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Bayesian Inference for Inverse Power Exponentiated Pareto Distribution Using Progressive Type-II Censoring with Application to Flood-Level Data Analysis

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Abstract: Progressive type-II (Prog-II) censoring schemes are gaining traction in estimating the parameters, and reliability characteristics of lifetime distributions. The focus of this paper is to enhance the accuracy and reliability of such estimations for the inverse power exponentiated Pareto (IPEP) distribution, a flexible extension of the exponentiated Pareto distribution suitable for modeling engineering and medical data. We aim to develop novel statistical inference methods applicable under Prog-II censoring, leading to a deeper understanding of failure time behavior, improved decision-making, and enhanced overall model reliability. Our investigation employs both classical and Bayesian approaches. The classical technique involves constructing maximum likelihood estimators of the model parameters and their bootstrap covariance intervals. Using the Gibbs process constructed by the Metropolis–Hasting sampler technique, the Markov chain Monte Carlo method provides Bayesian estimates of the unknown parameters. In addition, an actual data analysis is carried out to examine the estimation process's performance under this ideal scheme.

Keywords: inverse power exponentiated Pareto distribution; progressive type-II censoring; maximum likelihood estimation; Bayes estimation; bootstrap confidence interval; Metropolis–Hasting algorithm; statistical model; simulation

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1. Introduction

In survival analysis and reliability studies, censoring occurs when it is not feasible to collect comprehensive data on the sample. When analyzing lifetime data in such cases, censored samples are used. Analyzing data from various life-testing experiments is receiving more attention. In order to analyze lifetime data, several censoring schemes have been introduced in the literature. Type-I and type-II censoring schemes are the two most widely used applications in reliability analysis. However, during the experiment, none of these censoring schemes permit the removal of experimental units. Progressive Type-I and Type-II censoring schemes permit the removal of experimental units while the experiment is being performed. In the last ten to twelve years, statistical literature studies have focused on progressive censoring schemes due to their flexibility. Type-I censoring establishes an end time for experiments regardless of the number of failures, while Type-II censoring terminates an experiment following a particular number of failures. Type-II censoring is a method that 'stops' an experiment once a pre-established number of failures has occurred, regardless of the time that has passed since the failure occurred. There is no way to remove live units from the experiment at any other time than the point where it ends, which is one of the scheme's defects. As a solution to this drawback, Cohen [1] introduced progressive censoring schemes. For reliability experiments, progressive censoring has a significant impact on setting up duration experiments. It is common in industrial experiments to

immediately terminate experiments and limit the overall number of failures due to different causes (such as when expensive items have to be destroyed when it takes a long time to complete experiments). Progressive censoring techniques cut down on testing times and costs by permitting the removal of survival elements at various stages of the experiment.

In this study, three parameters for IPEP distribution were proposed by Khalifa et al. [2]. The random variable, y, has an IPEP distribution, given that its pdf and cdf are as follows:

$$f(y;\theta,\eta,\gamma) = \theta\,\eta\,\gamma\,y^{-\gamma-1}\left(1+y^{-\gamma}\right)^{-\eta-1}\left(1-\left(1+y^{-\gamma}\right)^{-\eta}\right)^{\theta-1};$$
$$y,\ \theta,\ \eta,\ \gamma > 0, \tag{1}$$

where θ and η are the shape parameters and γ is the scale parameter.

The cdf is defined as follows:

$$F(y;\theta,\eta,\gamma) = 1-\left(1-\left(1+y^{-\gamma}\right)^{-\eta}\right)^{\theta};\ y,\ \theta,\ \eta,\ \gamma > 0, \tag{2}$$

whereas the $S(y)$ and $h(y)$ functions for the IPEP distribution, respectively, are obtained by the following:

$$\begin{aligned} S(y;\theta,\eta,\gamma) &= 1-F(y) \\ &= \left(1-\left(1+y^{-\gamma}\right)^{-\eta}\right)^{\theta};\ y,\ \theta,\ \eta,\ \gamma > 0 \end{aligned} \tag{3}$$

and,

$$\begin{aligned} h(y;\theta,\eta,\gamma) &= \frac{f(y)}{S(y)} \\ &= \theta\eta\gamma y^{-\gamma-1}\left(1+y^{-\gamma}\right)^{-\eta-1}\left(1-\left(1+y^{-\gamma}\right)^{-\eta}\right)^{-1};\ y,\ \theta,\ \eta,\ \gamma > 0. \end{aligned} \tag{4}$$

Khalifa et al. [2] explained the IPEP model, a more accurate and flexible extension of the exponentiated Pareto distribution for fitting engineering and medical data, and introduced some statistical characteristics.

Furthermore, this dataset, which explores a novel area of application, has been evaluated for weather phenomena (rainfall, floods, droughts, etc.); flood data provided by Dumonceaux and Antle [3] have been used for application. The data are components of a sequence of inundation layers designed to establish a connection between the observations and forecasts obtained from the river gauge and a graphical depiction of the areas affected by high water levels. The purpose of this paper is to investigate how to select models that represent river floods. Our focus is on methods that use the maximum annual flood series, as well as some features of the most commonly used parametric models in hydrology. Classical goodness-of-fit procedures have limitations, and we propose a simple method based on the "flood rate" in order to screen alternative models.

As a result, the remainder of this work is arranged as follows: In Section 2, Prog-II censoring is presented. In Section 3, the literature review is discussed. In Section 4, the *MLEs* of θ, η and γ, and the Fisher information matrix are formulated. In Section 5, Bootstrap CIs are introduced. In Section 6, for various loss functions, including SEL and LINEX functions, Bayes estimates for the previously mentioned parameters and reliability functions are additionally obtained. In Section 7, the M-H algorithm is introduced. In Section 8, the dataset is studied. In Section 9, a simulation study is performed. Lastly, the conclusion is provided in Section 10.

2. Progressive Type-II Censoring

A brief description of a Prog-II censored experiment is as follows. Select $R_1,\ldots,R_m$, m non-negative integers so that $R_1+\cdots+R_m = n-m$, for $m < n$. Consider a trial where n identical items are tested. It is supposed that the n units' lifetime distributions have

the same distribution function (F) and are $i.d.d$ random variables. When the first failure occurs, let us say that $x_{1:m:n}$, R_1 surviving items are randomly selected to be eliminated. In the same way, the surviving items are eliminated at the time of the second failure, say $x_{2:m:n}$, R_2, and so on. Ultimately, all of the remaining items are removed at the time of the $m-th$ failure, let us say $x_{m:m:n}$; see Balakrishnan and Cramer [4].

As a result, for a Prog-II censoring scheme, $(n, m, R_1, \ldots, R_m)$, we observe the following sample: $x_{1:m:n} < x_{2:m:n} < \ldots < x_{m:m:n}$. The joint pdf for $x_{1:m:n} < x_{2:m:n} < \ldots < x_{m:m:n}$ is as follows:

$$f(x_{1:m:n} < x_{2:m:n} < \ldots < x_{m:m:n}) = C\prod_{i=1}^{m} f(x_{i:m:n})[1 - F(x_{i:m:n})]^{R_i}, 0 < x_1 < \ldots < x_m < \infty, \quad (5)$$

where C is a constant independent of the parameters and it is determined by $C = n(n - R_1 - 1)\ldots(n - R_1 - R_2 - \cdots - R_{m-1} - m + 1)$; R_i is the censoring scheme, and $F(x_{i:m:n})$ and $f(x_{i:m:n})$ are the cdf and probability density functions of $X_{1:m::n}, X_{2:m:n}, \ldots, X_{m:m:n}$, respectively, see Figure 1.

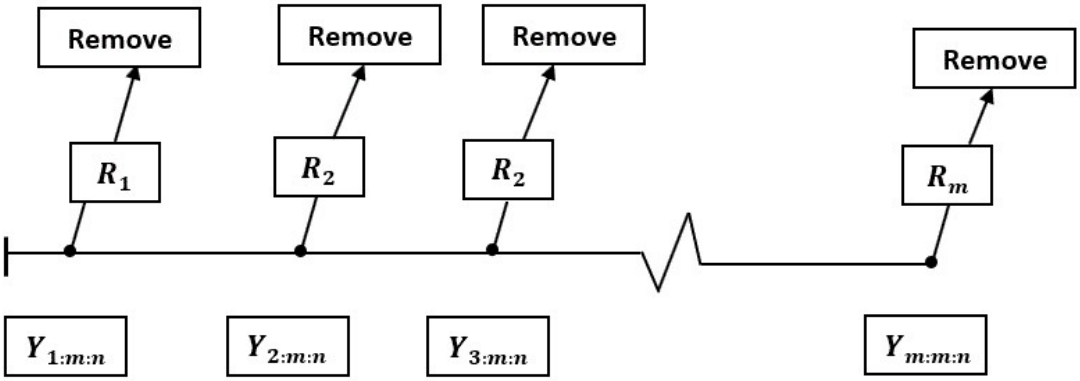


Figure 1. The Prog-II censoring scheme is represented schematically.

3. Literature Review

Many studies, such as Mait and Kaya [5], Meeker and Escobar [6], and Cai et al. [7], have explored Prog-II censoring with differed failure time breakdowns. Additionally, Panah and Asadi [8] and Mousa and Jaheen [9] have applied Prog-II censoring to estimate parameters from different lifetime distributions. Balakrishnan [10] investigated the characteristics under progressively censored samples and reported that a collection of several advances in the inferential procedures depend on types-I and -II censored samples, Almetwally et al. [11] studied the MLE and BE method to determine the parameters of the EGE distribution under progressive censoring. Buzaridah et al. [12] and Khalifa et al. [13] studied estimations of the distribution parameters under Prog-II censored data and Chen and Gui [14] presented statistical inference of the joint Prog-II censored generalized inverted exponential distribution.

This study seeks to fill this need by pursuing three objectives. Estimating parameters: We estimate the parameters, $S(y)$ and $h(y)$, of the IPEP using the standard MLE technique. Further, we examine ACI and two CIs based on bootstrap methods, concerning the model parameters. As far as we know, there is currently no study available on the estimation of model parameters or reliability features of the IPEP using Prog-II censoring schemes. Bayesian estimation (BE): We calculate the BEs for the unknown IPEP parameters, together with the corresponding $S(y)$ and $h(y)$ functions. The Bayesian methodology is utilized for both LFs, considering two commonly used LFs in Bayesian estimation, which are the SEL and the LINEX. To determine the unknown parameters' BEs, we utilize MCMC methods. Specifically, we apply the Gibbs process inside the M-H approach. We evaluate the effectiveness of the offered approaches by conducting a comprehensive simulation analysis, specifically focusing on their simulated MSE. Another criterion is employed to compare the 95% confidence intervals derived using the asymptotic MLE and CRI.

