\arctan(x/z))}{(x^2+z^2)^{s/2}(e^{2\pi x}-1)}dx. \tag{2}$$

For Re $z < 0$, the relation

$$\zeta(s,z) = \frac{1}{z^s} + \zeta(s,z+1), \tag{3}$$

With its evident subsequent applications, like $\zeta(s,z) = \frac{1}{z^s} + \frac{1}{(z+1)^s} + \zeta(s,z+2)$, etc., are exploited. Being an example of zeta functions, $\zeta(s,z)$ obeys the functional equation

(Hurwitz formula) $\zeta(s,z) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\left(\sin(\frac{\pi s}{2})\sum_{n=1}^{\infty}\frac{\cos(2\pi na)}{n^{1-s}} + \cos(\frac{\pi s}{2})\sum_{n=1}^{\infty}\frac{\sin(2\pi na)}{n^{1-s}}\right)$, valid for Re $s < 0$ and $0 < a \leq 1$. For rational values of a, this equation acquires a rather compact form known as Rademacher's formula; see below for its concrete expressions.

The properties of the Hurwitz zeta function have been much studied due to its great importance in number theory, statistics, and physics; for the latter, see especially [6]. Of course, the particular question of the location of zeros of Hurwitz zeta function has also been much studied, which is not surprising, given the celebrated Riemann hypothesis and the circumstance that $\zeta(s,1) = \zeta(s)$. Probably, the theorem of Davenport and Heilbronn [7] remains the most interesting result here: they proved that, for any rational or transcendent irrational $0 < z < 1$, $z \neq 1/2$, there are infinitely many zeros of $\zeta(s,z)$ in any strip $1 < \mathrm{Re}s(s,z) < 1+\varepsilon$ for any real positive ε; later on, the same was proven by Cassel for algebraic irrational z [8]. There are also a number of both analytic and numerical studies dealing with the (mainly real) zeros of $\zeta(s,z)$; see, e.g., [9–14]. In particular, for real $0 < x \leq 1$, Spira [9] established the absence of zeros to the right of $\mathrm{Re}s \geq 1+x$, as well as the absence of zeros for $|\mathrm{Im}s| \geq 1$ if $\mathrm{Re}s < -1$, and showed that for $|\mathrm{Im}s| \leq 1$ and $\mathrm{Re}s \leq (-4x+1+2[1-2x])$, the only zeros are (analogs of) trivial zeros, one in each interval $-2n-4x\pm 1$; n is an integer and $n \geq 1-2x$. He also indicated the formula describing the number of zeros of $\zeta(s,x)$ with $0 < \mathrm{Re}s \leq T$:

$$N(x,T) = \frac{T}{2\pi}\ln T - T(\frac{1+\ln(2\pi x)}{2\pi}) + O(\ln T). \quad (4)$$

(This relation was not explicitly proven by Spira; he presented only the statement that it can be proven using a method of Berndt's [15]. The same formula was conjectured and tested numerically in [14]).

Nevertheless, despite these findings, the picture is far from fully clear, so the question concerning the sum over inverse powers of Hurwitz zeta function zeros remains actual. The calculations of such sums (and functions related to them) have a long tradition in the theory of the special functions, apparently started already by Rayleigh for the case of Bessel's function in the XIX century [16] (for a good review of the Rayleigh function and its applications, see, e.g., [17]); for other examples, see our recent publications [18,19]). In many cases, such sums can improve our understanding of the function studied and enable us to make certain statements concerning its zeros: in our opinion, the present work well illustrates this circumstance, sf. Below, the discussion of the possibility of claiming the infinite number of s-zeros of the $\zeta(s,z)$ function close to $s = 0$ when z approaches zero from the analysis of the sums $\sum_{\substack{zeroes\ of\ \zeta(s,z), \\ z\to 0}} \frac{k_i}{\rho_i^n}$.

We study the sums over inverse powers of the Hurwitz zeta function zeros, together with some related questions, in the present paper. Our method is mainly based on the generalized Littlewood theorem, and recently, analogous studies were undertaken for the polygamma, incomplete gamma, and Riemann zeta functions [18], as well as elliptical functions [19]. These applications followed our earlier applications to the Riemann zeta function [20–22]; here, we would like to underline that one of them has been highlighted in a recent encyclopedia of mathematics entry [23]. For this reason, only a very short discussion of the method itself is given below. All details can be found in the aforementioned papers, especially [18].

2. The Generalized Littlewood Theorem

The generalized Littlewood theorem concerning contour integrals of the logarithm of the analytical function is stated as follows:

Theorem 1 (The Generalized Littlewood theorem)**.** *Let C denote the rectangle bounded by the lines $x = X_1, x = X_2, y = Y_1, y = Y_2$, where $X_1 < X_2, Y_1 < Y_2$, and let f(z) be analytic and* *non-zero on C and meromorphic inside it, and let also g(z) be analytic on C and meromorphic inside it. Let F(z) = ln(f(z)) be the logarithm defined as follows: we start with a particular determination at* $x = X_2$*, and we obtain the value at other points through continuous variation along y = const from* $\ln(X_2 + iy)$*. If, however, this path would cross a zero or pole of f(z), we take F(z) to be* $F(z \pm 0)$*, according to whether we approach the path from above or below. Let also* $\widetilde{F}(z) = \ln(f(z))$ *be the logarithm defined through continuous variation along any smooth curve fully lying inside the contour, which avoids all poles and zeros of f(z) and starts from the same particular determination at* $x = X_2$*. Suppose also that the poles and zeros of the functions f(z) and g(z) do not coincide.*

Then,

$$\int_C F(z)g(z)dz = 2\pi i\left(\sum_{\rho_g} res(g(\rho_g)\cdot\widetilde{F}(\rho_g)) - \sum_{\rho_f^0}\int_{X_1+iY_\rho^0}^{X_\rho^0+iY_\rho^0} g(z)dz + \sum_{\rho_f^{pol}}\int_{X_1+iY_\rho^{pole}}^{X_\rho^{pole}+iY_\rho^{pole}} g(z)dz\right). \tag{5}$$

where the sum is over all ρ_g*, which are poles of the function g(z) lying inside C, all* $\rho_f^0 = X_\rho^0 + iY_\rho^0$*, which are zeros of the function f(z), both counted while taking into account their multiplicities (that is, the corresponding term is multiplied by m for a zero of the order m) and which lie inside C, and all* $\rho_f^{pole} = X_\rho^{pole} + iY_\rho^{pole}$*, which are poles of the function f(z), counted while taking into account their multiplicities, and which lie inside C. The assumption is that all relevant integrals on the right-hand side of the equality exist.*

The proof of this theorem [21] is very close to the proof of the standard Littlewood theorem corresponding to the case *g(z)* = 1; see, e.g., [24]. Below, with the exception of Section 4, we apply this theorem to certain particular cases when the contour integral $\int_C F(z)g(z)dz$ disappears (tends to zero) if the contour tends to infinity—that is, when $X_1, Y_1 \to -\infty$ and $X_2, Y_2 \to +\infty$ This means that Equation (5) takes the form

$$\sum_{\rho_f^0}\int_{-\infty+iY_\rho^0}^{X_\rho^0+iY_\rho^0} g(z)dz - \sum_{\rho_f^{pol}}\int_{-\infty+iY_\rho^{pole}}^{X_\rho^{pole}+iY_\rho^{pole}} g(z)dz = \sum_{\rho_g} res(g(\rho_g)\cdot F(\rho_g)). \tag{6}$$

3. Sums over Inverse Powers of Zeros for the Hurwitz Zeta Function

3.1. General Formulae

At the point *s* = 1, the Hurwitz zeta function possesses absolutely and uniformly converging Laurent expansion with the generalized Stieltjes constants [1–5]

$$\gamma_n(z) = \lim_{N\to\infty}\left(\sum_{k=0}^{N}\frac{\ln^n(k+z)}{k+z} - \frac{\ln^{n+1}(N+z)}{n+1}\right): \tag{7}$$

$$\zeta(s,z) = \frac{1}{s-1} + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_n(z)(s-1)^n. \tag{8}$$

This expansion is evidently analogous to that of the Riemann zeta function $\zeta(s) = \frac{1}{s-1} + \sum\limits_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_n(s-1)^n$. Here, $\gamma_0 := \gamma$ is the Euler–Mascheroni constant.

Thus,

$$(s-1)\zeta(s,z) = 1 - \psi(z)(s-1) - \gamma_1(z)(s-1)^2 + \frac{1}{2}\gamma_2(z)(s-1)^3 + O((s-1)^4), \tag{9}$$

and this solves the problem of the sum of inverse powers of "zeros minus one"; compare with the similar results in our previous works [18,19]. We need just to add that the asymptotic of the Hurwitz zeta function, similar to the Riemann zeta function, for large $|s|$ is *O*(sln*(s))* [1–5], so we can use the generalized Littlewood theorem for $n \geq 3$.

Let us slightly enlarge the scope of our elementary Lemma 1 from [18].

Lemma 1. *Let f(z) be an analytical function defined on the whole complex plane except possibly a countable set of points. Let also this function be represented in some* neighborhood *of the point z = 0 via the Taylor expansion* $f(z) = 1 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots$*, and the contour integral* $\int_C \frac{\ln f(z)}{z^3} dz$ *tends to zero when contour C tends to infinity (see Theorem 1 for the details). Then, for the sum over zeros* $\rho_{i,0}$ *, having order* k_i*, and poles* $\rho_{i,pole}$*, having order* l_i*, of the function f(z), we have*

$$\sum \left(\frac{k_i}{\rho_{i,0}^2} - \frac{l_i}{\rho_{i,pole}^2}\right) = a_1^2 - 2a_2, \tag{10}$$

$$\sum \left(\frac{k_i}{\rho_{i,0}^3} - \frac{l_i}{\rho_{i,pole}^3}\right) = -a_1^3 + 3a_1 a_2 - 3a_3, \tag{11}$$

$$\sum \left(\frac{k_i}{\rho_{i,0}^4} - \frac{l_i}{\rho_{i,pole}^4}\right) = a_1^4 - 4a_1 a_3 + 2a_2^2 - 4a_4. \tag{12}$$

Proof. We trivially have, in some *neighborhood* of the point $z = 0$, the Taylor expansion $\ln(f(z)) = a_1 z + (a_2 - \frac{1}{2}a_1^2)z^2 + (a_3 - a_1 a_2 + \frac{1}{3}a_1^3)z^3 + (a_4 - \frac{1}{2}a_2^2 + a_1 a_3 - \frac{1}{4}a_1^4)z^4 + \ldots$, and now the direct application of Theorem 1 to the integrals $\int_C \frac{\ln f(z)}{z^n} dz$ with $n = 3$, 4, and 5 gives the statement of the current lemma. □

Remark 1. *In such a particular form, the lemma is presented for the ease of subsequent applications. Of course, this is possible to establish the difficult-to-use general relation between the coefficients of the Taylor expansions of the functions f(z) and* $\ln(f(z)) = b_1 z + b_2 z^2 + b_3 z^3 + \ldots$*, and this actually has been done in [25]. For completeness, we reproduce here Lemma 2.1 from that work (given there for one particular function with the alternating signs in the Taylor expression, but this is not important. Note also the misprint in the formulation of the lemma; the final formula in its proof is correct).*

Lemma 2. *Let the function f(z) have the following Taylor expansion in the neighborhood of z = 0:*

$$f(z) = 1 - \sum_{n=1}^{\infty} a_n z^n. \text{ Then, } \ln(f(z)) = \sum_{n=1}^{\infty} b_n z^n, \textit{where}$$

$$b_n = -\sum_{j_1+2j_2+3j_3+\ldots=n} \frac{(j_1 + j_2 + j_3 + \ldots - 1)!}{j_1! j_2! j_3! \ldots} a_1^{j_1} a_2^{j_2} a_3^{j_3} \ldots$$

Proof.

$$\ln(f(z)) = -\sum_{m=1}^{\infty} \frac{1}{m} \left(\sum_{k=1}^{\infty} a_k z^k\right)^m = -\sum_{m=1}^{\infty} \frac{1}{m} \sum_{j_1+j_2+j_3+\ldots=m} \frac{m!}{j_1! j_2! \ldots} z^{j_1+2j_2+3j_3+\ldots} a_1^{j_1} a_2^{j_2} a_3^{j_3} \ldots$$

$$= -\sum_{m=1}^{\infty} \sum_{j_1+j_2+j_3+\ldots=m} \frac{(m-1)!}{j_1! j_2! \ldots} z^{j_1+2j_2+3j_3+\ldots} a_1^{j_1} a_2^{j_2} a_3^{j_3} \ldots$$

$$= -\sum_{n=1}^{\infty} z^n \sum_{j_1+2j_2+3j_3+\ldots=n} \frac{(j_1+j_2+j_3+\ldots-1)!}{j_1! j_2! j_3! \ldots} a_1^{j_1} a_2^{j_2} a_3^{j_3} \ldots$$

□

Using Lemma 1 and just substituting the appropriate values of a_i into it, we obtain the following:

$$\sum_{zeroes\ of\ \zeta(s,z)} \frac{k_i}{(\rho_i-1)^2} = \psi^2(z) + 2\gamma_1(z), \tag{13}$$

$$\sum_{zeroes\ of\ \zeta(s,z)} \frac{k_i}{(\rho_i-1)^3} = \psi^3(z) + 3\psi(z)\gamma_1(z) - \frac{3}{2}\gamma_2(z), \tag{14}$$

$$\sum_{zeroes\ of\ \zeta(s,z)} \frac{k_i}{(\rho_i-1)^4} = \psi^4(z) + 2\psi(z)\gamma_2(z) + 2\gamma_1^2(z) + \frac{2}{3}\gamma_3(z). \tag{15}$$

Here, we exploited $\gamma_0(z) = -\psi(z)$ [1–5], where $\psi(z)$ is the digamma function (see, e.g., [3–5]) for discussion of this function.

