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Article

Canonical Equations of Hamilton with Symmetry and Their Applications

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Abstract: Two systems of mathematical physics are defined by us, which are the first-order differential system (FODS) and the second-order differential system (SODS). Basing on the conventional Legendre transformation, we obtain a new kind of canonical equations of Hamilton (CEH) with some kind of symmetry. We show that the FODS can only be expressed by the new CEH, but do not by the conventional CEH, while the SODS can be done by both the new and the conventional CEHs, on basis of the same conventional Legendre transformation. As an example, we prove that the nonlinear Schrödinger equation can be expressed with the new CEH in a consistent way. Based on the new CEH, the approximate soliton solution of the nonlocal nonlinear Schrödinger equation is obtained, and the soliton stability is analysed analytically as well. Furthermore, because the symmetry of a system is closely connected with certain conservation theorem of the system, the new CEH may be useful in some complicated systems when the symmetry considerations are used.

Keywords: canonical equations of Hamilton; nonlinear Schrödinger equation

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1. Introduction

The Hamiltonian viewpoint offers a theoretical framework in lots of physical fields. In classical mechanics, it constitutes the basis for further developments, including the Hamilton-Jacobi theory, the perturbation approaches and the chaos [1]. The canonical equations of Hamilton (CEH) in the field of classical mechanics are expressed as [1]

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad -\dot{p}_i = \frac{\partial H}{\partial q_i} \quad (i = 1, \cdots, n) \tag{1}$$

where q_i and p_i are respectively the generalized coordinate and momentum, while $\dot{q}_i = dq_i/dt$ is the generalized velocity. The generalized momentum p_i is defined as $p_i = \frac{\partial L}{\partial \dot{q}_i}$ with L being the Lagrangian. By the Legendre transformation $H = \sum_{i=1}^{n} \dot{q}_i p_i - L$, the Hamiltonian H is then obtained.

The CEH (1) can be extended to the continuous system [1]

$$\dot{q}_s = \frac{\delta h}{\delta p_s}, \quad -\dot{p}_s = \frac{\delta h}{\delta q_s}, \quad (s = 1, \cdots, N) \tag{2}$$

where the subscript s indicates the components of the quantity of the continuous system [1], $\frac{\delta h}{\delta q_s} = \frac{\partial h}{\partial q_s} - \frac{\partial}{\partial x}\frac{\partial h}{\partial q_{s,x}}$ and $\frac{\delta h}{\delta p_s} = \frac{\partial h}{\partial p_s} - \frac{\partial}{\partial x}\frac{\partial h}{\partial p_{s,x}}$ denote the functional derivatives of h with respect to q_s and p_s with $q_{s,x} = \frac{\partial q_s}{\partial x}$ and $p_{s,x} = \frac{\partial p_s}{\partial x}$, q_s and p_s are respectively the generalized coordinate and momentum, and h is the Hamiltonian density. The generalized momentum p_s is defined as

$$p_s = \frac{\partial l}{\partial \dot{q}_s}. \tag{3}$$

The Hamiltonian density h is acquired by the following Legendre transformation

$$h = \sum_{s=1}^{N} \dot{q}_s p_s - l, \tag{4}$$

where l is the Lagrangian density. But it is significantly different for the continuous system that q_s and p_s are now the functions of both the time coordinate t and the spatial coordinate x. It should be noted that the spatial coordinate x is not the generalized coordinate, but is only the continuous index replacing the discrete i in Equation (1). To avoid confusion, we refer to time t as the evolution coordinate. h is a function of q_s, p_s and $q_{s,x}$ but not $p_{s,x}$ [1], so $\frac{\delta h}{\delta p_s} = \frac{\partial h}{\partial p_s}$, then the first equation of Equation (2) can be also expressed by

$$\dot{q}_s = \frac{\partial h}{\partial p_s}. \tag{5}$$

To our knowledge, the CEH in all current literatures are of the form (2), which are constructed on the basis of the second-order differential system (SODS). Besides the SODS, there are a lot of the first-order differential systems (FODS) to describe physical phenomena. Among them, the nonlinear Schrödinger equation (NLSE) is just the universal FODS. A question is raised in the nature of things: what is the form of the CEH applicable for the FODS? Are the conventional CEH, Equation (2), still valid for the FODS? In this paper, we gain a new CEH of the formal symmetry valid for the FODS, from which the NLSE can be expressed in a consistent manner. We also prove that the symmetric CEH is equivalent to the conventional CEH for the SODS. However, the conventional CEH can not model the FODS.