This comparison is based on a distinct criterion. To make a comparison, *CP* and *ACLs* are utilized.

4. Inference Procedures

In this part, we will develop the inference processes for the parameters and reliability function of the IPEP distribution using the scheme $R = (R_1, R_2, \ldots, R_m)$.

4.1. Maximum Likelihood Estimator

Let us suppose that n components are tested with corresponding lifetimes that are equally distributed using the cdf from (2) and the pdf from (1). If $y_1, y_2, y_3, \ldots, y_m$ are denoted as Prog-II censored samples of size m, then the likelihood function is provided by using the Prog-II censored data.

$$L(\underset{\sim}{y}|\theta,\eta,\gamma) = C\prod_{i=1}^{m} f(y_{i:m:n}|\theta,\eta,\ \gamma)[1 - F(y_{i:m:n}|\theta,\eta,\ \gamma)]^{R_i}, \tag{6}$$

the constant $C = n\ (n - R_1 - 1)\cdots(n - \sum_{i=1}^{n-1}(R_i + 1))$ is independent of the parameters, and $f(y_{i:m:n}|\theta,\eta,\ \gamma)$ is the pdf of y in (1), while $F(y_{i:m:n}|\theta,\eta,\ \gamma)$ is the cdf of y in (2).

Consequently, the joint pdf is as follows:

$$\begin{aligned} L(y;\theta,\eta,\gamma) = C\theta^n\eta^n\gamma^n \prod_{i=1}^{n} y_i^{-\gamma-1}\left(1+y_i^{-\gamma}\right)^{-\eta-1}\left(1-\left(1+y_i^{-\gamma}\right)^{-\eta}\right)^{\theta-1} \\ \times \prod_{i=1}^{n}\left(1-\left(1+y_i^{-\gamma}\right)^{-\eta}\right)^{\theta R_i}. \end{aligned} \tag{7}$$

Following that, the natural logarithm of the likelihood function, without the normalization constant, is as follows:

$$\begin{aligned} \ell(y;\theta,\eta,\gamma) = n\ln(\theta) + n\ln(\eta) + n\ln(\gamma) - (\gamma+1)\sum_{i=0}^{n}\ln(y_i) - (\eta+1)\sum_{i=0}^{n}\ln\left(1+y_i^{-\gamma}\right) \\ + (\theta-1)\sum_{i=0}^{n}\ln\left(1-\left(1+y_i^{-\gamma}\right)^{-\eta}\right) + \theta\sum_{i=0}^{n} R_i\ln\left(1-\left(1+y_i^{-\gamma}\right)^{-\eta}\right). \end{aligned} \tag{8}$$

Using the log-likelihood (8) to construct the nonlinear equations, the *MLEs* of the parameters $\theta,\ \eta$ and γ are obtained as follows:

$$\frac{\partial\ell}{\partial\theta} = \frac{n}{\theta} + \sum_{i=0}^{n}\ln\left(1-\left(1+y_i^{-\gamma}\right)^{-\eta}\right) + \sum_{i=0}^{n} R_i\ln\left(1-\left(1+y_i^{-\gamma}\right)^{-\eta}\right), \tag{9}$$

$$\begin{aligned} \frac{\partial\ell}{\partial\eta} = \frac{n}{\eta} - \sum_{i=0}^{n}\ln\left(1+y_i^{-\gamma}\right) + \frac{(\theta-1)\sum_{i=0}^{n}\left(1+y_i^{-\gamma}\right)^{-\eta}\ln\left(1+y_i^{-\gamma}\right)}{\sum_{i=0}^{n}\left(1-\left(1+y_i^{-\gamma}\right)^{-\eta}\right)} \\ - \frac{\theta\sum_{i=0}^{n} R_i\left(1+y_i^{-\gamma}\right)^{-\eta}\ln\left(1+y_i^{-\gamma}\right)}{\sum_{i=0}^{n}\left(1-\left(1+y_i^{-\gamma}\right)^{-\eta}\right)} \end{aligned} \tag{10}$$

and

$$\frac{\partial \ell}{\partial \gamma} = \frac{n}{\gamma} - \sum_{i=0}^{n} \ln(y_i) - \frac{(\eta+1)\sum_{i=0}^{n} y_i^{-\theta}\ln(y_i)}{\sum_{i=0}^{n}\left(1+y_i^{-\gamma}\right)} + \frac{\eta(\theta-1)\sum_{i=0}^{n} y_i^{-\gamma}\ln(y_i)\left(1+y_i^{-\gamma}\right)^{-\eta-1}}{\sum_{i=0}^{n}\left(1-\left(1+y_i^{-\gamma}\right)^{-\eta}\right)}$$
$$+ \frac{\eta\theta\sum_{i=0}^{n} R_i\, y_i^{-\gamma}\ln(y_i)\left(1+y_i^{-\gamma}\right)^{-\eta-1}}{\sum_{i=0}^{n}\left(1-\left(1+y_i^{-\gamma}\right)^{-\eta}\right)}. \quad (11)$$

The estimates should be obtained using a numerical method like Newton–Raphson since (9)–(11) cannot be solved analytically. Ahmed [15] provides a clear illustration of the algorithm.

4.2. Fisher's Information Matrix

The normal approximation of the *MLEs* can help construct the approximate CI and conduct hypothesis testing for the parameters θ, η , and γ. $I_n(\theta, \eta, \gamma)$, and the corresponding 3×3 observed information matrix is given as follows:

$$I_n = \begin{pmatrix} \frac{\partial^2 \ell}{\partial \theta^2} & \frac{\partial^2 \ell}{\partial \theta \partial \eta} & \frac{\partial^2 \ell}{\partial \theta \partial \gamma} \\ \frac{\partial^2 \ell}{\partial \eta \partial \theta} & \frac{\partial^2 \ell}{\partial \eta^2} & \frac{\partial^2 \ell}{\partial \eta \partial \gamma} \\ \frac{\partial^2 \ell}{\partial \gamma \partial \theta} & \frac{\partial^2 \ell}{\partial \gamma \partial \eta} & \frac{\partial^2 \ell}{\partial \gamma^2} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}.$$

Considering that the *MLE* of $\hat{\phi} = (\hat{\theta}, \hat{\eta}, \hat{\gamma})$ illustrates asymptotic normality, we may express it as $\hat{\phi} \sim N(\phi, I_n^{-1}(\hat{\phi}))$. As a result, the $100(1-\vartheta)\%$ approximate *CIs* are defined by the following:

$$\hat{\phi}_L = \hat{\phi} - z_{\frac{\vartheta}{2}}\sqrt{var(\hat{\phi})}\ and\ \hat{\phi}_U = \hat{\phi} + z_{\frac{\vartheta}{2}}\sqrt{var(\hat{\phi})},$$

where $0 < \vartheta < 1$ and $z_{\frac{\vartheta}{2}}$ represent the upper $\frac{\vartheta}{2}$ percentiles of the standard normal, respectively.

So, a major diagonal of $I_n^{-1}(\hat{\phi})$ is formed by the elements $var(\hat{\theta})$, $var(\hat{\eta})$ and $var(\hat{\gamma})$. It is also necessary to know the variance of the parameters θ, η , and γ, to clarify the ACI of $S(t)$ and $h(t)$ functions. See, Greene [16] for more information on using the delta method to approximately estimate the variance of $\hat{S}(t)$ and $\hat{h}(t)$. Following this technique, it can determine the variance of $\hat{S}(t)$ and $\hat{h}(t)$, respectively

$$var(\hat{S}(t)) = \left[\nabla \hat{S}(t)\right]^T \left[\hat{V}\right] \left[\nabla \hat{S}(t)\right]$$

and

$$var(\hat{h}(t)) = \left[\nabla \hat{h}(t)\right]^T \left[\hat{V}\right] \left[\nabla \hat{h}(t)\right],$$

where $\hat{S}(t)$ and $\hat{h}(t)$ for θ, η , and γ are denoted by $\nabla \hat{S}(t)$ and $\nabla \hat{h}(t)$.

Following that, $S(t)$ and $h(t)$ can have a $100(1-\vartheta)\%$ two-sided confidence estimate.

$$\hat{S}(t) \pm z_{\frac{\vartheta}{2}}\sqrt{var(\hat{S}(t))}$$

and

$$\hat{h}(t) \pm z_{\frac{\vartheta}{2}}\sqrt{var(\hat{h}(t))}.$$

5. Bootstrap Confidence Intervals

The bootstrap techniques are more likely to give CIs that are more approximative. As a consequence, we present CIs based on bootstrap techniques.

5.1. Boot-p Method

(1) Given the initial data $y = y_{1:m:n}, y_{2:m:n}, \ldots, y_{m:m:n}$, *and* $\hat{\theta}, \hat{\eta}, \hat{\gamma}$, these are found by minimizing Equations (9)–(11).

(2) Utilizing the algorithm explained in Balakrishnan and Sandhu [17], a Prog-II censoring sample $y^* = y^*_{1:m:n}, y^*_{2:m:n}, \ldots, y^*_{m:m:n}$ from the IPEP distribution, with the Prog-II censoring scheme $(R_1, R_2, \ldots, R_m)$.

(3) The bootstrap sample is used to derive the $MLEs$ and $\hat{\omega}^*$ (in our case, $\omega = \theta$, η, γ, $S(t)$ and $h(t)$).

(4) Steps (2) and (3) occur frequently during N boot times, and the result is $\hat{\omega}^*_1, \hat{\omega}^*_2, \ldots, \hat{\omega}^*_{N\ Boot}$, where $\hat{\omega}^*_i = \left(\hat{\theta}^*_i, \hat{\eta}^*_i, \hat{\gamma}^*_i, \hat{S}^*_i(t), \hat{h}^*_i(t)\right)$.

(5) Ascending orders are applied to $\hat{\omega}^*_i$, where $i = 1, 2, \ldots, N\ boot$, and the results are $\hat{\omega}^*_{(1)}, \hat{\omega}^*_{(2)}, \ldots, \hat{\omega}^*_{(N\ Boot)}$.

Let $G_1(z) = P(\hat{\omega}^* \leq z)$ be the cdf of $\hat{\omega}^*$. For the value of z, $\hat{\omega}_{boot-P} = G_1^{-1}(z)$ is determined. For the value of $\hat{\omega}$, the estimated bootstrap-p, $100(1-\vartheta)\%\ CI$ is denoted by the following:

$$\left[\hat{\omega}_{boot-P}\left(\frac{\vartheta}{2}\right),\ \hat{\vartheta}_{boot-P}\left(1-\frac{\vartheta}{2}\right)\right]. \tag{12}$$

5.2. Boot-t Method

(1)–(3) The parametric Boot-p is the same.