To find the sums $\sum\limits_{zeroes\ of\ \zeta(s,z)} \frac{k_i}{\rho_i^2}$, $\sum\limits_{zeroes\ of\ \zeta(s,z)} \frac{k_i}{\rho_i^3}$, etc., we also use our standard approach to write if $z \neq 1/2; 0, -1, -2, -3\ldots$:

$$\frac{\zeta(s,z)}{\zeta(0,z)} = 1 + \frac{1}{\zeta(0,z)}\zeta'(0,z)s + \frac{1}{2\zeta(0,z)}\zeta''(0,z)s^2 + \frac{1}{6\zeta(0,z)}\zeta'''(0,z)s^3 + O(s^4) \tag{16}$$

(all derivatives in the paper are over the variable s), whence, per Lemma 1,

$$\sum_{zeroes\ of\ \zeta(s,z)} \frac{k_i}{\rho_i^2} = 1 + \frac{\left[\ln\Gamma(z) - \frac{1}{2}\ln 2\pi\right]^2}{(1/2-z)^2} - \frac{\zeta''(0,z)}{1/2-z}, \tag{17}$$

$$\begin{aligned}
&\sum_{zeroes\ of\ \zeta(0,z)} \frac{k_i}{\rho_i^3} = 1 - \tfrac{1}{(1/2-z)^3}(\ln\Gamma(z) - \ln 2\pi)^3 + \tfrac{3}{2(1/2-z)^2}(\ln\Gamma(z) - \ln 2\pi)\zeta''(0,z) \\
&-\tfrac{1}{2(1/2-z)}\zeta'''(0,z) \\
&\sum_{zeroes\ of\ \zeta(0,z)} \frac{k_i}{\rho_i^4} = 1 + \tfrac{1}{(1/2-z)^4}(\ln\Gamma(z) - \ln 2\pi)^4 - \tfrac{2}{3(1/2-z)^3}(\ln\Gamma(z) - \ln 2\pi)\zeta'''(0,z) \\
&+\tfrac{1}{2(1/2-z)^2}[\zeta''(0,z)]2 - \tfrac{\zeta'''\prime(0,z)}{6(1/2-z)}
\end{aligned} \tag{18}$$

Here, we used the relations [1–5]

$$\zeta(0,z) = \frac{1}{2} - z \tag{19}$$

and

$$\zeta'(0,z) = \ln\Gamma(z) - \frac{1}{2}\ln 2\pi, \tag{20}$$

readily following from the integral representation (1) and the second Binet's integral formula $\ln\Gamma(z) = (z-1/2)\ln z + \frac{1}{2}\ln(2\pi) + 2\int\limits_0^\infty \frac{\arctan(x/z)}{e^{2\pi x}-1}dx$ [3–5] for Re $z > 0$. In these equations, 1 is the contribution of the simple pole at $s = 1$.

Similarly, for any $p \neq 1$, we have

$$\sum_{zeroes\ of\ \zeta(s,z)} \frac{k_i}{(\rho_i-p)^2} = \frac{1}{(1-p)^2} + \frac{\left[\zeta'(s,z)|_{s=p}\right]^2}{\zeta^2(p,z)} - \frac{\zeta''(s,z)|_{s=p}}{\zeta(p,z)}, \tag{21}$$

provided that $\zeta(p,z) \neq 0$ and $z \neq 0, -1, -2, -3\ldots$, and analogous relations can be without difficulties established for the sums over larger inverse powers.

3.2. Behavior of the s-Zeros When z Tends to Infinity

The behavior of the sums over inverse powers of zeros of the Hurwitz zeta function when z tends to infinity is trivial and not interesting: all such sums tend to zero.

We know that, for any $s \neq 1$, as $z \to \infty$ in the sector $\arg(z) \leq \pi - \delta$ with an arbitrary, small, positive, fixed δ, the following asymptotic holds [1,2]:

$$\zeta(s,z) \sim \frac{z^{1-s}}{s-1} + \frac{1}{2}z^{-s} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}(s)_{2k-1} z^{1-s-2k}. \tag{22}$$

Here, B_{2k} are Bernoulli numbers, and Pochhammer symbol notation is employed. Corresponding asymptotics of the Hurwitz zeta function derivatives have also been much studied; see, e.g., [26–28], with the main conclusion that the "naïve" differentiation of (22) suffices. Substitution of these asymptotics to (21) readily shows that, for $z \to \infty$ in the sector $\arg(z) \leq \pi - \delta$ with an arbitrary small positive fixed δ and any $p \neq 1$, the sum $\sum\limits_{\substack{zeroes\ of\ s, \\ \zeta(s,z)}} \frac{k_i}{(\rho_i - p)^2}$ tends to zero for any such p, such that $\rho_i \neq p$.

The necessary technical details can easily be provided, but indeed, the statement that all $\sum\limits_{zeroes\ of\ \zeta(s,z)} \frac{k_i}{(\rho_i - p)^n}$, $n \geq 2$, for the Hurwitz zeta function to tend to zero for any p not equal to one or zero when z tends to infinity follows just from the circumstance that the used "asymptotic function" (first term in (22)) $\frac{z^{1-s}}{s-1}$ has no zeros (and one simple pole at $s = 1$), there is nothing to prove and study. (If we consider two terms of the asymptotic development (22), we do obtain one zero at $s = 1-2z$, where $\frac{z^{1-s}}{s-1} + \frac{1}{2}z^{-s} = 0$—but clearly, for any p, $\frac{1}{(1-2z-p)^n} \to 0$ when $|z|$ tends to infinity, etc.).

For $p = 1$, we have the same picture of the disappearance of the sums, provided that the simple pole at $z = 1$ is removed; see Formulae (13–15). For $n = 2$, we can illustrate this, applying the known asymptotics $\psi(z) \sim \ln z + O(1/z)$ [3–5] and $\gamma_1(z) \sim -\frac{1}{2}\ln^2 z + O(\ln z/z)$ [29] when $z \to \infty$ in the sector $\arg(z) \leq \pi - \delta$ with an arbitrary small positive fixed δ. We are unaware of the studies of the corresponding asymptotics for larger Stieltjes constants. They can be inferred from the requirement that the sum in question tends to zero. For example, from (17), we have $\gamma_2(z) \sim -\frac{1}{3}\ln^3 z + o(\ln^3 z)$.

This asymptotic behavior merely reflects the fact that there is no small in-module s-zeros when z tends to infinity except possibly the case of a large by-module negative real z, in which case the question should be studied separately.

3.3. Behavior of the s-Zeros When z Tends to Zero

Quite the contrary, the behavior of the sums over inverse powers of zeros (that is, of course, also the behavior of the zeros themselves) is interesting and complicated when z tends to zero. Let us start with the analysis of $\sum\limits_{\substack{zeroes\ of\ \zeta(s,z), \\ z \to 0}} \frac{k_i}{\rho_i^2}$.

We know $\Gamma(z) = \frac{1}{z} - \gamma + O(z)$ [3–5]. From $\zeta^{(k)}(0,1) = \zeta^{(k)}(0) = O(1)$ and $\zeta(s,z) = \frac{1}{z^s} + \zeta(s, z+1)$, we have for small z

$$\zeta^{(k)}(0,z) = (-1)^k \ln^k z + \zeta^{(k)}(0) + O(z). \tag{23}$$

(In particular, from (20), $\zeta'(0,z) = -\ln z - \frac{1}{2}\ln 2\pi - \gamma z + O(z^2)$. Sf. Also the paper of Deniger, who showed that, for real positive z,

$\zeta''(0,z) = \zeta''(0) + \gamma_1 z + \ln^2 z + \sum\limits_{n=1}^{\infty} (\ln^2(z+n) - \ln^2 n - 2z\frac{\ln n}{n})$, the series converges absolutely and uniformly on any compact subset of R^+ [30], and the result is analytically continued to all z except $z = 0, -1, -2\ldots$). Thus, from (23), it immediately follows that, when $z \to 0$, $(\ln\Gamma(z) - \frac{1}{2}\ln(2\pi))^2 = \ln^2 z + \ln(2\pi)\cdot\ln z + \frac{1}{4}\ln^2(2\pi) + O(z)$,

$\zeta''(z) = \ln^2 z + \zeta''(0) + O(z)$, and thus, the asymptotic of $\sum\limits_{zeroes\ of\ \zeta(s,z)} \frac{k_i}{\rho_i^2}$ with the $O(z)$ precision is $\sum\limits_{\substack{zeroes\ of\ \zeta(s,z),\\ z\to 0}} \frac{k_i}{\rho_i^2} = 2\ln^2 z + 4\ln(2\pi)\cdot\ln z + 1 + \ln^2(2\pi) - 2\zeta''(0) + O(z)$.

For completeness, we present the value of $\zeta''(0)$ here (see, e.g., [30]): $\zeta''(0) = \frac{1}{2}(-\ln^2 2\pi - \frac{\pi^2}{12} + \gamma^2 + \gamma_1) \simeq -2.006$. Thus,

$$\sum_{\substack{zeroes\ of\ \zeta(s,z),\\ z\to 0}} \frac{k_i}{\rho_i^2} = 2\ln^2 z + 4\ln(2\pi)\cdot\ln z + 1 + 2\ln^2(2\pi) + \frac{\pi^2}{6} - 2\gamma^2 - 2\gamma_1 + O(z). \tag{24}$$

Certainly, this sum tends to the plus infinity, so attesting the presence of s-zeros tending to zero when $z \to 0$.

We have $\zeta(s,z) = z^{-s} + \zeta(s, 1+z)$; thus, s-zeros of $\zeta(s,z)$ are the solutions of the equation $z^{-s} = -\zeta(s, 1+z)$. In the first approximation, the "asymptotic equation" is $z^{-s} = \frac{1}{2}$, whose solutions are $s = -\frac{\ln 2}{\ln(1/z)} + \frac{2\pi i n}{\ln(1/z)}$; n is an arbitrary integer. The next approximation is given via $\exp(-\ln z(\frac{\ln 2}{\ln z} + \delta s)) = \frac{1}{2} + \frac{1}{2}\ln 2\pi\cdot\frac{\ln 2}{\ln z}$, so $s_0 = \frac{\ln 2}{\ln z} - \frac{\ln 2\pi\cdot\ln 2}{\ln^2 z} + o(\frac{1}{\ln^2 z})$, and similarly, for all complex solutions, the following applies: $\exp(-\ln z(\frac{\ln 2}{\ln z} + \delta s)) = \frac{1}{2} + \frac{1}{2}\ln 2\pi\cdot(\frac{\ln 2}{\ln z} + \frac{2n\pi i}{\ln z})$, thus

$$s_n = \frac{\ln 2}{\ln z} - \frac{\ln 2\pi\cdot\ln 2}{\ln^2 z} + \frac{2n\pi i}{\ln z} - \frac{2n\pi i\cdot\ln 2\pi}{\ln^2 z} + o(\frac{1}{\ln^2 z}). \tag{25}$$

Here, we used $\zeta'(0) = -\frac{1}{2}\ln 2\pi$. Of course, such a series in the inverse powers of the logarithm converges extremely slowly. It is also clear that not all n, but only their finite number, is indeed the solution for any concrete finite z.

To enrich our consideration, we invoke Rouché's theorem about zeros of the sums of analytical functions; see, e.g., [31]. The searched zeros of the function $\zeta(s,z) = z^{-s} + \zeta(s, 1+z)$ can be seen as zeros of the sum of the two functions: $\zeta(s,z) = f(s,z) + g(s,z)$ with $f(s,z) = z^{-s} - 1/2$ and $g(s,z) = 1/2 + \zeta(s, 1+z)$, both holomorphic in the region $|s| < 1$ (remember that we are working with the function of s here; z is just some complex number).

