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Uniqueness of Finite Exceptional Orthogonal Polynomial Sequences Composed of Wronskian Transforms of Romanovski-Routh Polynomials

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Abstract: This paper exploits two remarkable features of the translationally form-invariant (TFI) canonical Sturm–Liouville equation (CSLE) transfigured by Liouville transformation into the Schrödinger equation with the shape-invariant Gendenshtein (Scarf II) potential. First, the Darboux–Crum net of rationally extended Gendenshtein potentials can be specified by a single series of Maya diagrams. Second, the exponent differences for the poles of the CSLE in the finite plane are energy-independent. The cornerstone of the presented analysis is the reformulation of the conventional supersymmetric (SUSY) quantum mechanics of exactly solvable rational potentials in terms of 'generalized Darboux transformations' of canonical Sturm–Liouville equations introduced by Rudyak and Zakhariev at the end of the last century. It has been proven by the author that the first feature assures that all the eigenfunctions of the TFI CSLE are expressible in terms of Wronskians of seed solutions of the same type, while the second feature makes it possible to represent each of the mentioned Wronskians as a weighted Wronskian of Routh polynomials. It is shown that the numerators of the polynomial fractions in question form the exceptional orthogonal polynomial (EOP) sequences composed of Wronskian transforms of the given finite set of Romanovski–Routh polynomials excluding their juxtaposed pairs, which have already been used as seed polynomials.

Keywords: canonical Sturm–Liouville equation; Liouville transformation; shape-invariant potential; Darboux–Crum transformations; Maya diagrams; polynomial Wronskians; Routh polynomials; Romanovski–Routh polynomials

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1. Introduction

In a recent work [1], the author introduced the concept of the translationally form-invariant (TFI) rational canonical Sturm–Liouville equation (RCSLE) converted by Liouville transformation [2–4] to the Schrödinger equation with a translationally shape-invariant (TSI) Liouville potential [5,6]. It is essential that all the eigenfunctions have a 'quasi-rational' [7] form, being expressible either in terms of classical orthogonal (Jacobi or Laguerre) polynomials [8,9] or via Romanovski/pseudo-Jacobi polynomials in Lesky's terms [10,11] (simply referred to as Romanovski polynomials [12] in [13–17]) with degree-dependent indexes in most cases. (Keeping in mind that Lesky was simply unaware of Routh's revolutionary treatise [18], we [19–21] prefer to term the aforementioned finite orthogonal subset of Routh polynomials as 'Romanovski–Routh' (R-Routh) polynomials.) We refer to these three families of rational CSLEs (RCSLEs) as 'Jacobi-reference' ($\mathfrak{J}$Ref), 'Laguerre-reference' ($\mathfrak{L}$Ref), and 'Routh-reference' ($\mathfrak{R}$Ref) respectively. The CSLEs from the same family share the same reference polynomial fraction (RefPF) combined with different density functions.

It was shown in [1] that the RCSLEs under consideration can be divided into two groups (A and B) similarly to the classification scheme suggested by Odake and Sasaki [22,23] for rational TSI potentials. (While combining the CSLEs into the two groups is unique, it was found that this is generally not true for the TSI potentials and that two TSI potentials exactly solvable in terms of hypergeometric or confluent hypergeometric functions may be included into both groups depending on their rational representation.) If the density function for the $\mathfrak{J}$Ref, $\mathfrak{L}$Ref, or $\mathfrak{R}$Ref CSLE requires only simple poles in the finite plane, then the resultant CSLE can be converted by gauge transformation to the three real reductions of the complex Bochner-type differential equations [24] with polynomial solutions forming Jacobi, Laguerre and Routh (twisted Jacobi [25], or pseudo-Jacobi [26]) differential polynomial systems [27,28].

The common feature of these CSLEs is that they can be converted to the so-called [29] 'prime' form such that their eigenfunctions obey the Dirichlet boundary conditions (DBCs) at the ends of the quantization interval. One can then take advantage of powerful theorems proven in [30] for zeros of principal solutions of SLEs solved under the DBCs at singular ends.

The next important development was the reformulation of the conventional supersymmetric (SUSY) theory of exactly solvable rational potentials in terms of the so-called [31] 'generalized Darboux transformations' (GDTs) introduced by Rudyak and Zakhariev [32] at the end of the last century. Since various authors give completely different meaning to the latter term, we (for the reason scrupulously explained in Section 2) prefer to refer to the mentioned operations as Liouville–Darboux transformations' (LDTs).

In a sharp contrast with Quesne's breakthrough paper [17] starting from a rational SUSY partner of the Scarf II potential, then converting the corresponding Schrödinger equation to the RCSLE, we directly apply a rational LDT (RLDT) to the $\mathfrak{R}$Ref CSLE and then convert the resultant RCSLE to its prime form. We then prove that the rational Liouville–Darboux transforms (RLD$\mathfrak{T}$s) of the eigenfunctions of the prime $\mathfrak{R}$Ref SLE obey the DBCs and thereby represent the eigenfunctions of the transformed RSLE. Moreover, it is proven that the new SLE may not have any other eigenfunctions, which implies that the Dirichlet problem in question is exactly solvable. This important result is commonly taken for granted in conventional SUSY quantum mechanics [5,6].

The concept of the translational form-invariance of RCSLEs is based on the existence of the so-called [1] 'basic solutions' such that their analytical continuations into the complex plane remain finite in any regular point. The RCSLE is referred to as TFI iff the LDT using one of these basic solutions as the transformation function (TF) simply shifts each of the translational parameters by one.

Both $\mathfrak{J}$Ref and $\mathfrak{L}$Ref CSLEs with simple-pole density functions have a quartet of basic solutions, and as a result, their rational Darboux–Crum [33,34] transforms (RDC$\mathfrak{T}$s) are specified by two series of Maya diagrams [35], so the corresponding exceptional DPSs (X-DPSs) are formed by pseudo-Wronskians of Jacobi or Laguerre polynomials with the same absolute values of the polynomial indexes (as well as with the same absolute value of the argument in the latter case). (In following [36–38], we term the given DPS 'exceptional' if it either does not start from a constant or lacks the first-order polynomials and thereby does not obey the prerequisites of the Bochner theorem [24].) We direct the reader to the recent review article by Durán [39] for a detailed discussion of this non-trivial issue, as well as for relevant references.

On other hand, the $\mathfrak{R}$Ref CSLE with the simple-pole density function has only two basic solutions and as a result its RDC$\mathfrak{T}$s can be specified by a single series of Maya diagrams [1]. As proven by the author [1], these RDC$\mathfrak{T}$s can be constructed using only Wronskians of Routh polynomials. If the polynomial Wronskian in question does not have real zeros, then the eigenfunctions of the transformed RCSLE are expressible in terms of a *finite* exceptional orthogonal polynomial (EOP) sequence in Quesne's terms [17].

While the exceptional orthogonal polynomial systems (X-OPSs) have attracted the broad attention of both mathematicians and physicists (see, e.g., [36–40] and the references therein), nearly all of the cited works overlook the revolutionary discovery [23] of the Darboux–Crum nets of rational potentials composed of RDC$\mathfrak{T}$s of the three TSI potentials:

i. Hyperbolic Pöschl–Teller (h-PT) potential [41];
ii. Gendenshtein potential [42,43] (Scarf II potential in the classification scheme of Cooper et al. [5,6];
iii. Morse potential [44].

As it has been pointed out by the author [1], the corresponding eigenfunctions for these three families of solvable rational potentials are expressible in terms of the RDC$\mathfrak{T}$s of the Romanovski–Jacobi (R-Jacobi), already mentioned R-Routh, and Romanovski–Bessel (R-Bessel) polynomials.

Since the finite EOP sequences formed by the RDC$\mathfrak{T}$s of the R-Jacobi polynomials constitute some orthogonal subsets of X-Jacobi DPSs, they are generally formed by pseudo-Wronskians of Jacobi polynomials with the same absolute value for each polynomial index. In [45], we constructed the subnet of this DC net, which is formed solely of the Wronskians of the Jacobi polynomials with the same indexes. However, the analysis of the whole DC net of the solvable rational potentials specified by two series of Maya diagrams, with additional restrictions on the polynomial indexes, constitutes a much more serious problem, which we plan to address in a separate publication.

It was Alhaidari [46] who drew our attention to the alternative rational representation of the Morse potential, making it possible to quantize this potential in terms of a finite sequence of orthogonal polynomials [47–49], which were identified in [50] as R-Bessel polynomials (a fact already mentioned in passing by Quesne [17]). This observation helped the author to fully appreciate the significance of the mentioned paper by Odake and Sasaki [23], who did not realized that the eigenfunctions of the three TSI potentials listed above are composed of three families of Romanovski polynomials [12]. Similarly to the RDC$\mathfrak{T}$s of the R-Routh polynomials, the DC net of finite X-Bessel orthogonal polynomial sequences is specified by a single series of Maya diagrams, so each finite EOP sequence can be represented in the form of the Wronskian transforms (W$\mathfrak{T}$s) of the R-Bessel polynomials.

This paper is focused on the detailed analysis of the finite EOP sequences formed by the W$\mathfrak{T}$s of R-Routh polynomials, which (to our best knowledge) have never been discussed in the literature so far.

Again, in contrast with [23], all the arguments below are presented with no mentioning of the Schrödinger equation. To formulate the corresponding quantum mechanical problem, one can perform the Liouville transformation of the constructed RCSLE, which results in a rational potential exactly solvable in terms of the corresponding finite EOP sequence. But this is simply a physical application of the developed formalism unrelated to our discussion.

2. Translational Form Invariance of $\Re$Ref CSLE with the Simple-Pole Density

Let us start our analysis with the $\Re$Ref CSLE:

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} + {}_i\mathrm{I}^{\mathrm{o}}[\eta;\lambda_{\mathrm{o}}] + \varepsilon_i\boldsymbol{\rho}_{\diamondsuit}[\eta]\right\}{}_i\Phi[\eta;\lambda_{\mathrm{o}};\varepsilon] \;=\; 0, \tag{1}$$

with the following RefPF [19–21]:

$${}_i\mathrm{I}^{\mathrm{o}}[\eta;\lambda_{\mathrm{o}}] \;=\; \frac{1-\lambda_{\mathrm{o}}^2}{4(\eta+i)^2} + \frac{1-(\lambda_{\mathrm{o}}^*)^2}{4(\eta-i)^2} + \frac{{}_i\mathrm{O}^{\mathrm{o}}(\lambda_{\mathrm{o}})}{4(\eta^2+1)} \tag{2}$$

and the following density function [45]:

$${}_i\boldsymbol{\rho}_{\diamondsuit}[\eta] = (\eta^2+1)^{-1} \tag{3}$$

having two simple poles at $\pm i$. (We use subscript i to stress that we deal with the RCSLEs always having two poles at $\pm i$, while diamond indicates that the density function in question has only simple poles. By definition, all the parameters marked by a circle are chosen to be energy-independent.) The coefficient

$${}_i\mathrm{O}^{\mathrm{o}}(\lambda) := 2(Re^2\lambda - Im^2\lambda) + 1 \tag{4}$$

is chosen in such a way that the exponent difference (ExpDiff) for the pole of the $\Re$Ref CSLE (1) at infinity vanishes at $\varepsilon = 0$. As a result of such a specific choice of the density function, the $\Re$Ref CSLE becomes TFI [1,20,21], and the corresponding Liouville potential [13,17,19,42]

$$\mathrm{V_G}[\eta; a, b] = -(\eta^2 + 1)\ {}_i\mathrm{I^o}[\eta; a + ib] - 1/2\{\eta, \mathrm{x}\}, \tag{5}$$

with a and b standing for the real and imaginary parts of the complex parameter

$$\lambda_o \equiv a + ib \tag{6}$$

respectively, turns into the TSI Gendenshtein potential by the change of variable

$$\eta(\mathrm{x}) = \mathit{sinh}\,\mathrm{x} \tag{7}$$

satisfying the first-order ordinary differential equation (ODE)

$$\eta'(\mathrm{x}) = {}_i\boldsymbol{\rho}_{\Diamond}^{-1/2}\,[\eta(\mathrm{x})], \tag{8}$$

where prime stands for the derivative with respect to x, so the Schwatzian derivative expressed in terms of the variable η takes the following form:

$$\{\eta, \mathrm{x}\} = \frac{3}{2(\eta^2 + 1)} - 1/2. \tag{9}$$

Re-writing RefPF (2) as

$${}_i\mathrm{I^o}[\eta; a + ib] = \frac{a^2 - b^2 - 1 + 2ab\eta}{(\eta^2 + 1)^2} + \frac{1}{4(\eta^2 + 1)}, \tag{10}$$

we come to the Liouville potential

$$\mathrm{V_G}(\mathrm{x}; a, b) = \frac{b^2 - a^2 + 1/4 + 2ab\,\mathit{sinh}\,\mathrm{x}}{\mathit{cosh}^2\,\mathrm{x}} \tag{11}$$

commonly referred to in the literature as the Scarf II potential [5,6,13,17], with the conventional parameters A and B standing for $a - 1/2$ and b, respectively. As pointed out by Quesne [17], the potential remains unchanged under the simultaneous change of sign of a and b. Note that our parameter a differs by $1/2$ from the parameter $a \equiv \mathrm{A}$ in [13,42]). In addition, Gendenshtein [42] defines the Schrödinger equation with the coefficient of the second derivative being equal to $1/2$, so his formula for potential (5) differs by the mentioned factor.

The $\Re$Ref CSLE (1) has two infinite sequences of quasi-rational solutions (q-RSs), as follows:

$${}_i\phi_{\pm,\mathrm{m}}[\eta; \lambda_o] = (1 - i\eta)^{1/2(1 \pm \lambda_o)}(1 + i\eta)^{1/2(1 \pm \lambda_o^*)}\Re_{\mathrm{m}}^{(\pm\lambda_o)}(\eta) \tag{12}$$

$$\equiv (1 + \eta^2)^{\,1/2(1 \pm a)}\mathit{exp}(\mp 1/2b\,\mathit{arctan}\,\eta)\Re_{\mathrm{m}}^{(\pm\lambda_o)}[\eta] \tag{13}$$

where the Routh [18] polynomials, $\Re_{\mathrm{m}}^{(\lambda)}[\eta]$, are defined via (A2) in Appendix A under constraint (A8). Since each q-RS (12) has, by its definition [7], a *rational* logarithmic derivative, it can be used (see Section 3.1 below) as the TF for the RLDT of the $\Re$Ref CSLE (1), which gives rise to an exactly solvable RCSLE iff that the Routh polynomial in question does not have real roots [17,19]. We refer the reader to Section 3.1 below for a more thorough definition of the LDTs introduced in the cited paper of Rudyak and Zakhariev [32].

Keeping in mind that the ExpDiffs for the poles of the $\Re$Ref CSLE (1) in the finite complex plane are energy-independent, we [1] identified it as the CSLE of group A [23], which implies that the characteristic exponents (ChExps) of q-RSs (12) for these poles must be independent of the polynomial degree. As the direct corollary of the latter observation, we assert that the Wronskian of q-RSs (12) from the same sequence can be decomposed into the product of a quasi-rational weight and a Wronskian of Routh polynomials. Selecting polynomial Wronskians with no real zeros brings us to a net of exactly solvable RDC$\mathfrak{I}$s of the $\Re$Ref CSLE (1)—the main focus of this paper.

Examination of the asymptotic behavior of q-RSs (12) at infinity brings us to the following simple formula for two roots of the indicial equation:

$$ {}_{i}\rho_{\infty;\pm,\mathrm{m}} = \mp\, a - \mathrm{m} - 1 \tag{14}$$

which unambiguously determines the following solution energies:

$$ {}_{i}\varepsilon_{\pm,\mathrm{m}}(a) = -({}_{i}\rho_{\infty;\pm,\mathrm{m}} + 1/2)^2 \tag{15}$$

$$ = -(\pm a + \mathrm{m} + 1/2)^2. \tag{16}$$

Each of the sequences (12) starts from the basic solution mentioned in the Introduction, as follows:

$$ {}_{i}\phi_{\pm,0}[\eta;\lambda_{\mathrm{o}}] = (1 - i\eta)^{1/2(1\pm\lambda_{\mathrm{o}})}(1 + i\eta)^{1/2(1\pm\lambda^{*}_{\mathrm{o}})}. \tag{17}$$

As expected [1], the basic solutions satisfy the following generic TFI condition:

$$ {}_{i}\phi_{-,0}[\eta; a+1, b]\,{}_{i}\phi_{+;0}[\eta; a, b] = {}_{i}\boldsymbol{\rho}^{-1/2}_{\diamondsuit}[\eta]. \tag{18}$$

Note that the real-field reduction of the complex $\mathfrak{I}$Ref CSLE [45]

$$ \left\{ \frac{\mathrm{d}^2}{\mathrm{d}\eta^2} + \sum_{\aleph=\pm} \frac{1-\lambda^2_{\mathrm{o};\aleph}}{4(1-\aleph\eta)^2} + \frac{\mathrm{O^o} - \varepsilon}{4(\eta^2-1)} \right\} \Phi[\eta;\lambda_{\mathrm{o};\pm},\mathrm{O^o}\,;\varepsilon] = 0 \tag{19}$$

on the imaginary axis (with $\lambda_{\mathrm{o};\pm}$ and $\mathrm{O^o}$ standing for some complex parameters) retains only two of the four complex basic solutions, contrary to its reduction onto the real axis, with three parameters chosen to be real.

The gauge transformations

$$ {}_{i}\Phi[\eta; a+ib;\varepsilon] \equiv {}_{i}\Phi[\eta; a, b;\varepsilon] = {}_{i}\phi_{\pm;0}[\eta; a, b\big]\;{}_{i}\mathrm{F}_{\pm}[\eta;\ a+ib;\varepsilon] \tag{20}$$

convert the $\Re$Ref CSLE (1) to a pair of Bochner-type eigenequations

$$ \left\{ (\eta^2+1)\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} + {}_{i}\tau_1[\eta;\pm a \pm ib]\frac{\mathrm{d}}{\mathrm{d}\eta} + [\varepsilon - {}_{i}\varepsilon_{\pm,0}(a)] \right\} {}_{i}\mathrm{F}_{\pm}[\eta;\ a+ib;\varepsilon] = 0, \tag{21}$$

where

$$ {}_{i}\tau_1[\eta;\pm a \pm ib] := 2(\eta^2+1)\ ld\ {}_{i}\phi_{\pm,0}[\eta; a, b] = 2\Re_1^{(\pm a \pm ib)}(\eta) \tag{22}$$

(cf. (3.1) in [17] with $\alpha = 2b$, $\beta = 1 - a$ in our notation). Comparing (21) and (22) with (9.9.5) in [26], we find that the Routh polynomials forming q-RSs (12) are nothing but polynomial solutions of the Bochner-type eigenequations:

$$ \left\{ (\eta^2+1)\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} + 2[b - \mathrm{N}_{\pm}(a)\eta]\frac{\mathrm{d}}{\mathrm{d}\eta} + [{}_{i}\varepsilon_{\pm,\mathrm{m}}(a) - {}_{i}\varepsilon_{\pm,0}(a)] \right\} \mathrm{P_n}(\mathrm{x}; b, \mathrm{N}_{\pm}(a)) = 0 \tag{23}$$

at energies (16), with

$$ \mathrm{N}_{\pm}(a) = \mp a - 1. \tag{24}$$

Note that the pseudo-Jacobi polynomials defined via (9.9.1) in [26] are nothing but the monic Routh polynomials in our terms:

$$\mathrm{P_n}(\eta; b, \mathrm{N}) \equiv \hat{\mathfrak{R}}_n^{(-\mathrm{N}-1+ib)}(\eta) \tag{25}$$

(see Remarks on p. 233 in [26] for their definition) assuming, based on (A7), that 2N is not a non-negative integer smaller than n.

Setting

$$_{i}\not{p}[\eta] \equiv (\eta^2+1)^{1/2} \tag{26}$$

and performing the following gauge transformation

$$_{i}\Psi[\eta; \lambda_o; \varepsilon] = {}_{i}\not{p}^{-1/2}[\eta]\, {}_{i}\Phi[\eta; \lambda_o; \varepsilon] \tag{27}$$

we then convert the $\Re$Ref CSLE (1) into the 'prime' SLE [20,21]:

$$\left\{ \frac{d}{d\eta} {}_{i}\not{p}[\eta] \frac{d}{d\eta} - {}_{i}\mathcal{q}[\eta; \lambda_o] + \varepsilon\, {}_{i}w[\eta] \right\} {}_{i}\Psi[\eta; \lambda_o; \varepsilon] = 0 \tag{28}$$

with the following weight:

$$_{i}w[\eta] := {}_{i}\boldsymbol{\rho}_{\diamondsuit}[\eta]\; {}_{i}\not{p}[\eta] \tag{29}$$

$$= (\eta^2+1)^{-1/2}, \tag{30}$$

where the zero-energy free term is given by the following conventional formula [29]

$$_{i}\mathcal{q}[\eta; \lambda_o] := -\, {}_{i}\not{p}[\eta]\, \left({}_{i}\mathrm{I^o}[\eta; \lambda_o] + I\{ {}_{i}\not{p}[\eta] \} \right), \tag{31}$$

where

$$\mathfrak{I}\{\mathrm{f}[\eta]\} := 1/4\, \dot{\mathrm{f}}^2[\eta]/\mathrm{f}[\eta] - 1/2\, \ddot{\mathrm{f}}[\eta], \tag{32}$$

with dots denoting the derivatives of an arbitrary function (f[η]) with respect to η. It can be directly verified that [20,21]

$$\sqrt{\eta^2+1}\, \mathfrak{I}\{\sqrt{\eta^2+1}\} = \frac{3}{4(\eta^2+1)^{3/2}} - \frac{1}{4(\eta^2+1)^{1/2}}. \tag{33}$$

The main advantage of converting the $\Re Ref$ CSLE (1) to its prime form with respect to the regular singular point at infinity comes from our observation [29] that the ChExps for the pole in question have opposite signs; therefore, the corresponding principal Frobenius solution is unambiguously selected by the DBC at the given endpoint. (Remember that the energy reference point was chosen in such a way that the indicial equation for the pole at infinity has real roots at any negative energy) Reformulating the given spectral problem in such a way allows us to take advantage of powerful theorems proven in [30] for zeros of principal solutions of SLEs solved under the DBCs at singular end points. Our choice of leading coefficient function (26) assures that sum of the ChExps for the pole at infinity is equal to zero at any negative energy so that the DBCs in question unambiguously select the principal Frobenius solution (PFS) of the $\Re$Ref CSLE (1).