(4) For the variance–covariance matrix $I^{-1*} = \left(\frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial \eta}, \frac{\partial \ell}{\partial \gamma}\right)$, the estimates of the variances $S(t)$ and $h(t)$ are obtained using the asymptotic variance–covariance matrix.

(5) We calculate the statistic $I^{-1*\omega}$ as follows:

$$T^{*\omega} = \frac{(\hat{\omega}^* - \hat{\omega})}{\sqrt{var(\hat{\omega}^*)}},$$

where $\hat{\omega}$ denotes the MLEs of the unknown parameters while $var(\hat{\omega}^*)$ denotes the asymptotic variance.

(6) We repeat steps (2)–(5) $N\ Boot$ times and determine bootstrap estimates as $T^{*\omega}_{(1)}, T^{*\omega}_{(2)}, \ldots, T^{*\omega}_{(N\ Boot)}$.

Assume that $G_2(z) = P(T^* \leq z)$ represents the cdf of $\hat{T}^*$ for the specified z. $\hat{T}_{boot-t} = G_2^{-1}(z)$ is explained for the provided z. Approximately, the bootstrap-p $100(1-\vartheta)\%\ CI$ of $\hat{\omega} = \left(\hat{\theta}, \hat{\eta}, \hat{\gamma}, \hat{S}(t), \hat{h}(t)\right)$ is as follows:

$$\left[\hat{\omega}_{boot-t}\left(\frac{\vartheta}{2}\right),\ \hat{\omega}_{boot-p}\left(1-\frac{\vartheta}{2}\right)\right]. \tag{13}$$

6. Bayesian Estimation

Now, we derive BE under the assumption that the parameters are random. We describe parameter uncertainty by methods of a joint prior distribution that was constructed before the failure data were gathered. In this situation, the Bayesian method will be helpful since it enables us to use prior information when analyzing failure data. In Bayesian analysis, the parameters are updated using data to fit the unknown numbers as random variables. Compared to SEL and $LINEX$ $S(t)$ and $h(t)$, as well as some unknown parameters, are all estimated using the BE. We assume θ, η, and γ to be three independent parameters, following the gamma prior distributions, as follows:

$$\begin{cases} \pi_1(\theta) \propto \theta^{a_1-1}e^{-b_1\theta} & , \theta > 0, a_1 > 0, b_1 > 0, \\ \pi_2(\eta) \propto \eta^{a_2-1}e^{-b_2\eta} & , \eta > 0, a_2 > 0, b_2 > 0, \\ \pi_3(\gamma) \propto \gamma^{a_3-1}e^{-b_3\gamma} & , \gamma > 0, a_3 > 0, b_3 > 0, \end{cases} \tag{14}$$

where the parameters are assumed to be unknown and the hyperparameters a_i and b_i, $i = 1, 2, 3$ are selected to represent the prior assumption. Additionally, the posterior distribution of the parameters, indicated by $\pi^*\left(\theta,\ \eta,\ \gamma \mid \underline{y}\right)$, approximately the same, can be generated by combining the prior with the likelihood function in (7), and applying Bayes' theorem, which can be expressed as follows:

$$\pi^*\left(\theta,\ \eta,\ \gamma \mid \underline{y}\right) = \frac{\pi_1(\theta)\ \pi_2(\eta)\ \pi_3(\gamma)\ L(\theta,\eta,\gamma \mid \underline{y})}{\int_0^\infty \int_0^\infty \int_0^\infty \pi_1(\theta)\ \pi_2(\eta)\ \pi_3(\gamma)\ L(\theta,\eta,\gamma \mid \underline{y})\ d\theta d\eta d\gamma}. \tag{15}$$

The joint posterior density function from (15) is used to generate samples using the *MCMC* approach. Estimating the value of integrals is the major objective of the *MCMC* technique. Within M-H samplers, we perform the *MCMC* method using the Gibbs procedure. The joint posterior distribution is given as follows:

$$\begin{aligned} \pi^*\left(\theta,\eta,\gamma \mid \underline{y}\right) \propto\ & \theta^{a_1+m-1}\eta^{a_2+m-1}\gamma^{a_3+m-1}e^{-b_1\theta-b_2\eta-b_3\gamma}\prod_{i=1}^{m} y_i^{-\gamma-1} \\ & \prod_{i=1}^{m}\left(1+y_i^{-\gamma}\right)^{-\eta-1}\left(1-\left(1+y_i^{-\gamma}\right)^{-\eta}\right)^{\theta-1}\prod_{i=1}^{m}\left(1-\left(1+y_i^{-\gamma}\right)^{-\eta}\right)^{\theta R_i}. \end{aligned} \tag{16}$$

For $\theta,\ \eta$, and γ, the full conditionals are obtained as follows:

$$\pi_1^*\left(\theta \mid \eta,\gamma,\underline{y}\right) \propto \theta^{a_1+m-1}e^{-b_1\theta}\prod_{i=1}^{m}\left(1-\left(1+y_i^{-\gamma}\right)^{-\eta}\right)^{\theta-1}\left(1-\left(1+y_i^{-\gamma}\right)^{-\eta}\right)^{\theta R_i}, \tag{17}$$

$$\begin{aligned} \pi_2^*\left(\eta \mid \theta,\gamma,\underline{y}\right) \propto\ & \eta^{a_2+m-1}\ e^{-b_2\eta}\prod_{i=1}^{m}\left(1+y_i^{-\gamma}\right)^{-\eta-1}\left(1-\left(1+y_i^{-\gamma}\right)^{-\eta}\right)^{\theta-1} \\ & \times\prod_{i=1}^{m}\left(1-\left(1+y_i^{-\gamma}\right)^{-\eta}\right)^{\theta R_i} \end{aligned} \tag{18}$$

and

$$\begin{aligned} \pi_3^*\left(\gamma \mid \eta,\theta,\underline{y}\right) \propto\ & \gamma^{a_3+m-1}e^{-b_3\gamma}\prod_{i=1}^{m} y_i^{-\gamma-1}\left(1+y_i^{-\gamma}\right)^{-\eta-1}\left(1-\left(1+y_i^{-\gamma}\right)^{-\eta}\right)^{\theta-1} \\ & \times\prod_{i=1}^{m}\left(1-\left(1+y_i^{-\gamma}\right)^{-\eta}\right)^{\theta R_i}. \end{aligned} \tag{19}$$

It should be observed that the integral calculations in (17)–(19) cannot be resolved analytically. Therefore, samples use the *MCMC* approach, the joint posterior density function in (16). The plots shown in Figures 2–4 indicate that, despite the difficulty in directly sampling them using conventional methods, they are comparable to normal distributions.

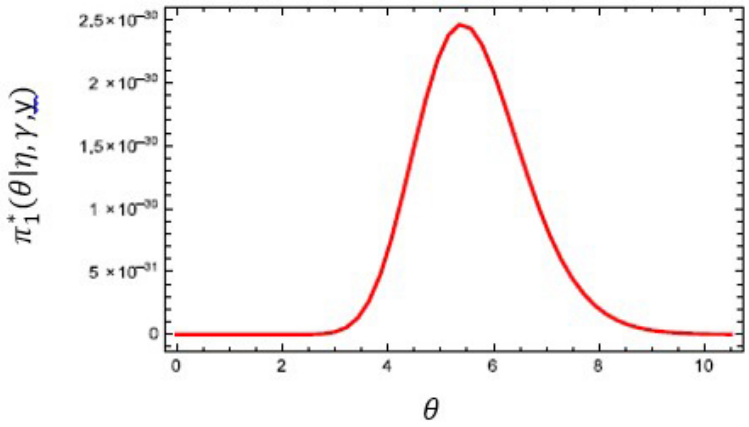


Figure 2. For θ, we display the conditional posterior density of MCMC.

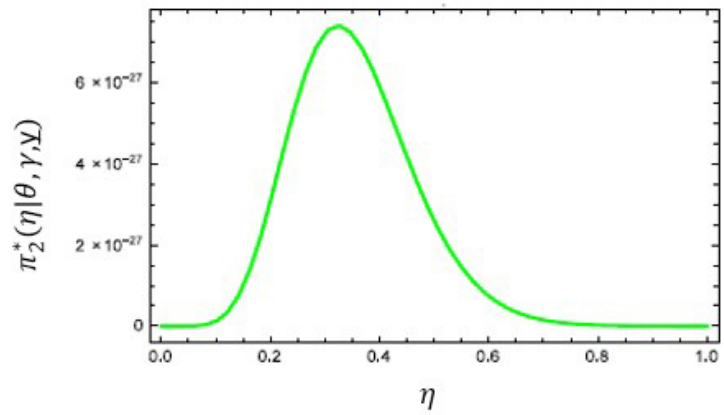


Figure 3. For η, we display the conditional posterior density of MCMC.

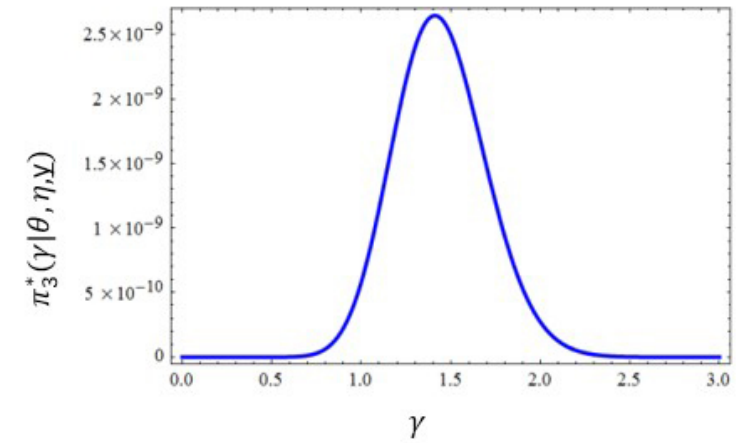


Figure 4. For γ, we display the conditional posterior density of MCMC.

6.1. Loss Function

In Bayesian analysis, loss function (LF) plays an essential role in reducing the risk connected to the estimator. We consider both symmetric and asymmetric LFs in this analysis. Compared to an asymmetric LF, which provides distinct Bayes estimators as the posterior distribution, a symmetric LF is straightforward and generally used.