It is well known that the symmetry plays an important role in theoretical physics [2]. The search for and the discovery of new symmetries promote the exploration of fundamental laws of physics. Based on the idea, the CEH with the symmetry found by us, although this symmetry is only formal, might find their appropriate position in modern theoretical physics.

2. The CEH for the FODS

The Newton's second law, the base of the Hamiltonian formulation, is modeled by a second-order differential equation of the evolution coordinate (the time coordinate). Here, we define the system governed by second-order partial differential equations of evolution coordinates as the SODS. Similarly, the FODS is the system governed by the first-order partial differential equations of evolution coordinates. The Lagrangian density of the SODS of the continuous systems is expressed in general as [1]

$$l = \sum_{s=1}^{N} \sum_{k=1}^{N} A_{sk} \dot{q}_s \dot{q}_k + \sum_{s=1}^{N} B_s \dot{q}_s + C, \tag{6}$$

where A_{sk}, B_s, C depend on not only q_s but also $q_{s,x}$ in general. The generalized momentum can be obtained by the definition (3) as

$$p_s = \sum_{k=1}^{N} (A_{sk} + A_{ks}) \dot{q}_k + B_s, \tag{7}$$

which is the function of q_s, $\dot{q}_s$ and $q_{s,x}$. Equation (7) in fact have N equations and contain $4N$ variables, which are q_s, $\dot{q}_s$, p_s and $q_{s,x}$. So the degree of freedom of Equation (7) is $3N$. Then we take q_s, p_s and $q_{s,x}$ as independent variables, and express the generalized velocities $\dot{q}_s$ by these independent variables.

Besides the SODSs, there are a number of the FODSs. The nonlinear Schrödinger equation (NLSE)

$$i\frac{\partial\varphi}{\partial t}+\frac{1}{2}\frac{\partial^2\varphi}{\partial x^2}+|\varphi|^2\varphi=0 \tag{8}$$

is just a universal model that can be applied to hydrodynamics [3], nonlinear optics [4–6], nonlinear acoustics [7], Bose-Einstein condensates [8]. In nonlinear optics [4–6], the NLSE (8) governs the propagation of the slowly-varying light-envelope with the evolution coordinate t being the propagation direction coordinate. The light-envelope φ is a cw paraxial beam in a planar waveguide [5] or a narrow spectral-width pulse in optical fibers [4,6]. x is a transverse space coordinate for the beam and a frame moving at the group velocity (the so-called retarded frame) for the pulse, respectively.

For the FODS, the Lagrangian density should be a linear function of the generalized velocities $\dot{q}_s$. If the Lagrangian density is a quadratic function of the generalized velocities like Equation (6), the equation of motion, i.e., the Euler-Lagrange equation

$$\frac{\partial}{\partial t}\frac{\partial l}{\partial\dot{q}_s}-\frac{\delta l}{\delta q_s}=0 \tag{9}$$

will be the second-order partial differential equation of the evolution coordinate t, which is in contradiction with the definition of the FODS. Therefore, the Lagrangian density of the FODS can only be expressed as

$$l=\sum_{s=1}^{N}F_s(q_s)\dot{q}_s+Q(q_s,q_{s,x}). \tag{10}$$

Besides, F_s in Equation (10) is not the function of $q_{s,x}$. If F_s is also the function of $q_{s,x}$, there will be such terms as $q_{s,x}\dot{q}_s$ appearing in Equation (10). Substitution of Equation (10) into Equation (9) leads to the appearance of the mixed partial derivative terms $\frac{\partial^2 q_s}{\partial x\partial t}$. Via the coordinate rotation transform, terms $\frac{\partial^2 q_s}{\partial t^2}$ and $\frac{\partial^2 q_s}{\partial x^2}$ will appear instead. Therefore, the Euler-Lagrange equation (9) expressed with the canonical form of the second-order partial differential equation [9] is the second-order partial differential equation about the evolution coordinate t. According to our definition, the system is the SODS. Consequently, the generalized momentum p_s, obtained by definition (3)

$$p_s=F_s(q_s), \tag{11}$$

is only a function of q_s. This is of significant difference from the case of the SODS, where the generalized momentum p_s is the function of not only q_s, but also $\dot{q}_s$ and $q_{s,x}$, as shown in Equation (7). Equation (11) have N equations, but contain $2N$ variables, q_s and p_s. Therefore, the degree of freedom of the system described by Equation (11) is N. We can take $q_1,\cdots,q_\nu$ and $p_1,\cdots,p_\mu$ as the independent variables, where $\nu+\mu=N$. The rest of generalized coordinates and momenta can be expressed by the independent variables $q_\alpha=f_\alpha(q_1,\cdots,q_\nu,p_1,\cdots,p_\mu)(\alpha=\nu+1,\cdots,N)$, and $p_\beta=g_\beta(q_1,\cdots,q_\nu,p_1,\cdots,p_\mu)$ $(\beta=\mu+1,\cdots,N)$.