Below, we specify the eigenfunctions of the given Dirichlet problem by subscript **c** following the labeling originally introduced by us for the transformation functions (TFs) of Darboux transformations of radial potentials [51] years before the birth of supersymmetric quantum mechanics [52–54]. One can directly verify that the eigenfunctions

$$_{i}\psi_{\mathbf{c},n}[\eta; a+ib] = (1+\eta^2)^{1/4-1/2a} \left(\frac{1+i\eta}{1-i\eta}\right)^{1/2ib} \mathrm{R}_n^{(-2b,-a+1)}[\eta] \tag{34}$$

with the R-Routh polynomial on the right defined via (A10) in Appendix A satisfy the DBCs

$$\lim_{\eta \to \pm\infty} {}_{i}\psi_{\mathbf{c},n}[\eta; \lambda_o] = 0 \; (n = 0, \ldots, n_{max}) \tag{35}$$

with

$$n_{max} = \lfloor a - 1/2 \rfloor \tag{36}$$

Examination of the integral

$$0 < \int_{-\infty}^{\infty} |{}_{i}\phi_{\mathbf{c},n}[\eta; \lambda_o]\,|^2 {}_{i}\boldsymbol{\rho}_{\Diamond}[\eta] d\eta = \int_{-\infty}^{\infty} |{}_{i}\psi_{\mathbf{c},n}[\eta; \lambda_o]\,|^2{}_{i}\,\psi[\eta] d\eta < \infty \tag{37}$$

reveals that q-RS (12) of the $\Re Ref$ CSLE (1) is squarely integrable with the weight ${}_{i}\boldsymbol{\rho}_{\Diamond}[\eta]$ iff the corresponding q-RS (34) of prime SLE (28) obeys DBCs (35).

To prove that the SLP problem in question is *exactly* solvable, the author [19] took advantage of Stevenson's idea [55] to express an analytically continued solution of the CSLE (19)—or, to be more accurate, an analytically continued solution of the prime SLE (28)—in terms of hypergeometric polynomials in a complex argument. It was just confirmed that the latter (formally complex) polynomials can be converted into real R-Routh polynomials (A10). The reader can argue that this proof is unnecessary, since the Gendenshtein potential is TSI [42]. However, as demonstrated in [19], Gendenshtein's arguments have to be accompanied by some additional assumptions, which drastically reduce the practical significance of his conclusions.

As a direct consequence of the disconjugacy theorem [40,56–59], we conclude that q-RS (12) may not have more than one node if

$${}_{i}\varepsilon_{\pm,m}(a) < {}_{i}\varepsilon_{\mathbf{c},0}(a) \equiv {}_{i}\varepsilon_{-,0}(a), \tag{38}$$

and hence, it is necessarily nodeless iff its asymptotic values at the ends of the quantization interval have the same sign. Compared with more complicated examples discussed in [40,56–59], the DBCs for the prime SLE (28) are imposed at $\pm\infty$, so the cited constraint holds if the given q-RS of type III (in Quesne's terms [17]) is formed by a Routh polynomial of an even degree.

Making use of (16), one finds

$${}_{i}\varepsilon_{-,0}(a) - {}_{i}\varepsilon_{+,m}(a) = (m+1)(2\,a+m) > 0 \tag{39}$$

and

$${}_{i}\varepsilon_{-,0}(a) - {}_{i}\varepsilon_{-,m}(a) = m(m+1-2\,a) > 0 \text{ for } m > 2\,a - 1 > 0. \tag{40}$$

Keeping in mind that leading coefficient (A7) of Routh polynomial (A6) differs from zero for any $a > 0$; therefore, we conclude that all q-RSs (12) with the upper label (+) and *even* non-negative m must be nodeless [17]. This is also true for q-RSs (12) with the lower label (−) and even

$$m = 2j \; > \; 2\,a - 1. \tag{41}$$

(Note that the rising factorial in (A7) in Appendix A starts from a positive factor in the latter case and therefore must be positive.).

3. Quantization of RDCTs of Gendenshtein Potentials by Finite Sequences of EOPs

It was proven in [1] that any RDCT of the TSI SLE with two basic solutions can be obtained using seed solutions of the same type (+ or −). Since the eigenfunctions of the CSLE (1) belong to the sequence $|-, m)$, we focus solely on the RDCTs that use, as their seed functions, only the q-RSs from the latter sequence. These restrictions allow us to express all the EOP sequences of our interest as Wronskians of Routh polynomials with exactly the same complex index, $-\lambda_o$.

3.1. Liouville–Darboux Transformations

As discussed in the end of previous section, q-RSs (12) with the lower label $(-)$ are nodeless for even values of m larger than $2\,a-1$. The notation

$$ {}_{i}\phi_{\pm,2\mathbf{j}}[\eta;\lambda_{o}] = {}_{i}\phi[\eta;\pm\lambda_{o}]\,\Re_{2\mathbf{j}}^{(\pm\lambda_{o})}[\eta] \tag{42} $$

with

$$ {}_{i}\phi[\eta;\lambda] := (1-i\eta)^{1/2(1+\lambda)}(1+i\eta)^{1/2(1+\lambda*)} \tag{43} $$

always assumes that the degree of the Routh polynomial obeys the mentioned constraint for any q-RS $|-,2\mathbf{j})$ with $j > a - 1/2$, while $\mathbf{j}$ is allowed to be any positive integer for the q-RSs $|+,2\mathbf{j})$. Any nodeless q-RS (42) can be used as the TF for the RLDT [20,21] such that Rudyak and Zahariev's reciprocal function [32]

$$ {}_{i}\phi_{\mathbf{c},0}[\eta;\lambda_{o}|\pm,2\mathbf{j}] = \frac{{}_{i}\boldsymbol{\rho}_{\diamondsuit}^{-1/2}[\eta]}{{}_{i}\phi_{\pm,2\mathbf{j}}[\eta;\lambda_{o}]} \tag{44} $$

$$ = \frac{{}_{i}\phi[\eta;-1\mp\lambda_{o}]}{\Re_{2\mathbf{j}}^{(\pm\lambda_{o})}[\eta]} \tag{45} $$

represents a q-RS of the transformed RCSLE as follows:

$$ \left\{\frac{d^{2}}{d\eta^{2}} + {}_{i}I^{o}[\eta;\lambda_{o}|\pm,2\mathbf{j}] + {}_{i}\varepsilon_{\pm,2\mathbf{j}}(a)\,{}_{i}\boldsymbol{\rho}_{\diamondsuit}[\eta]\right\}{}_{i}\phi_{\mathbf{c},0}[\eta;\lambda_{o}|\pm,2\mathbf{j}] = 0 \tag{46} $$

at the energy ${}_{i}\varepsilon_{\pm,2\mathbf{j}}(a)$, where

$$ \begin{aligned} {}_{i}I^{o}[\eta;\lambda_{o}|\pm,2\mathbf{j}] = -ld^{2}{}_{i}\phi_{\mathbf{c},0}[\eta;\lambda_{o}|\pm,2\mathbf{j}] - \dot{ld}\,{}_{i}\phi_{\mathbf{c},0}[\eta;\lambda_{o}|\pm,2\mathbf{j}] \\ - {}_{i}\varepsilon_{\pm,2\mathbf{j}}(a)\,{}_{i}\boldsymbol{\rho}_{\diamondsuit}[\eta], \end{aligned} \tag{47} $$

with ld standing for the logarithmic derivative of a function with respect to η. It will be proven below that q-RS (43) represents the zero-energy eigenfunction of the given Sturm–Liouville problem as indicated by the label.

We [29,60] suggested the term 'LDT' to stress that we deal with the following three-step operation:

(i) The Liouville transformation from the given SLE to the conventional Schrödinger equation;
(ii) The Darboux deformation of the corresponding Liouville potential;
(iii) The inverse Liouville transformation from the Schrödinger equation to the new SLE, which, by definition, preserves both the leading coefficient function and weight.

Note that the preservation of the leading coefficient function and weight is an additional constraint imposed on the 'Darboux transformations' of SLEs defined using the intertwining operators [37,38,61].

Representing the ℜRef CSLE (1) for the TF $({}_{i}\phi_{\pm,2\mathbf{j}}[\eta;\lambda_{o}])$ in the Riccati form as follows:

$$ -ld^{\,2}\,{}_{i}\phi_{\pm,2\mathbf{j}}[\eta;\lambda_{o}] - \dot{ld}\,{}_{i}\phi_{\pm,2\mathbf{j}}[\eta;\lambda_{o}] = {}_{i}I^{o}[\eta;\lambda_{o}] + {}_{i}\varepsilon_{\pm,2\mathbf{j}}(a)\,{}_{i}\boldsymbol{\rho}_{\diamondsuit}[\eta] \tag{48} $$

and taking into account that

$$ ld\,{}_{i}\phi_{\mathbf{c},0}[\eta;\lambda_{o}|\pm,2\mathbf{j}] = -ld\,{}_{i}\phi_{\pm,2\mathbf{j}}[\eta;\lambda_{o}] - 1/2\,ld\,{}_{i}\boldsymbol{\rho}_{\diamondsuit}[\eta], \tag{49} $$

we can alternatively represent RefPF (47) as [31]

$$ {}_i\mathrm{I}^{\mathrm{o}}[\eta;\lambda_{\mathrm{o}}|\pm,2\mathbf{j}] = {}_i\mathrm{I}^{\mathrm{o}}[\eta;\lambda_{\mathrm{o}}] + 2\sqrt{{}_i\boldsymbol{\rho}_{\Diamond}[\eta]}\,\frac{\mathrm{d}}{\mathrm{d}\eta}\frac{ld_i\phi_{\pm,2\mathbf{j}}[\eta;\lambda_{\mathrm{o}}]}{\sqrt{{}_i\boldsymbol{\rho}_{\Diamond}[\eta]}} + \mathcal{I}\{{}_i\boldsymbol{\rho}_{\Diamond}[\eta]\}, \tag{50} $$

where the so-called [29] 'universal correction' is defined via the following generic formula:

$$ \mathcal{I}\{\mathrm{f}[\eta]\} := 1/2\sqrt{\mathrm{f}[\eta]}\,\frac{\mathrm{d}}{\mathrm{d}\eta}\frac{ld\,\mathrm{f}[\eta]}{\sqrt{\mathrm{f}[\eta]}} \tag{51} $$

$$ \equiv 1/2\,\dot{ld}\,\mathrm{f}[\eta] - 1/4\,ld^2\,\mathrm{f}[\eta] \tag{52} $$

$$ = -\mathrm{f}^{1/2}[\eta]\,\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\,\mathrm{f}^{-1/2}[\eta] \tag{53} $$

Note that the logarithmic derivatives of both functions, ${}_i\phi_{\pm,2\mathbf{j}}[\eta;\lambda_{\mathrm{o}}]$ and ${}_i\boldsymbol{\rho}_{\Diamond}[\eta]$, behave as follows with large values of η:

$$ ld\,\mathrm{f}[\eta] = \mathrm{f}_{-1}\eta^{-1} + \mathrm{f}_{-2}\eta^{-2} + 0(\eta^{-3}) \tag{54} $$

where both constants, f_{-1} and f_{-2}, may generally depend on the sign of η. As the direct consequence of (54), we find that

$$ \frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}(\eta\,ld\,\mathrm{f}[\eta]) \approx \mathrm{f}_{-2}\eta^{-3} + 0(\eta^{-4}) \tag{55} $$

Keeping in mind the asymptotic behavior of the density function with large values of $|\eta|$,

$$ {}_i\boldsymbol{\rho}_{\Diamond}[\eta] \approx \eta^{-2} \text{ for } |\eta| >> 1, \tag{56} $$

the analysis of the right-hand side of (50) with large values of $|\eta|$ shows that

$$ \lim_{|\eta|\to\infty}(\eta^2\,{}_i\mathrm{I}^{\mathrm{o}}[\eta;\lambda_{\mathrm{o}}|\pm,2\mathbf{j}]) = \lim_{|\eta|\to\infty}(\eta^2\,{}_i\mathrm{I}^{\mathrm{o}}[\eta;\lambda_{\mathrm{o}}]). \tag{57} $$

We have thus proven that the RLDT in question does not change the ExpDiff of the CSLE at infinity—the common feature of the Fuchsian CSLEs with the density functions decreasing as $1/\eta^2$ with large values of $|\eta|$.

Substituting the logarithmic derivative

$$ ld_i\phi_{\mathbf{c},0}[\eta;\lambda_{\mathrm{o}}|\pm,2\mathbf{j}] = ld_i\phi[\eta;-1\mp\lambda_{\mathrm{o}}] - ld\,\Re^{(\pm\lambda_{\mathrm{o}})}_{2\mathbf{j}}[\eta] \tag{58} $$

into (47) and taking into account that

$$ \begin{gathered} {}_i\phi^{-1}[\eta;\lambda]\,{}_i\ddot{\phi}[\eta;\lambda] = \\ ld^2\,{}_i\phi[\eta;\lambda] + \dot{ld}\,{}_i\phi[\eta;\lambda] = -{}_i\mathrm{I}^{\mathrm{o}}[\eta;\lambda] - {}_i\varepsilon_{+,0}(Re\lambda)\,{}_i\boldsymbol{\rho}_{\Diamond}[\eta] \end{gathered} \tag{59} $$

with

$$ {}_i\varepsilon_{+,0}(Re\lambda) := -(Re\lambda + 1/2)^2 \equiv {}_i\varepsilon_{-,0}(-Re\lambda), \tag{60} $$

and also making use of (21), we come to the alternative representation for the RefPFs of CSLEs (46):

$$ {}_i\mathrm{I}^{\mathrm{o}}[\eta;\lambda_{\mathrm{o}}|\pm,2\mathbf{j}] = {}_i\mathrm{I}^{\mathrm{o}}[\eta;1\pm\lambda_{\mathrm{o}}] + 2\widehat{Q}[\eta;\overline{\eta}_{2\mathbf{j}}(\pm\lambda_{\mathrm{o}})] + \frac{{}_i\widehat{O}^{\,\mathrm{o}}_{2\mathbf{j}}[\eta;\lambda_{\mathrm{o}}|\pm,2\mathbf{j}]}{4(\eta^2+1)\Re^{(\pm\lambda_{\mathrm{o}})}_{2\mathbf{j}}[\eta]}, \tag{61} $$

where we set

$$\hat{\Re}_{\mathrm{m}}^{(\lambda)}[\eta] = \Pi[\eta;\overline{\eta}_{\mathrm{m}}(\lambda)] := \prod_{\mathrm{k}=1}^{\mathrm{m}}[\eta - \eta_{\mathrm{m;k}}(\lambda)] \tag{62}$$

$$\widehat{Q}[\eta;\overline{\eta}_{\mathrm{m}}] := -1/2\Pi[\eta;\overline{\eta}_{\mathrm{m}}]\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\Pi^{-1}[\eta;\overline{\eta}_{\mathrm{m}}] \tag{63}$$

$$= 1/2\left\{\overset{\bullet}{ld}\,\Pi[\eta;\overline{\eta}_{\mathrm{m}}] - ld^2\,\Pi[\eta;\overline{\eta}_{\mathrm{m}}]\right\}, \tag{64}$$

and

$$\begin{gathered}{}_{i}\widehat{\mathrm{O}}^{\,\mathrm{o}}_{2\mathbf{j}}[\eta;\lambda_{\mathrm{o}}|\pm,2\mathbf{j}] := 8\Re_1^{(-1\mp\lambda_{\mathrm{o}})}(\eta)\,\overset{\bullet}{\Re}{}^{(\pm\lambda_{\mathrm{o}})}_{2\mathbf{j}}[\eta] - \\ 4[{}_{i}\varepsilon_{\pm,2\mathbf{j}}(a) - {}_{i}\varepsilon_{\pm,0}(a)]\Re_{2\mathbf{j}}^{(\pm\lambda_{\mathrm{o}})}[\eta].\end{gathered} \tag{65}$$

Taking into account that

$$\Re_1^{(-1\mp\lambda_{\mathrm{o}})}[\eta] = -\Re_1^{(\pm\lambda_{\mathrm{o}})}[\eta] + 2\eta, \tag{66}$$

it is convenient to re-write polynomial (65) as

$${}_{i}\widehat{\mathrm{O}}^{\,\mathrm{o}}_{2\mathbf{j}}[\eta;\lambda_{\mathrm{o}}|\pm,2\mathbf{j}] := 4(\eta^2+1)\,\overset{\bullet\bullet}{\Re}{}^{(\pm\lambda_{\mathrm{o}})}_{2\mathbf{j}}[\eta] + 8\eta. \tag{67}$$

Examination of RefPFs (61) reveals that the RLDTs in question change by one the ExpDiffs for the poles at $\pm i$ while creating the new second-order pole at each zero of the Routh polynomial.

Again, we can convert CSLE (46) to its prime form

$$\begin{gathered}\left\{\tfrac{\mathrm{d}}{\mathrm{d}\eta}(\eta^2+1)^{1/2}\tfrac{\mathrm{d}}{\mathrm{d}\eta} - {}_{i}\mathcal{q}\,[\eta;\lambda_{\mathrm{o}}|\pm,2\mathbf{j}] + {}_{i}\varepsilon_{\mathbf{c},\mathrm{n}}(\lambda_{\mathrm{o}})\,{}_{i}\wp[\eta]\right\}\times \\ {}_{i}\psi_{\mathbf{c},\mathrm{n}}[\eta;\lambda_{\mathrm{o}}|\pm,2\mathbf{j}] = 0\end{gathered} \tag{68}$$

solved under the DBCs

$$\lim_{|\eta|\to\infty}{}_{i}\psi_{\mathbf{c},n}[\eta;\lambda_{\mathrm{o}}|\pm,2\mathbf{j}] = 0, \tag{69}$$

where [29]

$${}_{i}\mathcal{q}\,[\eta;\lambda_{\mathrm{o}}|\pm,2\mathbf{j}] := -\sqrt{\eta^2+1}\,({}_{i}\mathrm{I}^{\mathrm{o}}\,[\eta;\lambda_{\mathrm{o}}|\pm,2\mathbf{j}] + \mathfrak{I}\{\sqrt{\eta^2+1}\}) \tag{70}$$

and

$${}_{i}\psi_{\mathbf{c},\mathrm{n}}[\eta;\lambda_{\mathrm{o}}|\pm,2\mathbf{j}] := (\eta^2+1)^{-1/4}{}_{i}\phi_{\mathbf{c},\mathrm{n}}[\eta;\lambda_{\mathrm{o}}|\pm,2\mathbf{j}] \tag{71}$$

by definition. It then directly follows from asymptotic Formula (57), coupled with (31) and (33), that

$$\lim_{|\eta|\to\infty}|\eta\,{}_{i}\mathcal{q}\,[\eta;\lambda_{\mathrm{o}}|\pm,2\mathbf{j}]| = \lim_{|\eta|\to\infty}|\eta_{i}\mathcal{q}\,[\eta;\lambda_{\mathrm{o}}]| < \infty, \tag{72}$$

which confirms that the two ChExps for the pole of prime SLE (68) at infinity differ only by sign and, therefore, OBCs unambiguously select the PFS near the given singularity. In particular, keeping in mind that the absolute value of the q-RS

$${}_{i}\psi_{\pm,2\mathbf{j}}[\eta;\lambda_{\mathrm{o}}] := (\eta^2+1)^{-1/4}{}_{i}\phi_{\pm,2\mathbf{j}}[\eta;\lambda_{\mathrm{o}}] \equiv {}_{i}\psi[\eta;\pm\lambda_{\mathrm{o}}]\Re_{2\mathbf{j}}^{(\pm\lambda_{\mathrm{o}})}(\eta) \tag{73}$$

infinitely grows as $|\eta|\to\infty$ we conclude that the nodeless solution of prime SLE (68),

$${}_{i}\psi_{\mathbf{c},0}[\eta;\lambda_{\mathrm{o}}|\pm,2\mathbf{j}] := (\eta^2+1)^{-1/4}{}_{i}\phi_{\mathbf{c},0}[\eta;\lambda_{\mathrm{o}}|\pm,2\mathbf{j}] = {}_{i}\psi^{-1}_{\pm,2\mathbf{j}}[\eta;\lambda_{\mathrm{o}}], \tag{74}$$

vanishes at both quantization ends and therefore represents the lowest-energy eigenfunction. Any other eigenfunction of prime SLE (68) has the following generic form [32]:

$$ {}_i\psi_{\mathbf{c},n+1}[\eta;\lambda_o|\pm,2\mathbf{j}] = \frac{W\{{}_i\psi_{\pm,2\mathbf{j}}[\eta;\lambda_o],{}_i\psi_{\mathbf{c},n}[\eta;\lambda_o]\}}{{}_i\rho_{\diamondsuit}^{1/2}[\eta]\,{}_i\psi_{\pm,2\mathbf{j}}[\eta;\lambda_o]} \tag{75} $$

As discussed in more detail in Appendix B, all the EOP sequences forming the eigenfunctions of prime SLE (68) for TF $|+,2\mathbf{j})$ can be represented as Wronskians of R-Routh polynomials of sequential degrees starting from the first-degree polynomial.