6.1.1. Symmetric Bayes Estimation

To obtain the *SEL* for estimates, the parameters θ, η , and γ are as follows:

$$\hat{\theta}_{SB} = E\Big(\theta \mid \eta,\ \gamma,\ \underline{y}\Big)$$
$$= K^{-1}\int_0^\infty\int_0^\infty\int_0^\infty \theta^{a_1+m-1}e^{-b_1\theta}\prod_{i=1}^{m}\Big(1-\Big(1+y_i^{-\gamma}\Big)^{-\eta}\Big)^{\theta-1}\prod_{i=1}^{m}\Big(1-\Big(1+y_i^{-\gamma}\Big)^{-\eta}\Big)^{\theta R_i}d\theta d\eta d\gamma, \tag{20}$$

$$\hat{\eta}_{SB} = E\Big(\eta \mid \theta,\ \gamma,\ \underline{y}\Big)$$
$$= K^{-1}\int_0^\infty\int_0^\infty\int_0^\infty \eta^{a_2+m-1}\,e^{-b_2\eta}\prod_{i=1}^{m}\Big(1+y_i^{-\gamma}\Big)^{-\eta-1}\Big(1-\Big(1+y_i^{-\gamma}\Big)^{-\eta}\Big)^{\theta-1}$$
$$\times\prod_{i=1}^{m}\Big(1-\Big(1+y_i^{-\gamma}\Big)^{-\eta}\Big)^{\theta R_i}d\theta d\eta d\gamma \tag{21}$$

and

$$\hat{\gamma}_{SB} = E\Big(\gamma \mid \theta,\ \eta,\ \underline{y}\Big)$$
$$= K^{-1}\int_0^\infty\int_0^\infty\int_0^\infty \gamma^{a_3+m-1}e^{-b_3\gamma}\prod_{i=1}^{m}y_i^{-\gamma-1}\Big(1+y_i^{-\gamma}\Big)^{-\eta-1}\Big(1-\Big(1+y_i^{-\gamma}\Big)^{-\eta}\Big)^{\theta-1}$$
$$\times\prod_{i=1}^{m}\Big(1-\Big(1+y_i^{-\gamma}\Big)^{-\eta}\Big)^{\theta R_i}d\theta d\eta d\gamma, \tag{22}$$

where

$$k = \int_0^\infty\int_0^\infty\int_0^\infty \pi(\theta,\eta,\gamma)\ L(\theta,\eta,\gamma \mid \underline{y})\ d\theta d\eta d\gamma$$
$$= \int_0^\infty\int_0^\infty\int_0^\infty \theta^{a_1+m-1}\eta^{a_2+m-1}\gamma^{a_3+m-1}e^{-b_1\theta-b_2\eta-b_3\gamma}\prod_{i=1}^{m}y_i^{-\gamma-1}$$
$$\prod_{i=1}^{m}\Big(1+y_i^{-\gamma}\Big)^{-\eta-1}\Big(1-\Big(1+y_i^{-\gamma}\Big)^{-\eta}\Big)^{\theta-1}\prod_{i=1}^{m}\Big(1-\Big(1+y_i^{-\gamma}\Big)^{-\eta}\Big)^{\theta R_i}d\theta d\eta d\gamma.$$

6.1.2. A Symmetric Bayes Estimation

The $LINEX$ can be used to derive the BE of $\theta,\ \eta$ and γ, as the following:

$$\hat{\theta}_{LB} = \frac{-1}{c}\log\Big(E\left[e^{-c\,\theta}\mid \underline{y}\right]\Big), c \neq 0,$$

where

$$E\left[e^{-c\,\theta}\mid \underline{y}\right] = K^{-1}\int_0^\infty\int_0^\infty\int_0^\infty \theta^{a_1+m-1}e^{-\,\theta(c+b_1)}$$
$$\prod_{i=1}^{m}\Big(1-\Big(1+y_i^{-\gamma}\Big)^{-\eta}\Big)^{\theta-1}\Big(1-\Big(1+y_i^{-\gamma}\Big)^{-\eta}\Big)^{\theta R_i}d\theta d\eta d\gamma, \tag{23}$$

$$\hat{\eta}_{LB} = \frac{-1}{c}\log\Big(E\left[e^{-c\,\eta}\mid \underline{y}\right]\Big), c \neq 0,$$

where

$$\left(E\left[e^{-c\,\eta}\mid \underline{y}\right]\right)=K^{-1}\int_0^\infty\int_0^\infty\int_0^\infty \eta^{a_2+m-1}e^{-\,\eta(c+b_2)}\prod_{i=1}^{m}\left(1+y_i^{-\gamma}\right)^{-\eta-1}\left(1-\left(1+y_i^{-\gamma}\right)^{-\eta}\right)^{\theta-1}$$
$$\times\prod_{i=1}^{m}\left(1-\left(1+y_i^{-\gamma}\right)^{-\eta}\right)^{\theta R_i}d\theta d\eta d\gamma \tag{24}$$

and

$$\hat{\gamma}_{LB}=\frac{-1}{c}\log\left(E\left[e^{-c\,\gamma}\mid \underline{y}\right]\right), c\neq 0,$$

where

$$\left(E\left[e^{-c\,\gamma}\mid \underline{y}\right]\right)=K^{-1}\int_0^\infty\int_0^\infty\int_0^\infty \sigma^{a_3+m-1}\,e^{-\,\gamma(c+b_3)}\prod_{i=1}^{m}y_i^{-\gamma-1}\left(1+y_i^{-\gamma}\right)^{-\eta-1}$$
$$\left(1-\left(1+y_i^{-\gamma}\right)^{-\eta}\right)^{\theta-1}\times\prod_{i=1}^{m}\left(1-\left(1+y_i^{-\gamma}\right)^{-\eta}\right)^{\theta R_i}d\theta d\eta d\gamma. \tag{25}$$

In order to obtain posterior distribution samples for computing the BE of θ, η and γ of IPEP(θ, η, γ), with distribution under Prog-II censored, we suggest using the *MCMC* approach. Additional Metropolis-in-Gibbs samplers and Gibbs sampling are essentially classes of *MCMC* techniques.

7. Metropolis–Hastings Algorithm

A general-purpose *MCMC* approach is the M-H algorithm, originally presented by Metropolis et al. [18] and extended by Hastings [19]. The M-H algorithm can select random samples from the known target distribution, regardless of how complicated it is. We can produce samples from the posterior distribution for unknown parameters to estimate interval approximations and Bayesian point estimators.

Then, the M–H algorithm is employed as follows:

(1) We begin with a first estimate of $\left(\theta^{(0)},\ \eta^{(0)},\ \gamma^{(0)}\right)$.

(2) At $j=1$.

(3) We utilize the subsequent $M-H$ method and produce the values $\theta^{(j)}$, $\eta^{(j)}$, and $\gamma^{(j)}$ from

$$\pi_1^*\left(\theta^{(j-1)}\mid\eta^{(j-1)},\gamma^{(j-1)},\underline{y}\right),\pi_2^*\left(\eta^{(j-1)}\mid\ \theta^{(j)},\gamma^{(j-1)},\underline{y}\right)\text{ and }\pi_3^*\left(\gamma^{(j-1)}\mid\theta^{(j)},\eta^{(j)},\underline{y}\right).$$

Using the standard propositional distributions

$$N(\theta^{(j-1)},\ var(\theta)),N(\eta^{(j-1)},var(\eta))\text{ and }N(\gamma^{(j-1)},\ var(\gamma)).$$

(i) We produce a proposal θ^* from $N(\theta^{(j-1)},\ var(\theta))$, η^* from $N(\eta^{(j-1)},var(\eta))$ and γ^* from $N(\gamma^{(j-1)},\ var(\gamma))$.

(ii) We estimate the probabilities of approval

$$\psi_\theta=\left[1,\frac{\pi_1^*\left(\theta^*\mid\eta^{(j-1)},\ \gamma^{(j-1)},\ \underline{y}\right)}{\pi_1^*\left(\theta^{(j-1)}\mid\eta^{(j-1)},\ \gamma^{(j-1)},\ \underline{y}\right)}\right],$$

$$\psi_\eta=\left[1,\frac{\pi_2^*\left(\eta^*\mid\ \theta^{(j)},\ \gamma^{(j-1)},\ \underline{y}\right)}{\pi_2^*\left(\eta^{(j-1)}\mid\ \theta^{(j)},\ \gamma^{(j-1)},\ \underline{y}\right)}\right]$$

and

$$\psi_\gamma = \left[1, \frac{\pi_3^*\left(\gamma^* \mid \theta^{(j)}, \eta^{(j)}, \underline{y}\right)}{\pi_3^*\left(\gamma^{(j-1)} \mid \theta^{(j)}, \eta^{(j)}, \underline{y}\right)}\right].$$

(iii) From a Uniform $(0,1)$ distribution, we produce u_1, u_2 and u_3.

(iv) We implement the proposition and place $\theta^{(j)} = \theta^*$; otherwise, we place $\theta^{(j)} = \theta^{(j-1)}$.

(v) We implement the proposition and place $\eta^{(j)} = \eta^*$; otherwise, we place $\eta^{(j)} = \eta^{(j-1)}$.

(vi) We implement the proposition and place $\gamma^{(j)} = \gamma^*$; otherwise, we place $\gamma^{(j)} = \gamma^{(j-1)}$.

(vii) The $S(t)$ and $h(t)$ calculations are as follows:

$$S^{(j)}(t) = \left(1 - \left(1 + t^{-\gamma^j}\right)^{-\eta^j}\right)^{\theta^j};\ t,\ \theta,\ \eta,\ \gamma > 0,$$

$$h^{(j)}(t) = \frac{\theta^j \eta^j \gamma^j t^{-\gamma^j - 1}\left(1 + t^{-\gamma^j}\right)^{-\eta^j - 1}\left(1 - \left(1 + t^{-\gamma^j}\right)^{-\eta^j}\right)^{\theta^j - 1}}{\left(1 - \left(1 + t^{-\gamma^j}\right)^{-\eta^j}\right)^{\theta^j}};\ t,\ \theta,\ \eta,\ \gamma > 0.$$

(4) At $j = j + 1$

(5) Steps (3)–(4) N times should be replicated.

(6) The initial M simulated variants are removed after producing the convergence to remove the initial value selection. The sample that was chosen is $\omega^{(j)} = (\theta^{(j)}, \eta^{(j)}, \gamma^{(j)}, S^{(j)}(t), h^{(j)}(t))$, where $j = M+1, \ldots, N$, for a sufficiently sized N generates an approximate posterior sample that may be utilized to obtain the BE of

$$\hat{\omega}_M = \frac{1}{N - M}\sum_{j=M+1}^{N} \omega^{(j)}.$$

where the Markov chain's burn-in time of ($N = 1000$) is used to compute the HPD interval of $\omega = (\theta, \eta, \gamma, S(t), h(t))$, the $MCMC$ sample of $\omega^{(j)}$=($\theta^{(j)}$, $\eta^{(j)}$, $\gamma^{(j)}$, and the $S^{(j)}(t)$,$h^{(j)}(t)$) order $(\omega_1, \omega_2, \omega_3, \ldots, \omega_N$ as $\omega_{[1]} < \omega_{[2]} \cdots < \omega_{[N]})$.

Then, we construct the $100(1-\nu)$ % CI of $((\omega_{(1)}, \omega_{[N(1-\nu)+1]}), \ldots, (\omega_{[N\nu]}, \omega_{[N]}))$.