Theorem 2. *For an arbitrary, large, positive integer N and arbitrary, small, real* ε, *we can find such a real value of* $z_0(N,\varepsilon)$ *that the function* $\zeta(s,z)$ *with* $|z| \le z_0$ *has at least N zeros in the area* $|s| < \varepsilon$.

Proof. Let $0 < |z| \le 3/4$ (with the upper limit, we avoid the non-existence of the function $\zeta(s,z)$ for $z = -1$), and consider the following close area D: a circle with its interior defined as $|s| \le 1-\delta$ for an arbitrary small fixed positive $\delta < 1$. (The most interesting case is, of course, $|s| \le \varepsilon$ with an arbitrary small fixed positive ε). We, evidently, can select the value of z (possibly with the very small module), such that the following applies:

(*i*) There are no zeros of the function $f(s,z) = z^{-s} - 1/2$ on ∂D (i.e., on the circle $|s| = 1-\delta$; trivial);

(*ii*) On ∂D $|f(s,z)| > |g(s,z|$, because the module of $g(s,z) = 1/2 + \zeta(s, 1+z)$ is bounded there (although it can be very large when δ is small due to the presence of the pole at $s = 1$), $|z^{-s}|$ is not.

Thus, Rouché's theorem states that, inside D, the function $\zeta(s,z)$ has the same number of zeros, when taking into account their orders, as the function $f(s,z)$. The zeros of the latter were described above. Taking smaller and smaller values of $|z|$, we will get larger and larger numbers of zeros of $\zeta(s,z)$ lying in D. □

The existence of such close-to-0 s-zeros is, in a sense, "predicted" through the counting Formula (4), which states that the number of zeros of $\zeta(s,x)$ logarithmically tends to infinity when $x \to 0$ for any finite T. Note that the main asymptotic term of the sum (24) is exactly the sum of zeros over the inverse squares of the solutions of the "asymptotic" equation $z^{-s} = \frac{1}{2}$, viz. $s = -\frac{\ln 2}{\ln(1/z)} + \frac{2\pi i n}{\ln(1/z)}$. This case has already been considered by us for the roots of the equation $e^z = a$ in [18], so we will not include its evident slight generalization here again, and we will limit ourselves to the following remark.

Remark 2. *The application of the generalized Littlewood theorem to zeros of $f(z) = e^{bz} - a = 0$, having for $a \neq 1$ the Taylor expansion $\frac{f(z)}{1-a} = 1 + \frac{b}{1-a}z + \frac{b^2}{2(1-a)}z^2 + O(z^3)$, immediately gives$\sum \frac{k_i}{\rho_i^2} = \frac{b^2}{(1-a)^2} - \frac{b^2}{1-a}$—this can be written knowing nothing about the exact values of zeros. Actually, we know that they all are simple (and equal to $\rho_n = \frac{\ln a}{b} + \frac{2\pi i n}{b}$, provided that$a \neq 0$; otherwise, there are no zeros). Hence, k_i can be omitted. If a = 1 and , we have$\frac{f(z)}{bz} = 1 + \frac{1}{2}bz + \frac{1}{6}b^2z^2 + \frac{1}{24}b^3z^3 + \frac{1}{120}b^4z^4 + O(z^5)$; thus,$\sum' \frac{1}{\rho_i^2} = \frac{b^2}{4} - \frac{b^2}{3} = -\frac{b^2}{12}$. Here, as usual, the prime sign in the sum means that the value z = 0 should be omitted during the summing. This is simply the statement $\sum_{n=-\infty}^{n=\infty}{}' \frac{b^2}{(2\pi n i)^2} = -\frac{b^2}{12}$(Basel problem solution). Quite similarly, exploiting Equation (12), we can establish$\zeta(4) = \frac{\pi^4}{90}$[32], etc.*

Just for curiosity, we can find the sum over the inverse second powers of the roots $\rho_i$of the equation$f(z) := \exp(bz) - 1 - bz = 0$. We have $\frac{2f(z)}{b^2z^2} = 1 + \frac{1}{3}bz + \frac{1}{12}b^2z^2 + O(z^3)$, whence $\sum' \frac{1}{\rho_i^2} = \frac{b^2}{9} - \frac{b^2}{6} = -\frac{b^2}{18}$. *(Can this be named "the general Basel problem"?) See also the discussion of the general problem concerning the sums over inverse powers of roots of the equation $f(z) = a$in [18].*

In addition to those described above, there are also other s-zeros of $\zeta(s,z)$ when z tends to zero, including that close to 1 (it is discussed below), zeros with Re $s > 1$, whose existence was proven by Davenport and Heilbronn [7] and Cassel [8], and infinitely many zeros in the critical strip $0 \leq \text{Re} s \leq 1$, where the function $\zeta(s)$ is unbounded. There are also zeros close to the trivial zeros of the Riemann zeta function lying at $s = -2$, -4, -6... To find such a zero at $s = -2k + \delta$ with a small $|\delta|$, we have the equation $\zeta(-2k+\delta, z) = z^{2k-\delta} + \zeta(-2k+\delta, 1+z) = 0$, whose solution is $\delta \sim -\frac{z^{2k}}{\zeta'(-2k)}$, i.e.,

$$s \sim -2k - \frac{z^{2k}}{\zeta'(-2k)}. \tag{26}$$

For completeness, let us recall that $\zeta'(-2k) = \frac{(-1)^k \zeta(2k+1)(2k)!}{2^{2k+1}\pi^{2k}}$ [32].

The following simple proposition holds.

Proposition 1. *For any p with Rep < 0 and p not equal to -2, -4-, -6..., as well as not equal to any zero of$\zeta(s,z)$, the following applies:*

$$\sum_{\substack{zeroes\ of\ \zeta(s,z),\\ z \to 0}} \frac{k_i}{(\rho_i - p)^2} = \frac{1}{(1-p)^2} + \frac{[\zeta'(p)]^2}{\zeta^2(p)} - \frac{\zeta''(p)}{\zeta(p)} + O(z^{-\text{Re}p} \ln^2 z). \tag{27}$$

For any p with Re$p > 0$, *when p is not equal to 1, as well as not equal to any zero of* $\zeta(s,z)$, *the following applies:*

$$\sum_{\substack{zeroes\ of\ \zeta(s,z),\\ z\to 0}} \frac{k_i}{(\rho_i - p)^2} = \frac{1}{(1-p)^2} + O(z^{\mathrm{Re}p}). \tag{28}$$

Proof. From (21), we see, applying $\zeta(p,z) = \frac{1}{z^p} + \zeta(p) + O(z)$, that in the neighborhood of $z = 0$ for p not equal to one, $\zeta'(p,z) = -\frac{\ln z}{z^p} + \zeta'(p) + O(z)$, and $\zeta''(p,z) = \frac{\ln^2 z}{z^p} + \zeta''(p) + O(z)$. Thus, if Re $p < 0$ and $p \neq -2, -4, -6\ldots$, and it is also not equal to any zero of $\zeta(s,z)$, there are no peculiarities in the following sum: $\sum_{\substack{zeroes\ of\ \zeta(s,z),\\ z\to 0}} \frac{k_i}{(\rho_i-p)^2} = \frac{1}{(1-p)^2} + \frac{[\zeta\prime(p)]^2}{\zeta^2(p)} - \frac{\zeta''(p)}{\zeta(p)} + O(z^{-\mathrm{Re}p}\ln^2 z)$.

If Re$p > 0$, $p \neq 1$, and it is also not equal to any zero of $\zeta(s,z)$, asymptotically, we have

$$\sum_{\substack{zeroes\ of\ \zeta(s,z),\\ z\to 0}} \frac{k_i}{(\rho_i-p)^2} = \frac{1}{(1-p)^2} + \left(\frac{z^{-p}\ln z + \zeta'(1+z)}{z^{-p} + \zeta(p,1+z)}\right)^2 - \frac{z^{-p}\ln^2 p + \zeta''(p,1+z)}{z^{-p}+\zeta(p,1+z)} =$$
$$\frac{1}{(1-p)^2} + \left(\frac{\ln z + z^p\zeta\prime(1+z)}{1+z^p\zeta(p,1+z)}\right)^2 - \frac{\ln^2 p + z^p\zeta''(p,1+z)}{1+z^p\zeta(p,1+z)} = \frac{1}{(1-p)^2} + O(z^{\mathrm{Re}p}).$$

□

Similar statements hold for larger powers of zeros in the sums.

If $p = 1$, we need to use Formulae (13)–(15) and their analogs for a larger n. For $z \to 0$, the module of the sum $\sum_{\substack{zeroes\ of\ \zeta(s,z),\\ z\to 0}} \frac{k_i}{(\rho_i-1)^2} = \psi^2(z) + 2\gamma_1(z)$ becomes infinitely large, attesting the presence of zero close to $s = 1$ for the case. The asymptotics are well known: $\psi(z) = -\frac{1}{z} + O(1)$ [3–5], while, for $\gamma_1(z)$, we use the equations $\gamma_1(z) = \gamma_1(z+1) - \frac{\ln z}{z}$ (following again from $\zeta(s,z) = \frac{1}{z^s} + \zeta(s,z+1)$) and $\gamma_1(1+z) = O(1)$ to write $\gamma_1(z) = -\frac{\ln z}{z} + O(1)$. Similarly, $\gamma_l(z) = \frac{(-1)^l \ln^l z}{z} + \gamma_l$. Thus, $\sum_{\substack{zeroes\ of\ \zeta(s,z),\\ z\to 0}} \frac{k_i}{(\rho_i-1)^2} \sim \frac{1}{z^2} - 2\frac{\ln z}{z} + O(1)$, so we can deduce the existence of a zero at $s = 1 - z + z^2\ln z + O(z^2)$ when $z \to 0$. Asymptotic of all subsequent sums $\sum_{\substack{zeroes\ of\ \zeta(s,z),\\ z\to 0}} \frac{k_i}{(\rho_i-1)^n}$, $n > 2$, is quite consistent with the existence of such single s-zero when z tends to zero.

Again, using Rouché's theorem, we can show that this zero is isolated and simple. Stronger versions of the theorem given below can be proven, but for our purposes, the following one seems sufficient.

Theorem 3. *For any z, such that and* $|z| \leq 0.01$ *, the circle with its interior* $D|s-1| \leq 0.1$ *contains only one simple s-zero of the Hurwitz zeta function* $\zeta(s,z)$. *This zero approaches 1 as* $s = 1 - z + z^2\ln z + O(z^2)$ *when z approaches zero.*

Proof. The consideration includes the holomorphic on the whole complex plane function $\delta\cdot\zeta(1+\delta,z)$, where $\delta = s-1$. We write $\delta\cdot\zeta(1+\delta,z) = f(\delta) + g(\delta)$ with $f(\delta) = \frac{\delta}{z^{1+\delta}} + 1$

and $g(\delta) = \delta \cdot \zeta(1+\delta, 1+z) - 1$. The numerical analysis readily shows that, in the chosen area of s, z, $|g(\delta)| = |\delta \cdot \zeta(1+\delta, 1+z) - 1| < 1$. It is also clear that the chosen circle D contains only one simple zero of the function *f(s)* (there exists only one simple solution of $\delta = -z^{1+\delta}$)—viz., that which approaches 1 as $s = 1 - z + z^2 \ln z + O(z^2)$ when z approaches zero. Finally, for ∂D,

$$\left|f(\delta)\right| = \left|\frac{\delta}{z^{1+\delta}} + 1\right| \geq \left|\frac{\delta}{z^{1+\delta}}\right| - 1 \geq \frac{0.1}{0.01^{1-0.1}} - 1 > 5 > \left|g(\delta)\right|. \ \square$$

For a small real positive z, the existence of a single simple zero tending to 1 as 1-z + z^2ln$z$ + $O(z^2)$ has been proven by Endo and Suzuki [10]. Note that, for such a z, Res < 1; hence, the series representation (1) cannot be used to search for its value.

3.4. Behavior of the s-Zeros When z Tends to -n

Quite analogously, the sums over inverse powers of zeros become infinitely large when $z \to -n$ for any non-negative n; compare with (16–18). Again, from $\zeta^{(k)}(0,1) = O(1)$ and $\zeta(s,z) = \frac{1}{z^s} + \zeta(s, z+1)$, that is, $\zeta^{(k)}(s,z) = \frac{(-1)^k \ln^k z}{z^s} + \zeta^{(k)}(s, z+1)$, we have, through induction ($\zeta(s, z-n) = \frac{1}{(z-n)^s} + \frac{1}{(z-n+1)^s} + \ldots + \frac{1}{(z-1)^s} + \frac{1}{z^s} + \zeta(s, z+1)$, etc.), that for any $-n$, the following applies: $\zeta^{(k)}(0, -n+z) = (-1)^k \ln^k z + O(1)$. We have, from (16), that for $s, \delta \to 0$ $\sum\limits_{zeroes\ of\ \zeta(s,-n+\delta)} \frac{k_i}{\rho_i^2} = 1 + \frac{\ln^2 \delta}{(1/2+n)^2} - \frac{\ln^2 \delta}{1/2+n} + o(\ln^2 \delta)$—the sum, which again reflects the presence of s-zeros tending to 0. And, indeed, we have for $z \to -n$, $n \geq 1$, the following "asymptotic equation": $\zeta(-n+\delta) \sim n + \delta^{-s} - \frac{1}{2} = 0$; hence, $\delta^{-s} = -n + \frac{1}{2}$, whose solutions are $s = \frac{\ln(-n+1/2)}{\ln(1/\delta)} + \frac{(2n+1)\pi i}{\ln(1/\delta)} + O(\frac{1}{\ln^2 \delta})$, sf. the corresponding discussion for the case $z \to 0$ given above. Note that, when $n \neq 0$, we do not have real solutions anymore; all zeros are complex, and the real part of the corresponding solutions for $n \geq 2$ is positive, contrary to the cases n = 0, 1.