Let us now prove the cornerstone of the theory of the RLDs developed in [29]:

Theorem 1. *Prime SLEs (68) are exactly solvable under the DBCs at infinity.*

Proof of Theorem 1. Our purpose is thus to show that all the eigenfunctions of prime SLEs (68), other than a nodeless eigenfunction (74), can be written in quasi-rational form (75). Indeed, suppose that one of prime SLEs (68) has another eigenfunction ${}_i\psi_{\mathbf{c},n}[\eta;\lambda_o|\pm,2\mathbf{j}]$, which, by definition, obeys the DBCs

$$ \lim_{|\eta|\to\infty} {}_i\psi_{\mathbf{c},n}[\eta;\lambda_o|\pm,2\mathbf{j}] = 0. \tag{76} $$

Since this eigenfunction must be a PFS near each singular end point at infinity,

$$ {}_i\psi_{\mathbf{c},n}[\eta;\lambda_o|\pm,2\mathbf{j}] \propto |\eta|^{-\rho_\pm} \text{ for } |\eta| >> 1; \tag{77} $$

therefore,

$$ \lim_{|\eta|\to\infty} (\eta\ {}_i\dot{\psi}_{\mathbf{c},n}[\eta;\lambda_o|\pm,2\mathbf{j}]) = 0. \tag{78} $$

Taking into account that the lowest-energy eigenfunction (74) has a quasi-rational form,

$$ \lim_{|\eta|\to\infty} |\eta\ ld_i\psi_{\mathbf{c},0}[\eta;\lambda_o|\pm,2\mathbf{j}]\,| < \infty, \tag{79} $$

we find that the function

$$ {}_i\psi_{\pm}[\eta;\lambda_o] := \frac{W\{{}_i\psi_{\mathbf{c},0}[\eta;\lambda_o|\pm,2\mathbf{j}],{}_i\psi_{\mathbf{c},n}[\eta;\lambda_o|\pm,2\mathbf{j}]\}}{{}_i\rho_{\diamondsuit}^{1/2}[\eta]\,{}_i\psi_{\mathbf{c},0}[\eta;\lambda_o|\pm,2\mathbf{j}]} \tag{80} $$

$$ \begin{aligned} &= \sqrt{\eta^2+1}\ {}_i\dot{\psi}_{\mathbf{c},n}[\eta;\lambda_o|\pm,2\mathbf{j}] \\ &- \sqrt{\eta^2+1}\ ld_i\psi_{\mathbf{c},0}[\eta;\lambda_o|\pm,2\mathbf{j}]\ {}_i\psi_{\mathbf{c},n}[\eta;\lambda_o|\pm,2\mathbf{j}] \end{aligned} \tag{81} $$

vanishes at both limits ($\eta \to \pm\infty$) and, therefore, must be an eigenfunction of prime SLE (28), in contradiction with the fact that any eigenfunction of the $\Re$Ref CSLE (1) can be written in the form of (12). We thus assert that q-RSs (75) represent all possible eigenfunctions of prime SLE (68), which completes the proof of Theorem 1. □

For the RD$\mathfrak{I}$ using the TF $|-,2\mathbf{j})$ eigenfunctions (75) with $n \geq 0$ can be represented as the weighted Wronskians of Routh and R-Routh polynomials with the common quasi-rational weight

$$ {}_i\psi[\eta;\lambda_o|-,2\mathbf{j}] \equiv {}_i\rho_{\diamondsuit}^{-1/4}[\eta]\,{}_i\phi[\eta;\lambda_o|-,2\mathbf{j}] := \frac{{}_i\psi[\eta;1-\lambda_o]}{\Re_{2\mathbf{j}}^{(-\lambda_o)}(\eta)}, \tag{82} $$

namely

$$ {}_i\psi_{\mathbf{c},n+1}[\eta;\lambda_o|-,2\mathbf{j}] = -{}_i\psi[\eta;\lambda_o|-,2\mathbf{j}]\ {}_i\mathfrak{W}_{2\mathbf{j}+n-1}[\eta;\lambda_o|-\vdots n,2\mathbf{j}], \tag{83} $$

where

$$_{i}\mathcal{W}_{\mathbf{2j}+n-1}[\eta; a+ib|-\vdots \mathrm{n}, \mathbf{2j}] := \mathrm{W}\left\{\mathrm{R}_{\mathrm{n}}^{(-2b,-a+1)}[\eta], \Re_{\mathbf{2j}}^{(-a-ib)}[\eta]\right\} \tag{84}$$

is the polynomial of degree $\mathbf{2j}+\mathrm{n}-1$, with the R-Routh polynomial defined via (A10) in Appendix A. One can directly verify that q-RSs (74) of prime SLEs (68) obey the DBCs iff $\mathrm{n} < a - 1/2$.

By analogy with (37), we conclude that q-RSs (74) are normalizable with the weight $_{i}\psi[\eta]$ as follows:

$$0 < \int_{-\infty}^{+\infty} |_{i}\psi_{\mathbf{c},\mathrm{n}}[\eta;\lambda_{\mathrm{o}}|-,\mathbf{2j}]|^{2}{}_{i}\psi[\eta]d\eta \equiv \int_{-\infty}^{+\infty} |_{i}\phi_{\mathbf{c},\mathrm{n}}[\eta;\lambda_{\mathrm{o}}|-,\mathbf{2j}]|^{2}{}_{i}\rho_{\Diamond}[\eta]\, d\eta < \infty. \tag{85}$$

It directly follows from the analysis presented by Gestesy et al. [30] for the generic SLE solved under the DBCs that eigenfunctions (74) must be mutually orthogonal with the weight, $_{i}\psi[\eta]$. Again, as the direct corollary of this orthogonality, we conclude that polynomial Wronskians (84) are necessarily orthogonal with the weight,

$$_{i}\mathrm{W}[\eta;\lambda_{\mathrm{o}}|-,\mathbf{2j}] := {}_{i}\psi^{2}[\eta;\lambda_{\mathrm{o}}|-,\mathbf{2j}]_{i}\psi[\eta]; \tag{86}$$

namely,

$$\begin{gathered}\int_{-\infty}^{\infty} {}_{i}\mathcal{W}_{\mathbf{2j}+\mathrm{n}-1}[\eta;\lambda_{\mathrm{o}}|-\vdots \mathrm{n},\mathbf{2j}]\, {}_{i}\mathcal{W}_{\mathbf{2j}+\mathrm{n}'-1}[\eta;\lambda_{\mathrm{o}}|-\vdots \mathrm{n}',\mathbf{2j}]\, {}_{i}\mathrm{W}[\eta;\lambda_{\mathrm{o}}|-,\mathbf{2j}]\, \mathrm{d}\eta = 0 \\ \text{for } 0 \le \mathrm{n}' < \mathrm{n} \le \lfloor a - 1/2 \rfloor < j\end{gathered}. \tag{87}$$

The polynomial sequence under consideration starts from the polynomial

$$_{i}\mathcal{W}_{\mathbf{2j}-1}[\eta;\lambda_{\mathrm{o}}|-\vdots 0,\mathbf{2j}] = -\overset{\bullet}{R}{}_{\mathbf{2j}}^{(-\lambda_{\mathrm{o}})}[\eta]. \tag{88}$$

Note that the polynomial degree of $\mathbf{2j}-1$ is larger than 1 if the given sequence contains at least two polynomials, so the sequence lacks the first-degree polynomial regardless of the value of $\mathbf{j}$.

It then directly follows from Theorem 2.1 in [30] (Theorem 14.10 in [62]) that the polynomial Wronskian of the degree $\mathbf{2j}+\mathrm{n}-1$ has exactly $\mathrm{n}+1$ (all simple) zeros for any $\mathrm{n} < a - 1/2$. In particular, this implies that the polynomial Wronskian of the degree $\mathbf{2j}-1$ starting the given sequence must have a single simple real zero. Indeed, combining (9.9.6) in [26] with (A6) and (A7) in Appendix A, we can re-write (88) as

$$_{i}\mathcal{W}_{\mathbf{2j}-1}[\eta;\lambda_{\mathrm{o}}|-\vdots 0,\mathbf{2j}] = -\mathbf{2j}\, \mathrm{K}_{\mathbf{2j}}(-a)\, \mathrm{P}_{\mathbf{2j}-1}(\eta; b, a-2). \tag{89}$$

Since the q-RS $_{i}\phi_{-,2j-1}[\eta;\lambda_{\mathrm{o}}-1]$ lies below the lowest eigenvalue, $_{i}\varepsilon_{-,0}(\lambda_{\mathrm{o}}-1)$, the disconjugacy theorem states that this solution may not have more than one node. On the other hand, any polynomial of odd degree necessarily has at least one real zero.

Let us finally prove that polynomial Wronskians (84) satisfy a Bochner-type ODE with polynomial coefficients and thereby form an EOP sequence. As a matter of fact, we will prove a more general result that the Wronskians of two Routh polynomials of degrees m and n, with fixed m > 0 and n running through all non-negative integer values other than m, form an X-DPS starting from a polynomial of degree of m − 1 if m > 1 or lacking the first-degree polynomial if m = 1, so the infinite polynomial sequence in question does not obey the prerequisites of the Bochner theorem [24].

Substituting the q-RSs

$$_{i}\phi_{-,\mathrm{n}+1}[\eta;\lambda_{\mathrm{o}}|-,\mathrm{m}] := {}_{i}\phi[\eta;\lambda_{\mathrm{o}}|-,\mathrm{m}]\, {}_{i}\mathcal{W}_{\mathrm{m}+\mathrm{n}-1}[\eta;\lambda_{\mathrm{o}}|-\vdots \mathrm{m},\mathrm{n}] \tag{90}$$

into CSLE (68) with the RefPF represented in form (61) and also taking into account (59) and (63), we find that polynomial Wronskians (84) obey the following Bochner-type ODE:

$$\left\{{}_i\mathbf{D}_{-;\mathrm{m}}[\eta;\lambda_o] + {}_iC_\mathrm{m}[\eta;\lambda_o;{}_i\varepsilon_{-,\mathrm{n}}(\lambda_o)|-,\mathrm{m}]\right\}{}_i\mathfrak{W}_{\mathrm{m+n-1}}[\eta;\lambda_o|-\vdots\mathrm{n},\mathrm{m}] = 0 \quad (\mathrm{n}=0,1,\ldots,\mathrm{m}-1,\mathrm{m}+1,\ldots), \tag{91}$$

where

$${}_i\mathbf{D}_{-;\mathrm{m}}[\eta;\lambda_o] := (\eta^2+1)\Re_\mathrm{m}^{(-\lambda_o)}[\eta]\frac{d^2}{d\eta^2} + {}_i\tau_{\mathrm{m}+1}[\eta;\lambda_o|-,\mathrm{m}]\frac{d}{d\eta} \tag{92}$$

is the second-order differential operator with the energy-independent polynomial coefficient of the first derivative

$${}_i\tau_{\mathrm{m}+1}[\eta;\lambda_o|-,\mathrm{m}] := 2(\eta^2+1)\, ld\, {}_i\phi[\eta;\lambda_o|-,\mathrm{m}] \tag{93}$$

$$= 2\Re_1^{(1-\lambda_o)}[\eta]\Re_\mathrm{m}^{(-\lambda_o)}[\eta] - 2(\eta^2+1)\dot{\Re}_\mathrm{m}^{(-\lambda_o)}[\eta]. \tag{94}$$

by analogy with (22). The energy-dependent free term of ODE (91) is the m-degree polynomial linear in ε, as follows:

$${}_iC_\mathrm{m}[\eta;\lambda_o;\varepsilon|-,\mathrm{m}] = {}_iC_\mathrm{m}[\eta;\lambda_o-,\mathrm{m}] + \varepsilon\Re_\mathrm{m}^{(-\lambda_o)}(\eta) \tag{95}$$

where

$$C_\mathrm{m}[\eta;\lambda_o|-,\mathrm{m}] = 2\Re_1^{(-\lambda_o)}(\eta)\,\dot{\Re}_\mathrm{m}^{(-\lambda_o)}[\eta] - {}_i\varepsilon_{-,\mathrm{m}}(Re\lambda_o)\Re_\mathrm{m}^{(-\lambda_o)}[\eta]. \tag{96}$$

For $\mathrm{m} = 2\mathbf{j}$ and $\mathrm{n} < a - 1/2$, the polynomial Wronskians in question form a finite EOP subset of the X-DPS in question.

3.2. Finite EOP Sequences as Truncations of X-DPSs Formed by Wronskians of Routh Polynomials

Let $-\vdots\overline{\mathrm{M}}_\mathrm{p}$ be a finite set of the q-RSs ${}_i\phi_{-,\mathrm{m_k}}[\eta;\lambda_o]$, with

$$\overline{\mathrm{M}}_\mathrm{p} := \mathrm{m}_{\mathrm{k}=1,\,\ldots,\,\mathrm{p}}(\mathrm{m}_{\mathrm{k}+1} > \mathrm{m}_\mathrm{k} \geq 0 \text{ for } \mathrm{k} < \mathrm{p}) \tag{97}$$

standing for a monotonic sequence of non-negative integers. Again, we assume that $2a$ is not a positive integer; therefore, the leading coefficient, $\mathrm{K_m}(-a)$, of the Routh polynomial $\Re_\mathrm{m}^{(-a-i\,\mathrm{b})}[\eta]$ differs from zero regardless of the polynomial degree (m). Writing the leading coefficient of the polynomial Wronskian

$${}_i\mathfrak{W}_{\mathfrak{N}(\overline{\mathrm{M}}_\mathrm{p})}[\eta;\lambda_o|-\vdots\overline{\mathrm{M}}_\mathrm{p}] := W\{\Re^{(-\lambda_o)}_{\mathrm{m}_{\mathrm{k}=1,\,\ldots,\,\mathrm{p}}}[\eta]\} \tag{98}$$

as

$${}_i\mathfrak{W}_{\mathfrak{N}(\overline{\mathrm{M}}_\mathrm{p});\mathfrak{N}(\overline{\mathrm{M}}_\mathrm{p})}(\lambda_o|-\vdots\overline{\mathrm{M}}_\mathrm{p}) = \prod_{\mathrm{k}=1}^{\mathrm{p}} \mathrm{K}_{\mathrm{m_k}}(-a)\mathrm{D}(\overline{\mathrm{M}}_\mathrm{p}), \tag{99}$$

where [63]

$$\mathrm{D}(\mathrm{m}_1,\ldots,\mathrm{m}_p) := \begin{vmatrix} 1 & 1 & \ldots & 1 & 1 \\ \mathrm{m}_1 & \mathrm{m}_2 & \ldots & \mathrm{m_p} & \mathrm{m_{p+1}} \\ \mathrm{m}_1^{\mathrm{p}-1} & \mathrm{m}_2^{\mathrm{p}-1} & \ldots & \mathrm{m_p^{p-1}} & \mathrm{m_{p+1}^{p-1}} \\ \mathrm{m}_1^\mathrm{p} & \mathrm{m}_2^\mathrm{p} & \ldots & \mathrm{m_p^p} & \mathrm{m_{p+1}^p} \end{vmatrix} \tag{100}$$

$$= \prod_{\mathrm{k}=1}^{\mathrm{p}} \prod_{\mathrm{k'}=\mathrm{k}+1}^{\mathrm{p}} (\mathrm{m_{k'}} - \mathrm{m_k}) > 0, \tag{101}$$

one can verify that the cited restriction assures that the degree of polynomial Wronskian (98) is equal to [63]

$$\mathfrak{N}(\overline{M}_p) = \sum_{k=1}^{p} m_k - {}^{1}/_{2}p(p-1). \tag{102}$$

Substituting the Wronskians of the q-RSs ${}_{i}\phi_{-,m_k}[\eta;\lambda_o]$,

$$W\{{}_{i}\phi_{-,m_{k=1,\,\dots,\,p}}[\eta;\lambda_o]\} = {}_{i}\phi^{p}[\eta;-\lambda_o]_{i}\mathfrak{W}_{\mathfrak{N}(\overline{M}_p)}[\eta;\lambda_o|-\vdots\overline{M}_p], \tag{103}$$

into Schulze-Halberg's [64] general formula for the DCT of the zero-energy free term of the generic CSLE,

$$\begin{gathered}{}_{i}I^{o}[\eta;\lambda_o|-\vdots\overline{M}_p] = {}_{i}I^{o}[\eta;\lambda_o]+\\ 2\sqrt{{}_{i}\boldsymbol{\rho}_{\diamondsuit}[\eta]}\frac{d}{d\eta}\frac{ld\, W\{{}_{i}\phi_{-,m_{k=1,\,\dots,\,p}}[\eta;\lambda_o]\}}{\sqrt{{}_{i}\boldsymbol{\rho}_{\diamondsuit}[\eta]}} - p(p-2)\mathfrak{I}\{{}_{i}\boldsymbol{\rho}_{\diamondsuit}[\eta]\},\end{gathered} \tag{104}$$

coupled with (51), then combining all the second-order poles at $\eta \neq \pm 1$ into Quesne PF (A19) with the monomial product replaced with polynomial Wronskian (98), one finds

$$\begin{gathered}{}_{i}I^{o}[\eta;\lambda_o|-\vdots\overline{M}_p] = {}_{i}I^{o}[\eta;\lambda_o] + 2p\sqrt{{}_{i}\boldsymbol{\rho}_{\diamondsuit}[\eta]}\frac{d}{d\eta}\frac{ld\,{}_{i}\phi[\eta;-\lambda_o+{}^{1}/_{2}p-1]}{\sqrt{{}_{i}\boldsymbol{\rho}_{\diamondsuit}[\eta]}}+\\ 2Q[\eta;\lambda_o|-\vdots\overline{M}_p] + p\,ld(\eta^2+1)ld\,{}_{i}\mathfrak{W}_{\mathfrak{N}(\overline{M}_p)}[\eta;\lambda_o|-\vdots\overline{M}_p].\end{gathered} \tag{105}$$

Setting

$$\widehat{Q}[\eta;\lambda_o|-\vdots\overline{M}_p] := -{}^{1}/_{2}\,{}_{i}\mathfrak{W}_{\mathfrak{N}(\overline{M}_p)}[\eta;\lambda_o|-\vdots\overline{M}_p]\frac{d^2}{d\eta^2}\,{}_{i}\mathfrak{W}^{-1}_{\mathfrak{N}(\overline{M}_p)}[\eta;\overline{\eta}_m], \tag{106}$$

again replacing the monomial product in the right-hand side of (A22) for polynomial Wronskian (98) and also making use of (2), (4), (A13) and (A14), coupled with

$${}^{1}/_{4}[{}_{i}O^{o}(\lambda_o) - {}_{i}O^{o}(\lambda_o-p)] = p(a-{}^{1}/_{2}p), \tag{107}$$

we come to the following sought-for expression:

$$\begin{gathered}{}_{i}I^{o}[\eta;\lambda_o|-\vdots\overline{M}_p] = {}_{i}I^{o}[\eta;\lambda_o-p] + 2\widehat{Q}[\eta;\lambda_o|-\vdots\overline{M}_p]\\ +\frac{{}_{i}\widehat{O}_{\mathfrak{N}(\overline{M}_p)}[\eta;\lambda_o|-\vdots\overline{M}_p]}{4(\eta^2+1)\,{}_{i}\mathfrak{W}_{\mathfrak{N}(\overline{M}_p)}[\eta;\lambda_o|-\vdots\overline{M}_p]}\end{gathered} \tag{108}$$

with

$$\begin{gathered}{}_{i}\widehat{O}_{\mathfrak{N}(\overline{M}_p)}[\eta;\lambda_o|-\vdots\overline{M}_p] := 4p_{i}\ddot{\mathfrak{W}}_{\mathfrak{N}(\overline{M}_p)}[\eta;\lambda_o|-\vdots\overline{M}_p]\\ +8p\eta\,{}_{i}\dot{\mathfrak{W}}_{\mathfrak{N}(\overline{M}_p)}[\eta;\lambda_o|-\vdots\overline{M}_p].\end{gathered} \tag{109}$$

For $p = 1$, polynomial Wronskian (98) and polynomial (109) turn into the Routh polynomial and polynomial (67), respectively, and we come back to RefPF (61).

