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On the Analytic Extension of Lauricella–Saran's Hypergeometric Function F_K to Symmetric Domains

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Abstract: In this paper, we consider the representation and extension of the analytic functions of three variables by special families of functions, namely branched continued fractions. In particular, we establish new symmetric domains of the analytical continuation of Lauricella–Saran's hypergeometric function F_K with certain conditions on real and complex parameters using their branched continued fraction representations. We use a technique that extends the convergence, which is already known for a small domain, to a larger domain to obtain domains of convergence of branched continued fractions and the PC method to prove that they are also domains of analytical continuation. In addition, we discuss some applicable special cases and vital remarks.

Keywords: Lauricella–Saran's hypergeometric function; branched continued fraction; holomorphic functions of several complex variables; analytic continuation; convergence

MSC: 33C65; 32A17; 32A10; 30B40; 40A99

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1. Introduction

Special functions, including Lauricella–Saran's hypergeometric functions, occur naturally in various problems in mathematics, statistics, physics, chemistry, and engineering. This paper discusses the representation and analytical extension of these functions. Domains of analytical continuation will be symmetric domains of convergence of special families of functions, namely branched continued fractions. Note that here, the domain is an open connected subset of $\mathbb{C}^3$.

Lauricella–Saran's family of 14 functions (F_A, F_B, $\ldots$, F_T or F_1, F_1, $\ldots$, F_{14}) owes its appearance mainly to two papers [1,2]. These functions are defined by triple power series, particularly

$$F_K(\alpha_1,\alpha_2,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_3;\mathbf{z})=\sum_{p,q,r=0}^{+\infty}\frac{(\alpha_1)_p(\alpha_2)_{q+r}(\beta_1)_{p+r}(\beta_2)_q}{(\gamma_1)_p(\gamma_2)_q(\gamma_3)_r}\frac{z_1^p}{p!}\frac{z_2^q}{q!}\frac{z_3^r}{r!},$$

where $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2,\gamma_3\in\mathbb{C}$; $\gamma_1,\gamma_2,\gamma_3\notin\{0,-1,-2,\ldots\}$; $(\alpha)_k=\alpha(\alpha+1)_{k-1}$, $k\geq 1$, $(\alpha)_0=1$; and $\mathbf{z}=(z_1,z_2,z_3)\in\Theta$, where

$$\Theta=\{\mathbf{z}\in\mathbb{C}^3:\ |z_1|<1,\ |z_2|<1,\ |z_3|<(1-|z_1|)(1-|z_2|)\}.$$

Various applications and studies of different properties of Lauricella–Saran's functions are discussed in many scientific works. In particular, Lauricella–Saran's hypergeometric function F_K was used to compute the canonical partition function of the model of heteropolymers in the form of a freely jointed chain [3,4] and the generalized Nordsieck integral [5] to investigate the compound gamma bivariate distribution [6,7] and the propagator seagull diagram [8]. The problem of analytical continuation and asymptotics for the F_K function using the integral representation was considered in [9,10]. Asymptotic expansions of the function F_D were studied in [11].

The authors of [12] gave the formal expansion

$$\frac{F_K(\alpha_1,\alpha_2,\beta_1,\beta_2;\alpha_1,\beta_2,\gamma_3;\mathbf{z})}{F_K(\alpha_1,\alpha_2,\beta_1+1,\beta_2;\alpha_1,\beta_2,\gamma_3+1;\mathbf{z})} = 1 - z_1 - \cfrac{u_1 z_3}{1 - z_2 - \cfrac{u_2 z_3}{1 - z_1 - \cfrac{u_3 z_3}{1 - z_2 - \cfrac{u_4 z_3}{1 - \ddots}}}}, \quad (1)$$

where

$$u_{2k-1} = \frac{(\alpha_2+k-1)(\gamma_3+k-1-\beta_1)}{(\gamma_3+2k-2)(\gamma_3+2k-1)}, \quad u_{2k} = \frac{(\beta_1+k)(\gamma_3+k-\alpha_2)}{(\gamma_3+2k-1)(\gamma_3+2k)}, \quad k \geq 1, \quad (2)$$

as well as the following expansion, which is symmetrical to it:

$$\frac{F_K(\alpha_1,\alpha_2,\beta_1,\beta_2;\alpha_1,\beta_2,\gamma_3;\mathbf{z})}{F_K(\alpha_1,\alpha_2+1,\beta_1,\beta_2;\alpha_1,\beta_2,\gamma_3+1;\mathbf{z})} = 1 - z_2 - \cfrac{v_1 z_3}{1 - z_1 - \cfrac{v_2 z_3}{1 - z_2 - \cfrac{v_3 z_3}{1 - z_1 - \cfrac{v_4 z_3}{1 - \ddots}}}}, \quad (3)$$

where

$$v_{2k-1} = \frac{(\beta_1+k-1)(\gamma_3+k-1-\alpha_2)}{(\gamma_3+2k-2)(\gamma_3+2k-1)}, \quad v_{2k} = \frac{(\alpha_2+k)(\gamma_3+k-\beta_1)}{(\gamma_3+2k-1)(\gamma_3+2k)}, \quad k \geq 1. \quad (4)$$

It is also established here that

$$\Omega_{\tau,r} = \left\{ \mathbf{z} \in \mathbb{C}^3 : \; z_k \notin [1-r,+\infty), \; k = 1,2, \; z_3 \notin \left[\frac{r}{4\tau},+\infty\right) \right\}, \quad 0 < r < 1, \quad (5)$$

is the domain of the analytical continuation of the function on the left side of Equation (1) (or Equation (3)), provided that $0 < u_k \leq \tau$ (or $0 < v_k \leq \tau$), where $k \geq 1$. The problem of representing and extending Lauricella–Saran's hypergeometric functions F_D and F_S through branched continued fractions was considered in [13–15], respectively.