Quite similarly, when $z \to -n$, for any non-negative n, we have $\psi(-n+z) = -\frac{1}{z} + O(1)$ and $\gamma_l(-n+z) = \frac{(-1)^l \ln^l z}{z} + O(1)$ (compare with the discussion of the case p = 1 in the previous section), and thus, similarly close to one zero, $s = 1 - \delta + \delta^2 \ln \delta + O(\delta^2)$ also exists for the case.

Also similar to the case n = 0, for Rep > 0, $p \neq 1$, asymptotically, we have $\sum\limits_{\substack{zeroes\ of\ \zeta(s,z),\\ z \to -n}} \frac{k_i}{(\rho_i - p)^m} \sim \frac{1}{(1-p)^m}$. But when $z \to -n$, analogs of the simple zeros do not lie anymore in the close neighborhood of s = −2, −4, −6...

Remark 3. *1. Using Rouché's theorem, the theorems analogous to those two of the previous subsection can be easily proven when $z \to -n$ for any non-negative n.*

2. Note, that for the s-zero tending to 1, as 1-z + z^2lnz + O(z^2), existing when $z \to 0$, the sums $\sum\limits_{\substack{zeroes\ of\ \zeta(s,z),\\ z \to 0,-1,-2,-3\ldots}} \frac{k_i}{(\rho_i - p)^n}$ *for all $n \geq 2$ are, of course, consistent with the only one such s-zero. Quite the contrary, for the s-zeros tending to 0 when $z \to 0$, the sums* $\sum\limits_{\substack{zeroes\ of\ \zeta(s,z),\\ z \to 0,-1,-2,-3\ldots}} \frac{k_i}{\rho_i^n}$ *are inconsistent with the only one, or even some finite number of zeros. For example, from* $\sum\limits_{\substack{zeroes\ of\ \zeta(s,z),\\ z \to 0}} \frac{k_i}{\rho_i^2} = 2\ln^2 z + o(\ln^2 z)$*, supposing that there is only one zero, we should anticipate* $\sum\limits_{\substack{zeroes\ of\ \zeta(s,z),\\ z \to 0}} \frac{k_i}{\rho_i^3} = 2^{3/2} \ln^3 z + o(\ln^3 z)$*(or* $\sum\limits_{\substack{zeroes\ of\ \zeta(s,z),\\ z \to 0}} \frac{k_i}{\rho_i^3} =$

$-2^{3/2}\ln^3 z + o(\ln^3 z))$ *and* $\sum\limits_{\substack{zeroes\ of\ \zeta(s,z), \\ z\to 0}} \frac{k_i}{\rho_i^4} = 4\ln^3 z + o(\ln^3 z)$, *while, from (17), we have* $\sum\limits_{\substack{zeroes\ of\ \zeta(s,z), \\ z\to 0}} \frac{k_i}{\rho_i^3} = 3\ln^3 z + o(\ln^3 z)$, *and from (18),* $\sum\limits_{\substack{zeroes\ of\ \zeta(s,z), \\ z\to 0}} \frac{k_i}{\rho_i^4} = \frac{37}{3}\ln^4 z + o(\ln^4 z)$.

4. The Sums over Inverse Zeros of the Hurwitz Zeta Function

The sums over first powers of the inverse zeros of the Hurwitz zeta function, $\sum\limits_{zeroes\ of\ \zeta(s,z)} \frac{k_i}{\rho_i - p}$, cannot be directly obtained through our method because the contour integral $\int\limits_C \frac{\ln\zeta(s,z)}{(s-p)^2} ds$ does not tend to zero in the limit of infinitely large contours C; it diverges. But the same situation occurs for the Riemann zeta function—however, the "symmetric" sum $\lim_{T\to\infty} \sum\limits_{\substack{non-trivial\ zeroes\ of\ \zeta(s), \\ |\mathrm{Im}\rho| < T}} \frac{k_i}{\rho_i - p}$ well exists. It is known that the following applies [32]:

$$\lim_{T\to\infty} \sum_{\substack{non-trivial\ zeroes\ of\ \zeta(s), \\ |\mathrm{Im}\rho| < T}} \frac{k_i}{\rho_i} = \frac{1}{2}\gamma + 1 - \frac{1}{2}\ln 4\pi, \tag{29}$$

etc. Indeed, the existence of such sums over non-trivial zeros readily follows from the counting function for such zeros, the circumstance that their real parts are bounded (they all are lying inside the critical strip), and the possibility of pairing complex conjugate zeros, thus obtaining $\frac{1}{\sigma+iT} + \frac{1}{\sigma-iT} = \frac{2\sigma}{\sigma^2+T^2}$.

The Riemann *xi*-function $\xi(s) = (s-1)\pi^{-s/2}\Gamma(1+s/2)\zeta(s)$ is entire, and its only zeros are non-trivial zeros of $\zeta(s)$ [32], but in the frame of our method, we still cannot use the integrals $\int\limits_C \frac{\ln\xi(s)}{(s-p)^2} ds$ because the asymptotic of $\ln\xi(s)$ is $O(s\ln s)$. However, actually, we *can* use this integral, and below, we will demonstrate how.

First, with our approach, we easily get $\sum\limits_{zeroes of \xi(s)} \frac{k_i}{(\rho_i-p)^2} = -\frac{d}{ds}\frac{\xi'}{\xi}\Big|_{s=p}$ and then formally integrate this relation, obtaining $\lim_{T\to\infty} \sum\limits_{\substack{zeroes\ of\ \xi(s), \\ |\mathrm{Im}\rho| < T}} \frac{k_i}{\rho_i - p} = -\frac{\xi'}{\xi}(p) + C_1$—provided, of course, that p is not equal to any non-trivial zero and that the corresponding constant C_1 exists. To find the latter, we can use the value $p = 1/2$, where, evidently, $\lim_{T\to\infty} \sum\limits_{\substack{zeroes\ of\ \xi(s), \\ |\mathrm{Im}\rho| < T}} \frac{k_i}{\rho_i - 1/2} = 0$ and $\frac{\xi'}{\xi}(1/2) = 0$. Hence, $C_1 = 0$ and, thus,

$$\lim_{T\to\infty} \sum_{\substack{non-trivial\ zeroes\ of\ \zeta(s), \\ |\mathrm{Im}\rho| < T}} \frac{k_i}{\rho_i - p} = -\frac{\xi'}{\xi}(p). \tag{30}$$

But this actually suggests that the limit of the value of the contour integral $\int\limits_C \frac{\ln \xi(s)}{(s-p)^2} ds$ for properly chosen infinitely large contours C exists (and, in particular, it is equal to zero for $p = 1/2$), and now we will analyze similar contour integrals directly.

Let us introduce the function $f(s) := \zeta(s)\Gamma(1+s/2)$, which has the simple pole at $s = 1$, and whose only zeros are non-trivial zeros of the Riemann zeta function. We consider the contour integral $\int\limits_C \frac{\ln f(s)}{(s-p)^2} ds$, dividing the contour C into two parts as follows: the left, where Re $s < 1/2$, and the right, where Re $s > 1/2$. And "the paired symmetrical contributions" $-\ln(f(s))ds$ and $\ln(f(1-s))ds$ are summed; see Figure 1. For the function $\Gamma(1+s/2)\zeta(s)$, its "symmetric partner" (reflection $s \mapsto 1-s$) is $\Gamma(3/2-s/2)\zeta(1-s)$. Now, we substitute in the functional equation $\zeta(1-s) = 2^{1-s}\pi^{-s}\sin\left(\frac{\pi(1-s)}{2}\right)\Gamma(s)\zeta(s)$ [32] the expressions $\sin\left(\frac{\pi(1-s)}{2}\right) = \frac{\pi}{\Gamma(\frac{1-s}{2})\Gamma(\frac{1+s}{2})}$ and $\Gamma(s) = \pi^{-1/2}2^{s-1}\Gamma(\frac{s}{2})\Gamma(\frac{1+s}{2})$ (reflection and duplication rules for the gamma function [3–5]) to write $\zeta(1-s)\Gamma(\frac{1-s}{2}) = \pi^{-s+1/2}\Gamma(\frac{s}{2})\zeta(s)$. (Of course, this is nothing other than the quite known other form of the functional equation [32], and thus, it can be written immediately without any calculations. We present these calculations because they will be useful later on, during the proof of Theorem 4.) Finally, from $\Gamma(s+1) = s\Gamma(s)$, we have the following: $\frac{2}{s}\left(\frac{1}{2}-\frac{s}{2}\right)^{-1}\pi^{s-1/2}\zeta(1-s)\Gamma(3/2-s/2) = \Gamma(1+s/2)\zeta(s)$. In terms of the logarithms, $\ln[\frac{2}{s}\left(\frac{1}{2}-\frac{s}{2}\right)^{-1}\pi^{s-1/2}] = -\ln[\zeta(1-s)\Gamma(3/2-s/2)] + \ln[\Gamma(1+s/2)\zeta(s)]$, we explicitly expressed the difference of the *ln(f(s))* for both the "left and right parts of the contour".

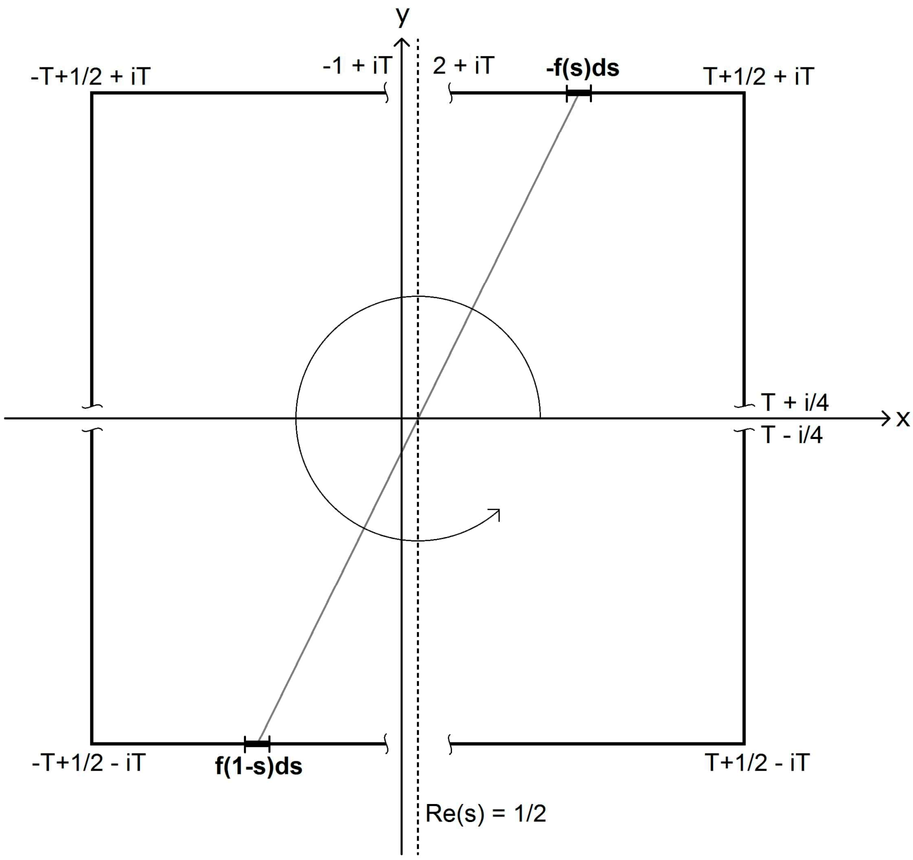

Figure 1. Illustrating the division of the contour C, using in the integral $\int\limits_C f(s)ds$, into left and right parts with the vertical line Res = 1/2. The integral value is then calculated by pairing the "symmetric contributions" at s and $1-s$: $-f(s)ds + f(1-s)ds$. The segments, removed from consideration during the calculations (their contributions clearly tend to zero in the limit of infinitely large contours), are also shown.