8. Application to Real Data and Simulation Study

To illustrate the practical applications of the provided techniques on estimation and prediction problems, we provide numerical outcomes for the estimation of IPEP model parameters using Prog-II censoring on actual datasets. The analysis involves comparing the performance of BE with MLE by analyzing both simulated and actual data.

Flood-Level Data

Using real datasets provided by Dumonceaux and Antle [12], the 20 flood levels represent a real dataset of Susquehanna River's maximum flood levels at Harrisburg, Pennsylvania, between 1890 and 1969. The data are as follows:

Prog-II samples were generated from the complete dataset, with $m = 10$ and a censoring scheme of $R = (5^{(2)}, 0^{(18)})$, as presented in Table 1. Prior to conducting the analysis, it is crucial to evaluate the adequacy of the presented model to represent the data. The K-S distance measures the difference between both the fitted and empirical distribution functions, yielding a K-S value of 0.084 and an associated p-value of 0.95. These findings indicate that the distribution of the IPEP provides a satisfactory fit to the given dataset. Figure 5 represents the empirical quantile function of IPEP distribution for the flood-level data.

Table 1. Progressively censored sample based on data of 20 flood levels.

y_i	0.645	0.613	0.315	0.449	0.297
R_i	5	5	0	0	0
y_i	0.402	0.379	0.423	0.379	0.324
R_i	0	0	0	0	0
y_i	0.269	0.740	0.218	0.412	0.494
R_i	0	0	0	0	0
y_i	0.416	0.338	0.392	0.484	0.265
R_i	0	0	0	0	0

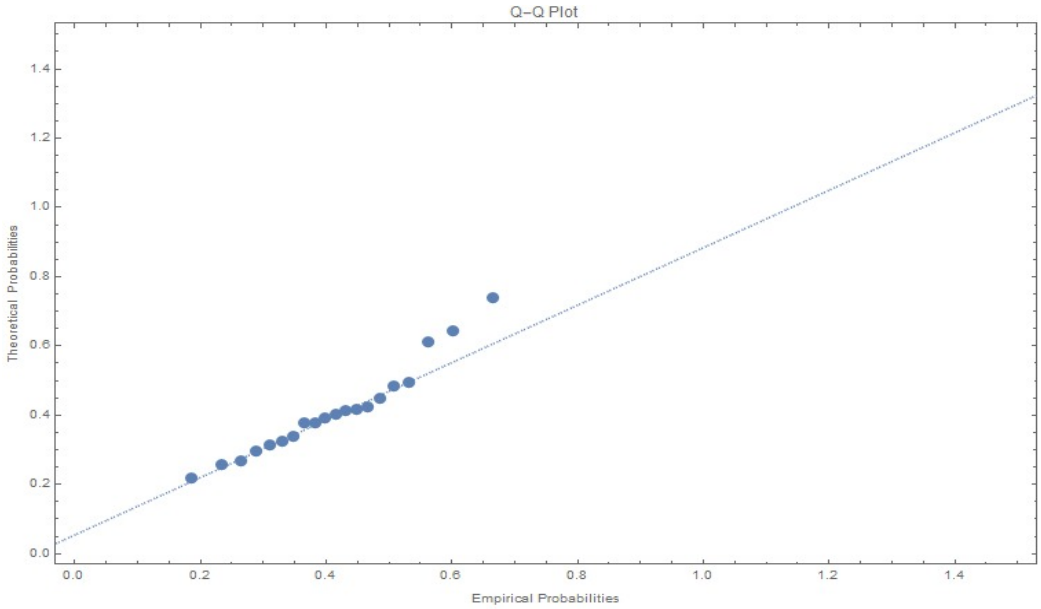


Figure 5. The IPEPD fitted to the real dataset.

Point estimates for the parameters θ, η, and γ, along with $S(t)$ and $h(t)$ at time $t = 0.4$, were obtained using real data on Prog-II censoring schemes presented in Table 1. All these point estimates, including MLE, Boot-p , Boot-t, and BE under SEL and $LINEX$, are presented in Table 2. The 95% confidence intervals (CIs) for the BE estimators and MLEs are reported in Table 3.

Table 2. MLE and Bayes $MCMC$ estimations for real data under SEL and $LINEX$.

Parameters	***MLE***	**Boot-p**	**Boot-t**	***SEL***	***LINEX***		
					$c = -0.2$	$c = 0.2$	$c = 0.0001$
θ	18.4965	21.451	19.752	26.7214	31.1442	20.924	26.7209
η	3.24641	3.532	3.125	3.19389	3.34409	2.86674	3.19388
γ	1.19367	1.354	1.124	1.06675	1.08007	1.05268	1.06675
$S(0.4)$	0.675392	0.732	0.643	0.475241	0.496605	0.447277	0.47524
$h(0.4)$	2.13903	2.975	2.562	3.75233	9.59553	2.75506	3.75213

Table 3. The 95% *CIs* of the parameters and reliability functions for real data.

Parameters	*MCMC*	*ACIs*
θ	$(19.8089, 31.4986)$	$(-140.335, 177.328)$
η	$(1.87614, 3.70367)$	$(-8.0912, 14.584)$
γ	$(0.846438, 1.23629)$	$(-2.04768, 4.43503)$
$S(0.4)$	$(0.0481525, 0.670927)$	$(0.500616, 0.850169)$
$h(0.4)$	$(1.86308, 10.4146)$	$(0.640487, 3.63758)$

It is well established that the $LINEX$ approximates the behavior of the SEL function due to its symmetrical nature for values of c near to zero. Figures 6–10 display the $MCMC$ method's posterior density function plots as well as the trace plots of the unknown parameters θ, η, γ, and $S(t)$ and $h(t)$ functions. Furthermore, the MLE for the parameters θ, η, γ as well as the $S(t)$ and $h(t)$ are unique and represent the actual maximums, as shown in Figure 11.

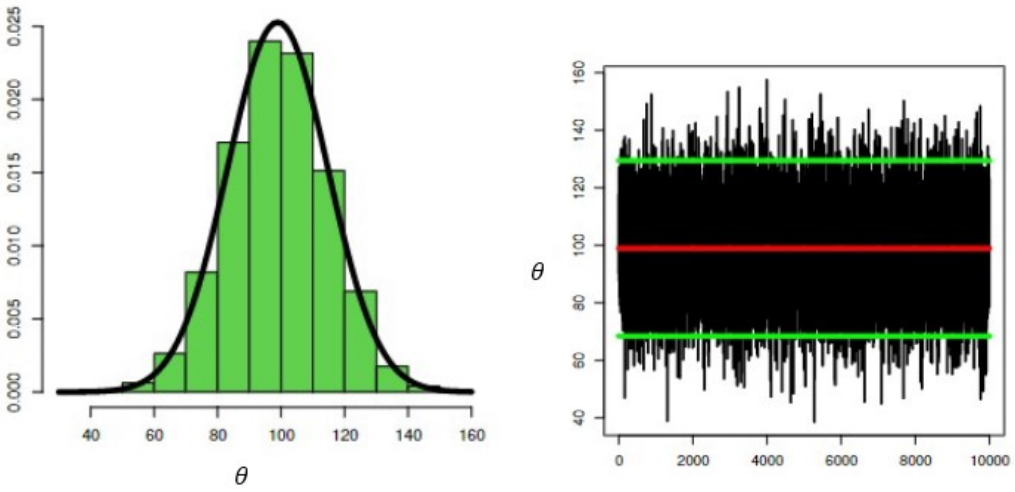


Figure 6. For θ, we display the posterior density function and trace plots.

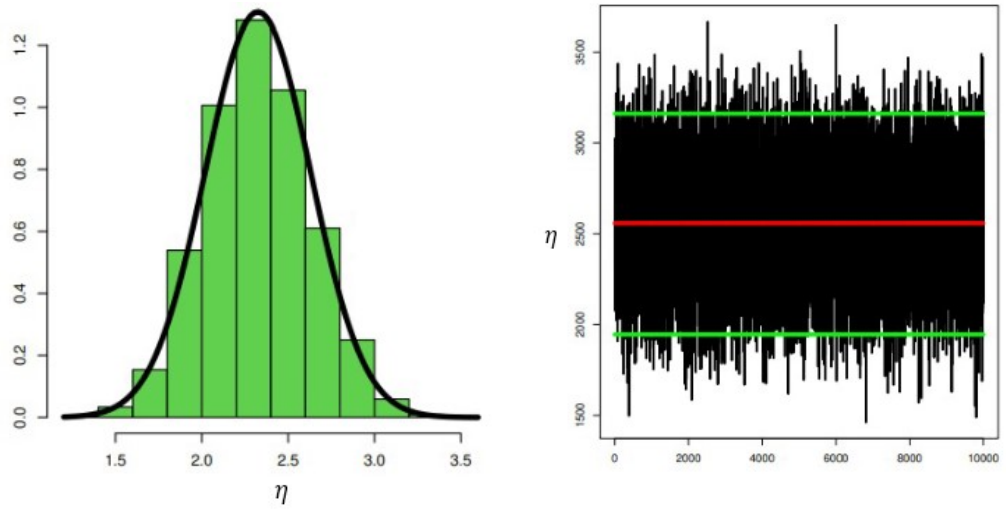


Figure 7. For η, we display the posterior density function and trace plots.

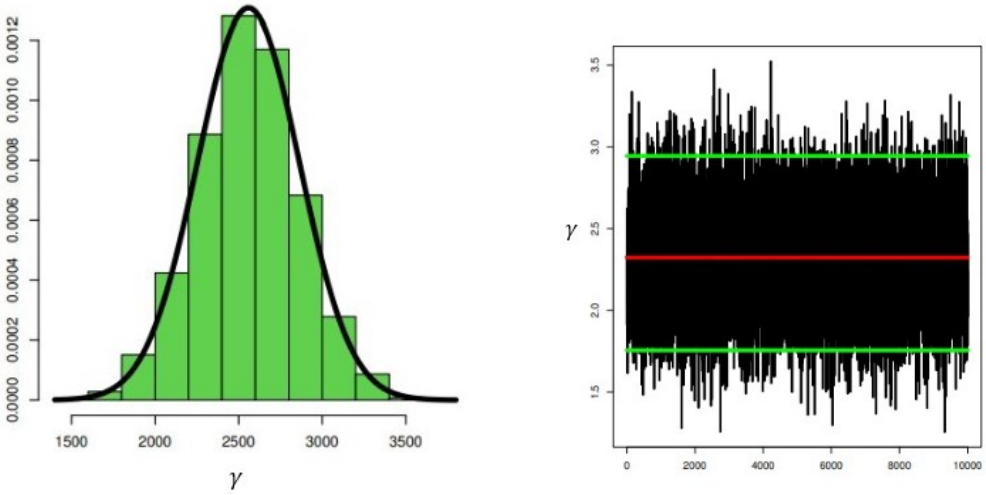


Figure 8. For γ, we display the posterior density function and trace plots.