We now derive the CEH for the FODS. The total differential of the Hamiltonian density h can be obtained by using Equation (4)

$$dh=\sum_{s=1}^{N}p_s d\dot{q}_s+\sum_{\eta=1}^{\mu}\dot{q}_\eta dp_\eta+\sum_{\beta=\mu+1}^{N}\dot{q}_\beta\left(\sum_{\lambda=1}^{\nu}\frac{\partial g_\beta}{\partial q_\lambda}dq_\lambda+\sum_{\eta=1}^{\mu}\frac{\partial g_\beta}{\partial p_\eta}dp_\eta\right)-dl, \tag{12}$$

while the total differential of the Lagrangian density $l(q_s,\dot{q}_s,q_{s,x})$ with respect to its arguments is

$$dl=\sum_{\lambda=1}^{\nu}\frac{\partial l}{\partial q_\lambda}dq_\lambda+\sum_{\alpha=\nu+1}^{N}\frac{\partial l}{\partial q_\alpha}\left(\sum_{\lambda=1}^{\nu}\frac{\partial f_\alpha}{\partial q_\lambda}dq_\lambda+\sum_{\eta=1}^{\mu}\frac{\partial f_\alpha}{\partial p_\eta}dp_\eta\right)+\sum_{s=1}^{N}\frac{\partial l}{\partial\dot{q}_s}d\dot{q}_s+\sum_{s=1}^{N}\frac{\partial l}{\partial q_{s,x}}dq_{s,x}. \tag{13}$$

Substitution of Equation (13) into Equation (12) yields

$$\begin{aligned} dh = & \sum_{\lambda=1}^{\nu}\left(\sum_{\beta=\mu+1}^{N}\dot{q}_\beta\frac{\partial g_\beta}{\partial q_\lambda}-\sum_{\alpha=\nu+1}^{N}\frac{\partial l}{\partial q_\alpha}\frac{\partial f_\alpha}{\partial q_\lambda}-\frac{\partial l}{\partial q_\lambda}\right)dq_\lambda \\ & +\sum_{\eta=1}^{\mu}\left(\dot{q}_\eta+\sum_{\beta=\mu+1}^{N}\dot{q}_\beta\frac{\partial g_\beta}{\partial p_\eta}-\sum_{\alpha=\nu+1}^{N}\frac{\partial l}{\partial q_\alpha}\frac{\partial f_\alpha}{\partial p_\eta}\right)dp_\eta-\sum_{s=1}^{N}\frac{\partial l}{\partial q_{s,x}}dq_{s,x}. \end{aligned} \tag{14}$$

Since the total differential of $h(q_1,\cdots,q_\nu,p_1,\cdots,p_\mu,q_{s,x})$ with respect to its arguments can be written as $dh=\sum_{\lambda=1}^{\nu}\frac{\partial h}{\partial q_\lambda}dq_\lambda+\sum_{\eta=1}^{\mu}\frac{\partial h}{\partial p_\eta}dp_\eta+\sum_{s=1}^{N}\frac{\partial h}{\partial q_{s,x}}dq_{s,x}$, by comparing this equation with Equation (14), we obtain $2N$ equations

$$\frac{\partial h}{\partial q_\lambda} = \sum_{\beta=\mu+1}^{N}\dot{q}_\beta\frac{\partial g_\beta}{\partial q_\lambda}-\sum_{\alpha=\nu+1}^{N}\frac{\partial l}{\partial q_\alpha}\frac{\partial f_\alpha}{\partial q_\lambda}-\frac{\partial l}{\partial q_\lambda}, \tag{15}$$

$$\frac{\partial h}{\partial p_\eta} = \dot{q}_\eta+\sum_{\beta=\mu+1}^{N}\dot{q}_\beta\frac{\partial g_\beta}{\partial p_\eta}-\sum_{\alpha=\nu+1}^{N}\frac{\partial l}{\partial q_\alpha}\frac{\partial f_\alpha}{\partial p_\eta}, \tag{16}$$

$$\frac{\partial h}{\partial q_{s,x}} = -\frac{\partial l}{\partial q_{s,x}}, \tag{17}$$