For the integral $\int\limits_C \frac{A(s)}{(s-p_1)^2}ds$ with the function $A(s)$, such that $A(s) = A(1-s)$, we have, when considering the contour integral as the sum of its left and right parts, the following: $\int\limits_C \frac{A(s)}{(s-p_1)^2}ds = \int\limits_{C/2}[\frac{A(s)}{(s-p_1)^2} - \frac{A(s)}{(1-s-p_1)^2}]ds$. Here and below, $C/2$ under the integral sign denotes the integration over "left half of the contour", viz. the joined segments $[1/2+iT, -T+1/2+iT]$, $[-T+1/2+iT, -T+1/2-iT]$ and $[-T+1/2-iT, 1/2-iT]$; see Figure 1. Evidently, $\int\limits_{C/2}[\frac{A(s)}{(s-p_1)^2} - \frac{A(s)}{(s-p_2)^2}]ds = \int\limits_{C/2}\frac{A(s)(2s-p_1-p_2)(p_1-p_2)}{(s-p_1)^2(s-p_2)^2}ds = 0$ for any p_1, p_2, and $A(s)$ having asymptotic $o(s)$; thus, only the factor $\ln[\frac{2}{s}(\frac{1}{2}-\frac{s}{2})^{-1}\pi^{s-1/2}]$, presented in the $C/2$, contributes to the contour integral value. The contribution of $\ln[\frac{2\pi^{-1/2}}{s}(\frac{1}{2}-\frac{s}{2})^{-1}]$, which is $O(\ln s)$, tends to zero, so only the term $\ln[\pi^s] = s\ln\pi$ contributes. Its contribution is just one-half of the value of the contour integral $\int\limits_C \frac{\ln\pi^s}{(s-p)^2}ds = 2\pi i\cdot\ln\pi$.

Thus, the generalized Littlewood theorem statement (see (5)) is as follows: $\frac{1}{2}\ln\pi = \lim_{T\to\infty}(-\frac{1}{1-p} + \sum\limits_{\substack{non-trivial\ zeroes\ of\ \zeta(s),\\ |\mathrm{Im}(\rho_i)|<T}} \frac{k_i}{\rho_i-p} + \frac{\zeta'}{\zeta}(p) + \frac{1}{2}\psi(1+p/2))$. And we have proven that

$$\lim_{T\to\infty}\sum_{\substack{non-trivial\ zeroes\ of\ \zeta(s),\\ |\mathrm{Im}(\rho_i)|<T}} \frac{k_i}{\rho_i-p} = \frac{1}{2}\ln\pi + \frac{1}{1-p} - \frac{\zeta'}{\zeta}(p) - \frac{1}{2}\psi(1+p/2), \tag{31}$$

provided, of course, that p is not equal to one or any trivial or non-trivial zero of the Riemann zeta function. For $p = 0$, using the quite known $\frac{\zeta'}{\zeta}(0) = \ln(2\pi)$ and $\psi(1) = -\gamma$ [32], we immediately restore Equation (29).

Certainly, this is much easier to prove (31) from the definition of the Riemann *xi*-function and (30), but the above consideration will guide us through the proof of the following theorem. First, let us give the necessary definition.

Definition 1. *We will name zeros ρ_j of the Hurwitz zeta function $\zeta(s,z)$ with $\mathrm{Re}\rho \le -1$ trivial, and we will order them in such a way that $\mathrm{Re}\rho_0 \ge \mathrm{Re}\rho_1 \ge \mathrm{Re}\rho_2 \ge \ldots$.*

Theorem 4. *Let m and n be positive integers and $0 < m/n \le 1$. The following formula holds for p not equal to 1, any zero of the $\zeta(s,\frac{m}{n})$, and an arbitrary complex number u such that u+p/2 is not a pole of the digamma function $\psi(u+p/2)$:*

$$\lim_{T\to\infty}\sum_{\substack{non-trivial\ zeroes\ of\ \zeta(s,m/n),\\ |\mathrm{Im}(\rho_i)|<T}} \frac{k_i}{\rho_i-p} = \frac{1}{2}\ln\pi + \frac{1}{2}\ln(n/m) - \frac{\zeta'(s,m/n)}{\zeta(s,m/n)}(p) - \frac{1}{2}\psi(u+p/2) + \frac{1}{1-p} - \sum_{\substack{trivial\ zeroes\ of\ \zeta(s,m/n),\\ j=0}}^{\infty}[\frac{1}{\rho_j-p} + \frac{1}{2j+p+2u}] \tag{32}$$

In other words, the symmetric sum over inverse non-trivial zeros of the Hurwitz zeta function $\zeta(s,\frac{m}{n})$ can be expressed via the logarithmical derivative of this function, elementary functions (we count digamma function in this class), and the sum $\sum\limits_{\substack{trivial\ zeroes\ of\ \zeta(s,m/n),\\ j=0}}^{\infty}[\frac{1}{\rho_j-p} + \frac{1}{2j+p+2u}]$. The latter is needed to handle the diverging sum over inverse trivial zeros of the Hurwitz zeta function. It might be instructive to compare this with a similar approach to polygamma functions, where inverse zeros and poles are paired when summing [18].

Proof. Our first aim is to evaluate the contour integral $\int\limits_C \frac{\ln[\zeta(s,m/n)\Gamma(s/2)]}{(s-p)^2}ds$ in the limit of infinitely large "symmetric" contour C and then to exclude the simple poles of $\Gamma(s/2)$ from the sums, thus obtaining the result pertinent to the Hurwitz zeta function. Clearly, due to the asymptotic of the functions that occur here, we can omit some finite segments of the contour, say $[-1+iT, 2+iT]$, $[-1-iT, 2-iT]$ and $[T+i/4, T-i/4]$, $[-T+i/4, -T-i/4]$, from the calculations; they contribute nothing within the limit in question.

To start with, we note that the results of Spira [9], briefly reviewed in the introduction, show that, except the "trivial zeros", all zeros of the function $\zeta(s,x)$ with real $0 < x \leq 1$ are contained in the strip $-1 < \mathrm{Re}s < 1+x$. They are complex-conjugated, and the counting function of zeros Equation (4) implies the convergence of the "symmetric" sum $\lim_{T\to\infty} \sum\limits_{\substack{zeroes\ of\ \zeta(s,z),\\ \mathrm{Re}s>-1, |\mathrm{Im}s|<T.}} \frac{k_i}{\rho_i - p}$ if p is not a non-trivial zero of the function $\zeta(s,x)$. Another result of Spira's, that for $|\mathrm{Im}s| \leq 1$ and $\mathrm{Re}s \leq (-4x+1+2[1-2x])$, the only zeros are (analogs of) trivial zeros, one in each interval $-2n-4x\pm1$, n is an integer and $n \geq 1-2x$, guarantees the convergence of the sum $\sum\limits_{\substack{trivial\ zeroes\ of\ \zeta(s,z),\\ j=1}}^{\infty} [\frac{1}{\rho_j - p} + \frac{1}{2j+p+2k}]$—provided, of course, that p is not a trivial zero of the function $\zeta(s,z)$ and also does not coincide with any pole of the digamma function $\psi(k+p/2)$, i.e., $p \neq -2k-2j$, where j is any non-negative integer.

We have the following functional equation (Rademacher's formula) [1–5]:

$$\zeta(1-s, \frac{m}{n}) = \frac{2\Gamma(s)}{(2\pi n)^s} \sum_{k=1}^{n} [\cos(\frac{\pi s}{2} - \frac{2\pi k m}{n})\zeta(s, \frac{k}{n})], \tag{33}$$

so, using reflection and duplication rules for the gamma function (see above) we obtain the following:

$$\begin{aligned}
\zeta(1-s, \tfrac{m}{n})\Gamma(\tfrac{1-s}{2}) &= 2\Gamma(s)\Gamma(\tfrac{1-s}{2})(2\pi n)^{-s}\sum_{k=1}^{n}\cos\Big(\tfrac{\pi s}{2} - \tfrac{2\pi k m}{n}\Big)\zeta(s, \tfrac{k}{n}) \\
&= 2\pi^{-1/2}2^{s-1}\Gamma(\tfrac{s}{2})\Gamma(\tfrac{1+s}{2})\Gamma(\tfrac{1-s}{2})(2\pi n)^{-s}\sum_{k=1}^{n}\cos\Big(\tfrac{\pi s}{2} - \tfrac{2\pi k m}{n}\Big)\zeta(s, \tfrac{k}{n}) \\
&= \pi^{1/2}2^{s}\Gamma(\tfrac{s}{2})\frac{1}{\sin(\pi/2 - \pi s/2)}(2\pi n)^{-s}\sum_{k=1}^{n}\cos\Big(\tfrac{\pi s}{2} - \tfrac{2\pi k m}{n}\Big)\zeta(s, \tfrac{k}{n}) \\
&= \pi^{1/2}\Gamma(s/2)(\pi n)^{-s}\sum_{k=1}^{n}[\cos(2\pi k m/n) + \tan(\pi s/2)\sin(2\pi k m/n)]\zeta(s, \tfrac{k}{n}).
\end{aligned}$$

We write, supposing $\zeta(s, m/n) \neq 0$, the following:

$$\zeta(1-s, \tfrac{m}{n})\Gamma\Big(\tfrac{1-s}{2}\Big) = \pi^{1/2}\Gamma(s/2)(\pi n) - s\zeta(s, \tfrac{m}{n}) \times$$
$$\sum_{k=1}^{n}[\cos(2\pi k m/n) + \tan(\pi s/2)\sin(2\pi k m/n)]\zeta(s, \tfrac{k}{n})/\zeta(s, \tfrac{m}{n})$$

That is, we have established the relation between the functions in the right and left parts of the contour in the contour integral $\int\limits_C \frac{\ln[\zeta(s,m/n)\Gamma(s/2)]}{(s-p)^2}ds$. From the previous consideration for the Riemann zeta function, we know that only those factors in such a relation whose logarithms asymptotically are at least $O(s)$ are important for the integral value. Thus, rather rough estimations suffice, and below, in this section, the writing $f(s) \sim g(s)$ means $\ln f(s) = \ln g(s) + o(s)$ for a large $|s|$. We can use, for $\mathrm{Re}\, s > 1$ (see the note in the beginning of the proof that we exclude certain finite segments of the contour from the consideration), $\zeta(s, \frac{m}{n}) = \frac{n^s}{m^s} + \zeta(s, 1 + \frac{m}{n})$, where $\zeta(s, 1 + \frac{m}{n}) = O(1)$, and the circumstance that all trigonometrical functions appearing here are *O(1)* (for estimation of the $\tan(\pi s/2)$, remember that we exclude the segments $[T + i/4, T - i/4]$, $[-T + i/4, -T - i/4]$ from the calculations; see again the note in the beginning of the proof). We get

$$\zeta(1-s, \frac{m}{n})\Gamma\left(\frac{1-s}{2}\right) \sim \Gamma(s/2)(\pi n)^{-s}\zeta(s, \frac{m}{n}) \times \sum_{k=1}^{n} [\cos(2\pi km/n) + \tan(\pi s/2)\sin(2\pi km/n)]\frac{m^s}{k^s},$$

where actually, in the sum, only the term with $k = 1$ is important:

$$\zeta(1-s, \frac{m}{n})\Gamma\left(\frac{1-s}{2}\right) \sim \Gamma(s/2)(\pi n/m)^{-s}\zeta(s, \frac{m}{n}) \tag{34}$$

because, for any complex numbers a_k, $\sum_{k=1}^{n} a_k \frac{m^s}{k^s} = a_1 m^s \cdot (1 + \sum_{k=2}^{n} \frac{a_k}{a_1}\frac{1}{k^s})$ with $\ln(1 + \sum_{k=1}^{n} \frac{a_k}{a_2}\frac{1}{k^s}) = O(1)$—for our case, definitely $a_1 \neq 0$. Thus, only the factor $(\pi n/m)^{-s}$ contributes to the contour integral value. Again, this contribution is one-half of the contour integral value $\int_C \frac{\ln(\pi n/m)^s}{(s-p)^2} ds = 2\pi i \cdot (\ln \pi + \ln(n/m))$. Thus, the application of the generalized Littlewood theorem to the integral over the symmetric contour C, $\int_C \frac{\ln[(\pi n/m)^{-s/2}\zeta(s,m/n)\Gamma(s/2)]}{(s-p)^2} ds$, reads as follows:

$$\tfrac{1}{2}\ln\pi + \tfrac{1}{2}\ln(n/m) = \lim_{T\to\infty}\Big(\sum_{\substack{non-trivial\ zeroes\ of\ \zeta(s,m/n),\\ |\mathrm{Im}(\rho_i)|<T}} \tfrac{k_i}{\rho_i - p}\Big) + \tfrac{\zeta'(s,m/n)}{\zeta(s,m/n)}(p) + \tfrac{1}{2}\psi(p/2) - \tfrac{1}{1-p} + \sum_{\substack{trivial\ zeroes\ of\ \zeta(s,m/n),\\ j=0}}^{\infty} \Big[\tfrac{1}{\rho_j - p} + \tfrac{1}{2j+p}\Big]$$