Theorem 2. W$\mathfrak{I}$s *of infinitely many Routh polynomials with a common index form an X-DPS.*

Proof of Theorem 2: Let us first show that the gauge transformation

$${}_{i}\Phi[\eta;\lambda_o;\varepsilon|-\vdots\overline{M}_p] = {}_{i}\phi[\eta;\lambda_o|-\vdots\overline{M}_p]\,{}_{i}F[\eta;\lambda_o;\varepsilon|-\vdots\overline{M}_p] \tag{110}$$

with

$$ {}_{i}\phi[\eta;\lambda_\mathrm{o}|-\vdots\overline{\mathrm{M}}_\mathrm{p}] := \frac{{}_{i}\phi[\eta;\mathrm{p}-\lambda_\mathrm{o}]}{{}_{i}\mathfrak{W}_{\mathcal{N}(\overline{\mathrm{M}}_\mathrm{p})}[\eta;\lambda_\mathrm{o}|-\vdots\overline{\mathrm{M}}_\mathrm{p}]} \tag{111} $$

converts the CSLE

$$ \left\{\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} + {}_{i}\mathrm{I}^\mathrm{o}[\eta;\lambda_\mathrm{o}|-\vdots\overline{\mathrm{M}}_\mathrm{p}] + \varepsilon\, {}_{i}\boldsymbol{\rho}_\Diamond[\eta]\right\} {}_{i}\Phi[\eta;\lambda_\mathrm{o};\varepsilon|-\vdots\overline{\mathrm{M}}_\mathrm{p}] = 0 \tag{112} $$

into the second-order ODE

$$ \{\mathbf{D}[\eta;\lambda_\mathrm{o}|-\vdots\overline{\mathrm{M}}_\mathrm{p}] + {}_{i}\mathrm{C}^\mathrm{o}_{\mathcal{N}(\overline{\mathrm{M}}_\mathrm{p})}[\eta;\lambda_\mathrm{o};\varepsilon|-\vdots\overline{\mathrm{M}}_\mathrm{p}]\}{}_{i}\mathrm{F}[\eta;\lambda_\mathrm{o};\varepsilon|-\vdots\overline{\mathrm{M}}_\mathrm{p}] = 0 \tag{113} $$

with polynomial coefficients. Namely, taking into account that

$$ \begin{aligned} {}_{i}\ddot{\phi}[\eta;\lambda_\mathrm{o}|-\vdots\overline{\mathrm{M}}_\mathrm{p}]/{}_{i}\phi[\eta;\lambda_\mathrm{o}|-\vdots\overline{\mathrm{M}}_\mathrm{p}] &= \tfrac{(\lambda_\mathrm{o}-\mathrm{p})^2-1}{4(\eta+i)^2} + \tfrac{(\lambda_\mathrm{o}^*-\mathrm{p})^2-1}{4(\eta-i)^2} \\ &+ \frac{2\Re_1^{(\mathrm{p}-\lambda_\mathrm{o})}[\eta]_{i}\dot{\mathfrak{W}}_{\mathcal{N}(\overline{\mathrm{M}}_\mathrm{p})}[\eta;\lambda_\mathrm{o}|-\vdots\overline{\mathrm{M}}_\mathrm{p}]}{(\eta^2+1)_{i}\mathfrak{W}_{\mathcal{N}(\overline{\mathrm{M}}_\mathrm{p})}[\eta;\lambda_\mathrm{o}|-\vdots\overline{\mathrm{M}}_\mathrm{p}]} \\ &-2\overset{\frown}{\mathrm{Q}}[\eta;\lambda_\mathrm{o}|-\vdots\overline{\mathrm{M}}_\mathrm{p}], \end{aligned} \tag{114} $$

one finds

$$ \begin{aligned} \mathbf{D}[\eta;\lambda_\mathrm{o}|-\vdots\overline{\mathrm{M}}_\mathrm{p}] &:= (\eta^2+1)\,{}_{i}\mathfrak{W}_{\mathcal{N}(\overline{\mathrm{M}}_\mathrm{p})}[\eta;\lambda_\mathrm{o}|-\vdots\overline{\mathrm{M}}_\mathrm{p}]\tfrac{\mathrm{d}^2}{\mathrm{d}\eta^2} \\ &+{}_{i}\tau_{\mathcal{N}(\overline{\mathrm{M}}_\mathrm{p})+1}[\eta;\lambda_\mathrm{o}|-\vdots\overline{\mathrm{M}}_\mathrm{p}]\tfrac{\mathrm{d}}{\mathrm{d}\eta}, \end{aligned} \tag{115} $$

where

$$ \begin{aligned} {}_{i}\tau_{\mathcal{N}(\overline{\mathrm{M}}_\mathrm{p})+1}[\eta;\lambda_\mathrm{o}|-\vdots\overline{\mathrm{M}}_\mathrm{p}] &= 2\Re_1^{(\mathrm{p}-\lambda_\mathrm{o})}[\eta]_{i}\mathfrak{W}_{\mathcal{N}(\overline{\mathrm{M}}_\mathrm{p})}[\eta;a+ib|-\vdots\overline{\mathrm{M}}_\mathrm{p}] \\ &-2(\eta^2+1)_{i}\dot{\mathfrak{W}}_{\mathcal{N}(\overline{\mathrm{M}}_\mathrm{p})}[\eta;a+ib|-\vdots\overline{\mathrm{M}}_\mathrm{p}], \end{aligned} \tag{116} $$

$$ \begin{aligned} {}_{i}C^\mathrm{o}_{\mathcal{N}(\overline{\mathrm{M}}_\mathrm{p})}[\eta;\lambda_\mathrm{o};\varepsilon|-\vdots\overline{\mathrm{M}}_\mathrm{p}] &= {}_{i}C^\mathrm{o}_{\mathcal{N}(\overline{\mathrm{M}}_\mathrm{p})}[\eta;\lambda_\mathrm{o}|-\vdots\overline{\mathrm{M}}_\mathrm{p}]+ \\ &\varepsilon\,{}_{i}\mathfrak{W}_{\mathcal{N}(\overline{\mathrm{M}}_\mathrm{p})}[\eta;\lambda_\mathrm{o}|-\vdots\overline{\mathrm{M}}_\mathrm{p}], \end{aligned} \tag{117} $$

and

$$ \begin{aligned} {}_{i}C^\mathrm{o}_{\mathcal{N}(\overline{\mathrm{M}}_\mathrm{p})}[\eta;\lambda_\mathrm{o}|-\vdots\overline{\mathrm{M}}_\mathrm{p}] &= 1/4\,{}_{i}\overset{\frown}{\mathrm{O}}{}^\mathrm{o}_{\mathcal{N}(\overline{\mathrm{M}}_\mathrm{p})}[\eta;\lambda_\mathrm{o}|-\vdots\overline{\mathrm{M}}_\mathrm{p}] \\ &-2\Re_1^{(\mathrm{p}-\lambda_\mathrm{o})}[\eta]_{i}\dot{\mathfrak{W}}_{\mathcal{N}(\overline{\mathrm{M}}_\mathrm{p})}[\eta;a+ib|-\vdots\overline{\mathrm{M}}_\mathrm{p}] \end{aligned} \tag{118} $$

$$ = \mathrm{p}_{i}\ddot{\mathfrak{W}}_{\mathcal{N}(\overline{\mathrm{M}}_\mathrm{p})}[\eta;\lambda_\mathrm{o}|-\vdots\overline{\mathrm{M}}_\mathrm{p}] - 2\Re_1^{(-\lambda_\mathrm{o})}[\eta]_{i}\dot{\mathfrak{W}}_{\mathcal{N}(\overline{\mathrm{M}}_\mathrm{p})}[\eta;a+ib|-\vdots\overline{\mathrm{M}}_\mathrm{p}]. \tag{119} $$

Choosing

$$ \begin{gathered} {}_{i}\Phi[\eta;\lambda_\mathrm{o};{}_{i}\varepsilon_{-,\mathrm{n}}(\lambda_\mathrm{o})|-\vdots\overline{\mathrm{M}}_\mathrm{p},n] = {}_{i}\phi[\eta;\lambda_\mathrm{o}|-\vdots\overline{\mathrm{M}}_\mathrm{p}]\mathfrak{W}_{\mathcal{N}(\overline{\mathrm{M}}_\mathrm{p},\mathrm{n})}[\eta;\lambda_\mathrm{o}|-\vdots\overline{\mathrm{M}}_\mathrm{p},\mathrm{n}] \\ \text{for anyn} \notin \overline{\mathrm{M}}_\mathrm{p}, \end{gathered} \tag{120} $$

we find that the polynomial Wronskians under consideration satisfy the ODE of the Bochner type as follows:

$$\left\{\mathbf{D}[\eta;\lambda_o|-\vdots\overline{M}_p]+{}_iC^o_{\mathfrak{N}(\overline{M}_p)}[\eta;\lambda_o;{}_i\varepsilon_{-,n}(\lambda_o)|-\vdots\overline{M}_p]\right\}\times \\ \mathfrak{W}_{\mathfrak{N}(\overline{M}_p,n)}[\eta;\lambda_o|-\vdots\overline{M}_p,n]=0 \\ \text{for any} n\notin\overline{M}_p \tag{121}$$

and thereby form an X-DPS, which completes the proof of Theorem 2. □

Here, we are only interested in X-DPSs containing finite EOP sequences. If polynomial Wronskian (99) does not have real zeros, we term such a set of seed Routh polynomials 'admissible' and mark it with the symbolic expression $\overline{\mathbf{M}}_p$.

Theorem 3. *Every X-DPS* $(-\vdots\overline{\boldsymbol{M}}_p, n((n\notin\overline{\boldsymbol{M}}_p))$ *contains a finite EOP sequence starting from a polynomial of the degree* $\mathfrak{N}(\overline{\boldsymbol{M}}_p)-p$.

Proof of Theorem 3: By analogy with the discussion presented by us in the previous subsection, we represent CSLE (112) in the prime form as

$$\left\{\frac{d}{d\eta}(\eta^2+1)^{1/2}\frac{d}{d\eta}-{}_i\mathcal{q}\left[\eta;\lambda_o|-\vdots\overline{\mathbf{M}}_p\right]+\varepsilon\,{}_i\mathcal{w}[\eta]\right\}{}_i\Psi[\eta;\lambda_o\,;\varepsilon|-\vdots\overline{\mathbf{M}}_p]=0, \tag{122}$$

where

$$ {}_i\mathcal{q}\,[\eta;\lambda_o|-\vdots\overline{\mathbf{M}}_p]=-\sqrt{\eta^2+1}\,({}_iI^o\,[\eta;\lambda_o|-\vdots\overline{m}_p]+\mathfrak{I}\left\{\sqrt{\eta^2+1}\right\}). \tag{123}$$

Prime SLE (122) is then solved under the DBCs

$$\lim_{\eta\to\pm\infty}{}_i\Psi[\eta;a+ib\,;{}_i\varepsilon_n(a)|-\vdots\overline{\mathbf{M}}_p]=0. \tag{124}$$

Setting

$$f[\eta]=W\left\{{}_i\phi_{-,m_{k=1,\,\dots,\,p}}[\eta;\lambda_o]\right\} \tag{125}$$

in (51), (54) and (55) shows that

$$\lim_{|\eta|\to\infty}(\eta^2{}_iI^o[\eta;\lambda_o|-\vdots\overline{\mathbf{M}}_p])=\lim_{|\eta|\to\infty}(\eta^2{}_iI^o[\eta;\lambda_o]) \tag{126}$$

by analogy with (57). We have thus proven that the RDCT in question does not change the ExpDiff of the CSLE at infinity. As it has been already stressed in Section 3.1, this is the common feature of the Fuchsian CSLEs with the density functions decreasing as $1/\eta^2$ with large values of $|\eta|$.

It directly follows from the latter relation that the zero-energy free term (123) in prime SLE (122) satisfies the following asymptotic relation:

$$\lim_{\eta\to\pm\infty}|\eta\,{}_i\mathcal{q}\,[\eta;\lambda_o|-\vdots\overline{\mathbf{M}}_p]|\;=\lim_{\eta\to\pm\infty}|\eta\,{}_i\mathcal{q}\,[\eta;\lambda_o]|<\infty, \tag{127}$$

similar to (72), which confirms that the two ChExps for the pole of prime SLE (68) at infinity differ only by sign and, therefore, OBCs unambiguously select the PFS near the given singularity.

Keeping in mind that

$$\mathfrak{N}(\overline{\mathbf{M}}_p,n)-\mathfrak{N}(\overline{\mathbf{M}}_p)=n-p, \tag{128}$$

we conclude that each Dirichlet problem formulated in such a way has a finite subset of quasi-rational eigenfunctions,

$$ {}_{i}\psi_{\mathbf{c},\underset{\sim}{\mathrm{n}}(\overline{\mathbf{M}}_{\mathrm{p}},\mathrm{n})}[\eta;\lambda_{\mathrm{o}}|-\vdots\overline{\mathbf{M}}_{\mathrm{p}}] = {}_{i}\psi_{\mathbf{c},0}[\eta;\lambda_{\mathrm{o}}-\mathrm{p}]\,\frac{{}_{i}\mathcal{W}_{\mathcal{N}(\overline{\mathbf{M}}_{\mathrm{p}},\mathrm{n})}[\eta;\lambda_{\mathrm{o}}|-\vdots\overline{\mathbf{M}}_{\mathrm{p}},\mathrm{n}]}{{}_{i}\mathcal{W}_{\mathcal{N}(\overline{\mathbf{M}}_{\mathrm{p}})}[\eta;\lambda_{\mathrm{o}}|-\vdots\overline{\mathbf{M}}_{\mathrm{p}}]}, \tag{129} $$

which obey the DBCs

$$ \lim_{\eta\to\pm\infty} {}_{i}\psi_{\mathbf{c},\underset{\sim}{\mathrm{n}}(\overline{\mathbf{M}}_{\mathrm{p}},\mathrm{n})}[\eta;\lambda_{\mathrm{o}}|-\vdots\overline{\mathbf{M}}_{\mathrm{p}}] = 0 \tag{130} $$

for $0 \le \mathrm{n} < a - 1/2$ and, therefore, are mutually orthogonal with the weight ${}_{i}\mathcal{w}[\eta]$ as follow:

$$ \begin{gathered} \int_{-\infty}^{\infty} {}_{i}\psi_{\mathbf{c},\underset{\sim}{\mathrm{n}}(\overline{\mathbf{M}}_{\mathrm{p}},\mathrm{n})}[\eta;\lambda_{\mathrm{o}}|-\vdots\overline{\mathbf{M}}_{\mathrm{p}}]\; {}_{i}\psi_{\mathbf{c},\underset{\sim}{\mathrm{n}}(\overline{\mathbf{M}}_{\mathrm{p}},\mathrm{n}')}[\eta;\lambda_{\mathrm{o}}|-\vdots\overline{\mathbf{M}}_{\mathrm{p}}]\,{}_{i}\mathcal{w}[\eta]\,d\eta = 0 \\ \text{for } 0 \le \mathrm{n} < \mathrm{n}' < a - 1/2. \end{gathered} \tag{131} $$

Their order number, $\underset{\sim}{\mathrm{n}}(\overline{\mathbf{M}}_{\mathrm{p}},\mathrm{n})$, for $\mathrm{p} > 1$ will be explicitly specified in next three subsections, while, as demonstrated earlier,

$$ \underset{\sim}{\mathrm{n}}(\overline{\mathbf{M}}_{1},\mathrm{n}) = \mathrm{n}+1. \tag{132} $$

The polynomial Wronskians in the numerator of the PF in the right-hand side of (129) are thereby orthogonal with the weight

$$ {}_{i}W[\eta;\lambda_{\mathrm{o}}|-\vdots\overline{\mathbf{M}}_{\mathrm{p}}] \equiv \frac{{}_{i}\phi^{2}_{\mathbf{c},0}[\eta;\lambda_{\mathrm{o}}-\mathrm{p}]}{(\eta^{2}+1)\,{}_{i}\mathcal{W}^{2}_{\mathcal{N}(\overline{\mathbf{M}}_{\mathrm{p}})}[\eta;\lambda_{\mathrm{o}}|-\vdots\overline{\mathbf{M}}_{\mathrm{p}}]}, \tag{133} $$

i.e.,

$$ \begin{gathered} \int_{-\infty}^{\infty} {}_{i}\mathcal{W}_{\mathcal{N}(\overline{\mathbf{M}}_{\mathrm{p}},\mathrm{n})}[\eta;\lambda_{\mathrm{o}}|-\vdots\overline{\mathbf{M}}_{\mathrm{p}},\mathrm{n}\Big\}\,{}_{i}\mathcal{W}_{\mathcal{N}(\overline{\mathbf{M}}_{\mathrm{p}},\mathrm{n}')}[\eta;\lambda_{\mathrm{o}}|-\vdots\overline{\mathbf{M}}_{\mathrm{p}},\mathrm{n}'\Big\}\,{}_{i}W[\eta;\lambda_{\mathrm{o}}|-\vdots\overline{\mathbf{M}}_{\mathrm{p}}]\,d\eta = 0, \\ \text{for } 0 \le \mathrm{n} < \mathrm{n}' < a - 1/2 \end{gathered} \tag{134} $$

Therefore, these polynomials form a finite EOP sequence starting from the polynomial of the positive degree

$$ \begin{gathered} \mathcal{N}(\overline{\mathbf{M}}_{\mathrm{p}}) - \mathrm{p} = \sum_{\mathrm{k}=1}^{\mathrm{p}} \mathrm{m}_{\mathrm{k}} - 1/2\,\mathrm{p}(\mathrm{p}+1) > 0 \\ \text{if } \mathrm{m}_{\mathrm{k}} > \mathrm{p} \end{gathered} \tag{135} $$

which completes the proof of Theorem 3. □

Like any other q-RS of this type, the eigenfunctions of prime SLE (122) satisfy the following asymptotic relation:

$$ \lim_{|\eta|\to\infty} |\eta\; ld\,{}_{i}\psi_{\mathbf{c},\mathrm{n}}[\eta;\lambda_{\mathrm{o}}|-\vdots\overline{\mathbf{M}}_{\mathrm{p}}]| < \infty, \tag{136} $$

which will be used in Appendix C to prove that the so-called 'generalized' [65] Wronskian ($\mathcal{g}$ − W) vanishes at $\pm\infty$.

In the next subsection, we discuss the subnets of EOP sequences with exactly the same number of polynomials in every sequence from the given subnet. Each subnet starts from one of the EOS sequences introduced in the previous subsection.

3.3. Subnets of Finite EOP Sequences with No Gaps in Polynomial Degrees

Let $\overline{\mathbf{M}}_{\mathrm{p}}^{\mathrm{e,o}}$ specify a monotonically increasing set of alternating even an odd degrees of seed Routh polynomials as follows:

$$\overline{\mathbf{M}}_{\mathrm{p}}^{\mathrm{e,o}} = 2\mathrm{j}_1, 2\mathrm{j}_2 + 1, 2\mathrm{j}_3, \ldots, 2\mathrm{j}_{\mathrm{p}-1} + 1 - \ell,\ 2\mathrm{j}_{\mathrm{p}} + \ell, \tag{137}$$

where

$$a - 1/2 <\ \mathrm{j}_1 \leq \mathrm{j}_2 < \mathrm{j}_3 \leq \cdots \leq \mathrm{j}_{\mathrm{J}+\ell-1} \leq \mathrm{j}_{\mathrm{J}+\ell}$$

or, to be more precise,

$$\mathrm{j}_{\mathrm{J}+\ell-2} \leq\ \mathrm{j}_{\mathrm{J}+\ell-1} < \mathrm{j}_{\mathrm{J}+\ell}\ \ if\ \ \ell = 0\ \ \ or\ \ \mathrm{j}_{\mathrm{J}+\ell-2} < \mathrm{j}_{\mathrm{J}+\ell-1} \leq \mathrm{j}_{\mathrm{J}+\ell}\ \ if\ \ \ell = 1,$$

so the corresponding polynomial Wronskian (98) for partition (137) has the even degree

$$\mathfrak{N}(\overline{\mathbf{M}}_{\mathrm{p}}^{\mathrm{e,o}}) = \sum_{\mathrm{k}=1}^{\mathrm{p}} \mathrm{m}_{\mathrm{k}} - 1/2\mathrm{p}(\mathrm{p}-1). \tag{138}$$

(Note that inequality (A8) necessarily holds, since the degree of each Routh polynomial is larger than $a - 1/2$).