where $\lambda=1,\cdots,\nu,\eta=1,\cdots,\mu,s=1,\cdots,N$. From the Euler-Lagrange Equation (9), we obtain $\frac{\partial l}{\partial q_s}=\frac{\partial}{\partial t}\frac{\partial l}{\partial \dot{q}_s}+\frac{\partial}{\partial x}\frac{\partial l}{\partial q_{s,x}}$, substitution of which into Equations (15) and (16) yields

$$\frac{\partial h}{\partial q_\lambda}=-\dot{p}_\lambda+\sum_{\beta=\mu+1}^{N}\dot{q}_\beta\frac{\partial g_\beta}{\partial q_\lambda}-\sum_{\alpha=\nu+1}^{N}\dot{p}_\alpha\frac{\partial f_\alpha}{\partial q_\lambda}-\sum_{\alpha=\nu+1}^{N}\frac{\partial}{\partial x}\frac{\partial l}{\partial q_{\alpha,x}}\frac{\partial f_\alpha}{\partial q_\lambda}-\frac{\partial}{\partial x}\frac{\partial l}{\partial q_{\lambda,x}}, \tag{18}$$

$$\frac{\partial h}{\partial p_\eta}=\dot{q}_\eta+\sum_{\beta=\mu+1}^{N}\dot{q}_\beta\frac{\partial g_\beta}{\partial p_\eta}-\sum_{\alpha=\nu+1}^{N}\dot{p}_\alpha\frac{\partial f_\alpha}{\partial p_\eta}-\sum_{\alpha=\nu+1}^{N}\frac{\partial}{\partial x}\frac{\partial l}{\partial q_{\alpha,x}}\frac{\partial f_\alpha}{\partial p_\eta}. \tag{19}$$

Then substituting Equation (17) into Equations (18) and (19), we obtain N CEHs for the FODS

$$\frac{\delta h}{\delta q_\lambda}=-\dot{p}_\lambda+\sum_{\beta=\mu+1}^{N}\dot{q}_\beta\frac{\partial g_\beta}{\partial q_\lambda}-\sum_{\alpha=\nu+1}^{N}\dot{p}_\alpha\frac{\partial f_\alpha}{\partial q_\lambda}+\sum_{\alpha=\nu+1}^{N}\frac{\partial}{\partial x}\frac{\partial h}{\partial q_{\alpha,x}}\frac{\partial f_\alpha}{\partial q_\lambda}, \tag{20}$$

$$\frac{\delta h}{\delta p_\eta}=\dot{q}_\eta+\sum_{\beta=\mu+1}^{N}\dot{q}_\beta\frac{\partial g_\beta}{\partial p_\eta}-\sum_{\alpha=\nu+1}^{N}\dot{p}_\alpha\frac{\partial f_\alpha}{\partial p_\eta}+\sum_{\alpha=\nu+1}^{N}\frac{\partial}{\partial x}\frac{\partial h}{\partial q_{\alpha,x}}\frac{\partial f_\alpha}{\partial p_\eta}. \tag{21}$$

To obtain Equation (21), we have used $\frac{\delta h}{\delta p_\eta}=\frac{\partial h}{\partial p_\eta}$, because h is not a function of $p_{\eta,x}$. The CEH, Equations (20) and (21), can be expressed in a symmetric form as

$$\frac{\delta h}{\delta q_\lambda}=\sum_{s=1}^{N}\left(\dot{q}_s\frac{\partial p_s}{\partial q_\lambda}-\dot{p}_s\frac{\partial q_s}{\partial q_\lambda}\right)+\sum_{\alpha=\nu+1}^{N}\frac{\partial}{\partial x}\frac{\partial h}{\partial q_{\alpha,x}}\frac{\partial f_\alpha}{\partial q_\lambda}, \tag{22}$$

$$\frac{\delta h}{\delta p_\eta}=\sum_{s=1}^{N}\left(\dot{q}_s\frac{\partial p_s}{\partial p_\eta}-\dot{p}_s\frac{\partial q_s}{\partial p_\eta}\right)+\sum_{\alpha=\nu+1}^{N}\frac{\partial}{\partial x}\frac{\partial h}{\partial q_{\alpha,x}}\frac{\partial f_\alpha}{\partial p_\eta} \tag{23}$$