In the last sum, the contributions of the trivial zeros are combined with the contributions of the simple poles of $\Gamma(s/2)$ lying at $-2j$, where j is any non-positive integer. Finally,

$$\lim_{T\to\infty}\sum_{\substack{non-trivial\ zeroes\ of\ \zeta(s,m/n),\\ |\mathrm{Im}(\rho_i)|<T}} \tfrac{k_i}{\rho_i - p} = \tfrac{1}{2}\ln\pi + \tfrac{1}{2}\ln(n/m) - \tfrac{\zeta'(s,m/n)}{\zeta(s,m/n)}(p) - \tfrac{1}{2}\psi(p/2) + \tfrac{1}{1-p} - \sum_{\substack{trivial\ zeroes\ of\ \zeta(s,m/n),\\ j=0}}^{\infty} \Big[\tfrac{1}{\rho_j - p} + \tfrac{1}{2j+p}\Big]$$

The essential part of the proof is finishing here. Now we just remind that $\psi(p + u) = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+p+u}\right)$ [3–5] so that, correspondingly, $\frac{1}{2}(\psi(p/2 + u) - \psi(p/2)) = \sum_{n=0}^{\infty}\left(\frac{1}{2n+p} - \frac{1}{2n+p+2u}\right)$. We substitute these relations into the above formula in such a way immediately obtaining Equation (32). This generalization is quite consistent with the known asymptotic of the ratio $\frac{\Gamma(s+u)}{\Gamma(s+v)}$, which is $\sum_{k=0}^{\infty} \frac{(-1)^k\Gamma(v-u+k)}{\Gamma(v-u)} B_k^{(v-u+1)} s^{u-v+k}$, where s tends to infinity with $|\arg(s+u)| < \pi$, $|\arg(s+v)| < \pi$, and B denotes the generalized Bernoulli polynomial [33]. For us it is sufficient to consider only the first term $k = 0$ here to see that, in the sense of our definition, $\Gamma(s+u) \sim \Gamma(s+v)$ and thus for any complex u $\zeta(s, \frac{m}{n})\Gamma\left(u + \frac{1-s}{2}\right) \sim \Gamma(s/2 + u)(\pi n/m)^{-s}\zeta(s, \frac{m}{n})$. $\square$

Remark 4. *1. The value m/n = 1, inserted into (32), restores the corresponding theorems for the Riemann zeta function. In particular, for u = 1 the sum* $\sum_{\substack{trivial\ zeroes\ of\ \zeta(s),\\ j=0}}^{\infty}[\frac{1}{\rho_j - p} + \frac{1}{2j+p+2}]$ *is exactly zero, and we obtain the formulae discussed just above the proof of Theorem 4.*

2. For m/n = 1/2, $\zeta(s,1/2) = (2^s - 1)\zeta(s)$ *[32]. Then, what is the meaning of an additional term* $\frac{1}{2}\ln 2$ *in (32)? Why does the derivative* $\frac{\zeta'}{\zeta}(p,1/2)$ *not suffice? It is instructive to see that this term reflects the appearance of an additional factor* $2^{s/2}$*, multiplied on the "symmetrical" function* $2\sinh(\frac{s\ln 2}{2})$*, describing the sum over new "non-trivial" (in the sense of our definition) zeros* $\rho_n = \pm\frac{2\pi i n}{\ln 2}$*, n = 0, 1, 2. . ., of the function* $\zeta(s,1/2)$*:* $2^s - 1 = 2^{s/2}\cdot 2\sinh(\frac{s\ln 2}{2})$*. With the term* $\frac{1}{2}\ln 2$*, the contribution of this additional "disturbing" factor is removed from* $\frac{\zeta'}{\zeta}(p,1/2)$ *during the calculation of* $\lim_{T\to\infty} \sum_{\substack{non-trivial\ zeroes\ of\ \zeta(s,1/2),\\ |\mathrm{Im}(\rho_i)|<T}} \frac{k_i}{\rho_i - p}$.

Remark 5. *It is hardly doubtful that the formula analogous to (32) is applicable to the function* $\zeta(s,z)$ *for any real 0 < z< 1 (very probably for a not-real z too), not only the rational z = m/n. However, the present author did not succeed in proving such a more general version of the theorem.*

5. Some Exact Values for the Sums Considered

Formulae obtained in the paper solve the problem of the calculation of the sums of the inverse powers of zeros of the Hurwitz zeta function, expressing them via the derivatives of this function at zero or the derivatives of the function $(s-1)\zeta(s,z)$ at $s = 1$, i.e., via the generalized Stieltjes coefficients. But for the Hurwitz zeta function, the specific question of the expression of the values of the derivatives at zero via the generalized Stieltjes coefficients received a lot of attention; see, e.g., [34–36] and the references therein, especially in [33]. This is due to the existence of the functional equation (Rademacher's formula) [1–5]: for the rational positive $z = m/n$, $1 \le m \le n$,

$$\zeta(s,\frac{m}{n}) = \frac{2\Gamma(1-s)}{(2\pi n)^{1-s}}\sum_{k=1}^{n}[\sin(\frac{\pi s}{2} + \frac{2\pi k m}{n})\zeta(1-s,\frac{k}{n})], \tag{35}$$

and, of course, also $\zeta(1-s,\frac{m}{n}) = \frac{2\Gamma(s)}{(2\pi n)^s}\sum_{k=1}^{n}[\cos(\frac{\pi s}{2} - \frac{2\pi k m}{n})\zeta(s,\frac{k}{n})]$, written above as Equation (33).

Following the tradition, a rather cumbersome expression of the second derivative of the Hurwitz zeta function $\zeta''(0,\frac{m}{n})$ via elementary functions (here, we count gamma function in this class), and the generalized Stieltjes coefficients $\gamma_1(\frac{k}{n}), \gamma_2(\frac{k}{n})$ (more precisely, via the sums $\sum_{k=1}^{n}\sin(\frac{2\pi k m}{n})\gamma_2(\frac{k}{n})$ and $\sum_{k=1}^{n}\cos(\frac{2\pi k m}{n})\gamma_1(\frac{k}{n})$) are given in the Appendix A.

As a result, numerous expressions are known for many (mostly first) generalized Stieltjes coefficients $\gamma_n(z)$ and their relations with the derivatives of the Hurwitz zeta function at zero; see, e.g., [34–36] and the references therein, especially the research results and references that can be found in Blagouchine's paper [34]. Many of these results might be used for the calculations of the sums over zeros considered here. For example, at least for the cases $\frac{m}{n} = \left\{\frac{1}{2},\frac{1}{4},\frac{3}{4},\frac{1}{3},\frac{2}{3},\frac{1}{6},\frac{5}{6}\right\}$, the coefficients $\gamma_1(\frac{m}{n})$ are known in close forms (see p. 100 of [35] and the references therein), which immediately enables to write rather elegant expressions for the corresponding sums $\sum_{zeroes\ of\ s}\frac{k_i}{(\rho_i - 1)^2}$. For instance,

$$\gamma_1(1/4) = 2\pi\ln\Gamma(1/4) - \tfrac{3\pi}{2}\ln\pi - \tfrac{7}{2}\ln^2 2 - (3\gamma + 2\pi)\ln 2 - \tfrac{\gamma\pi}{2} + \gamma_1 \cong -5.5181,$$
$$\gamma_1(3/4) = -2\pi\ln\Gamma(1/4) + \tfrac{3\pi}{2}\ln\pi - \tfrac{7}{2}\ln^2 2 - (3\gamma - 2\pi)\ln 2 + \tfrac{\gamma\pi}{2} + \gamma_1 \cong -0.391381,$$

etc.

For the sums $\sum\limits_{zeroes\ of\ s} \frac{k_i}{\rho_i^2}$, an interesting case appears for $n = 3$. We know $\sum\limits_{l=1}^{m-1} \zeta''\left(0, \frac{l}{m}\right) = -\frac{1}{2}\ln^2 m - \ln m \cdot \ln 2\pi$, a Formula (58) of [34]. Thus, $\zeta''(0, 1/3) + \zeta''(0, 2/3) = -\frac{1}{2}\ln^2 3 - \ln 3 \cdot \ln 2\pi$, and this enables us to express the difference in the sums over the inverse square of zeros of two Hurwitz zeta functions in a rather elegant form, which does not include any generalized Stieltjes coefficient:

$$\sum_{zeroes\ \zeta(s,1/3)} \frac{k_i}{\rho_i^2} - \sum_{zeroes\ \zeta(s,2/3)} \frac{k_i}{\rho_i^2} = 36[\ln^2 \Gamma(1/3) - \ln^2 \Gamma(2/3) - \ln \Gamma(1/3) \cdot \ln 2\pi + \ln \Gamma(2/3) \cdot \ln 2\pi] + \ln^2 3 + 6 \ln 3 \cdot \ln 2\pi \cong 2.2436 .$$

This equality has been tested: the numerical application of Formula (16) gives $\sum\limits_{zeroes\ \zeta(s,1/3)} \frac{k_i}{\rho_i^2} \cong 5.9583$ and $\sum\limits_{zeroes\ \zeta(s,2/3)} \frac{k_i}{\rho_i^2} \cong 3.7146$, so the difference in question is around 2.2436, indeed.

The relation $\zeta''(0, p) + \zeta''(0, 1-p) = -2(\gamma + \ln 2\pi)\ln(2 \sin \pi p) + 2\sum\limits_{n=1}^{\infty} \frac{\cos 2\pi pn \cdot \ln n}{n}$ (p. 575 of [34]) is also worthwhile to note: it enables us to calculate $\sum\limits_{zeroes\ \zeta(s,p)} \frac{k_i}{\rho_i^2} - \sum\limits_{zeroes\ \zeta(s,1-p)} \frac{k_i}{\rho_i^2}$ for any positive rational p of less than 1 without the recourse to any generalized Stieltjes coefficient or second derivatives of the Hurwitz zeta function (but with the recourse to infinite series).

6. Numerical Illustration

It seems really interesting and, we believe, illuminative to see how s-zeros of the function $\zeta(s, z)$ evolve when z moves from a small, real, positive value to $z = 1/2$ and then to $z = 1$; sf. some data in this direction in [14]. As a prototype of such research, we can indicate the very interesting papers by Kölbig [37,38] about the incomplete Riemann and gamma functions; it seems that the questions put forward by him there still remain unanswered. For example, the following questions about s-zeros of the Hurwitz zeta function arise. Do its "numerous" zeros existing in the neighborhood of $s = 0$ for a small z evolve to "false" $s_k = 2\pi i n/\ln 2$ zeros of the function $\zeta(s, 1/2)$ when z increases from (almost) zero to 1/2? What happens with them after, when z further moves to 1 (i.e., only the Riemann zeta function's zeros rest), and so on?

These questions evidently require extensive numerical research and are not investigated in the present paper (however, see some simple observations at the end of the section). Our aim is much more modest: just to exploit the circumstance that the above formulae are quite fit for numerical calculations due to the availability of the option to calculate derivatives of the Hurwitz zeta function, e.g., *hurwitzZeta*(n, s, z) in MATLAB, which returns $\frac{d^n \zeta(s,z)}{ds^n}$, or *StieltjesGamma[n,a]* in Matematica, which returns generalized Stieltjes coefficients $\gamma_n(a)$. (But take caution; in Matematica, the function $\zeta(s, z) = \sum\limits_{n=0}^{\infty} \frac{1}{\left((n+z)^2\right)^{s/2}}$ is implemented as the Hurwitz zeta function).

In Figure 2, we present the numerical results obtained by exploiting the aforementioned *hurwitzZeta*$(1, s, z)$ and *hurwitzZeta*$(2, s, z)$ functions.

Our other numerical observations are the following. We confirm the presence of small by-module real negative zeros of the Hurwitz zeta function when real positive $z \to 0$, the absence of small by-module positive real zeros for such a case, and the absence of small by-module real zeros when $z \to -1, -2, -3$. For example, for $z = 10^{-5}$, $\zeta(s, z) = 0$ for $s \approx -0.071$; for $z = 10^{-4}$, $\zeta(s, z) = 0$ for $s \approx -0.094$; for $z = 10^{-3}$,$s \approx -0.135$; for $z = 0.1$, $s \approx -0.623$, and so on, we have tested that this zero moves to the first trivial zero of $\zeta(s, z)$ lying at $s = -2$ when z moves to ½. We also confirm the presence of small by-module complex conjugate complex zeros when $z \to 0$.