The TFs for the sequential LDTs generating each subnet of the solvable CSLEs under consideration have the following quasi-rational form:

$$\begin{aligned}\phi[\eta;\boldsymbol{\lambda}_{\mathrm{o}}| - \vdots\overline{\mathbf{M}}_{\mathrm{p}}^{\mathrm{e,o}}, 2\mathbf{j}_{\mathrm{p}+1} + 1 - \ell_{\mathrm{p}}] = {}_{i}\phi_{\mathbf{c},0}[\eta;\lambda_{\mathrm{o}} - \mathrm{p}] \\ \times \frac{{}_{i}\mathcal{W}_{\mathfrak{N}(\overline{\mathrm{m}}_{\mathrm{p}}^{\mathrm{e,o}};2\mathrm{j}_{\mathrm{p}+1}+1-\ell_{\mathrm{p}})}[\eta;\lambda_{\mathrm{o}}|-\vdots\overline{\mathbf{M}}_{\mathrm{p}}^{\mathrm{e,o}};2\mathbf{j}_{\mathrm{p}+1}+1-\ell_{\mathrm{p}}]}{{}_{i}\mathcal{W}_{\mathfrak{N}(\overline{\mathbf{M}}_{\mathrm{p}}^{\mathrm{e,o}})}[\eta;\lambda_{\mathrm{o}}|-\vdots\overline{\mathbf{M}}_{\mathrm{p}}^{\mathrm{e,o}}]}\end{aligned} \tag{139}$$

with

$${}_{i}\phi_{\mathbf{c},0}[\eta;\lambda_{\mathrm{o}} - \mathrm{p}] \equiv {}_{i}\boldsymbol{\rho}_{\diamondsuit}^{-1/2\mathrm{p}}[\eta]\ {}_{i}\phi_{\mathbf{c},0}[\eta;\lambda_{\mathrm{o}}]. \tag{140}$$

By definition, each q-RS (139) lies at the energy,

$${}_{i}\varepsilon_{-,\,\mathrm{j}_{\mathrm{p}+1}+\ell_{\mathrm{p}+1}}(a) = -(2\mathbf{j}_{\mathrm{p}+1} + \ell_{\mathrm{p}+1} + 1/2 -\ a)^2, \tag{141}$$

below the lowest eigenvalue,

$${}_{i}\varepsilon_{-,\,\mathrm{j}_{\mathrm{p}}+\ell_{\mathrm{k}}}(a) = -(2\mathbf{j}_{\mathrm{p}} + 1/2 + \ell_{\mathrm{p}} -\ a)^2, \tag{142}$$

of RCSLE (112) with

$$\mathrm{M}_{\mathrm{p}} = \overline{\mathbf{M}}_{\mathrm{p}}^{\mathrm{e,o}} \tag{143}$$

or, to be more precise,

$$\begin{gathered}{}_{i}\varepsilon_{-,\,\mathrm{j}_{\mathrm{p}+1}+\ell_{\mathrm{p}+1}}(a) - {}_{i}\varepsilon_{-,\,\mathrm{j}_{\mathrm{p}}+\ell_{\mathrm{p}}}(a) = \\ -4\,(\mathrm{j}_{\mathrm{p}+1} - \mathrm{j}_{\mathrm{p}} + 1/2)(\mathrm{j}_{\mathrm{p}} + \mathrm{j}_{\mathrm{p}+1} + 1 -\ a) < 0,\end{gathered} \tag{144}$$

keeping in mind that

$$a - 1/2 <\ \mathrm{j}_{\mathrm{p}} \leq j_{\mathrm{p}+1}. \tag{145}$$

As the direct consequence of the disconjugacy theorem, we conclude that q-RS (139) may not have more than one node. On other hand, the polynomial Wronskian in the numerator of the PF in the right-hand side of (139) has an even degree by its definition and, therefore, may have only an even number of real zeros. We have thus proven that TFs (139) for the sequential RLDTs of RCSLE (112) using the finite sets of seed solutions specified by the partitions (137) are all nodeless; therefore, the polynomial Wronskians forming eigenfunctions (129) are mutually orthogonal with weight (133).

Theorem 4. *When applied to an exactly solvable prime SLE (122), an LDT with a finite quasi-rational TF diverging at infinity leads to the exactly solvable prime SLE with the inserted lowest-energy eigenvalue at the energy of the given q-RS.*

Proof of Theorem 4: First, let us remind the reader that the function

$$\psi_{\mathbf{c},0}[\eta;\lambda_o| - \vdots\overline{\mathbf{M}}^{e,o}_{p+1}] = {}_i\psi^{-1}_{\mathbf{c},0}[\eta;\lambda_o - p] \times \frac{{}_i\mathcal{W}_{\mathcal{N}(\overline{\mathbf{M}}^{e,o}_p)}[\eta;\lambda_o|-\vdots\overline{\mathbf{M}}^{e,o}_p]}{{}_i\mathcal{W}_{\mathcal{N}(\overline{\mathbf{M}}^{e,o}_{p+1})}[\eta;\lambda_o|-\vdots\overline{\mathbf{M}}^{e,o}_{p+1}]} \tag{146}$$

is a solution of prime SLE (122) assuming that partition (143) is replaced with $\overline{\mathbf{M}}^{e,o}_{p+1}$. Also note that the difference between the polynomial degrees

$$\mathcal{N}(\overline{\mathbf{M}}^{e,o}_p) - \mathcal{N}(\overline{\mathbf{M}}^{e,o}_{p+1}) = 2j_p - 2j_{p+1} - p + \ell_p - \ell_{p+1} \leq 0 \tag{147}$$

is non-positive, since $\ell_p = 0$, $\ell_{p+1} = 1$ if $j_{p+1} = j_p$, whereas $|\ell_p - \ell_{p+1}| = 1$ regardless of the value of $j_{p+1} \geq j_p$. We thus conclude that nodeless function (146) obeys the DBCs at infinity and, therefore, represents the lowest-energy eigenfunction of the prime SLE in question.

Now, we can simply reproduce the arguments presented in Section 3.1 in support of Theorem 1, with TF (42) and RCSLE (46) replaced by TF (139) and RCSLE (112), respectively, for partition (143). □

Note that Theorem 4 was formulated by us in more general terms than the proof itself, which was formally applied to prime SLE (122) with $\overline{\mathbf{M}}_p$ defined via (137). It is worth stressing that the presented arguments are based on the following two presumptions:

(i) The LDT applied to prime SLE (122) has a quasi-rational TF, with no zeros on the real axis;
(ii) Its reciprocal obeys the DBCs at infinity and, therefore, represents the lowest- energy eigenfunction of the new Sturm–Liouville problem.

In next subsection, we re-use Theorem 4 for a different choice of admissible TFs covered by the above presumptions.

3.4. Exact Solvability of RDTs of $\Re$Ref CSLE Constructed Using Only Seed Functions Composed of R-Routh Polynomials

Let $\overline{\mathbf{N}}_{2J,L}$ be J 'juxtaposed' [66–68] pairs of seed solutions,

$$\begin{gathered}{}_i\phi_{-,n}[\eta;a+ib] = {}_i\phi_{-,0}[\eta;a+ib]R^{(-2b,-a)}_n(\eta) \\ \text{for } n = 1,\ldots,\ \lfloor a - 1/2 \rfloor,\end{gathered} \tag{148}$$

with polynomial degrees (n) forming L segments of even lengths as follows:

$$\begin{gathered}\overline{\mathbf{N}}_{2J,L} = n_1 : n_1 + 2j_1 - 1, n_2 : n_2 + 2j_2 - 1, \ldots, n_L : n_{2j} \\ (n_1 > 0,\ \ n_{2j} \leq \lfloor a - 1/2 \rfloor).\end{gathered} \tag{149}$$

Polynomial Wronskian (98) turns into the following Wronskian of R-Routh polynomials:

$$\begin{gathered}{}_i\mathcal{W}_{\mathcal{N}(\overline{\mathbf{N}}_{2J,L})}[\eta;a+ib| - \vdots\overline{\mathbf{N}}_{2J,L}] = W\left\{R^{(-2b,-a)}_{n_{k=1,2,\ldots,2J}}[\eta]\right\} \\ \text{for } n_{k=1,\ \ldots,\ 2J} \in \overline{\mathbf{N}}_{2J,L},\end{gathered} \tag{150}$$

which is expected to have more specific features specified by conjectures in [63] for zeros of the generic Wronskian of orthogonal polynomials. It has been proven by Karlin and Szegő [69] that the Wronskian of an even number of orthogonal polynomials with sequential degrees,

$$\overline{\mathbf{N}}_{2j_1,1} = \mathrm{n}_1 : \mathrm{n}_1 + 2\mathrm{j}_1 - 1, \tag{151}$$

may not have real zeros, and here we extend this proof to the Wronskians of the R-Routh polynomials for the 'admissible' [63] partitions (149).

If polynomial Wronskian (150) does not have real zeros, then each q-RS,

$$\begin{aligned} {}_i\psi_{\mathbf{c},\underset{\sim}{\mathrm{n}}(\overline{\mathbf{N}}_{2J,L},\mathrm{n})}[\eta;\lambda_o|-\vdots\overline{\mathbf{N}}_{2J,L}] = {}_i\psi_{\mathbf{c},0}[\eta;\lambda_o-2J]\times \\ \frac{{}_i\mathcal{W}_{\mathfrak{N}(\overline{\mathbf{N}}_{2J,L},\mathrm{n})}[\eta;\lambda_o|-\vdots\overline{\mathbf{N}}_{2J,L},\mathrm{n}]}{{}_i\mathcal{W}_{\mathfrak{N}(\overline{\mathbf{N}}_{2J,L})}[\eta;\lambda_o|-\vdots\overline{\mathbf{N}}_{2J,L}]} \quad (\mathrm{n}\notin\overline{\mathbf{N}}_{2J,L}), \end{aligned} \tag{152}$$

satisfies the DBCs at infinity and, therefore, represents the eigenfunction of the prime SLE,

$$\left\{\frac{d}{d\eta}(\eta^2+1)^{1/2}\frac{d}{d\eta} - {}_i\mathcal{q}\left[\eta;\lambda_o|-\vdots\overline{\mathbf{N}}_{2J,L}\right] + \varepsilon\,{}_i\mathcal{w}[\eta]\right\}{}_i\Psi[\eta;\lambda_o;\varepsilon|-\vdots\overline{\mathbf{N}}_{2J,L}] = 0 \tag{153}$$

for the energy of $\varepsilon = \varepsilon_{\mathbf{c},\mathrm{n}}(a)$. However, to proceed with mathematical induction, we also need to prove that the Dirichlet problem in question does not have eigenfunctions other than q-RSs (152); this is the most challenging part of the proof presented below. In particular, we need to confirm that the eigenfunction at the lowest energy of $\varepsilon_{\mathbf{c},0}(a)$ is nodeless.

First, taking into account that

$$\dot{\Re}_{\mathrm{m}}^{(-a-ib)}[\eta] = 1/2(\mathrm{m}+1-2a)\Re_{\mathrm{m}-1}^{(1-a-ib)}[\eta], \tag{154}$$

we can represent the aforementioned eigenfunction as

$$\begin{aligned} {}_i\psi_{\mathbf{c},\underset{\sim}{\mathrm{n}}(\overline{\mathbf{N}}_{2J,L},0)}[\eta;\lambda_o|-\vdots\overline{\mathbf{N}}_{2J,L}] = \frac{1}{4^J}\prod_{\mathrm{m}\in\overline{\mathbf{N}}_{2J,L}}(\mathrm{m}+\mathrm{n}_1-2a)\times \\ {}_i\psi_{\mathbf{c},0}[\eta;\lambda_o-2J]\,\frac{{}_i\mathcal{W}_{\mathfrak{N}(\overline{\mathbf{N}}_{2J,L})-2J}[\eta;\lambda_o-1|-\vdots\overline{\mathbf{N}}_{2J,L}-\overline{1}_{2J}]}{{}_i\mathcal{W}_{\mathfrak{N}(\overline{\mathbf{N}}_{2J,L})}[\eta;\lambda_o|-\vdots\overline{\mathbf{N}}_{2J,L}]}, \end{aligned} \tag{155}$$

where the symbolic expression $\overline{1}_{2J}$ stands for the 2J-element row formed by ones. The crucial point is that the partition $\overline{\mathbf{N}}_{2J,L}-\overline{1}_{2J}$ starts from the positive integer $\mathrm{n}_1 - 1$ for any $\mathrm{n}_1 > 1$.

Let us now use the eigenfunction

$$\begin{aligned} {}_i\phi_{\mathbf{c},\underset{\sim}{\mathrm{n}}(\overline{\mathbf{N}}_{2J,L},0)}[\eta;\lambda_o|-\vdots\overline{\mathbf{N}}_{2J,L}] = \\ (\eta^2+1)^{-1/4}{}_i\psi_{\mathbf{c},\underset{\sim}{\mathrm{n}}(\overline{\mathbf{N}}_{2J,L},0)}[\eta;\lambda_o|-\vdots\overline{\mathbf{N}}_{2J,L}] \end{aligned} \tag{156}$$

as the TF for the RLDT and show that

$$\begin{aligned} {}_i\mathrm{I}^o[\eta;\lambda_o|-\vdots\overline{\mathbf{N}}_{2J,L}] + 2\sqrt{{}_i\rho_\diamondsuit[\eta]}\,\frac{d}{d\eta}\frac{ld_i\phi_{\mathbf{c},\underset{\sim}{\mathrm{n}}(\overline{\mathbf{N}}_{2J,L},0)}[\eta;\lambda_o|-\vdots\overline{\mathbf{N}}_{2J,L}]}{\sqrt{{}_i\rho_\diamondsuit[\eta]}} \\ +\mathcal{S}\left\{{}_i\rho_\diamondsuit[\eta]\right\} = {}_i\mathrm{I}^o[\eta;\lambda_o-1|-\vdots\overline{\mathbf{N}}_{2J,L}-\overline{1}_{2J}], \end{aligned} \tag{157}$$

where, according to (103) and (104),

$$\begin{aligned} {}_i\mathrm{I}^{o}[\eta;\lambda|-\vdots\overline{\mathrm{M}}_{\mathrm{p}}] = {}_i\mathrm{I}^{o}[\eta;\lambda] + 2\,\mathrm{p}\,\sqrt{{}_i\boldsymbol{\rho}_{\Diamond}[\eta]}\frac{\mathrm{d}}{\mathrm{d}\eta}\frac{ld\;{}_i\phi[\eta;-\lambda]}{\sqrt{{}_i\boldsymbol{\rho}_{\Diamond}[\eta]}} + \\ 2\,\sqrt{{}_i\boldsymbol{\rho}_{\Diamond}[\eta]}\frac{\mathrm{d}}{\mathrm{d}\eta}\frac{ld\;{}_i\mathcal{W}_{\mathfrak{N}(\overline{\mathrm{M}}_{\mathrm{p}})}[\eta;\lambda|-\vdots\overline{\mathrm{M}}_{\mathrm{p}}]}{\sqrt{{}_i\boldsymbol{\rho}_{\Diamond}[\eta]}} - \mathrm{p}(\mathrm{p}-2)\mathfrak{I}\Big\{{}_i\boldsymbol{\rho}_{\Diamond}[\eta]\Big\}. \end{aligned} \tag{158}$$

Substituting (156) into the left-hand side of (157) thus gives

$$\begin{aligned} {}_i\mathrm{I}^{o}[\eta;\lambda_{\mathrm{o}}|-\vdots\overline{\mathbf{N}}_{2\mathrm{J,L}}] = {}_i\mathrm{I}^{o}[\eta;\lambda_{\mathrm{o}}] + 4\mathrm{J}\,\sqrt{{}_i\boldsymbol{\rho}_{\Diamond}[\eta]}\frac{\mathrm{d}}{\mathrm{d}\eta}\frac{ld\;{}_i\phi[\eta;-\lambda_{\mathrm{o}}]}{\sqrt{{}_i\boldsymbol{\rho}_{\Diamond}[\eta]}} + \\ 2\,\sqrt{{}_i\boldsymbol{\rho}_{\Diamond}[\eta]}\frac{\mathrm{d}}{\mathrm{d}\eta}\frac{ld\;{}_i\mathcal{W}_{\mathfrak{N}(\overline{\mathrm{M}}_{\mathrm{p}})}[\eta;\lambda_{\mathrm{o}}|-\vdots\overline{\mathbf{N}}_{2\mathrm{J,L}}]}{\sqrt{{}_i\boldsymbol{\rho}_{\Diamond}[\eta]}} - 4\mathrm{J}(\mathrm{J}-1)\mathfrak{I}\Big\{{}_i\boldsymbol{\rho}_{\Diamond}[\eta]\Big\}, \end{aligned} \tag{159}$$

where we also take into account that

$$ld\;{}_i\phi[\eta;\mathrm{p}-\lambda_{\mathrm{o}}] = ld\;{}_i\phi[\eta;-\lambda_{\mathrm{o}}] - {}^{1}\!/_{2}\mathrm{p}\;ld\;{}_i\boldsymbol{\rho}_{\Diamond}[\eta] \tag{160}$$

and form-invariance condition (A24) for the $\Re$Ref CSLE (1) as follows:

$${}_i\mathrm{I}^{o}[\eta;\lambda_{\mathrm{o}}-1] - {}_i\mathrm{I}^{o}[\eta;\lambda_{\mathrm{o}}] = 2\sqrt{{}_i\boldsymbol{\rho}_{\Diamond}[\eta]}\frac{\mathrm{d}}{\mathrm{d}\eta}\frac{ld_i\phi[\eta;-\lambda_{\mathrm{o}}]}{\sqrt{{}_i\boldsymbol{\rho}_{\Diamond}[\eta]}} + \mathfrak{I}\{{}_i\boldsymbol{\rho}_{\Diamond}[\eta]\}. \tag{161}$$

We have thus proven that the RLDT using eigenfunction (156) as its TF converts the given RCSLE with RefPF (159) into the RCSLE with RefPF (157).

To proceed with mathematical induction, we assume that polynomial Wronskian (150) does not have real zeros for any partition starting from $\mathrm{n}_1 - 1$ and also that the corresponding prime SLE is exactly solvable in terms of the q-RSs of our interest.

In particular, this assumption is valid for prime SLE (153) with $\overline{\mathbf{N}}_{2\mathrm{J,L}}$ and λ_{o} replaced by $\overline{\mathbf{N}}_{2\mathrm{J,L}} - \overline{1}_{2\mathrm{J}}$ and $\lambda_{\mathrm{o}} - 1$, respectively. Let us now apply the RLDT to this SLE using the reciprocal of eigenfunction (155) as its TF. Since the latter quasi-rational TF obeys both prepositions summarized in the end of Section 3.4, we come up with the following result:

Lemma 1. *For any $n_1 > 1$, all the eigenfunctions of prime SLE (153) with no poles on the real axis have the quasi-rational form (152) if this is true for any partition starting from the positive integer $n_1 - 1$.*

Based on Lemma 1, we can restrict our analysis only to partitions (149) with $\mathrm{n}_1 = 1$ as follows:

$$\overline{\mathbf{N}}^{0}_{2\mathrm{J,L}} = 1 : 2\mathrm{j}_1,\;\; \overline{\mathbf{N}}_{2\mathrm{J}-2\mathrm{j}_1,\mathrm{L}-1}, \tag{162}$$

where

$$\overline{\mathbf{N}}_{2\mathrm{J}-2\mathrm{j}_1,\mathrm{L}-1} := \mathrm{n}_2 : \mathrm{n}_2 + 2\mathrm{j}_2 - 1, \ldots, \mathrm{n}_{\mathrm{L}} : \mathrm{n}_{2\mathrm{J}}. \tag{163}$$

Let us start from the simplest case, as follows:

$$\overline{\mathbf{N}}^{0}_{2\mathrm{j}_1,1,\mathrm{j}_1} := 1 : 2\mathrm{j}_1; \tag{164}$$

therefore,

$$\mathfrak{N}(\overline{\mathbf{N}}^{0}_{2\mathrm{j},1,j}) = j(2\mathrm{j}+1) - j(2\mathrm{j}-1) = 2\mathrm{j}. \tag{165}$$

It was proven in [1] that all the RLD$\mathfrak{I}$s of the TFI SLE of group A described by a single series of Maya diagrams can be constructed using the sequences of the TFs formed by Wronskians of seed polynomials. In particular, it can be shown that

$$ {}_{i}\mathfrak{W}_{2\mathbf{j}_1}[\eta; a+ib| - \vdots\overline{\mathbf{N}}^{0}_{2\mathbf{j}_1,1,j_1}] = D(1{:}\ 2\mathrm{j}_1)\hat{\mathfrak{R}}^{(a+ib)}_{2\mathrm{j}_1}[\eta], \tag{166} $$

keeping in mind that the row and column of the same length represent the conjugated Young diagrams. (In the simplest case $j_1 = 1$, the cited formula has been explicitly proven in [70].) It was proven in Section 3.1 that the Routh polynomial in question (see Appendix B for more details) and, therefore, Wronskian (166) may not have real zeros. This confirms the Karlin and Szegő's theorem [69] for $n_1 = 1$. Now that we have proven that the corresponding prime SLE may not have poles on the real axis, directly following from the arguments presented in support of Lemma 1, we propose the following inference:

Lemma 2. *All the eigenfunctions of prime SLE (153) specified by partition (164) have the quasi-rational form (152).*

Repeating the arguments presented above, we can then extend this assertion to an arbitrary partition (151), which brings us to the following corollary of Karlin and Szegő's theorem [69]:

Theorem 5. *The Wronskian of an even number of R-Routh polynomials with sequential degrees may not have real zeros.*

Proof of Theorem 5. For J = 1, the theorem directly follows from our extension of Adler's renowned results [71] to RCSLE (112) for partition (148)—termed by us in Appendix C as the 'enhanced Adler theorem'. Since polynomial Wronskian (150) does not have zeros on the real axis, the corresponding prime SLE may not have poles on the real axis. In addition, Lemma 2 assures that the corresponding Dirichlet problem is exactly solvable in terms of q-RSs (152) for $\mathrm{n}_1 = 1$.