$(\lambda=1,\cdots,\nu,\eta=1,\cdots,\mu$, and $\nu+\mu=N)$, because $\dot{p}_\lambda=\sum_{\lambda'=1}^{\nu}\dot{p}_{\lambda'}\frac{\partial q_{\lambda'}}{\partial q_\lambda}$, $\dot{q}_\eta=\sum_{\eta'=1}^{\mu}\dot{q}_{\eta'}\frac{\partial p_{\eta'}}{\partial p_\eta}$, $\sum_{\eta=1}^{\mu}\dot{q}_\eta\frac{\partial p_\eta}{\partial q_\lambda}=0$, and $\sum_{\lambda=1}^{\nu}\dot{p}_\lambda\frac{\partial q_\lambda}{\partial p_\eta}=0$. The CEH above can be extended to the discrete system

$$\frac{\partial H}{\partial q_\lambda}=\sum_{s=1}^{N}\left(\dot{q}_s\frac{\partial p_s}{\partial q_\lambda}-\dot{p}_s\frac{\partial q_s}{\partial q_\lambda}\right),\quad \frac{\partial H}{\partial p_\eta}=\sum_{s=1}^{N}\left(\dot{q}_s\frac{\partial p_s}{\partial p_\eta}-\dot{p}_s\frac{\partial q_s}{\partial p_\eta}\right), \tag{24}$$

where $\lambda=1,\cdots,\nu,\eta=1,\cdots,\mu$, and $\nu+\mu=N$.

We will prove that the symmetric CEH (22) and (23) can also describe the SODS. In Equations (22) and (23), if all the generalized coordinates q_s and momenta p_s are independent, we can obtain that $\frac{\partial g_\beta}{\partial q_\lambda} = \frac{\partial f_\alpha}{\partial q_\lambda} = \frac{\partial g_\beta}{\partial p_\eta} = \frac{\partial f_\alpha}{\partial p_\eta} = 0$ $(\alpha = \nu + 1, \cdots, N, \beta = \mu + 1, \cdots, N)$, then the CEH will be reduced to Equation (2). It does be the case of the SODS, where all the generalized coordinates and momenta are independent. So, the symmetric CEH obtained by us can express both the FODS and the SODS. In other words, the new CEH and the conventional CEH are equivalent when describing the SODS, but the former are of some formally symmetry. The conventional CEH, Equation (2), can only be used to expresses the SODS.

3. Application of the Symmetric CEH for Continuous Systems to the NLSE

In this section, we will apply the new CEH with symmetry, Equations (22) and (23), to the NLSE. The Lagrangian density for the NLSE is stated as [10] $l = -\frac{i}{2}(\varphi^* \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \varphi^*}{\partial t}) + \frac{1}{2}|\frac{\partial \varphi}{\partial x}|^2 - \frac{1}{2}|\varphi|^4$. The NLSE is complex, so it is in fact an equation of two real functions, one is of its real part and the other is of its imaginary part. Alternatively, we can take the fields φ and φ^* as two independent functions. In this sense, the components of the quantity N for the NLSE is equal to two. In other words, for the NLSE there are two generalized coordinates, $q_1 = \varphi^*$ and $q_2 = \varphi$, and two generalized momenta

$$p_1 = \frac{i}{2}\varphi, p_2 = -\frac{i}{2}\varphi^*. \tag{25}$$

By using Equation (4) [4], we can obtain the Hamiltonian density

$$h = -\frac{1}{2}\left|\frac{\partial \varphi}{\partial x}\right|^2 + \frac{1}{2}|\varphi|^4. \tag{26}$$

If we take q_1 and p_1 as the independent variables, q_2 and p_2 can be expressed by the relations (25) as $q_2 = -2ip_1$ and $p_2 = -\frac{i}{2}q_1$, respectively. It also should be noted that h is also the function of $q_{s,x}$. Then, we can express the Hamiltonian density (26) with independent variables $q_1, p_1, q_{1,x}$ and $q_{2,x}$ as

$$h = -\frac{1}{2}q_{1,x}q_{2,x} - 2q_1^2 p_1^2. \tag{27}$$

For the NLSE, $\nu = \mu = 1$ and $N = 2$, therefore the Equation (22) in fact have only one equation. It is the same case for Equation (23). Consequently, for the NLSE, the CEH (22) and (23) can produce two equations. From the left side of Equation (22), we can obtain $\frac{\delta h}{\delta q_1} = \frac{1}{2}\frac{\partial^2 \varphi}{\partial x^2} + |\varphi|^2\varphi$. From its right side, we can have $-\dot{p}_1 + \dot{q}_2 \frac{\partial p_2}{\partial q_1} = -i\dot{\varphi}$. Then the NLSE (8) is obtained. While, for the other CEH (23), the left side is

$$\frac{\delta h}{\delta p_2} = -4q_1^2 p_1 = -2i|\varphi|^2\varphi^*, \tag{28}$$

and the right side is $\dot{q}_1 - \dot{p}_2 \frac{\partial q_2}{\partial p_1} + \frac{\partial}{\partial x}\frac{\partial h}{\partial q_{2,x}}\frac{\partial q_2}{\partial p_1} = 2\dot{\varphi}^* + i\frac{\partial^2 \varphi^*}{\partial x^2}$, which results in the generation of the complex conjugate of the NLSE. As a result, the CEHs (22) and (23) are consistent. From one of the two CEHs, the NLSE is expressed; from the other, the complex conjugate of the NLSE is expressed.