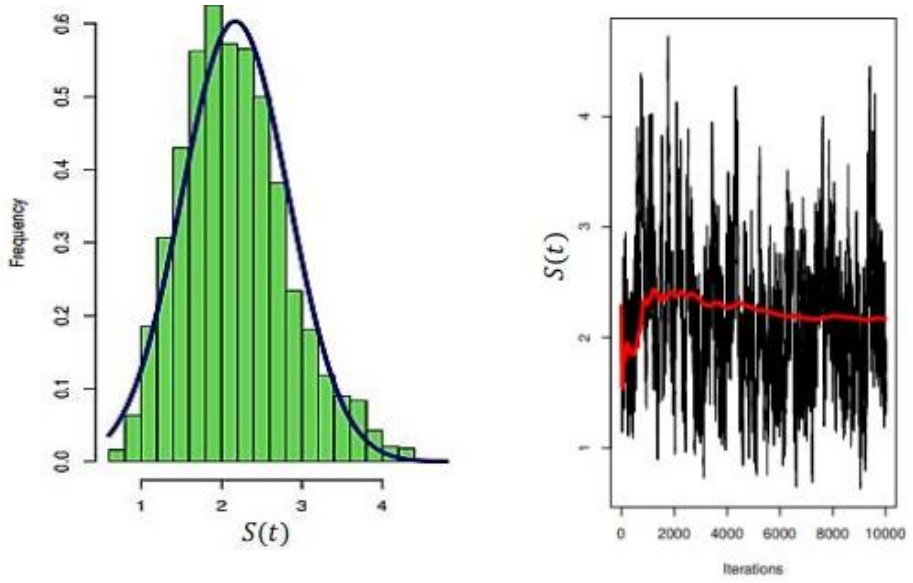


Figure 9. For $S(t)$, we display the posterior density function and trace plots.

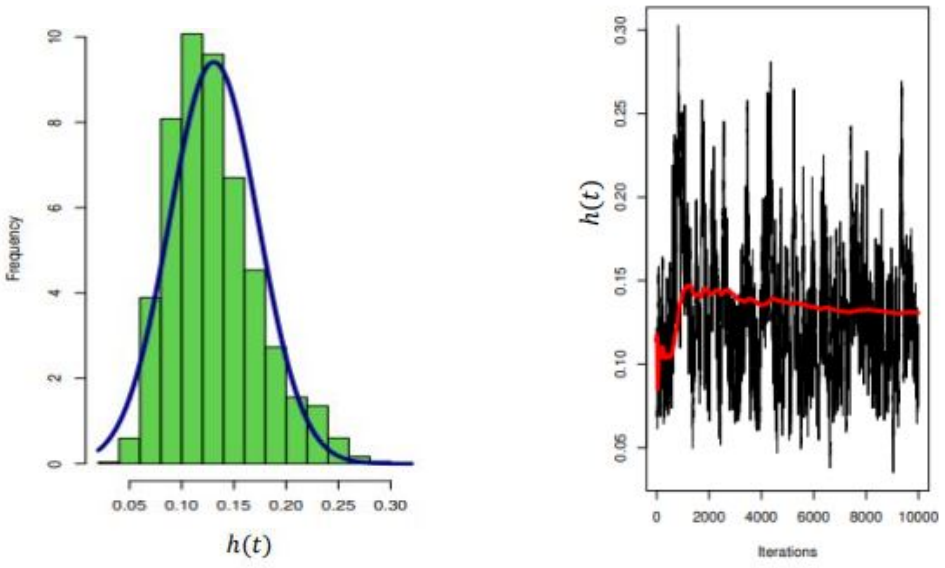


Figure 10. For $h(t)$, we display the posterior density function and trace plots.

Furthermore, a comparison of the CIs for MLEs and BEs in Table 3 demonstrates that Bayes estimators possess narrower CIs compared to their MLEs for all three parameters (θ, η , and γ). For the parameters and reliability functions, Table 3 shows the 95% ACIs for MLE and CRIs for MCMC (BE). Figure 11 visualizes the profile's likelihood, illustrating the uniqueness and similarity of the maximum values with the MLE results. In Tables 2 and 3, it is seen that BEs outperform MLEs in the Prog-II censoring samples. Compared to the approximation CIs, the Bayes CRIs have the shortest confidence lengths. This study may effectively assist in the failure analysis of the airplane's air conditioning dataset since the distribution of IPED is appropriate to the applicable data.

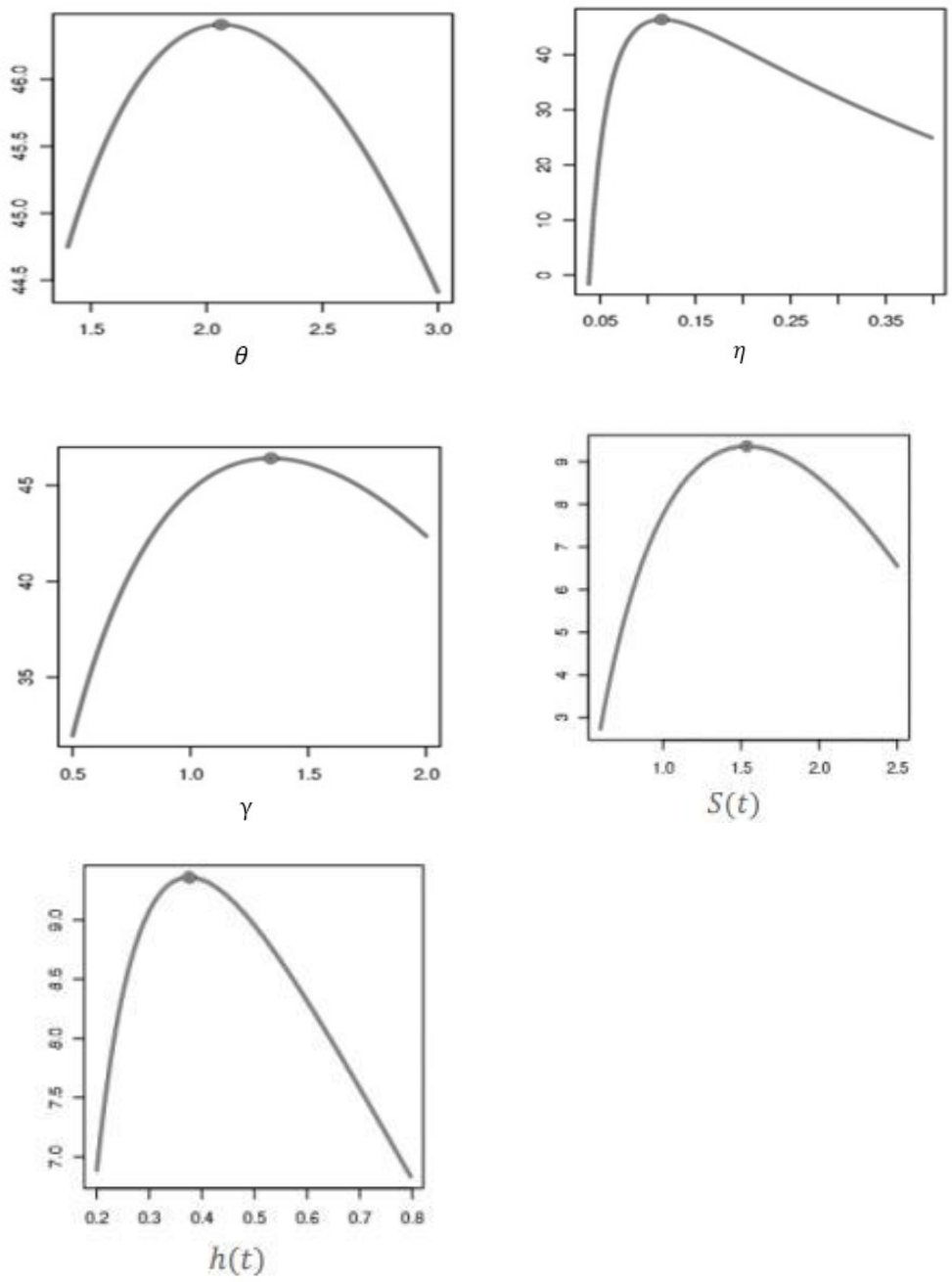


Figure 11. The profile log-likelihood plots for MLE for the parameters and reliability functions.

9. Simulation Study

To produce the censored Prog-II illustration with initial values $\theta_0 = 1.2$, $\eta_0 = 1$ and $\gamma_0 = 0.5$ from the IPEP distribution, a comparison of the various techniques used by the estimators of results θ, η, γ, $S(t)$ and $h(t)$, at $t = 0.4$, has been discussed in calculating their MSE for $k = 1,2,3,4,5$, $t = 0.4$ and $MLEs(\omega_k) = 1/M \sum_{i=1}^{M} \left(\hat{\omega}_k^{(i)} - \omega_k\right)^2$, where $M = 10{,}000$ represents the number of generated samples. Another criterion is used to compare the 95% CI obtained using the asymptotic MLE and CRI, compared using a different criterion. For comparison, CP and ACL are used. The CP of an interval of confidence is the percentage of times that the initial significance value is found in the interval. Under the following data created from the IPEP distribution using the quantile function of y, Monte Carlo experiments were conducted, where y follows IPEP distribution and is obtained by using $F_{IPEP}(Y, \theta, \eta, \gamma) = U$ as follows:

$$y = \left(\left(1 - (1-U)^{\frac{1}{\theta}} \right)^{\frac{-1}{\eta}} - 1 \right)^{\frac{-1}{\gamma}}. \tag{26}$$

The random variable $Y = Q(U)$ is given by Equation (26), if U is a uniform variate on the unit interval $(0,1)$.

The subsequent progressive approaches are generated as follows:

I: $R_1 = n - m$, $R_i = 0$ for $i \neq 1$.

II: $R_{(m/2)} = R_{(m/2)+1} = ((n-m)/2)$, $R_i = 0$ for $i \neq (m/2)$ and $i \neq (m/2)+1$.

III: $R_m = n - m$, $R_i = 0$ for $i \neq m$.

Tables 4–8 show the results of the estimate parameters and their $MSEs$, while Table 9 displays the values of the ACL and CP of CIs.

Table 4. The biased and MSE (in brackets) of the ML and BE for θ.