Let us now assume that any eigenfunction of the given prime SLE has the quasi-rational form (152) for the partition $\overline{\mathbf{N}}_{2,1}$, starting from the positive integer $\mathrm{n}_1 - 1$. Then, according to Lemma 1, this should be also true for the partition $\overline{\mathbf{N}}_{2,1}$ starting from n_1.

We can now repeat these arguments for J > 1, assuming any eigenfunction of the prime SLE for the partition $\overline{\mathbf{N}}_{2\mathrm{J},1}$ has the quasi-rational form (152). Note that this is necessarily true for partition (164) with $\mathrm{j}_1 = \mathrm{J}$, and we can thus use Lemma 1 to extend this assertion to an arbitrary positive value of n_1.

To confirm that the theorem holds for any partition (151) with $\mathrm{j}_1 = \mathrm{J}+1$, we can again take advantage of the enhanced Adler theorem but, this time, coupled with the generic chain formula for Wronskians [72] as follows:

$$ \begin{gathered} {}_{i}\mathfrak{W}_{\mathfrak{N}(\overline{\mathrm{M}}_{2\mathrm{J}},\mathrm{n},\mathrm{n}+1)}[\eta;\lambda| - \vdots\overline{\mathrm{M}}_{2\mathrm{J}},\mathrm{n},\mathrm{n}+1] = \\ \frac{\mathrm{W}\left\{{}_{i}\mathfrak{W}_{\mathfrak{N}(\overline{\mathrm{M}}_{2\mathrm{J}},\mathrm{n})}[\eta;\lambda|-\vdots\overline{\mathrm{M}}_{2\mathrm{J}},\mathrm{n}],{}_{i}\mathfrak{W}_{\mathfrak{N}(\overline{\mathrm{m}}_{2\mathrm{J}},\mathrm{n}+1)}[\eta;\lambda|-\vdots\overline{\mathrm{M}}_{2\mathrm{J}},\mathrm{n}+1]\right\}}{{}_{i}\mathfrak{W}^{2\mathrm{J}-1}_{\mathfrak{N}(\overline{\mathrm{M}}_{2\mathrm{J}})}[\eta;\lambda|-\vdots\overline{\mathrm{M}}_{2\mathrm{J}}]}, \end{gathered} \tag{167} $$

where we set

$$ \overline{\mathrm{M}}_{2\mathrm{J}} = \overline{\mathbf{N}}_{2\mathrm{J},1}, \quad \mathrm{n} = \mathrm{n}_1 + 2\mathrm{J}. \tag{168} $$

We can then use mathematical induction to complete the proof. □

Combining Theorem 5 with Lemma 1 brings one to the following intermediate result:

Lemma 3. *Any eigenfunction of the prime SLE (153) specified by the single-segment partition (151) has the quasi-rational form (152).*

We are finally ready to prove the most important result of this section.

Theorem 6. *Wronskians of R-Routh polynomials with degrees forming even-length segments may not have real zeros.*

Proof of Theorem 6. According to Theorem 5, this assertion is valid for any single segment of an even length. Let us now assume that the theorem is valid for any partition formed by L-1 even-length segments formed by positive integers and, in addition, that any eigenfunction of the corresponding prime SLE has the quasi-rational form (152) with $\overline{\mathbf{N}}_{2J,L}$ replaced with $\overline{\mathbf{N}}_{2J,L-1}$. It directly follows from Lemma 3 that the assumption is valid for any single-segment partition of an even length (L = 1). Apparently, we can focus solely on the partitions with $n_1 = 1$.

If at least one segment of an L-segment partition has the length of 2, then we can again take advantage of the enhanced Adler theorem coupled with the cited chain formula with $\overline{M}_{2J}$ in (167) substituted by $\overline{\mathbf{N}}_{2J-2,L-1}$ and

$$\overline{\mathbf{N}}_{2J,L} = \overline{\mathbf{N}}_{2J-2,L-1}, n, n+1 \quad (n, n+1 \notin \overline{\mathbf{N}}_{2J-2,L-1}). \tag{169}$$

However, we cannot proceed with mathematical induction before proving that any eigenfunction of the prime SLE (153) for the set of seed solutions specified by the partition (169) can be represented in the quasi-rational form (152).

Note that, by analogy with the proof of Theorem 5, we use mathematical induction in two distinguished way. First, we increase J by one and take advantage of the enhanced Adler theorem coupled with the cited chain relation to prove that the corresponding polynomial Wronskian does not have real zeros. We then repeatedly increase n_1 by one (starting from $n_1 = 1$) to utilize Lemma 1.

Whether or not polynomial Wronskian (150) has real zeros, one can apply the RLDT to the corresponding RCSLE using the q-RS

$$\begin{aligned} {}_{i}\phi^{0}[\eta;\lambda_{o}| - \vdots\overline{\mathbf{N}}^{0}_{2J,L,j_1}] &= {}_{i}\phi[\eta; 2J-\lambda_{o}] \times \\ &\frac{{}_{i}\mathfrak{W}_{\mathfrak{N}(\overline{\mathbf{N}}^{0}_{2J,L,j_1},0)}[\eta;\lambda_{o}| - \overline{\mathbf{N}}^{0}_{2J,L,j_1},0]}{{}_{i}\mathfrak{W}_{\mathfrak{N}(\overline{\mathbf{N}}^{0}_{2J,L,j_1})}[\eta;\lambda_{o}| - \vdots\overline{\mathbf{N}}^{0}_{2J,L,j_1}]} \end{aligned} \tag{170}$$

as its TF, where we set

$$\overline{\mathbf{N}}^{0}_{2J,L,j_1} := 1 : 2j_1, \overline{\mathbf{N}}_{2J-2j_1,L-1}, \tag{171}$$

assuming that the second segment of this partition starts from a positive integer larger than $2j_1 + 1$. Taking into account that

$$\frac{d^{2j_1}}{d\eta^{2j_1}} \Re^{(2J-a-ib)}_{m}[\eta] = 4^{-j_1} \prod_{k=1}^{2j_1} (m+k-2a)\, \Re^{(2J+2j_1-a-ib)}_{m-2j_1}[\eta], \tag{172}$$

as a direct corollary of (154), one finds that

$$\begin{aligned} &{}_{i}\mathfrak{W}_{\mathfrak{N}(0,\overline{\mathbf{N}}^{0}_{2J,L,j_1})}[\eta;\lambda_{o}-2J| - \vdots 0, \overline{\mathbf{N}}^{0}_{2J,L,j_1}] \\ &\propto {}_{i}\mathfrak{W}_{\mathfrak{N}(\overline{\mathbf{N}}'_{2J-2j_1,L-1})}[\eta;\lambda_{o}-2J-2j_1| - \vdots\overline{\mathbf{N}}'_{2J-2j_1,L-1}], \end{aligned} \tag{173}$$

where for briefness, we set

$$\overline{\mathbf{N}}'_{2J-2j_1,L-1} := \overline{\mathbf{N}}_{2J-2j_1,L-1} - 2j_1\overline{1}_{2J-2j_1}. \tag{174}$$

Note that $a - 2J - 2j_1$ must be larger than 1/2, since the $\Re$Ref CSLE (1) has at least $2J + 2j_1$ eigenfunctions.

By analogy with (157), we now need to confirm that

$$\begin{gathered} {}_i\mathrm{I}^o[\eta;\lambda_o|-\vdots\overline{\mathbf{N}}^0_{2J,L,j_1}] + 2\sqrt{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]}\frac{d}{d\eta}\frac{{}_i\phi^0[\eta;\lambda_o|-\vdots\overline{\mathbf{N}}^0_{2J,L,j_1}]}{\sqrt{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]}} \\ +\mathfrak{I}\left\{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]\right\} = {}_i\mathrm{I}^o[\eta;\lambda_o - 2\mathbf{j}_1|-\vdots\overline{\mathbf{N}}'_{2J-2j_1,L-1}], \end{gathered} \tag{175}$$

where

$$\begin{gathered} {}_i\mathrm{I}^o[\eta;\lambda_o - 2j_1|-\vdots\overline{\mathbf{N}}_{2J-2j_1,L-1} - 2\mathbf{j}_1\overline{1}_{2J-2j_1}] = {}_i\mathrm{I}^o[\eta;\lambda_o - 2j_1] + \\ 4\,(J-j_1)\sqrt{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]}\frac{d}{d\eta}\frac{ld\;{}_i\phi[\eta;2j_1-\lambda_o]}{\sqrt{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]}} - 4\,(J-j_1)(J-j_1-1)J\left\{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]\right\} \\ 2\sqrt{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]}\frac{d}{d\eta}\frac{ld\;{}_i\mathfrak{W}_{\mathfrak{N}(\overline{M}_p)}[\eta;\lambda|-\vdots\overline{\mathbf{N}}'_{2J-2j_1,L-1}]}{\sqrt{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]}}. \end{gathered} \tag{176}$$

Indeed, making use of (160), one can verify that

$$\begin{gathered} 4\,(J-j_1)\sqrt{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]}\frac{d}{d\eta}\frac{ld\;{}_i\phi[\eta;2j_1-\lambda_o]}{\sqrt{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]}} - 4\,(J-j_1)(J-j_1-1)\mathfrak{I}\left\{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]\right\} = \\ 4\,(J-j_1)\sqrt{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]}\frac{d}{d\eta}\frac{ld\;{}_i\phi[\eta;-\lambda_o]}{\sqrt{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]}} + 4[j_1(j_1-1) - J(J-1)]\mathfrak{I}\left\{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]\right\}, \end{gathered} \tag{177}$$

whereas form-invariance condition (161) can be generalized as follows:

$${}_i\mathrm{I}^o[\eta;\lambda_o - p] = {}_i\mathrm{I}^o[\eta;\lambda_o] + 2p\sqrt{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]}\frac{d}{d\eta}\frac{ld_i\phi[\eta;-\lambda_o]}{\sqrt{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]}}, \tag{178}$$

which gives

$$\begin{gathered} {}_i\mathrm{I}^o[\eta;\lambda_o|-\vdots\overline{\mathbf{N}}^0_{2J,L,j_1}] = {}_i\mathrm{I}^o[\eta;\lambda_o] + 4\,J\sqrt{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]}\frac{d}{d\eta}\frac{ld\;{}_i\phi[\eta;-\lambda_o]}{\sqrt{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]}} \\ -4J(J-1)\mathfrak{I}\{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]\} + 2\sqrt{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]}\frac{d}{d\eta}\frac{{}_i\mathfrak{W}_{\mathfrak{N}(\overline{\mathbf{N}}^0_{2J,L,j_1})}[\eta;\lambda_o|-\vdots\overline{\mathbf{N}}^0_{2J,L,j_1}]}{\sqrt{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]}}. \end{gathered} \tag{179}$$

Substituting (170) into the left-hand side of (175) and taking advantage of (160), coupled with (176), we come back to the following conventional formula:

$$\begin{gathered} {}_i\mathrm{I}^o[\eta;\lambda_o|-\vdots\overline{\mathbf{N}}^0_{2J,L,j_1}] = {}_i\mathrm{I}^o[\eta;\lambda_o] + 4\,J\sqrt{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]}\frac{d}{d\eta}\frac{ld\;{}_i\phi[\eta;-\lambda_o]}{\sqrt{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]}} \\ -4J(J-1)\mathfrak{I}\{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]\} + 2\sqrt{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]}\frac{d}{d\eta}\frac{{}_i\mathfrak{W}_{\mathfrak{N}(\overline{\mathbf{N}}^0_{2J,L,j_1})}[\eta;\lambda_o|-\vdots\overline{\mathbf{N}}^0_{2J,L,j_1}]}{\sqrt{{}_i\boldsymbol{\rho}_\diamondsuit[\eta]}}, \end{gathered} \tag{180}$$

which completes the proof of (175). We have thus proven that the RLDT using eigenfunction (170) as its TF converts the given RCSLE with RefPF (180) into the RCSLE with RefPF (175). It is crucial that the partition specifying this RefPF is formed by the L-1 segments of even lengths and, therefore, each eigenfunction of prime SLE (153) with $\overline{\mathbf{N}}_{2J,L}$ and λo replaced respectively with (174) and $\lambda_o - 2j_1$, respectively, can be represented in the quasi-rational form of our interest.

Note that partition (174) is composed of L-1 segments of even lengths. We can thus proceed with mathematical induction, assuming that the theorem does hold for partition (174) and that all the eigenfunctions of prime SLE (153) with the partition $\overline{\mathbf{N}}_{2J,L}$ changed for (174) are expressible in the quasi-rational form under consideration.

We can finally prove that Theorem 6 holds for polynomial Wronskian (150) with $\overline{\mathbf{N}}_{2J,L}$ defined via (171). First, note that it is necessarily true for polynomial Wronskian (168) with $\overline{\mathrm{m}}_{2J-2}, \mathrm{n}, \mathrm{n}+1$ substituted by $\overline{\mathbf{N}}^{0}_{2J,L,1}$ as the straightforward corollary of the enhanced Adler theorem. This implies that q-RS (170) with $\mathrm{j}_1 = 1$ represents the lowest-energy eigenfunction of prime SLE (153) with $\overline{\mathbf{N}}_{2J,L}$ changed for $\overline{\mathbf{N}}^{0}_{2J,L,1}$. Since the reciprocal of this q-RS obeys the prerequisites for Theorem 4, we assert that all the eigenfunctions of prime SLE (153) with $\overline{\mathbf{N}}_{2J,L}$ changed as specified above have the quasi-rational form under consideration.

Now, let us assume that both the latter assertion and Theorem 6 hold for the partition

$$\overline{\mathbf{N}}^{0}_{2J,L,j} := 1 : 2j, \overline{\mathbf{N}}'_{2J-2j_1,L-1}, \tag{181}$$

bearing in mind that this is necessarily true for j =1. It then directly follows from the enhanced Adler theorem that polynomial Wronskian (167) with $\overline{\mathrm{M}}_{2J}$ substituted for $\overline{\mathbf{N}}^{0}_{2J,L,j}$ and

$$\overline{\mathrm{M}}_{2J}, \mathrm{n}, \mathrm{n}+1 = \overline{\mathbf{N}}^{0}_{2J,L,j}, 2j+1, 2j+2 \tag{182}$$

may not have real zeros. This confirms that prime SLE (153) specified by partition (182) has no poles on the real axis. It then directly follows from Lemma 1 that all the eigenfunctions of prime SLE (153) with $\overline{\mathbf{N}}_{2J,L}$ replaced with partition (182) can be represented in the quasi-rational form (152). We can then complete the proof of Theorem 6 by mathematical induction over j. □

In the next subsection, we will extend these arguments to prime SLE (122) obtained from the $\Re$Ref CSLE by the RDCT using the set of seed functions (144), in addition to pairs of the juxtaposed eigenfunctions.

3.5. General form of a Finite EOP Sequence Formed by W$\mathfrak{I}$s of R-Routh Polynomials

Let us now generalize the results of Sections 3.3 and 3.4 as follows:

Theorem 7. *Wronskians of Routh polynomials with degrees forming any compound partition* $(\overline{\mathbf{N}}_{2J}, \overline{\mathbf{M}}^{e,o}_{\Delta p})$ *may not have real zeros.*

Proof of Theorem 7. Let us consider the sequence of the LDTs with the TFs

$$*\psi_{\mathbf{c},0}[\eta;\lambda_o| - \vdots\overline{\mathbf{N}}_{2J}, \overline{\mathbf{M}}^{e,o}_{\Delta p}] = {}_{i}\psi_{\mathbf{c},0}[\eta;\lambda_o - p] \times \frac{{}_{i}\mathfrak{W}_{\mathfrak{N}(\overline{\mathbf{N}}_{2J},\overline{\mathbf{M}}^{e,o}_{\Delta p+1})}[\eta;\lambda_o| - \vdots\overline{\mathbf{N}}_{2J}, \overline{\mathbf{M}}^{e,o}_{\Delta p+1}]}{{}_{i}\mathfrak{W}_{\mathfrak{N}(\overline{\mathbf{N}}_{2J},\overline{\mathbf{M}}^{e,o}_{\Delta p})}[\eta;\lambda_o| - \vdots\overline{\mathbf{N}}_{2J}, \overline{\mathbf{M}}^{e,o}_{\Delta p}]} \quad (\mathrm{p} = 2\mathrm{J} + \Delta \mathrm{p}), \tag{183}$$

starting from prime SLE (122) with $\overline{\mathbf{M}}_{\mathrm{p}}$ substituted by $\overline{\mathbf{N}}_{2J}$. It was proven in Section 3.4 that each eigenfunction of this SLE has the quasi-rational form (129). Since the sequence in question starts from the RLDT with the TF

$$*\psi_{\mathbf{c},0}[\eta;\lambda_o| - \vdots\overline{\mathbf{N}}_{2J}, \overline{\mathbf{M}}^{e,o}_{1}] = {}_{i}\psi_{\mathbf{c},0}[\eta;\lambda_o] \times \frac{{}_{i}\mathfrak{W}_{\mathfrak{N}(\overline{\mathbf{N}}_{2J},\overline{\mathbf{M}}^{e,o}_{1})}[\eta;\lambda_o| - \vdots\overline{\mathbf{N}}_{2J}, \overline{\mathbf{M}}^{e,o}_{1}]}{{}_{i}\mathfrak{W}_{\mathfrak{N}(\overline{\mathbf{N}}_{2J})}[\eta;\lambda_o| - \vdots\overline{\mathbf{N}}_{2J}]} \quad (\overline{\mathbf{M}}^{e,o}_{1} \equiv 2\mathrm{j}_1 > a - 1/2), \tag{184}$$

both presumptions formulated at the end of Section 3.4 are satisfied. As the direct consequence of Theorem 4, we thus assert that Theorem 7 holds for p = 2J + 1.

Now, we can complete the proof by mathematical induction. Indeed, suppose that the denominator of the PF in the right-hand side of (129) with

$$\overline{\mathbf{M}}_{\mathrm{p}} = \overline{\mathbf{N}}_{2\mathrm{J}}, \overline{\mathbf{M}}^{\mathrm{e,o}}_{\Delta\mathrm{p}} \tag{185}$$

does not have real zeros and that the eigenvalues

$$\varepsilon_{\mathrm{m}_{\mathrm{k}=1,\,\ldots,\,\Delta\mathrm{p}}}(a),\ \varepsilon_{\mathrm{n}_{\mathrm{k}=1,\,\ldots,\,2\mathrm{J}}}(a), \tag{186}$$

where

$$\mathrm{m}_{\mathrm{k}=1,\,\ldots,\,\Delta\mathrm{p}} \equiv \overline{\mathbf{M}}^{\mathrm{e,o}}_{\Delta\mathrm{p}},\ \ \mathrm{n}_{\mathrm{k}=1,\,\ldots,\,2\mathrm{J}} \equiv \overline{\mathbf{N}}_{2\mathrm{J}}, \tag{187}$$

constitute the complete discrete energy spectrum of prime SLE (122) for partition (185).

It then directly follows from Theorem 4 that the RLDT using the seed solutions

$$\overline{\mathbf{M}}_{\mathrm{p}+1} = \overline{\mathbf{N}}_{2\mathrm{J}}, \overline{\mathbf{M}}^{\mathrm{e,o}}_{\Delta\mathrm{p}+1} \tag{188}$$

with

$$\overline{\mathbf{M}}^{\mathrm{e,o}}_{\Delta\mathrm{p}+1} = \overline{\mathbf{M}}^{\mathrm{e,o}}_{\Delta\mathrm{p}}, \mathrm{m}_{\mathrm{p}+1} \quad (\mathrm{m}_{\mathrm{p}+1} > \mathrm{m}_{\mathrm{p}}), \tag{189}$$

simply inserts the new eigenvalue, $\varepsilon_{\mathrm{m}_{\mathrm{p}+1}}(a)$, below energy spectrum (186) without affecting the rest of the energy spectrum. □

We have thus demonstrated that each X-DPS formed by Wronskians of Routh polynomials and specified by compound partition (188) contains the finite EOP sequence formed by the polynomial Wronskians

$$_{i}\mathfrak{W}_{\mathfrak{N}(\overline{\mathbf{N}}_{2\mathrm{J}},\overline{\mathbf{M}}^{\mathrm{e,o}}_{\Delta\mathrm{p}})}[\eta; a+ib| - \vdots\overline{\mathbf{N}}_{2\mathrm{J}}, \overline{\mathbf{M}}^{\mathrm{e,o}}_{\Delta\mathrm{p}}, \mathrm{n}] \text{ with } \mathrm{n} \notin \overline{\mathbf{N}}_{2\mathrm{J}} < a - 1/2. \tag{190}$$

4. Discussion

Although the current paper was focused solely on the RDC$\mathfrak{I}$s of the $\mathfrak{R}$Ref CSLE (1), the concept of LDTs sketched in Section 3.1 has much more general implications. As mentioned in the Introduction, this new concept forms the theoretical basis for the simplistic rules of SUSY quantum mechanics. For rational Liouville potentials with exponential tails at large absolute values of the argument (the subject of the current analysis), the main advantage of the suggested formalism (cf. [17], for example) manifests itself in our proofs, i.e., that the RDC$\mathfrak{I}$s of these potentials are exactly solvable in terms of q-RSs. For rational radial potentials, as well for rational potentials with singularities at both end points (like the t-PT potential), additional complications come up if the singular end point lies in the limit-circle region of the corresponding RCSLE, where the conventional rules of SUSY quantum mechanics become invalid [73].