We now demonstrate that the conventional CEH (2) can not be used to express the NLSE. By Equation (3), we have $p_\varphi = \partial l / \partial \dot{\varphi} = -i/2\varphi^*$. Substituting the Hamiltonian density (26) into the second equation of Equation (2) only yields $\frac{i}{2}\frac{\partial \varphi}{\partial t} + \frac{1}{2}\nabla_\perp^2 \varphi + |\varphi|^2\varphi = 0$, which does not be the NLSE (8). By substituting the Hamiltonian density (26) into Equation (5), we can obtain the left side $\frac{\partial \varphi^*}{\partial t}$, and the right side $\frac{\partial h}{\partial p_{\varphi^*}} = \frac{\partial h}{\partial \varphi}\frac{\partial \varphi}{\partial p_{\varphi^*}} = -2i|\varphi|^2\varphi^*$, where

$p_{\varphi^*} = \frac{\partial l}{\partial \dot{\varphi}^*}$. Then the equation $-\frac{i}{2}\frac{\partial \varphi^*}{\partial t} + |\varphi|^2\varphi^* = 0$ can be obtained. It is surely not the complex conjugate of the NLSE.

4. Application of the Symmetric CEH for Discrete Systems to Light-Envelope Propagations

In this section, we use the symmetric CEH for discrete systems, i.e., Equation (24), to discuss light-envelope propagations in nonlocally nonlinear media, which is modeled by the following (1+D)-dimensional nonlocal nonlinear Schrödinger equation (NNLSE) [11–14]

$$i\frac{\partial \varphi}{\partial z} + \nabla_\perp^2 \varphi + \Delta n \varphi = 0. \quad (29)$$

$\Delta n(\mathbf{r}, z)$ is the nonlinear refractive index, which can phenomenologically be expressed as a convolution between the response function $R(\mathbf{r})$ and the light intensity

$$\Delta n(\mathbf{r}, z) = \int_{-\infty}^{\infty} R(\mathbf{r} - \mathbf{r}')|\varphi(\mathbf{r}', z)|^2 \mathrm{d}^D\mathbf{r}'. \quad (30)$$

When R is the Dirac delta function, the NNLSE (29) is reduced to the NLSE (8).

We assume the trial solution of Equation (29)

$$\varphi(r, z) = q_A(z) \exp\left[-\frac{r^2}{q_w^2(z)}\right] \exp\left[iq_c(z)r^2 + iq_\theta(z)\right], \quad (31)$$

where q_A, q_θ are the amplitude and phase of the complex amplitude φ, respectively, q_w is the width, q_c is the phase-front curvature, and they all vary with z. We consider the response function

$$R(\mathbf{r}) = \frac{1}{(\sqrt{\pi}w_m)^D} \exp\left(-\frac{r^2}{w_m^2}\right). \quad (32)$$

Substituting the trial solution (31) into the Lagrangian density $l = \frac{i}{2}\left(\varphi^*\frac{\partial \varphi}{\partial z} - \varphi\frac{\partial \varphi^*}{\partial z}\right) - |\nabla_\perp \varphi|^2 + \frac{1}{2}|\varphi(\mathbf{r}, z)|^2 \Delta n(\mathbf{r}, z)$, and performing the integration $L = \int_{-\infty}^{\infty} l d^D\mathbf{r}$, we have

$$\begin{aligned} L \;=\; & -2^{-2-D}\pi^{D/2}q_A^2 q_w^{-2+D}(w_m^2 + q_w^2)^{-D/2}[-2q_A^2 q_w^{2+D} + 2^{D/2}(w_m^2 + q_w^2)^{D/2}(4D \\ & +4Dq_c^2 q_w^4 + Dq_w^4 \dot{q}_c + 4q_w^2 \dot{q}_\theta)], \end{aligned} \quad (33)$$

which is the function of generalized coordinates q_A, q_w, q_c and velocities $\dot{q}_c, \dot{q}_\theta$.