		MLE	***SEL***	***LINEX***		
(n,m)	***CS***			$c = -2.0$	$c = 2.0$	$c = 0.0001$
$(15,10)$	I	1.1804 (1.7371)	1.2234 (1.7403)	1.2238 (1.7412)	1.223 (1.7395)	1.2234 (1.7403)
	II	1.137 (1.4764)	1.1376 (1.4774)	1.1394 (1.4814)	1.1361 (1.4753)	1.1376 (1.4774)
	III	1.2754 (1.8757)	1.2746 (1.8731)	1.2747 (1.8732)	1.2745 (1.8729)	1.2746 (1.8731)
$(20,15)$	I	0.7293 (0.267)	0.7298 (0.2675)	0.7298 (0.2675)	0.7298 (0.2675)	0.7298 (0.2675)
	II	1.0663 (1.3621)	1.066 (1.3614)	1.0662 (1.3616)	1.0658 (1.3612)	1.066 (1.3614)
	III	1.2044 (1.6615)	1.2046 (1.6617)	1.2047 (1.6619)	1.2045 (1.6615)	1.2046 (1.6617)
$(30,20)$	I	0.9835 (1.0346)	0.9854 (1.041)	0.9863 (1.0464)	0.9844 (1.0368)	0.9854 (1.041)
	II	1.0083 (1.0445)	1.0089 (1.0454)	1.009 (1.0454)	1.0089 (1.0453)	1.0089 (1.0454)
	III	1.1336 (1.383)	1.134 (1.3833)	1.1342 (1.3837)	1.1338 (1.3829)	1.134 (1.3833)
$(40,20)$	I	1.104 (1.4097)	1.1043 (1.4095)	1.1044 (1.4098)	1.1042 (1.4093)	1.1043 (1.4095)
	II	1.0195 (1.1165)	1.0202 (1.119)	1.0207 (1.1211)	1.0198 (1.1179)	1.0202 (1.119)

Table 4. *Cont.*

		MLE	***SEL***		***LINEX***	
(n, m)	***CS***			$c = -2.0$	$c = 2.0$	$c = 0.0001$
	III	1.2004 (1.8254)	1.2012 (1.8297)	1.2013 (1.8299)	1.2011 (1.8295)	1.2012 (1.8297)
(50, 20)	*I*	1.1123 (1.3729)	1.1126 (1.3734)	1.1126 (1.3734)	1.1126 (1.3733)	1.1126 (1.3734)
	II	0.9788 (0.9609)	0.9799 (0.9643)	0.9803 (0.9655)	0.9795 (0.9633)	0.9799 (0.9643)
	III	1.1804 (1.8265)	1.18 (1.8246)	1.18 (1.8247)	1.18 (1.8244)	1.18 (1.8246)

Table 5. The biased and *MSE* (in brackets) of the ML and BE for η.

		MLE	***SEL***		***LINEX***	
(n, m)	***CS***			$c = -2.0$	$c = 2.0$	$c = 0.0001$
(15, 10)	*I*	1.1914 (1.0985)	1.1905 (1.0958)	1.1907 (1.0963)	1.1907 (1.0952)	1.1905 (1.0958)
	II	1.132 (1.0029)	1.1309 (1.0027)	1.132 (1.0053)	1.132 (1.0005)	1.1309 (1.0027)
	III	1.2519 (1.3043)	1.2513 (1.3032)	1.2514 (1.3034)	1.2514 (1.3031)	1.2513 (1.3032)
(20, 15)	*I*	0.9739 (0.8107)	0.9745 (0.8133)	0.9745 (0.8133)	0.9745 (0.8133)	0.9745 (0.8133)
	II	1.0529 (0.9099)	1.0536 (0.9114)	1.0536 (0.9116)	1.0536 (0.9113)	1.0536 (0.9114)
	III	1.1891 (1.1426)	1.1887 (1.142)	1.1889 (1.1423)	1.1889 (1.1417)	1.1887 (1.142)
(30, 20)	*I*	1.0363 (0.8282)	1.0362 (0.8275)	1.0364 (0.8281)	1.0364 (0.8271)	1.0362 (0.8275)
	II	1.0433 (0.8355)	1.0442 (0.837)	1.0444 (0.8378)	1.0444 (0.8364)	1.0442 (0.837)
	III	1.1396 (1.0202)	1.1394 (1.0206)	1.1397 (1.021)	1.1397 (1.0201)	1.1394 (1.0206)
(40, 20)	*I*	1.1084 (1.0142)	1.1082 (1.014)	1.1084 (1.0142)	1.1084 (1.0138)	1.1082 (1.014)
	II	1.0482 (0.8571)	1.0475 (0.8559)	1.0476 (0.8561)	1.0476 (0.8556)	1.0475 (0.8559)
	III	1.1136 (1.1825)	1.1132 (1.1816)	1.1132 (1.1817)	1.1132 (1.1815)	1.1132 (1.1816)
(50, 20)	*I*	1.1107 (0.9518)	1.1108 (0.9518)	1.1108 (0.9518)	1.1108 (0.9518)	1.1108 (0.9518)
	II	1.0042 (0.7413)	1.0057 (0.744)	1.0061 (0.7449)	1.0061 (0.7428)	1.0057 (0.744)
	III	1.057 (1.1936)	1.057 (1.1938)	1.0571 (1.1939)	1.0571 (1.1938)	1.057 (1.1938)

Table 6. The biased and *MSE* (in brackets) of the ML and BE for γ.

		MLE	***SEL***		***LINEX***	
(n, m)	***CS***			$c = -2.0$	$c = 2.0$	$c = 0.0001$
$(15, 10)$	*I*	0.6214 (0.426)	0.6217 (0.4261)	0.6217 (0.4261)	0.6217 (0.4261)	0.6217 (0.4261)
	II	0.6941 (0.6214)	0.6943 (0.6217)	0.6944 (0.6217)	0.6944 (0.6217)	0.6943 (0.6217)
	III	0.8014 (1.0307)	0.8013 (1.0302)	0.8013 (1.0302)	0.8013 (1.0302)	0.8013 (1.0302)
$(20, 15)$	*I*	0.7573 (0.6106)	0.7575 (0.6102)	0.7575 (0.6102)	0.7575 (0.6102)	0.7575 (0.6102)
	II	0.658 (0.3965)	0.6582 (0.3964)	0.6582 (0.3964)	0.6582 (0.3964)	0.6582 (0.3964)
	III	0.6391 (0.4083)	0.6389 (0.4084)	0.6389 (0.4084)	0.6389 (0.4084)	0.6389 (0.4084)
$(30, 20)$	*I*	0.5899 (0.2561)	0.5899 (0.2562)	0.5899 (0.2562)	0.5899 (0.2562)	0.5899 (0.2562)
	II	0.6055 (0.2851)	0.6054 (0.2851)	0.6054 (0.2851)	0.6054 (0.2851)	0.6054 (0.2851)
	III	0.6025 (0.3124)	0.6027 (0.3124)	0.6027 (0.3124)	0.6027 (0.3124)	0.6027 (0.3124)
$(40, 20)$	*I*	0.5828 (0.2621)	0.5828 (0.2622)	0.5828 (0.2622)	0.5828 (0.2622)	0.5828 (0.2622)
	II	0.5815 (0.2204)	0.5817 (0.2203)	0.5817 (0.2203)	0.5817 (0.2203)	0.5817 (0.2203)
	III	0.7225 (0.5905)	0.7225 (0.5907)	0.7225 (0.5907)	0.7225 (0.5907)	0.7225 (0.5907)
$(50, 20)$	*I*	0.557 (0.2226)	0.557 (0.2227)	0.557 (0.2227)	0.557 (0.2227)	0.557 (0.2227)
	II	0.5896 (0.2418)	0.5896 (0.2417)	0.5896 (0.2417)	0.5896 (0.2417)	0.5896 (0.2417)
	III	0.9735 (1.4031)	0.9735 (1.4029)	0.9735 (1.403)	0.9735 (1.4029)	0.9735 (1.4029)

Table 7. The biased and *MSE* (in brackets) of the ML and BE for $S(t)$.

		MLE	***SEL***		***LINEX***	
(n, m)	***CS***			$c = -2.0$	$c = 2.0$	$c = 0.0001$
$(15, 10)$	*I*	0.6588 (0.0125)	0.6586 (0.0125)	0.6586 (0.0125)	0.6586 (0.0125)	0.6586 (0.0125)
	II	0.6604 (0.0121)	0.6599 (0.0121)	0.6601 (0.0121)	0.6601 (0.0121)	0.6599 (0.0121)
	III	0.6689 (0.0121)	0.6688 (0.0121)	0.6688 (0.0121)	0.6688 (0.0121)	0.6688 (0.0121)
$(20, 15)$	*I*	0.6877 (0.0068)	0.6877 (0.0068)	0.6877 (0.0068)	0.6877 (0.0068)	0.6877 (0.0068)
	II	0.6608 (0.0081)	0.661 (0.0081)	0.661 (0.0081)	0.661 (0.0081)	0.661 (0.0081)
	III	0.6652 (0.0082)	0.665 (0.0082)	0.665 (0.0082)	0.665 (0.0082)	0.665 (0.0082)

Table 7. *Cont.*

		MLE	***SEL***		***LINEX***	
(n, m)	***CS***			$c = -2.0$	$c = 2.0$	$c = 0.0001$
$(30, 20)$	*I*	0.6678 (0.0065)	0.6675 (0.0066)	0.6676 (0.0065)	0.6676 (0.0066)	0.6675 (0.0066)
	II	0.6613 (0.0055)	0.6614 (0.0055)	0.6614 (0.0055)	0.6614 (0.0055)	0.6614 (0.0055)
	III	0.6641 (0.0051)	0.664 (0.0051)	0.664 (0.0051)	0.664 (0.0051)	0.664 (0.0051)
$(40, 20)$	*I*	0.6651 (0.0056)	0.6649 (0.0056)	0.6649 (0.0056)	0.6649 (0.0057)	0.6649 (0.0056)
	II	0.6598 (0.0049)	0.6594 (0.005)	0.6594 (0.005)	0.6594 (0.005)	0.6594 (0.005)
	III	0.6547 (0.0048)	0.6545 (0.0048)	0.6545 (0.0048)	0.6545 (0.0048)	0.6545 (0.0048)
$(50, 20)$	*I*	0.6582 (0.0055)	0.6581 (0.0055)	0.6581 (0.0055)	0.6581 (0.0055)	0.6581 (0.0055)
	II	0.6534 (0.0045)	0.6536 (0.0045)	0.6537 (0.0045)	0.6537 (0.0045)	0.6536 (0.0045)
	III	0.6449 (0.0064)	0.645 (0.0064)	0.645 (0.0064)	0.645 (0.0064)	0.645 (0.0064)

Table 8. The biased and *MSE* (in brackets) of the ML and BE for $h(t)$.