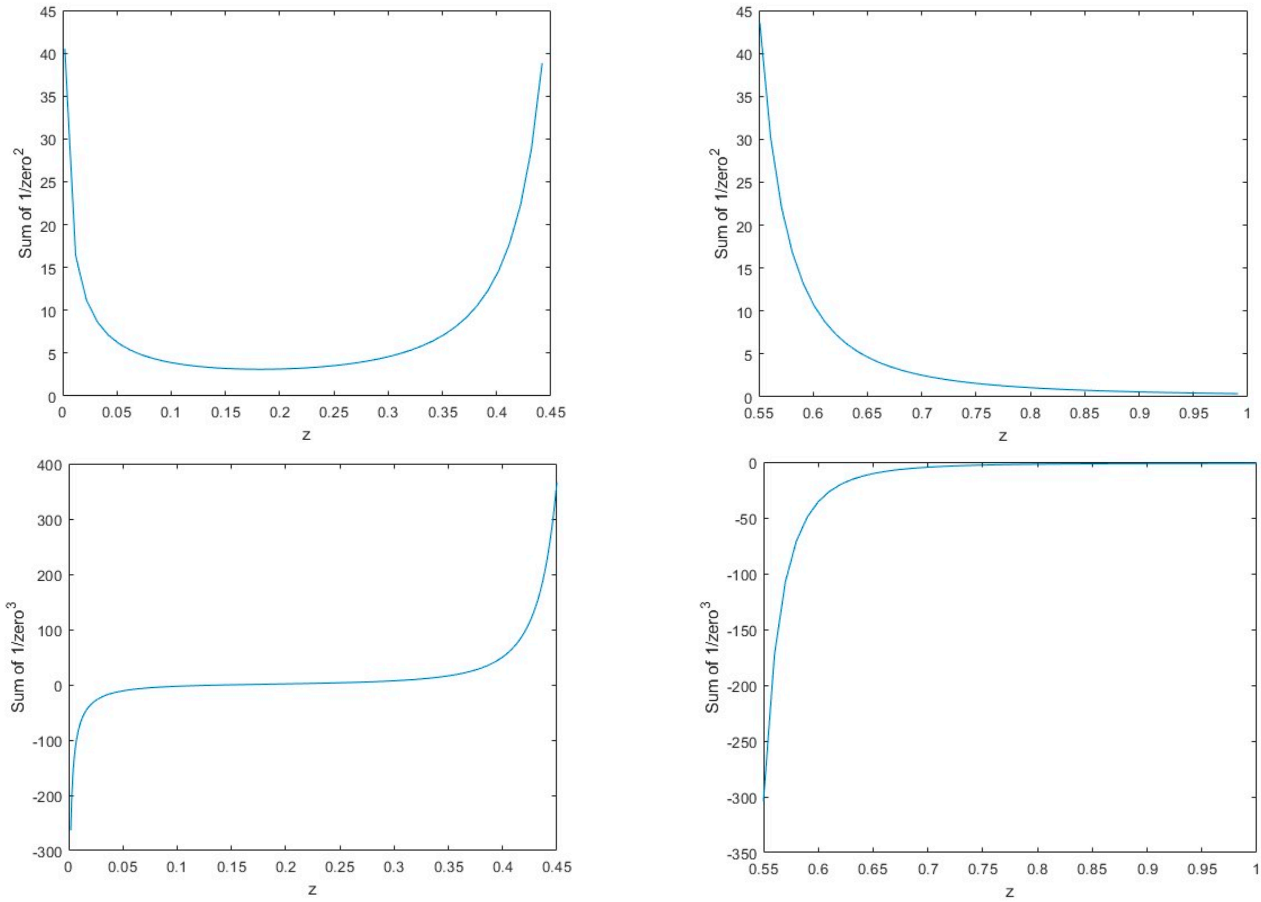

Figure 2. The values of $\sum_{zeroes\ \zeta(s,z)} \frac{k_i}{\rho_i^2}$ (**up**) and $\sum_{zeroes\ \zeta(s,z)} \frac{k_i}{\rho_i^3}$ (**down**).

Finally, the presence of s-zero tending to 0 as $s \sim -\frac{2\delta}{\ln 2} + o(\delta)$ when $z = \frac{1}{2} + \delta$ with real $\delta \to 0$ (a trivial consequence of Equations (16)–(18) in the neighborhood of $z = 1/2$), was also confirmed. We tested that it moves to the first trivial zero of $\zeta(s,z)$ lying at $s = -2$ when z moves to 1, and it moves to the zero close to 1 when z moves to 0.

7. Discussion and Conclusions

To finish, let us briefly discuss the question of the sum of inverse powers of zeros of the Hurwitz zeta function *as a function of* z *with fixed* s. Unfortunately, as of today, such a question cannot be put forward for an arbitrary s because, for a non-integer s, the expression $\zeta(s,z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^s}$ simply is *not* a continuous function of z. For example (we follow J. Lewittes's paper [39]), $\lim_{y\to 0+} = \sum_{n=0}^{\infty} \left(\frac{1}{2} + iy + n\right)^{-s} \neq \sum_{n=0}^{\infty} \left(\frac{1}{2} + n\right)^{-s}$ due to the discontinuity of the argument function, usually defined to be $-\pi \leq \arg(z) < \pi$, on the negative real axis. Thus, some restrictions of the permissible z values should be introduced for an arbitrary s, which makes our approach inapplicable. It seems that only the integer values of an s larger than one, that is $s = 2, 3, 4\ldots$, where the function is determined by the absolutely convergent series on the whole complex plane, can be considered. Then, this function is just the polygamma function, and the corresponding sums over inverse powers of zeros were earlier considered in [18].

The following question also seems natural to the present author. The aforementioned theorem of Davenport and Heilbronn [7] was proven using Kronecker's theorem about the Diophantine approximation and Rouché's theorem for a specially (ingeniously) constructed function *Z(s)*, which is quite close to $\zeta(s,x)$ and definitely has zeros with

Re $s > 1$, for all real *rational* $0 < x < 1$. Given that the rational numbers are everywhere in the segment $0 < x < 1$, and that the sums over zeros $\sum_{zeroes\ of\ \zeta(s,z)} \frac{k_i}{\rho_i^n}$ are certainly continuous, can a (kind of) topological proof for non-rational x be constructed? (Cassel's generalization [8] is not topological.) Can zeros with Re $s > 1$ "abruptly" disappear for all (or "topologically many") non-rational values of x? It seems not. (Certainly, there is no problem that such zeros "disappear" at certain *isolated* points such as, e.g., $x = 1/2$: this means only that $\lim_{x\to 1/2} \sum_{\substack{zeroes\ of\ \zeta(s,x),\\ \mathrm{Re}\, s > 1}} \frac{k_i}{\rho_i^n} = 0$ for all n, and in principle, there is nothing surprising here.) Similarly, we believe, some (kind of) topological proof of Theorem 4 for an arbitrary real $0 < z < 1$ might be possible.

Thus, in this paper, the generalized Littlewood theorem concerning contour integrals of the logarithm of analytical function was applied to find the sums over inverse powers of zeros of the Hurwitz zeta function, including appropriately arranged sums over the first powers of such zeros. Certain properties of zeros closely connected with these sums, e.g., the existence of the infinite number of tending to zero s-zeros of $\zeta(s,z)$ when $z \to 0$, were also studied.

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Appendix A

In a sense, the question of expression of the derivatives $\zeta'(0, \frac{m}{n}), \zeta''(0, \frac{m}{n})$ via the generalized Stieltjes coefficients $\gamma_1(\frac{k}{n})$ has been solved by J. Musser in his thesis work [36]. However, his final formulae, see pp. 27 and 21, contain parameters C_l ("the coefficients of the Taylor expansion of $\cos\left(\frac{\pi s}{2} - \frac{2\pi km}{n}\right)\frac{2}{(2\pi n)^s}$ about $s = 1$") and are not fully explicit. Here, we present the answer in the form similar to that given by Blagouchine [34], who has solved, in a sense, the inverse problem of expressing the generalized Stieltjes coefficients $\gamma_1(\frac{k}{n})$ via $\zeta'(0, \frac{m}{n}), \zeta''(0, \frac{m}{n})$.

Starting from (35), where m, n are natural numbers, $n \geq 1$ and $1 \leq m \leq n$, we have with the $O(s^3)$ precision the following Laurent expansion:

$$\begin{aligned}&\tfrac{1}{2} - \tfrac{m}{n} + s\zeta'(0, \tfrac{m}{n}) + \tfrac{s^2}{2}\zeta''(0, \tfrac{m}{n}) + O(s^3) = \tfrac{2}{2\pi n}[1 + \gamma s + \tfrac{\pi^2}{12}s^2][1 + s\ln(2\pi n) - \tfrac{s^2}{2}\ln^2(2\pi n)] \times \\ &\sum_{k=1}^{n}[\sin\tfrac{2\pi km}{n} + \tfrac{\pi s}{2}\cos\tfrac{2\pi km}{n} - \tfrac{\pi^2 s^2}{8}\sin\tfrac{2\pi km}{n} - \tfrac{\pi^3 s^3}{48}\cos\tfrac{2\pi km}{n}][-\tfrac{1}{s} - \psi(\tfrac{k}{n}) + \gamma_1(\tfrac{k}{n})s + \tfrac{1}{2}\gamma_2(\tfrac{k}{n})s^2]\end{aligned} \quad \text{(A1)}$$

The last factor in the square brackets (that under the sum sign) is equal to

$$\begin{aligned}&-\sum_{k=1}^{n}\sin(\tfrac{2\pi km}{n})\psi(\tfrac{k}{n}) + [\sum_{k=1}^{n}\sin(\tfrac{2\pi km}{n})\gamma_1(\tfrac{k}{n}) - \tfrac{\pi}{2}\sum_{k=1}^{n}\cos(\tfrac{2\pi km}{n})\psi(\tfrac{k}{n})]s \\ &+[\tfrac{1}{2}\sum_{k=1}^{n}\sin(\tfrac{2\pi km}{n})\gamma_2(\tfrac{k}{n}) + \tfrac{\pi}{2}\sum_{k=1}^{n}\cos(\tfrac{2\pi km}{n})\gamma_1(\tfrac{k}{n}) + \tfrac{\pi^2}{8}\sum_{k=1}^{n}\sin(\tfrac{2\pi km}{n})\psi(\tfrac{k}{n})]s^2,\end{aligned} \quad \text{(A2)}$$

the $O(1/s)$ term disappears due to the evident $\sum_{k=1}^{n}\sin(\frac{2\pi km}{n}) = 0$, and the contribution $\frac{\pi^3 s^2}{48}\sum_{k=1}^{n}\cos(\frac{2\pi km}{n}) = 0$ similarly disappears. In addition to these two, below, we will also use the summation rules $\sum_{k=1}^{n} k\cos(\frac{2\pi km}{n}) = n$ and $\sum_{k=1}^{n} k\sin(\frac{2\pi km}{n}) = -\frac{n}{2}\cot\frac{\pi m}{n}$. All of these classic rules are valid for $m = 1, 2, 3 \ldots n-1$ and readily follow from the sine and cosine representation via $e^{\pm ix}$, the sum of geometric progression formula $\sum\limits_{k=1}^{n} e^{ikx} = \frac{e^{ix(n+1)}-e^{ix}}{e^{ix}-1}$, and differentiation.

Now we apply the following, more complex, but still quite known summation rules, which are Gauss's identities $\sum\limits_{k=1}^{n-1} \cos(\frac{2\pi km}{n})\psi(\frac{m}{n}) = \gamma + n\ln(2\sin\frac{m\pi}{n})$, whence $\sum\limits_{k=1}^{n} \cos(\frac{2\pi km}{n})\psi(\frac{m}{n}) = n\ln(2\sin\frac{m\pi}{n})$ and $\sum\limits_{k=1}^{n} \sin(\frac{2\pi km}{n})\psi(\frac{k}{n}) = \frac{\pi}{2}(2m-n)$.

Thus, the third factor in A1 (that under the sum sign) is

$$\frac{\pi}{2}(n-2m) + [\sum_{k=1}^{n} \sin(\tfrac{2\pi km}{n})\gamma_1(\tfrac{k}{m}) - \tfrac{\pi n}{2}\ln(2\sin\tfrac{m\pi}{n})]s$$
$$+[\tfrac{1}{2}\sum_{k=1}^{n} \sin(\tfrac{2\pi km}{n})\gamma_2(\tfrac{k}{n}) + \tfrac{\pi}{2}\sum_{k=1}^{n} \cos(\tfrac{2\pi km}{n})\gamma_1(\tfrac{k}{n}) + \tfrac{\pi^3}{16}(2m-n)]s^2.$$

Equating the $O(s)$ terms should give the known $\zeta'(0,\frac{m}{n}) = \ln\Gamma(\frac{m}{n}) - \frac{1}{2}\ln 2\pi$. In such a way, we obtain the summation rule

$$\sum_{k=1}^{n} \sin(\frac{2\pi km}{n})\gamma_1(\frac{k}{n}) = \frac{\pi}{2}(\gamma + \ln(2\pi n))(2m-n) - \frac{\pi n}{2}(\ln\pi - \ln\sin\frac{m\pi}{n}) + n\pi\ln\Gamma(\frac{m}{n}), \tag{A3}$$

which is contained in Theorem 2 of Blagouchine [34]. In the same theorem, he also presents the sum

$$\sum_{k=1}^{n} \cos(\frac{2\pi km}{n})\gamma_1(\frac{k}{n}) = n(\gamma + \ln(2\pi n))\ln(2\sin\frac{m\pi}{n}) + \frac{n}{2}[\zeta''(0,\frac{m}{n}) + \zeta''(0,1-\frac{m}{n})]. \tag{A4}$$

(We added the term corresponding to $k = n$ to his original sum). We will use this rule later on, but at this stage, it is inapplicable due to the presence of second derivatives of the Hurwitz zeta function.