In [1], we (under the influence of Odake and Sasaki's breakthrough works [22,23]) applied the LDTs to the RCSLEs associated with the rational TSI potentials. It was shown that the RCSLEs in question have so-called 'basic' solutions satisfying the following translational form-invariance condition:

$$_{\iota}\phi_{-,0}[\eta; a+1, b]\ {}_{\iota}\phi_{+;0}[\eta; a,\ b] = {}_{\iota}\rho^{-1/2}[\eta], \tag{191}$$

with ${}_{\iota}\rho\,[\eta]$ standing for the density function of the corresponding TFI RCSLE. TFI condition (18) represents the particular case of (191) for the $\mathfrak{R}$Ref CSLE (1). This common property of TFI CSLEs directly leads to a subnet (or just the net here) of the RDC$\mathcal{T}$s specified by a single series of Maya diagrams [74].

It was observed that the RCSLEs with the q-RSs formed by Jacobi and Laguerre DPSs [25,27,28] have two pairs of basic solutions satisfying TFI condition (191). As a result, one needs two series of Maya diagrams [35] to specify their RDC*T*s solvable in terms of X-Jacobi and X-Laguerre DPSs. Note that the latter feature was related in [35] to the translational shape-invariance of the t-PT potential (trigonometric version [33] of the PT potential [41]) and the isotonic oscillator. However, we deal here with a much more general phenomenon, and the equivalence theorem proven in [35] can be re-formulated [1] whether or not the given X-DPS contains an X-OPS.

One also needs two series of Maya diagrams to identify all the RDC*T*s of the h-PT potential, since both trigonometric and hyperbolic versions of the PT potential can be obtained by the Liouville transformation of the same RSLE but on two different (finite or infinite) quantization intervals. The X-Jacobi DPSs containing RDC*T*s of R-Jacobi polynomials may or may not hold X-Jacobi OPSs.

In following Odake and Sasaki's classification of the TSI potentials under consideration, we include the TFI RCSLEs in group A if the ExpDiffs for the poles in the finite plane are energy-independent, otherwise referring to them as CSLEs of Group B. The common important feature of RCSLEs from group A is that Wronskians of quasi-rational seed solutions of the same type (+ or $-$) turn into weighted polynomial Wronskians. It was demonstrated that the grouping is the intrinsic characteristic of the TFI CSLE itself, while its Liouville potential can be, in some cases (h-PT and Morse potentials), included into either group, depending on the choice of the rational representation for the given TSI potential.

There are two TFI CSLEs of group A that have only single pairs of the basic solutions. One of them is the $\Re$Ref CSLE (1); the second is the Bessel-reference (*B*Ref) CSLE, which can be converted by Liouville transformation to the Schrödinger equation with Morse potential [50]. In both cases, the complete net of the DCTs of the given TFI CSLEs can be constructed using Wronskians of Routh or generalized Bessel [26,75] polynomials with the same index. This implies that an arbitrary EOP sequence for each net of the RCSLEs can be represented as a finite set of W$\mathfrak{I}$s of R-Routh or R-Bessel polynomials accordingly.

In Section 3.5, we constructed the net of Routh-seed (RS) EOP sequences specified by the compound partitions (188). We speculate that there is no other EOP sequence formed by W$\mathfrak{I}$s of R-Routh polynomials. This is the only conjecture that we have been unable to prove so far.

Note that the arguments presented in Section 3.3 to prove that the Wronskians of Routh polynomials specified by partition (137) do not have real zeros are automatically applied to polynomial Wronskians,

$$i\mathcal{W}_{\mathfrak{N}(\overline{\mathrm{M}}_{\mathrm{p}}^{\mathrm{e,o}})}[\eta;\lambda_{\mathrm{o}}|+\vdots\overline{\mathrm{M}}_{\mathrm{p}}^{\mathrm{e,o}}] := \mathrm{W}\left\{\Re^{(\lambda_{\mathrm{o}})}_{\mathrm{m}_{k=1,\,\ldots,\,\mathrm{p}}}[\eta]\right\} \quad \text{with } \mathrm{m}_k \in \overline{\mathrm{M}}_{\mathrm{p}}^{\mathrm{e,o}}, \tag{192}$$

although, this time, with no lower bound for the positive even integer $\mathrm{m}_1 = 2\mathrm{j}_1$. The common remarkable feature of the partitions selected in such a way is that they have even gaps between the segments. For the rational TSI potentials of our interest, the RDCTs of this kind were independently introduced in [23,35]. According to the theorems formulated in these papers, the state-adding RDCTs specified by partitions $|+\vdots\overline{\mathrm{M}}_{\mathrm{p}}^{\mathrm{e,o}})$ are equivalent to the RDCTs using the set $|-\vdots\overline{\mathbf{N}}_{2\mathrm{J,L}})$ of seed functions (148) such that the two equivalent transformations are described by the conjugated Young diagrams.

More specifically, the RDCTs using polynomial Wronskians (192) were explicitly introduced in [23], while, to the best of our knowledge, the EOP sequences introduced in Section 3.3 have not been discussed in the literature so far.

The RLD*T*s of the $\Re$Ref CSLE (1) with infinitely many TFs $|+\vdots 2\mathbf{j}_1)$ (which start the subnet of the RD*T*s using the seed functions $|+\vdots\overline{\mathrm{M}}_{\mathrm{p}}^{\mathrm{e,o}})$ were thoroughly analyzed by Quesne [17], and we refer the reader to Appendix B for more details.

5. Conclusions

As mentioned above, there are three TFI RCSLEs of group A quantized in terms of one of three families of the Romanovski polynomials [12]. Two of these RCSLEs have single pairs of the basic solutions, and as a result, all the possible EOP sequences associated with RDC$\mathfrak{T}$s of the given CSLE can be represented as the W$\mathfrak{T}$s of the R-Routh and R-Bessel polynomials. In this paper, we thoroughly discussed the EOP sequences composed of W$\mathfrak{T}$s of the R-Routh polynomials, and we refer the reader to the similar analysis presented by us in [46] for EOP sequences composed of W$\mathfrak{T}$s of R-Bessel polynomials.

Finite EOP sequences composed of RDC$\mathfrak{T}$s of R-Jacobi polynomials represent a more challenging problem, since the corresponding TFI CSLE has two pairs of the basic solutions. As a result, one needs two series of Maya diagrams [35] to specify all the possible RDC$\mathfrak{T}$s of this RCSLE. To construct the full net of the EOP sequences composed of RDCTs of R-Jacobi polynomials, one has to implement a more complicated algorithm, which consists of two steps:

(i) Quasi-rational representation of eigenfunctions in terms of either pseudo-Wronskians [35] or polynomial determinants [45];

(ii) Removal of zeros of the polynomial component of the resultant q-RS at the singular points of the Jacobi-reference ($\mathfrak{J}$Ref) CSLE in the finite plane.

Another complication comes from the fact that the classification of these q-RSs cannot be unambiguously performed using only Cases I, II, and III introduced by Quesne [76] for the *t*-PT potential and isotonic oscillator (types **a**, **b**, and **d**, respectively, in our terms [51,77]). Since the h-PT potential has a finite discrete energy spectrum, the sequence of q-RSs starting from the finite subset of eigenfunctions ends with the infinitely many solutions vanishing only at the origin. This is in addition to the conventional infinite 'primary' sequence ($\tilde{\mathbf{a}}$) of q-RSs formed by the classical Jacobi polynomials, where the tilde indicates that each q-RS vanishes at the lower end of the quantization interval (1,∞). (Since all the zeros of the classical Jacobi polynomials lie between −1 and +1 the q-RSs of this type may not have zero on the interval (1,∞).) The crucial distinction between these two sequences of the Case I q-RSs is that the 'secondary' sequence labelled by us as $\tilde{\mathbf{a}}'$ does not start from a basic solution [76].

The EOP sequences constructed using the primary TFs of type $\tilde{\mathbf{a}}$ were discovered in the celebrated paper by Odake and Sasaki [78] and were more cautiously examined in [79,80]. Since the corresponding eigenfunctions are formed by the Wronskians of the q-RSs composed of classical Jacobi polynomials and R-Jacobi polynomials, the polynomial sequences used for their construction turned out to be biorthogonal [81]. The q-RSs of this type were also used in [23] for the construction of multi-index EOP sequences.

The secondary sequence of the q-RSs of type $\tilde{\mathbf{a}}'$ can be used to generate multi-indexed X-DPSs composed of the Wronskians of the Jacobi polynomials. If all the seed solutions lie below the lowest-energy level, then the generated X-DPS contains a finite EOP sequence forming a complete set of the eigenfunctions for the isospectral CSLE constructed in such a way. Again, one can refine the enhanced Adler theorem for this particular case and then construct the RDC net of EOP sequences formed by the W*T*s of the R-Jacobi polynomials, by analogy with the analysis presented in Section 3.5. However, to construct finite EOP sequences using a mixture of the nodeless q-RSs of types $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{a}}'$, one needs to make use of the pseudo-Wronskians formally introduced by Gȯmez-Ullate et al. [35] in connection with the eigenfunctions of the rationally extended t-PT potentials quantized via multi-indexed X-Jacobi OPSs. A re-examination of the arguments presented in [35] reveals that the cited authors discuss all possible X-Jacobi DPSs rather than their infinite orthogonal subsystems.

In addition to the Case I Jacobi polynomials with no zeros on the interval (1,∞), the mentioned pseudo-Wronskians can include all the Jacobi polynomials associated with the Case II q-RSs below the lowest-energy level. One can also add the pairs of R-Jacobi polynomials of sequential degrees (again, after a more careful examination of the corresponding version of the enhanced Adler theorem). We plan to address these perplexing issues in an upcoming study.

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Appendix A. Routh DPS as a Supplementary Real-Field Reduction of Complex Jacobi Polynomials

The purpose of this Appendix is to summarize some general properties of the real DPS discovered by Routh in the revolutionary treatise [18] overlooked by mathematicians for more than a century. Although after being brought back to life in Ismail's monograph [82], Routh' paper was cited in numerous papers, those references are not sufficiently accurate in most cases.

In particular, contrary to the statement originally made by Ismail and later repeated by several authors, Routh [18] did not consider polynomial solutions in a complex plane. This novel and important step in the theory of Routh polynomials was made in [83] nearly a century after the publication of Routh' paper [18]. Namely, Cryer related the monic polynomials generated via the Rodrigues formula,

$$\hat{\mathfrak{R}}_{\mathrm{m}}^{(\alpha_{\mathrm{R}}+i\,\alpha_{\mathrm{I}})}(\mathrm{x}) = \frac{1}{\mathrm{K_m}(\alpha_{\mathrm{R}})\;\;\omega(\mathrm{x};\alpha_{\mathrm{R}},\alpha_{\mathrm{I}})}\frac{\mathrm{d^m}}{\mathrm{d^m x}}[(\mathrm{x}^2+1)^{\mathrm{m}}\omega(\mathrm{x};\alpha_{\mathrm{R}},\alpha_{\mathrm{I}})]\,, \tag{A1}$$

to Jacobi polynomials with complex conjugated indexes and an imaginary argument as follow:

$$\mathfrak{R}_{\mathrm{m}}^{(\alpha_{\mathrm{R}}+i\,\alpha_{\mathrm{I}})}(\mathrm{x}) := (-i)^{\mathrm{m}}\,\mathrm{P}_{\mathrm{m}}^{(\alpha_{\mathrm{R}}+i\,\alpha_{\mathrm{I}},\;\alpha_{\mathrm{R}}-i\,\alpha_{\mathrm{I}})}(i\mathrm{x}) \tag{A2}$$

$$= (i)^{\mathrm{m}}\,\mathrm{P}_{\mathrm{m}}^{(\alpha_{\mathrm{R}}-i\,\alpha_{\mathrm{I}},\;\alpha_{\mathrm{R}}+i\,\alpha_{\mathrm{I}})}(\mathrm{x}/i), \tag{A3}$$

denoted in the monograph [82] by the symbolic expression $\mathrm{P_m}(\mathrm{x};a,b)$, namely,

$$\mathfrak{R}_{\mathrm{m}}^{(\alpha_{\mathrm{R}}+i\,\alpha_{\mathrm{I}})}(\mathrm{x}) \equiv \mathrm{P_m}(\mathrm{x};\alpha_{\mathrm{R}},\alpha_{\mathrm{I}})\;\textit{in Ismails notation}. \tag{A4}$$

Here,

$$\omega(\mathrm{x};\alpha_{\mathrm{R}},\alpha_{\mathrm{I}}) \equiv (\mathrm{x}^2+1)^{\alpha_{\mathrm{R}}} e^{2\alpha_{\mathrm{I}}\,\arctan\,\mathrm{x}} \tag{A5}$$

stands for weight function (20.1.3) in [82] and the leading coefficient of the polynomial

$$\mathfrak{R}_{\mathrm{m}}^{(\alpha_{\mathrm{R}}+i\,\alpha_{\mathrm{I}})}(\mathrm{x}) = \mathrm{K_m}(\alpha_{\mathrm{R}})\,\hat{\mathfrak{R}}_{\mathrm{m}}^{(\alpha_{\mathrm{R}}+i\,\alpha_{\mathrm{I}})}(\mathrm{x}) \tag{A6}$$

is given by the conventional formula

$$\mathrm{K_m}(\alpha_{\mathrm{R}}) \equiv \frac{(2\alpha_{\mathrm{R}}+2\mathrm{m})_{\mathrm{m}}}{\mathrm{m}!\;\;2^{\mathrm{m}}} = \frac{\langle 2\alpha_{\mathrm{R}}+\mathrm{m}+1\rangle_{\mathrm{m}}}{\mathrm{m}!\,2^{\mathrm{m}}}, \tag{A7}$$

with $(\alpha)_{\mathrm{m}}$ and $\langle\alpha\rangle_{\mathrm{m}}$ standing for the falling and rising [84] factorials, respectively. Note that this leading coefficient differs from 0 iff

$$\alpha_{\mathrm{R}} \neq -1/2(\mathrm{m}+\mathrm{k}) \text{ for } 1 \leq \mathrm{k} \leq \mathrm{m} \tag{A8}$$

It directly follows from (4.22.3) in [85] that Routh polynomial (A2) turns (up to a constant multiplier) into the Routh polynomial of degree $\mathrm{k}-1$ if inequality (A8) fails. Alternatively, monic polynomials (A1) coincide with the polynomials ($\mathrm{P_m}(\mathrm{x};\nu,\mathrm{N})$) in the monograph [26] with $\mathrm{N} = -\alpha_{\mathrm{R}}-1$, $\nu = \alpha_{\mathrm{I}}$ as follows:

$$\mathrm{P_m}(\mathrm{x};\alpha_{\mathrm{I}},-\alpha_{\mathrm{R}}-1) \equiv \hat{\mathfrak{R}}_{\mathrm{m}}^{(\alpha_{\mathrm{R}}+i\,\alpha_{\mathrm{I}})}(\mathrm{x}) \tag{A9}$$

In our papers, we prefer to adopt the notation of this monograph to make use of the formulas listed in §9.9 in [26], with N changed for an arbitrary real number. (A similar suggestion was previously put forward in [86], although with the lower bound of $-1/2$ for N). Note that Jordaan and Toókos [87] use the term 'pseudo-Jacobi polynomial' in exactly the same way as we use 'Routh polynomials' here but adopt Ismail's formula (A4), not (A9), for their precise definition.

We thus use the term 'pseudo-Jacobi polynomials' for monic polynomials (A6) as a synonym for Routh (not R-Routh!) polynomials, assuming that $-2a$ is not a negative integer with an absolute value between n + 1 and 2n. We define the R-Routh polynomials via (3.5) and (3.6) in [17] as follows:

$$\begin{gathered} \mathrm{R}_{\mathrm{n}}^{(-2b,1-a)}(\mathrm{x}) = (-i)^{\mathrm{n}}\, \mathrm{P}_{\mathrm{n}}^{(-a-ib,-a+ib)}(i\mathrm{x}) \equiv \Re_{\mathrm{n}}^{(-a-ib)}(\mathrm{x}). \\ \text{for } \mathrm{n} < a - 1/2 \end{gathered} \tag{A10}$$

Note that the notation adopted by us from [17] differs from (97) in [14], as well as from the definition of 'Romanovski polynomials' in [13]. The polynomial degree is necessarily equal to n, since $\mathrm{n} + \mathrm{k} \le 2\mathrm{n} < 2a$ for $1 \le \mathrm{k} \le \mathrm{n}$.

Appendix B. Quesne Representation for Solvable RDTs of Gendenshtein (Scarf II) Potential

Let us explicitly relate representation (50) of the RefPF of RCSLE (46) to its alternative form originally suggested in [17] and therefore referred by us as 'Quesne representation'. It serves as a starting point for the subnet of the RDC*T*s of the $\Re$Ref CSLE (1) using q-RSs +,2j as the seed functions [17]. As pointed to in another innovative paper [23] published in the same year, the Liouville potentials associated with the latter subnet simply represent another form of the Krein–Adler [71,88] net of the rationally deformed Gendenshtein (Scarf II) potentials. In particular, the rationally deformed Scarf II potentials constructed by Quesne [17] can instead be converted to their Krein–Adler form, taking advantage of representation (166) of Routh polynomials as Wronskians of R-Routh polynomials of the sequential degrees starting from 1.

First, making use of (42), (43) and (51), let us re-write RefPF (50) as follows:

$$\begin{aligned} {}_i\mathrm{I}^{\mathrm{o}}[\eta;\lambda_{\mathrm{o}}|\pm,2\mathrm{j}] = {}_i\mathrm{I}^{\mathrm{o}}[\eta;\lambda_{\mathrm{o}}] &+ 2\sqrt{{}_i\rho_{\Diamond}[\eta]}\, \tfrac{\mathrm{d}}{\mathrm{d}\eta} \tfrac{ld_i\phi[\eta;\pm\lambda_{\mathrm{o}}-1/2]}{\sqrt{{}_i\rho_{\Diamond}[\eta]}} \\ &+ 2\sqrt{{}_i\rho_{\Diamond}[\eta]}\, \tfrac{\mathrm{d}}{\mathrm{d}\eta} \tfrac{ld\Re_{2\mathrm{j}}^{(\pm\lambda_{\mathrm{o}})}(\eta)}{\sqrt{{}_i\rho_{\Diamond}[\eta]}}. \end{aligned} \tag{A11}$$

Taking into account that

$$-ld_i\rho_{\Diamond}[\eta] = ld\,(\eta^2+1) = \frac{1}{\eta - i} + \frac{1}{\eta + i} \tag{A12}$$

$$ld\;{}_i\phi[\eta;\lambda] = \frac{1+\lambda}{2(\eta+i)} + \frac{1+\lambda*}{2(\eta-i)} \tag{A13}$$

and

$$2\dot{ld}\;{}_i\phi[\eta;\lambda] = -\frac{1+\lambda}{(\eta+i)^2} - \frac{1+\lambda*}{(\eta-i)^2}, \tag{A14}$$

we can then represent the second summand in sum (A11) as

$$\begin{aligned} 2\sqrt{{}_i\rho_{\Diamond}[\eta]}\, \tfrac{\mathrm{d}}{\mathrm{d}\eta} \tfrac{ld\;{}_i\phi[\eta;\lambda]}{\sqrt{{}_i\rho_{\Diamond}[\eta]}} &= 2\dot{ld}\;{}_i\phi[\eta;\lambda] - ld_i\rho_{\Diamond}[\eta]\,ld\;{}_i\phi[\eta;\lambda] \\ &= -\tfrac{\lambda+1}{2(\eta+i)^2} - \tfrac{\lambda*+1}{2(\eta-i)^2} + \tfrac{Re\lambda+1}{\eta^2+1}. \end{aligned} \tag{A15}$$

Setting

$$\lambda = \pm\lambda_{\mathrm{o}} - 1/2 \tag{A16}$$

and also taking into account that

$$ {}^{1}\!/\!_{4}[{}_{i}\mathrm{O^o}(\pm\lambda_o+1)-{}_{i}\mathrm{O^o}(\lambda_o)] = {}^{1}\!/\!_{2} \pm a, \tag{A17} $$

we finally come to the expression

$$ \begin{aligned} {}_{i}\mathrm{I^o}[\eta;\lambda_o|\pm,2\mathbf{j}] = {}_{i}\mathrm{I^o}[\eta;\lambda_o\pm 1] + 2\mathrm{Q}[\eta;\bar{\eta}_{2\mathrm{j}}(\pm\lambda_o)] + \\ ld(\eta^2+1)\; ld\; \Re^{(\pm\lambda_o)}_{2\mathrm{j}}[\eta], \end{aligned} \tag{A18} $$

adopted by us from [17], where

$$ \mathrm{Q}[\eta;\bar{\eta}_{2\mathrm{j}}] := l\,\dot{d}\;\;\Pi_{2\mathrm{j}}[\eta;\bar{\eta}_{2\mathrm{j}}] \tag{A19} $$

$$ = -\sum_{\mathrm{k}=1}^{2\mathrm{j}}(\eta-\eta_{2\mathrm{j};\mathrm{k}})^{-2} \tag{A20} $$

is nothing but an alternative form of the Quesne PF [17]

$$ \mathrm{Q}[\eta;\bar{\eta}_{2\mathrm{j}}] := \frac{\ddot{\Pi}_{2\mathrm{j}}\;[\eta;\bar{\eta}_{2\mathrm{j}}]}{\Pi_{2\mathrm{j}}[\eta;\bar{\eta}_{2\mathrm{j}}]} - \frac{\dot{\Pi}{}^{2}_{2\mathrm{j}}[\eta;\bar{\eta}_{2\mathrm{j}}]}{\Pi^{2}_{2\mathrm{j}}[\eta;\bar{\eta}_{2\mathrm{j}}]}. \tag{A21} $$

Note that the representation of the Routh polynomial in monomial form (62) is the direct consequence of the fact that the polynomial zeros are regular points of the $\Re$Ref CSLE (1) and, as a result, all simple. (In [60], we erroneously referred to (A18) as Quesne's 'partial' decomposition of the RefPF, wrongly assuming that the Quesne PF defined via (A19) has summands with simple poles.) We also take into account that PF (A21) is related to PF (63) via the following elementary formula:

$$ \mathrm{Q}[\eta;\bar{\eta}_{2\mathrm{j}}] = \widehat{\mathrm{Q}}[\eta;\bar{\eta}_{2\mathrm{j}}] + {}^{1}\!/\!_{2}\frac{\ddot{\Pi}_{2\mathrm{j}}\;[\eta;\bar{\eta}_{2\mathrm{j}}]}{\Pi_{2\mathrm{j}}[\eta;\bar{\eta}_{2\mathrm{j}}]}. \tag{A22} $$

Making use of Routh Equation (23), it is convenient to represent PF (A22) in a slightly different form as follows:

$$ \begin{aligned} 2\mathrm{Q}[\eta;\bar{\eta}_{2\mathrm{j}}(\pm\lambda_o)] = 2\widehat{\mathrm{Q}}[\eta;\bar{\eta}_{2\mathrm{j}}(\pm\lambda_o)] - \tfrac{2\eta}{\eta^2+1} ld\; \Re^{(\pm\lambda_o)}_{2\mathrm{j}}[\eta] + \\ -\frac{2\Re^{(\pm\lambda_o)}_{1}[\eta]}{\eta^2+1} ld\; \Re^{(\pm\lambda_o)}_{2\mathrm{j}}[\eta] + \frac{{}_{i}\varepsilon_{\pm,0}(a) - {}_{i}\varepsilon_{\pm,2j}(a)}{\eta^2+1}, \end{aligned} \tag{A23} $$

Substituting (A23) into (A18) then brings us back to RefPF (61).