		MLE	***SEL***		***LINEX***	
(n, m)	***CS***			$c = -2.0$	$c = 2.0$	$c = 0.0001$
$(15, 10)$	*I*	0.3279 (0.0191)	0.3285 (0.0192)	0.3286 (0.0192)	0.3286 (0.0192)	0.3285 (0.0192)
	II	0.348 (0.039)	0.3484 (0.0391)	0.3486 (0.0392)	0.3486 (0.0391)	0.3484 (0.0391)
	III	0.3653 (0.0801)	0.3653 (0.08)	0.3653 (0.08)	0.3653 (0.08)	0.3653 (0.08)
$(20, 15)$	*I*	0.3123 (0.0231)	0.3125 (0.0231)	0.3125 (0.0231)	0.3125 (0.0231)	0.3125 (0.0231)
	II	0.3256 (0.0141)	0.3255 (0.0141)	0.3255 (0.0141)	0.3255 (0.0141)	0.3255 (0.0141)
	III	0.3263 (0.0155)	0.3263 (0.0156)	0.3263 (0.0156)	0.3263 (0.0156)	0.3263 (0.0156)
$(30, 20)$	*I*	0.3016 (0.0062)	0.3019 (0.0062)	0.302 (0.0062)	0.302 (0.0062)	0.3019 (0.0062)
	II	0.3192 (0.0092)	0.3192 (0.0093)	0.3192 (0.0093)	0.3192 (0.0093)	0.3192 (0.0093)
	III	0.3212 (0.0087)	0.3214 (0.0087)	0.3214 (0.0087)	0.3214 (0.0087)	0.3214 (0.0087)

Table 8. *Cont.*

		MLE	***SEL***	***LINEX***		
(n, m)	***CS***			$c = -2.0$	$c = 2.0$	$c = 0.0001$
$(40, 2)$	*I*	0.3064 (0.0063)	0.3066 (0.0063)	0.3066 (0.0063)	0.3066 (0.0063)	0.3066 (0.0063)
	II	0.3245 (0.0124)	0.3252 (0.0128)	0.3252 (0.0128)	0.3252 (0.0128)	0.3252 (0.0128)
	III	0.3556 (0.0314)	0.3557 (0.0315)	0.3557 (0.0315)	0.3557 (0.0315)	0.3557 (0.0315)
$(50, 20)$	*I*	0.3101 (0.0066)	0.3102 (0.0066)	0.3102 (0.0066)	0.3102 (0.0066)	0.3102 (0.0066)
	II	0.3323 (0.0119)	0.3323 (0.0119)	0.3324 (0.0119)	0.3324 (0.0119)	0.3323 (0.0119)
	III	0.4108 (0.0743)	0.4107 (0.0743)	0.4107 (0.0743)	0.4107 (0.0743)	0.4107 (0.0743)

Table 9. The parameters and reliability functions that display the ACLs and CPs of 95% CIs.

		θ		η		γ		$S(t)$		$h(t)$	
(n,m)	CS	MLE	$MCMC$	MLE	$MCMC$	MLE	$MCMC$	MLE	$MCMC$	MLE	$MCMC$
$(15,10)$	I	17.5868 (0.9544)	0.0123 (0.9737)	14.7808 (0.93)	0.0095 (0.9311)	44.8695 (0.9427)	0.003 (0.9433)	0.5004 (0.9586)	0.0036 (0.925)	0.5997 (0.9456)	0.004 (0.956)
	II	29.5999 (0.9262)	0.0206 (0.9468)	30.8139 (0.9464)	0.018 (0.931)	368.392 (0.9613)	0.0047 (0.9679)	0.7324 (0.9568)	0.0068 (0.9682)	1.0057 (0.926)	0.0065 (0.9676)
	III	18.991 (0.9676)	0.0127 (0.953)	14.1414 (0.9307)	0.0095 (0.9268)	36.015 (0.9476)	0.0033 (0.9312)	0.4436 (0.9486)	0.0034 (0.9421)	0.6129 (0.9577)	0.0034 (0.9709)
$(20,15)$	I	4.7392 (0.9673)	0.0032 (0.9253)	4.9634 (0.9671)	0.003 (0.9576)	1.8623 (0.953)	0.0015 (0.9345)	0.434 (0.96)	0.001 (0.9362)	0.4675 (0.9373)	0.0012 (0.9562)
	II	12.933 (0.9645)	0.008 (0.9673)	10.5979 (0.9375)	0.0058 (0.9502)	51.1296 (0.9523)	0.0022 (0.9741)	0.4007 (0.926)	0.0023 (0.941)	0.4346 (0.9633)	0.0025 (0.9442)
	III	16.0775 (0.9539)	0.01 (0.9292)	12.5219 (0.9464)	0.0077 (0.9675)	37.8186 (0.963)	0.0022 (0.97)	0.3714 (0.9341)	0.0031 (0.931)	0.408 (0.9605)	0.0028 (0.949)
$(30,20)$	I	10.1025 (0.9602)	0.0084 (0.9504)	9.0456 (0.9551)	0.0056 (0.9301)	27.1896 (0.9259)	0.0018 (0.9547)	0.3698 (0.9708)	0.0026 (0.9601)	0.3686 (0.9461)	0.0024 (0.9723)
	II	9.2107 (0.9305)	0.0058 (0.9251)	8.2765 (0.9593)	0.0061 (0.9364)	64.0026 (0.9499)	0.0019 (0.9687)	0.3549 (0.9412)	0.0023 (0.9289)	0.4618 (0.9523)	0.0022 (0.9495)
	III	16.9095 (0.9603)	0.0114 (0.9468)	14.7537 (0.9279)	0.0097 (0.9253)	78.1061 (0.9649)	0.0024 (0.9354)	0.4086 (0.9272)	0.0037 (0.967)	0.5387 (0.9749)	0.0032 (0.9633)

Table 9. *Cont.*

		θ		η		γ		$S(t)$		$h(t)$	
(n,m)	CS	MLE	$MCMC$	MLE	$MCMC$	MLE	$MCMC$	MLE	$MCMC$	MLE	$MCMC$
$(40,20)$	I	11.0281 (0.9487)	0.007 (0.9568)	9.151 (0.9525)	0.0063 (0.9511)	36.255 (0.9374)	0.0018 (0.9493)	0.3995 (0.9602)	0.0027 (0.9396)	0.3577 (0.9555)	0.0025 (0.9591)
	II	15.0009 (0.9709)	0.0106 (0.9469)	14.0093 (0.9721)	0.0084 (0.9281)	49.6802 (0.9748)	0.0023 (0.9332)	0.2979 (0.9492)	0.0033 (0.9629)	0.4564 (0.9531)	0.0033 (0.966)
	III	16.2131 (0.9422)	0.0106 (0.9384)	11.1727 (0.9699)	0.0072 (0.931)	20.4228 (0.9561)	0.0025 (0.9254)	0.2753 (0.9499)	0.0028 (0.9683)	0.4936 (0.9697)	0.0025 (0.9289)
$(50,20)$	I	8.4469 (0.9474)	0.0058 (0.9484)	6.5389 (0.9671)	0.0045 (0.9646)	3.003 (0.966)	0.0016 (0.9447)	0.3366 (0.9305)	0.0019 (0.9365)	0.3503 (0.9274)	0.0018 (0.9386)
	II	14.27 (0.9707)	0.0102 (0.9353)	13.3122 (0.9712)	0.0088 (0.927)	108.914 (0.9532)	0.0023 (0.9442)	0.3081 (0.9662)	0.0033 (0.9515)	0.5134 (0.9434)	0.0034 (0.937)
	III	14.3368 (0.9707)	0.0097 (0.9501)	8.7387 (0.9386)	0.0058 (0.9325)	31.4154 (0.94)	0.0042 (0.9449)	0.2689 (0.9519)	0.0024 (0.9269)	0.7921 (0.9728)	0.0024 (0.9656)

Simulation Results

From the following observations, we note the following:

1. According to Tables 4–8, as sample sizes increase, $MSEs$ become smaller, and BEs have the smallest $MSEs$ for the parameters $\theta, \eta, \gamma,\ S(t)$ and $h(t)$. Therefore, when all variables are considered, BEs outperform $MLEs$.
2. The estimates from Bayes for $\theta, \eta, \gamma,\ S(t)$, and $h(t)$ are better in that the $MSEs$ are smaller.
3. For smaller $MSEs$ with $c = -2.0$ and 0.0001, the $LINEX$ estimates with $c = 2.0$ are greater.
4. The performance of Scheme I is greater than Schemes II and III due to the smaller $MSEs$ for the failure time intervals of fixed-value samples n and m.
5. The results from the $CRIs$ are more perfect than those from the $ACIs$ for identified failures, approaches, and sample sizes, as shown in Table 9.

10. Conclusions

In the field of distribution theory, there is a continuous effort to generalize current distributions. The aim of this work is to develop more reliable and adaptable models that may be used in a variety of censoring contexts. Many different kinds of techniques are being investigated to attain this goal, as indicated by the enormous amount of literature. The reliability and efficiency of the distribution used in fitting the given data have a considerable impact on the subsequent analysis and empirical results. This work focuses on the problem of estimating unknown parameters in the formulation of an IPEP distribution using a Prog-II censoring strategy. Our strategy combines both classical and Bayesian approaches. We calculated approximation CIs and bootstrap CIs for the IPEP distribution's unknown parameters. Furthermore, we used MCMC with the MH technique to calculate BEs for both LFs, across their related HPD interval estimations. According to the simulation results, BEs with the LINEX loss function exceed all of the traditional estimates. We also determined the best censoring approach for life-testing experiments using three criteria schemes, which is an important consideration for reliability researchers. As an actual data application, we used the flood dataset for all estimations in our research study. Future studies might investigate statistics applied to the IPEP distribution. In future works, we will use Joint Prog-II censored data to estimate the IPEP distribution parameters and compare them to all other censored algorithms that we will use.

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Abbreviations

IPEP	inverse power exponentiated Pareto
WGE	Weibull generalized exponential
Prog-II	progressive type-II
Y	continuous random variable
$i.d.d$	independent, identically distributed

cdf	cumulative distribution function
pdf	probability distribution function
$S(y)$	reliability function
$h(y)$	hazard function
MLE	maximum likelihood estimation
BE	Bayes estimation
LFs	symmetric and asymmetric loss functions
CIs	confidence intervals
ACI	asymptotic confidence intervals
M-H	Metropolis–Hastings
MCMC	Markov chain Monte Carlo
MSE	mean square error
CRI	credible interval
SEL	squared error loss function
LINEX	linear exponential loss function
ACL	average confidence interval length
CP	coverage probability
R_i	survival unit at time i
$\nabla \hat{S}(y)$	gradient of $\hat{S}(Y)$
$\nabla \hat{h}(y)$	gradient of $\hat{h}(Y)$

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