Then the generalized momenta can be obtained

$$p_A = p_w = 0, p_c = -2^{-2-\frac{D}{2}} D\pi^{D/2} q_A^2 q_w^{2+D}, p_\theta = -\left(\frac{\pi}{2}\right)^{D/2} q_A^2 q_w^D. \quad (34)$$

By Legendre transformation, the Hamiltonian can be determined

$$H = 2^{-1-D}\pi^{D/2}q_A^2 q_w^{-2+D}(w_m^2 + q_w^2)^{-D/2}[-q_A^2 q_w^{2+D} + 2^{1+\frac{D}{2}} D(w_m^2 + q_w^2)^{D/2}(1 + q_c^2 q_w^4)], \quad (35)$$

which will be proved to be a constant, that is $\dot{H} = 0$.

Four generalized coordinates and four generalized momenta are contained in the four equations (34). It indicates that Equation (34) have four degrees of freedom. Here, we can take q_c, q_θ, p_c and p_θ as independent variables. By solving Equation (34), we can express the generalized coordinates q_A and q_w by generalized momenta p_c and p_θ as $q_A = (-p_\theta)^{1/2}[Dp_\theta/(2\pi p_c)]^{D/4}$ and $q_w = [4p_c/(Dp_\theta)]^{1/2}$. Then, the Hamiltonian (35) is rewritten as

$$H = -\frac{D^2 p_\theta^2 + 16p_c^2 q_c^2}{4p_c} - \frac{1}{2}\pi^{-D/2}(\frac{4p_c}{Dp_\theta} + w_m^2)^{-D/2}. \quad (36)$$

By using the CEH (24) with $\mu = \nu = 2$ and $n = 4$, we can obtain four equations as

$$\dot{q}_c = \frac{D^2 p_\theta^2}{4p_c^2} - 4q_c^2 + \frac{D\pi^{-D/2} p_\theta^2 (\frac{4p_c}{Dp_\theta} + w_m^2)^{-D/2}}{4p_c + Dp_\theta w_m^2}, \tag{37}$$

$$\begin{aligned} \dot{q}_\theta = & -\frac{(4+D)\pi^{-D/2} p_c p_\theta (\frac{4p_c}{Dp_\theta} + w_m^2)^{-D/2}}{4p_c + Dp_\theta w_m^2} \\ & -\frac{D^2 p_\theta}{2p_c} - \frac{D\pi^{-D/2} p_\theta^2 w_m^2 (\frac{4p_c}{Dp_\theta} + w_m^2)^{-D/2}}{4p_c + Dp_\theta w_m^2}, \end{aligned} \tag{38}$$

$$\dot{p}_c = 8p_c q_c, \tag{39}$$

$$\dot{p}_\theta = 0. \tag{40}$$

One can find that the generalized coordinate q_θ does not appear in the Hamiltonian (36), therefore q_θ is a cyclic coordinate. Because the generalized momentum conjugate to a cyclic coordinate is conserved [1], the generalized momentum p_θ conjugate to the generalized coordinate q_θ is a constant, which can be confirmed by Equation (40). In fact, this represents that the power $P_0 = \int_{-\infty}^{\infty} |\varphi|^2 d^D\mathbf{r} = q_A^2 (\sqrt{\pi/2} q_w)^D$ is conservative. Then we can obtain

$$q_A^2 = P_0 (\sqrt{\pi/2} q_w)^{-D}. \tag{41}$$

Taking derivative with respect to z on two sides of the third equation of (34), then comparing it with Equation (39) we obtain $q_c = \frac{\dot{q}_w}{4q_w}$, the substitution of which into the Hamiltonian (35) yields $H = T + V$, where

$$T = \frac{1}{16} D P_0 \dot{q}_w^2, V = \frac{DP_0}{{q_w}^2} - \frac{1}{2}\pi^{-D/2} P_0^2 \left(w_m^2 + q_w^2\right)^{-D/2} \tag{42}$$

are the generalized kinetic energy and potential, respectively.

From the Hamiltonian point of view, the dynamics of light-envelopes in nonlinear media can be regarded as a problem of small oscillations of a Hamiltonian system about its equilibrium position. The equilibrium state of the system described by the Hamiltonian H corresponds to the soliton solutions of the NNLSE, which can be gained as the extremum points of generalized potential V. The equilibrium position is stable if a small disturbance from equilibrium leads to small bounded motion about the rest position. While, if an infinitesimal disturbance produces unbounded motion, the equilibrium is unstable [1]. In other words, the equilibrium must be stable when the extremum of the generalized potential is a minimum, and unstable otherwise. In this sense, therefore, the viewpoint in a few literatures [15–18], where solitons were taken for the extremum of the Hamiltonian rather than the generalized potential, might be somewhat ambiguous. In these literatures [15–18] the trial solution has an invariable profile (soliton profile), and the soliton state is the static state in fact. In this case, the kinetic energy is zero, and the Hamiltonian is just equal to the potential. In this connection, for the static system the extrema of the Hamiltonian and the generalized potential are equal only in value. Although in such literatures [15–18] the obtained soliton solutions are correct, it is more reasonable to regard the soliton solutions as the extremum points of the generalized potentials rather than the Hamiltonian.