Thus, finally, the factor under the sum sign in (A1) is

$$\tfrac{\pi}{2}(n-2m) + [\tfrac{\pi}{2}(\gamma + \ln 2\pi n)(2m-n) - \tfrac{\pi n}{2}(\ln\pi - \ln\sin\tfrac{m\pi}{n}) + n\pi\ln\Gamma(\tfrac{m}{n}) - \tfrac{\pi n}{2}\ln(2\sin\tfrac{m\pi}{n})]s$$
$$+[\tfrac{1}{2}\sum_{k=1}^{n} \sin(\tfrac{2\pi km}{n})\gamma_2(\tfrac{k}{n}) + \tfrac{\pi}{2}\sum_{k=1}^{n} \cos(\tfrac{2\pi km}{n})\gamma_1(\tfrac{k}{n}) + \tfrac{\pi^3}{16}(2m-n)]s^2,$$

and we have the following with $O(s^3)$ precision:

$$\begin{aligned}&\pi n[(\tfrac{1}{2} - \tfrac{m}{n} + s\zeta'(0,\tfrac{m}{n}) + \tfrac{s^2}{2}\zeta''(0,\tfrac{m}{n})] = [1 + \gamma s + \tfrac{\pi^2}{12}s^2][1 + s\ln(2\pi n) - \tfrac{s^2}{2}\ln^2(2\pi n)]\times\\ &\big\{[\tfrac{\pi}{2}(n-2m) + [\tfrac{\pi}{2}(\gamma + \ln 2\pi n)(2m-n) - \tfrac{\pi n}{2}\ln 2\pi + n\pi\ln\Gamma(\tfrac{m}{n})]s\\ &+[\tfrac{1}{2}\sum_{k=1}^{n} \sin(\tfrac{2\pi km}{n})\gamma_2(\tfrac{k}{n}) + \tfrac{\pi}{2}\sum_{k=1}^{n} \cos(\tfrac{2\pi km}{n})\gamma_1(\tfrac{k}{n}) + \tfrac{\pi^3}{16}(2m-n)]\big\}s^2.\end{aligned} \tag{A5}$$

Now we compare the $O(s^2)$ terms. To simplify the appearance of the subsequent formulae, let us denote $C_{mn} = \frac{\pi}{2}(\gamma + \ln 2\pi n)(2m-n) - \frac{\pi n}{2}\ln 2\pi + n\pi\ln\Gamma(\frac{m}{n})$. This gives

$$\tfrac{1}{2}\pi n\zeta''(0,\tfrac{m}{n}) = \tfrac{1}{2}\sum_{k=1}^{n} \sin(\tfrac{2\pi km}{n})\gamma_2(\tfrac{k}{n}) + \tfrac{\pi}{2}\sum_{k=1}^{n} \cos(\tfrac{2\pi km}{n})\gamma_1(\tfrac{k}{n}) + \tfrac{\pi^3}{16}(2m-n) +$$
$$\tfrac{\pi}{2}(n-2m)(\tfrac{\pi^2}{12} - \tfrac{1}{2}\ln^2 2\pi n) + C(m,n)(\gamma + \ln 2\pi n)$$

That is,

$$\frac{1}{2}\pi n\zeta''(0,\tfrac{m}{n}) = \frac{1}{2}\sum_{k=1}^{n}\sin(\tfrac{2\pi km}{n})\gamma_2(\tfrac{k}{n}) + \frac{\pi}{2}\sum_{k=1}^{n}\cos(\tfrac{2\pi km}{n})\gamma_1(\tfrac{k}{n}) + \frac{\pi^3}{16}(2m-n)+$$
$$\frac{\pi}{2}(n-2m)(\tfrac{\pi^2}{12}-\tfrac{1}{2}\ln^2 2\pi n) + [\tfrac{\pi}{2}(\gamma+\ln 2\pi n)(2m-n) - \tfrac{\pi n}{2}\ln 2\pi + n\pi\ln\Gamma(\tfrac{m}{n})](\gamma+\ln 2\pi n), \tag{A6}$$

which is our final formula. Remember that $m = 1, 2, 3, \ldots n-1$ here.

Remark A6. *Equation (A5), together with the sum A4, can be used to estimate*

$$\sum_{k=1}^{n}\sin(\tfrac{2\pi km}{n})\gamma_2(\tfrac{k}{n}):$$
$$\frac{1}{2}\sum_{k=1}^{n}\sin(\tfrac{2\pi km}{n})\gamma_2(\tfrac{k}{n}) = -\frac{\pi}{2}\sum_{k=1}^{n}\cos(\tfrac{2\pi km}{n})\gamma_1(\tfrac{k}{n}) - \frac{\pi^3}{16}(2m-n) + \frac{\pi n}{2}\zeta''(0,\tfrac{m}{n})-$$
$$\frac{\pi}{2}(n-2m)(\tfrac{\pi^2}{12}-\tfrac{1}{2}\ln^2 2\pi n) - [\tfrac{\pi}{2}(\gamma+\ln 2\pi n)(2m-n) - \tfrac{\pi n}{2}\ln 2\pi + n\pi\ln\Gamma(\tfrac{m}{n})](\gamma+\ln 2\pi n).$$

Thus,

$$\frac{1}{2}\sum_{k=1}^{n}\sin(\tfrac{2\pi km}{n})\gamma_2(\tfrac{k}{n}) = \frac{\pi n}{4}(\zeta''(0,\tfrac{m}{n}) - \zeta''(0,1-\tfrac{m}{n})) - \frac{\pi^3}{16}(2m-n) - \frac{\pi}{2}[n(\gamma+\ln(2\pi n))\ln(2\sin\tfrac{m\pi}{n})]-$$
$$\frac{\pi}{2}(n-2m)(\tfrac{\pi^2}{12}-\tfrac{1}{2}\ln^2 2\pi n) - [\tfrac{\pi}{2}(\gamma+\ln 2\pi n)(2m-n) - \tfrac{\pi n}{2}\ln 2\pi + n\pi\ln\Gamma(\tfrac{m}{n})](\gamma+\ln 2\pi n). \tag{A7}$$

Finally, we would like to note that the Laurent series expansion of Rademacher's formula written in the form (33), and the subsequent equating of the coefficients in front of the s terms, may be used to prove the main Theorem 1 of [34]. We believe that the following short exposition might still be useful.

From (33), we have the following with $O(s^2)$ precision:

$$-\frac{1}{s} - \psi(\tfrac{m}{n}) + \gamma_1(\tfrac{m}{n})s = 2[1 - s\ln 2\pi n + \tfrac{s^2}{2}\ln^2 2\pi n][\tfrac{1}{s} - \gamma + (\tfrac{1}{2}\gamma^2 + \tfrac{\pi^2}{12})s]\times$$
$$[\sum_{k=1}^{n}[\cos(\tfrac{2\pi km}{n}) + \tfrac{\pi}{2}\sin(\tfrac{2\pi km}{n})s - \tfrac{\pi^2}{8}\cos(\tfrac{2\pi km}{n})s^2][\tfrac{1}{2} - \tfrac{k}{n} + (\ln\Gamma(\tfrac{k}{n}) - \tfrac{1}{2}\ln 2\pi)s + \tfrac{1}{2}\zeta''(0,\tfrac{k}{m})s^2]$$

With the same precision, the last term in the square brackets under the sign of sum is

$$-\frac{1}{2} + [-\frac{\pi}{2n}\sum_{k=1}^{n}k\sin(\tfrac{2\pi km}{n}) + \sum_{k=1}^{n}\cos(\tfrac{2\pi km}{n})\ln\Gamma(\tfrac{k}{n})]s+$$
$$[\frac{1}{2}\sum_{k=1}^{n}\cos(\tfrac{2\pi km}{n})\zeta''(0,\tfrac{k}{m}) + \frac{\pi}{2}\sum_{k=1}^{n}\sin(\tfrac{2\pi km}{n})\ln\Gamma(\tfrac{k}{n}) + \frac{\pi^2}{8}]s^2,$$

Thus,

$$-\frac{1}{2s} - \frac{1}{2}\psi(\tfrac{m}{n}) + \frac{1}{2}\gamma_1(\tfrac{m}{n})s = [1 - s\ln 2\pi n + \tfrac{s^2}{2}\ln^2 2\pi n][\tfrac{1}{s} - \gamma + (\tfrac{1}{2}\gamma^2 + \tfrac{\pi^2}{12})s]\times$$
$$\Big\{-\frac{1}{2} + [\frac{\pi}{4}\cot\frac{\pi m}{n} + \sum_{k=1}^{n}\cos(\tfrac{2\pi km}{n})\ln\Gamma(\tfrac{k}{n})]s+$$
$$[\frac{1}{2}\sum_{k=1}^{n}\cos(\tfrac{2\pi km}{n})\zeta''(0,\tfrac{k}{m}) + \frac{\pi}{2}\sum_{k=1}^{n}\sin(\tfrac{2\pi km}{n})\ln\Gamma(\tfrac{k}{n}) + \frac{\pi^2}{8}]s^2\Big\},$$

where the aforementioned summation rule for $\sum_{k=1}^{n}k\sin(\tfrac{2\pi km}{n})$ was used.

Equating of $O(1)$ terms gives

$-\frac{1}{2}\psi(\frac{m}{n}) = \frac{\gamma}{2} + \frac{1}{2}\ln 2\pi n + \frac{\pi}{4}\cot\frac{\pi m}{n} + \sum_{k=1}^{n}\cos(\frac{2\pi km}{n})\ln\Gamma(\frac{k}{n})$, and then recalling $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin\pi z}$ and pairing the terms with $\ln\Gamma(\frac{m}{n})$ and $\ln\Gamma(1-\frac{m}{n})$, we arrive at the standard [3–5]

$$\psi(\frac{m}{n}) = -\gamma - \ln 2n - \frac{\pi}{2}\cot\frac{\pi m}{n} + 2\sum_{k=1}^{[\frac{n-1}{2}]}\cos(\frac{2\pi km}{n})\ln\sin(\frac{\pi k}{n}). \tag{A8}$$

Equating $O(s^2)$ terms leads to Blagouchine's first theorem [34]:

$$\gamma_1(\frac{m}{n}) = [\sum_{k=1}^{n}\cos(\frac{2\pi km}{n})\zeta''(0,\frac{k}{m}) + \pi\sum_{k=1}^{n}\sin(\frac{2\pi km}{n})\ln\Gamma(\frac{k}{n}) + \frac{\pi^2}{4}]+$$
$$[\frac{\pi}{2}\cot\frac{\pi m}{n} + 2\sum_{k=1}^{n}\cos(\frac{2\pi km}{n})\ln\Gamma(\frac{k}{n})][-\ln 2\pi n - \gamma]+$$
$$-\frac{1}{2}\ln^2 2\pi n - \gamma\ln 2\pi n - (\frac{1}{2}\gamma^2 + \frac{\pi^2}{12}).$$

He then made some additional manipulations: he substituted $\zeta''(0,1) = \zeta''(0) = \gamma_1 + \frac{1}{2}\gamma^2 - \frac{\pi^2}{24} - \frac{1}{2}\ln^2 2\pi$, paired terms with $\frac{k}{m}$ and $\frac{m-k}{m}$ in the sums, used explicit form for $\zeta''(0,1/2)$, etc.

The consideration of a larger quantity of terms in the aforementioned Taylor expansion will lead to the expressions of higher generalized Stieltjes coefficients $\gamma_n(\frac{m}{n})$ via higher derivatives $\zeta^{(m)}(0,\frac{m}{n})$ (cf. also the second part of [34]). In particular, due to the appearance of the values of the Riemann zeta function of odd integers in the Taylor expansion of $\Gamma(s)$, e.g., $\Gamma(s) = \frac{1}{s} - \gamma + (\frac{1}{2}\gamma^2 + \frac{\pi^2}{12})s + (-\frac{\pi^2}{12}\gamma - \frac{\gamma^3}{6} - \frac{1}{3}\zeta(3))s^2 + O(s^3)$ in such a way we may obtain probably not-without-the-interest expressions of $\zeta(2n+1)$ via the generalized Stieltjes coefficients $\gamma_k(\frac{m}{n})$ and derivatives $\zeta^{(k)}(0,\frac{m}{n})$.

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