Combining (A11) and (A18) for $\mathrm{j}=0$, we come the following form-invariance condition:

$$ \begin{aligned} {}_{i}\mathrm{I^o}[\eta;\lambda_o|\pm,0] = {}_{i}\mathrm{I^o}[\eta;\lambda_o\pm 1] \\ = {}_{i}\mathrm{I^o}[\eta;\lambda_o] \;+2\sqrt{{}_{i}\boldsymbol{\rho}_{\diamondsuit}\;[\eta]}\;\frac{\mathrm{d}}{\mathrm{d}\eta}\frac{ld_{i}\phi\,[\eta;\pm\lambda_o-{}^{1}\!/\!_{2}]}{\sqrt{{}_{i}\boldsymbol{\rho}_{\diamondsuit}\;[\eta]}}, \end{aligned} \tag{A24} $$

which has already been used by us to prove (157).

The examination of the PF appearing in (4.25) in [17], with

$$ \mathrm{g}^{(\mathrm{A,B})}_{\mathrm{m}}(\eta) \equiv \Re^{(\mathrm{A}-1/2+i\,\mathrm{B})}_{2\mathrm{j}}[\eta] \tag{A25} $$

in our notation, reveals that it is nothing but PF (A21). Note that Quesne changed the meaning of parameter A in subsection IVB in [17] compared with its definition in the preceding subsection IIIB (cf. her Equations (3.12) and (4.12) in [17]), i.e.,

$$\mathrm{A} \;=\; a + 1/2 \tag{A26}$$

in any formula for the rational SUSY partner of the 'Scarf II' potential.

The corresponding Liouville potentials take the following form:

$$_{i}\mathrm{V}[\eta;\lambda_{\mathrm{o}}|+,2\mathrm{j}] = {}_{i}\mathrm{V}_{\mathrm{G}}[\eta;\lambda_{\mathrm{o}}+1] - 2(\eta^2+1)\mathrm{Q}[\eta;\bar{\eta}_{2\mathrm{j}}(\lambda_{\mathrm{o}})] - 2\eta ld\, \Re_{2\mathrm{j}}^{(\lambda_{\mathrm{o}})}[\eta] \tag{A27}$$

Taking into account (A25) and (A26), we confirm that potential (A27) precisely matches Quesne's expression (4.11) in [17]. As a matter of fact, the simplified Formula (61) for the RefPF of RCSLE (46) was inspired by her work.

In her analysis, Quesne implicitly assumes that all the Routh polynomials of even degrees do not have real roots, which is, indeed, true, since all the q-RSs in question lie below the lowest-energy level. It is very possible that Quesne intuitively realized that this is a direct consequence of the disconjugacy theorem, keeping in mind that her study with Grandati [59] on applications of the disconjugacy theorem to the theory of rational potentials solvable by multi-indexed Jacobi and Laguerre X-OPSs.

Adopting the technique developed in [35] for the X-OPSs, we can convert the eigenfunctions of RCSLE (46) for the TF $|+, 2\mathbf{j})$ to the following quasi-rational form:

$$\begin{aligned} {}_{i}\phi_{\mathbf{c},\mathrm{n}+1}[\eta;\lambda_{\mathrm{o}}|+,2\mathbf{j}] &= -\frac{{}_{i}\phi_{\mathbf{c},0}[\eta;\lambda_{\mathrm{o}}]}{\sqrt{\eta^2+1}\,\Re_{2\mathrm{j}}^{(\lambda_{\mathrm{o}})}(\eta)}\,\mathcal{P}_{2\mathbf{j}+\mathrm{n}+1}[\eta;a+ib|-,\mathrm{n};+,2\mathbf{j}] \\ &\equiv -\frac{{}_{i}\phi_{\mathbf{c},0}[\eta;\lambda_{\mathrm{o}}+1]}{\Re_{2\mathrm{j}}^{(\lambda_{\mathrm{o}})}(\eta)}\,\mathcal{P}_{2\mathbf{j}+\mathrm{n}+1}[\eta;a+ib|-,\mathrm{n};+,2\mathbf{j}]\,(\mathrm{n}\geq 0) \end{aligned} \tag{A28}$$

with the polynomial component represented by the pseudo-Wronskian polynomials

$$\mathcal{P}_{2\mathbf{j}+\mathrm{n}+1}[\eta;a+ib|-,\mathrm{n};+,2\mathbf{j}] =: \begin{vmatrix} \mathrm{R}_{\mathrm{n}}^{(-2b,1-a)}[\eta] & \Re_{2\mathbf{j}+1}^{(a+ib)}[\eta] \\ \overset{\bullet}{\mathrm{R}}{}_{\mathrm{n}}^{(-2b,1-a)}[\eta] & \mathbb{S}_{2\mathbf{j}+1}^{(a+ib)}[\eta] \end{vmatrix}, \tag{A29}$$

where, by definition,

$$\begin{aligned} \mathbb{S}_{2\mathbf{j}+1}^{(a+ib)}[\eta] &:= (1+i\eta)^{1-a-ib}(1-i\eta)^{1+a+ib}\times \\ &\frac{d}{d\eta}\left[(1+i\eta)^{a+ib}(1-i\eta)^{a-ib}\Re_{2\mathbf{j}}^{(a+ib)}[\eta]\right] \end{aligned} \tag{A30}$$

$$= (\eta^2+1)\overset{\bullet}{\Re}{}_{2\mathbf{j}}^{(a+ib)}[\eta] - 2\Re_{1}^{(a-1+ib)}[\eta]\Re_{2\mathbf{j}}^{(a+ib)}[\eta]. \tag{A31}$$

Applying the complexified version of Gauss's contiguous relation (E.16) in [79] to the real-field reduction represented by Routh polynomials (A2) shows that

$$\mathbb{S}_{2\mathbf{j}+1}^{(a+ib)}[\eta] = -2(2\mathbf{j}+1)\Re_{2\mathbf{j}+1}^{(a-1+ib)}[\eta], \tag{A32}$$

which gives

$$\begin{aligned} \mathcal{P}_{2\mathbf{j}+\mathrm{n}+1}[\eta;a+ib|-,\mathrm{n};+,2\mathbf{j}] &= -2(2\mathbf{j}+1)\Re_{2\mathbf{j}+1}^{(a-1+ib)}[\eta]\mathrm{R}_{\mathrm{n}}^{(-2b,1-a)}[\eta] \\ &- \Re_{2\mathbf{j}}^{(a+ib)}[\eta]\,\overset{\bullet}{\mathrm{R}}{}_{\mathrm{n}}^{(-2b,1-a)}[\eta]\,. \end{aligned} \tag{A33}$$

Taking into account (166) and comparing (A28) with (170), we conclude that the EOP sequence in question must be another representation for the EOP sequence composed of the Wronskians of the R-Routh polynomials as follows:

$$\hat{\mathcal{P}}_{2\mathrm{j}+\mathrm{n}+1}[\eta; a+ib|-,\mathrm{n};+,2\mathrm{j}] = {}_{i}\mathcal{W}_{2\mathrm{j}+\mathrm{n}+1}[\eta; a-2\mathrm{j}-1+ib|-\vdots 1:2\mathrm{j},\mathrm{n}]. \quad \text{(A34)}$$

In particular, the latter relation for $\mathrm{n} = 0$ turns into (166) with $2\mathrm{j}_1$ substituted by $2\mathrm{j} + 1$.

More specifically, representation (166) of monic Routh polynomials as Wronskians of R-Routh polynomials of sequential degrees converted to their monic form is the direct consequence of the fact that the conjugated Young diagrams in this case are represented by rows and columns of the same length.

Quesne apparently used the complexified version of Gauss's contiguous relation (22.8.1) in [89] which brings us to her Equation (4.18) in [17], with $\alpha = -2b$, $\beta - 1 = -a$, $\nu = \mathrm{n}$, and $\mathrm{m} = 2\mathbf{j}$, namely,

$$\begin{aligned} &\mathcal{P}_{2\mathbf{j}+\mathrm{n}+1}[\eta; a+ib|-,\mathrm{n};+,2\mathbf{j}] = \\ &(\mathrm{n}-2\mathbf{j}-2a)\left[-\frac{ab}{(\mathrm{n}-a)(a+2\mathbf{j})}+\eta\right]\Re^{(a+ib)}_{2\mathbf{j}}[\eta]\mathrm{R}^{(-2b,1-a)}_{\mathrm{n}}[\eta] \\ &+\frac{(\mathrm{n}-a)^2+b^2}{\mathrm{n}-a}\Re^{(a+ib)}_{2\mathbf{j}}[\eta]\mathrm{R}^{(-2b,1-a)}_{\mathrm{n}-1}[\eta] \\ &-\frac{(a+2\mathbf{j})^2+b^2}{a+2\mathbf{j}}\Re^{(a+ib)}_{2\mathbf{j}-1}[\eta]\mathrm{R}^{(-2b,1-a)}_{\mathrm{n}}[\eta] \end{aligned} \quad \text{(A35)}$$

in our notation.

Appendix C. Enhanced Adler Theorem

The purpose of this Appendix is to extend Adler's Lemma 1 [71] to the eigenfunctions of prime SLE (122), namely,

Enhanced Adler Theorem. *The Wronskian of two sequential eigenfunctions of prime SLE (122) preserves its sign on the real axis.*

As demonstrated in Section 3.4, the proof of Theorem 6 for an arbitrary partition formed by segments of even lengths represents a more challenging problem. We thus postpone for future studies the discussion of the exact solvability of the Dirichlet problem obtained by means of the second-order DCT of the generic SLE with two 'juxtaposed' [66–68] eigenfunctions used as the seed functions.

Proof of Enhanced Adler Theorem. We thus want to confirm that the Wronskian, ${}_{i}\mathrm{W}_{\mathrm{N,N+1}}[\eta; \lambda_{\mathrm{o}}| - \vdots\overline{\mathbf{M}}_{\mathrm{p}}]$, of two sequential eigenfunctions, ${}_{i}\psi_{\mathbf{c},\mathrm{N}}[\eta; \lambda_{\mathrm{o}}| - \vdots\overline{\mathbf{M}}_{\mathrm{p}}]$ and ${}_{i}\psi_{\mathbf{c},\mathrm{N+1}}[\eta; \lambda_{\mathrm{o}}| - \vdots\overline{\mathbf{M}}_{\mathrm{p}}]$, of prime SLE (122) solved under DBCs (124) does not have real zeros. It was already been demonstrated in [30] that the eigenfunctions in question have N and N + 1 interlacing simple zeros ($\eta_{\mathrm{N,k}}$ and $\eta_{\mathrm{N+1,k'}}$) (k = 1, …, N and k' = 1, …, N + 1 accordingly), i.e.,

$$\eta_{\mathrm{N+1,1}} < \eta_{\mathrm{N,1}} < \eta_{\mathrm{N+1,2}} < \ldots < \eta_{\mathrm{N+1,N}} < \eta_{\mathrm{N,N}} < \eta_{\mathrm{N+1,N+1}}. \quad \text{(A36)}$$

The common important feature of all solutions of second-order linear ODEs is that the solution itself and its first derivative may not vanish at the same regular point.

This implies that the eigenfunctions $_i\psi_{\mathbf{c},\mathrm{N}}[\eta;\lambda_o|-\vdots\overline{\mathbf{M}}_p]$ and $_i\psi_{\mathbf{c},\mathrm{N}+1}[\eta;\lambda_o|-\vdots\overline{\mathbf{M}}_p]$ have nonzero first derivatives at their nodes. Furthermore, each function and its first derivative must have opposite sings at two sequential nodes as follows:

$$\psi_{\mathbf{c},\mathrm{N}+\kappa}[\eta_{\mathrm{N}+1-\kappa,k};\lambda_o|-\vdots\overline{\mathbf{M}}_p]\ \psi_{\mathbf{c},\mathrm{N}+\kappa}[\eta_{\mathrm{N}+1-\kappa,k+1};\lambda_o|-\vdots\overline{\mathbf{M}}_p]<0 \quad (\kappa=0,1) \tag{A37}$$

and

$$\dot{\psi}_{\mathbf{c},\mathrm{N}+\kappa}[\eta_{\mathrm{N}+\kappa,k};\lambda_o|-\vdots\overline{\mathbf{M}}_p]\ \dot{\psi}_{\mathbf{c},\mathrm{N}+\kappa}[\eta_{\mathrm{N}+\kappa,k+1};\lambda_o|-\vdots\overline{\mathbf{M}}_p]<0 \quad (\kappa=0,1), \tag{A38}$$

respectively.

Let us, in following [30], introduce the $g-\mathrm{W}$

$$_i\mathbf{W}_{\mathrm{N},\mathrm{N}+1}[\eta;\lambda_o|-\vdots\overline{\mathbf{M}}_p]:=(\eta^2+1)^{1/2}{}_i\mathrm{W}_{\mathrm{N},\mathrm{N}+1}[\eta;\lambda_o|-\vdots\overline{\mathbf{M}}_p] \tag{A39}$$

and generalize Adler's arguments [71] by proving that the $g-\mathrm{W}$ sign is preserved on the real axis. Indeed, representing $g-\mathrm{W}$ (C4) as

$$\begin{gathered}{}_i\mathbf{W}_{\mathrm{N},\mathrm{N}+1}[\eta;\lambda_o|-\vdots\overline{\mathbf{M}}_p]={}_i\psi_{\mathbf{c},\mathrm{N}}[\eta;\lambda_o|-\vdots\overline{\mathbf{M}}_p]{}_i\psi_{\mathbf{c},\mathrm{N}+1}[\eta;\lambda_o|-\vdots\overline{\mathbf{M}}_p]\times\\(\eta^2+1)^{1/2}\left\{ld_i\psi_{\mathbf{c},\mathrm{N}+1}[\eta;\lambda_o|-\vdots\overline{\mathbf{M}}_p]-ld_i\psi_{\mathbf{c},\mathrm{N}}[\eta;\lambda_o|-\vdots\overline{\mathbf{M}}_p]\right\}\end{gathered} \tag{A40}$$

and using DBCs (124) coupled with asymptotic relation (136), one finds that $g-\mathrm{W}$ (A39) vanishes at infinity as follows:

$$\lim_{|\eta|\to\infty}{}_i\mathbf{W}_{\mathrm{N},\mathrm{N}+1}[\eta;\lambda_o|-\vdots\overline{\mathbf{M}}_p]=0, \tag{A41}$$

Without loss of generality, we can choose both eigenfunctions to have the same sign within the interval $(-\infty,\eta_{\mathrm{N}+1,1})$, which assures that the derivative of the $g-\mathrm{W}$ [30],

$$\begin{gathered}{}_i\dot{\mathbf{W}}_{\mathrm{N},\mathrm{N}+1}[\eta;\lambda_o|-\vdots\overline{\mathbf{M}}_p]=-[\varepsilon_{\mathbf{c},\mathrm{N}+1}(a)-\varepsilon_{\mathbf{c},\mathrm{N}}(a)]\,{}_iw[\eta]\times\\\psi_{\mathbf{c},\mathrm{N}}[\eta;\lambda_o|-\vdots\overline{\mathbf{M}}_p]\ \psi_{\mathbf{c},\mathrm{N}+1}[\eta;\lambda_o|-\vdots\overline{\mathbf{M}}_p],\end{gathered} \tag{A42}$$

is negative there. Keeping in mind (A41), this should also be true for the $g-\mathrm{W}$; therefore,

$$\begin{gathered}\mathbf{W}_{\mathrm{N},\mathrm{N}+1}[\eta_{\mathbf{N}+1,1};\lambda_o|-\vdots\overline{\mathbf{M}}_p]=\sqrt{\eta^2_{\mathrm{N}+1,1}+1}\times\\\psi_{\mathbf{c},\mathrm{N}}[\eta_{\mathbf{N}+1,1};\lambda_o|-\vdots\overline{\mathbf{M}}_p]\ \dot{\psi}_{\mathbf{c},\mathrm{N}+1}[\eta_{\mathbf{N}+1,1};\lambda_o|-\vdots\overline{\mathbf{M}}_p]<0.\end{gathered} \tag{A43}$$

Note that, in contrast with Adler's arguments in support of his Lemma 1, the right-hand side of (A42) contains η-dependent weight (30), which further complicates the expression for the second derivative of the $g-\mathrm{W}$ compared with (13) in [71]. For this reason we have to slightly modify the proof of the lemma.

First, one can directly verify that

$$\psi_{\mathbf{c},\mathrm{N}+1}[\eta_{\mathbf{N},1};\lambda_o|-\vdots\overline{\mathbf{M}}_p]<0,\ \ \dot{\psi}_{\mathbf{c},\mathrm{N}}[\eta_{\mathbf{N},1};\lambda_o|-\vdots\overline{\mathbf{M}}_p]<0, \tag{A44}$$

so the $\mathcal{g}-\mathrm{W}$ is necessarily negative at the end of the interval $[\eta_{N+1,1}, \eta_{N,1}]$ as follows:

$$\begin{aligned} \mathbf{W}_{N,N+1}[\eta_{\mathbf{N},1};\lambda_o|-\vdots\overline{\mathbf{M}}_p] = -\sqrt{\eta_{N,1}^2+1}\times \\ \psi_{\mathbf{c},N+1}[\eta_{\mathbf{N},1};\lambda_o|-\vdots\overline{\mathbf{M}}_p]\ \dot{\psi}_{\mathbf{c},N}[\eta_{\mathbf{N},1};\lambda_o|-\vdots\overline{\mathbf{M}}_p] < 0. \end{aligned} \tag{A45}$$

Therefore, it must preserve its sign within the interval. It then directly follows from (A37) and (A38) that the $\mathcal{g}-\mathrm{W}$ must be also negative at the end of interval $[\eta_{N,1}, \eta_{N+1,2}]$ and, hence, must preserve its sign within the interval $(-\infty, \eta_{N+1,2}]$. One can then continue using (A37) and (A38) for other zeros (A36), including the last one, as follows:

$$\begin{aligned} \mathbf{W}_{N,N+1}[\eta_{N+1,N+1};\lambda_o|-\vdots\overline{\mathbf{M}}_p] = \sqrt{\eta_{N+1,N+1}^2+1}\times \\ \psi_{\mathbf{c},N}[\eta_{N+1,N+1};\lambda_o|-\vdots\overline{\mathbf{M}}_p]\ \dot{\psi}_{\mathbf{c},N+1}[\eta_{N+1,N+1};\lambda_o|-\vdots\overline{\mathbf{M}}_p] < 0. \end{aligned} \tag{A46}$$

Keeping in mind that the derivative of the $\mathcal{g}-\mathrm{W}$ must preserve its sign within the interval $[\eta_{N+1,N+1}, +\infty)$ and the $\mathcal{g}-\mathrm{W}$ itself vanishes in the limit $\eta \to \infty$, we conclude that the latter must remain negative on the real axis; therefore, this is also true for the conventional Wronskian of the two eigenfunctions. □

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