To find the equilibrium state (soliton solution), we let $\partial V/\partial q_w = 0$, then we obtain $-\frac{32}{q_w^3} + 8\pi^{-D/2} P_0 q_w \left(w_m^2 + q_w^2\right)^{-1-\frac{D}{2}} = 0$. The critical power is then obtained as

$$P_c = \frac{4\pi^{D/2} \left(w_m^2 + q_w^2\right)^{1+\frac{D}{2}}}{q_w^4}, \tag{43}$$

with which the light can propagate with a changeless profile. Besides, $P_0 = P_c$ also leads to $\dot{q}_c = q_c = 0$, which indicates that the soliton wavefront is a plane.

Then we address the soliton stability by discussing the generalized potential V. Performing the second-order derivative of V with respect to q_w, and substituting P_c into it, we have

$$\Upsilon \equiv \frac{\partial^2 V}{\partial q_w^2}\bigg|_{P_0=P_c} = \frac{64}{q_w^4}\left[2 - \frac{2+D}{2(1+\sigma^2)}\right], \tag{44}$$

where $\sigma = w_m/q_w$ represents the degree of nonlocality. When $\Upsilon > 0$, generalized potential V has a minimum, then the soliton is stable. From Equation (44) we can get the stability criterion of solitons

$$\sigma^2 > \frac{1}{4}(D-2), \tag{45}$$

which also agrees with the Vakhitov-Kolokolov criterion [19].

4.1. The Local Case

When $w_m \to 0$, $R(\mathbf{r}) \to \delta(\mathbf{r})$, and the NNLSE is reduced to the NLSE (8). Then, Equations (43) and (44) will be reduced to $P_c = 4\pi^{D/2}q_w^{D-2}$, $\Upsilon = \frac{32}{q_w^4}(2-D)$. In the case of $D=1$, $P_c = 4\sqrt{\pi}/q_w$, it is the same as Equation (42) of Ref. [10]. In the case of $D=2$, $P_c = 4\pi$, it is the same as Equation (16a) of Ref. [20]. We can obtain $\Upsilon > 0$ if $D < 2$, $\Upsilon < 0$ if $D > 2$, and $\Upsilon = 0$ if $D = 2$. Therefore, in the local case, the soliton is stable if $D = 1$, but unstable if $D > 2$. In the case of $D = 2$, it needs further analysis because $\Upsilon = 0$. In the case of $D = 2$, the generalized potential $V = (4\pi - P_0)P_0/2\pi q_w^2$, which does not have extreme when $P_0 \neq 4\pi$. When $P_0 = P_c = 4\pi$, we obtain that $V = 0$. It is the extreme rather than the minimum. Hence, (1+2)-dimensional local solitons are always unstable. When $P_0 = P_c$, the potential $V = 0$ is constant, the light-envelope without the external disturbance will stay in its initial state. But the ideal condition can not occur in experiment. If the external disturbance makes $P_0 > P_c$, the beam will become narrower and narrower, and the optical beam will eventually collapse. If the external disturbance makes $P_0 < P_c$, the optical beam will diffract at last. These conclusions all agree with those of Refs. [21–23].

4.2. The Nonlocal Case

When $w_m \neq 0$ and $D \leq 2$, the condition (45) is satisfied automatically. It means that the (1+1)-dimensional and the (1+2)-dimensional solitons are always stable in nonlocally nonlinear media of Gaussian response. It is consistent with the conclusion of Ref. [24]. When $D > 2$ the stabe solitons can exist when the degree of nonlocality should be strong enough that satisfies the criterion (45), which is also the same as Ref. [24].

5. Conclusions

We obtain a new CEH with symmetry in form. It can express both the FODS and the SODS, while the conventional CEH can only express the SODS but impossibly express the FODS on basis of the same conventional Legendre transformation. By using the new CEH we analytically obtain the approximate solution of the NNLSE and analytically discuss the stability of the solitons. We will use the new CEH to analytically deal with two important problems related to solitons in diverse nonlinear systems, that is, the existence and the stability, which will save amounts of time for the numerical explorations.

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