In Section 2 of this paper, we give the necessary definitions and preliminary results. New symmetric domains of the analytical continuation of Lauricella–Saran's hypergeometric function F_K with certain conditions for real and complex parameters, using their branched continued fraction representations, are established in Section 3.

General information on branched continued fractions can be found in [16–18].

2. Preliminary Definitions and Results

The concept of a branched continued fraction can be approached in different ways, particularly through the sequence of its approximants. A brief description follows.

Let $i(0) = 0$, $\mathcal{I}_0 = \{0\}$, and let

$$\mathcal{I}_k = \{ i(k) : \; i(k) = (i_1, i_2, \ldots, i_k), \; 1 \leq i_s \leq 3, \; 1 \leq s \leq k \}, \quad k \geq 1,$$

denote the sets of multiindices.

The ordered pair of sequences

$$\langle \{u_{i(k)}\}_{i(k)\in\mathcal{I}_k,\, k\in\mathbb{N}}, \{v_{i(k)}\}_{i(k)\in\mathcal{I}_k,\, k\in\mathbb{N}_0} \rangle$$

of complex numbers satisfies the following conditions:

(1) $u_{i(k)} \neq 0$ for $i(k) \in \mathcal{I}_k$; $k \geq 1$,

(2) If for $k \geq 1$ there exists a multiindex $i(k) \in \mathcal{I}_k$ such that $v_{i(k)} = 0$, then $v_{i(k-1),j} \neq 0$ for $1 \leq j \leq 3$ and $j \neq i_k$.

We then generate the sequence $\{f_k\}$ as follows:

$$\begin{aligned} f_0 &= v_0, \\ f_1 &= v_0 + \sum_{i_1=1}^{3} \frac{u_{i(1)}}{v_{i(1)}}, \\ f_2 &= v_0 + \sum_{i_1=1}^{3} \frac{u_{i(1)}}{v_{i(1)} + \sum\limits_{i_2=1}^{3} \frac{u_{i(2)}}{v_{i(2)}}}, \end{aligned}$$

and so on, in addition to

$$f_k = v_0 + \sum_{i_1=1}^{3} \cfrac{u_{i(1)}}{v_{i(1)} + \sum\limits_{i_2=1}^{3} \cfrac{u_{i(2)}}{v_{i(2)} + \ddots + \sum\limits_{i_k=1}^{3} \cfrac{u_{i(k)}}{v_{i(k)}}}},$$

and so on.

The ordered pair

$$\langle\langle\{u_{i(k)}\}_{i(k)\in\mathcal{I}_k, k\in\mathbb{N}}, \{v_{i(k)}\}_{i(k)\in\mathcal{I}_k, k\in\mathbb{N}_0}\rangle, \{f_k\}_{k\in\mathbb{N}_0}\rangle$$

is the *branched continued fraction* denoted by

$$v_0 + \sum_{i_1=1}^{3} \cfrac{u_{i(1)}}{v_{i(1)} + \sum\limits_{i_2=1}^{3} \cfrac{u_{i(2)}}{v_{i(2)} + \ddots + \sum\limits_{i_k=1}^{3} \cfrac{u_{i(k)}}{v_{i(k)} + \ddots}}}. \tag{6}$$

The values v_0, $u_{i(k)}$ and $v_{i(k)}$, $i(k) \in \mathcal{I}_k$, are called *elements* of Equation (6). The value f_k is called the kth *approximant*, where $k \geq 0$.

Furthermore, considering the branched continued fraction in Equation (6), we admit a confluent case where there are no constraints (1).

We shall need the following:

Definition 1. *A branched continued fraction (Equation (6)) converges if, at most, a finite number of its approximants do not make sense and if the limit of its sequence of approximants*

$$\lim_{n\to\infty} f_n$$

exists and is finite.

Note that the approximant makes sense if the $0/0$ uncertainty does not arise when computing its value. We assume that

$$\frac{1}{0} = \infty, \quad \frac{1}{\infty} = 0, \quad \text{and} \quad \frac{u}{0} + \frac{v}{0} = \frac{0}{0} \quad \text{for all} \quad u, v \in \mathbb{C}.$$

Definition 2. *A branched continued fraction (Equation (6)) converges absolutely if*

$$\sum_{n=1}^{\infty} |f_{n+1} - f_n| < \infty.$$

The convergence criteria for branched continued fractions are often given in terms of convergence sets. The definition is as follows:

Definition 3. *A convergence set Ω is a set where $\Omega \neq \emptyset$ and $\Omega \subseteq \mathbb{C} \times \mathbb{C}$ such that if $\langle u_{i(k)}, v_{i(k)} \rangle \in \Omega$ for all $i(k) \in \mathcal{I}_k$, where $k \geq 1$, then Equation (6) converges.*

Definition 4. *A uniform convergence set Ω is a convergence set to which there corresponds a sequence $\{\varepsilon_k\}$ of positive numbers depending only on Ω and converging to zero such that*

$$|f_{k+p} - f_k| \leq \varepsilon_k \quad \text{for all} \quad k, p \in \mathbb{N},$$

for every branched continued fraction (Equation (6)) with all $\langle u_{i(k)}, v_{i(k)} \rangle \in \Omega$.

Theorem 1 below was proven in [19]. Here, for convenience, we give its formulation using multiindex notation in the same way as in the triple power series. Note that this is possible because for any $k \geq 1$, there is a mapping $\varphi : \mathcal{J}_k \to \mathcal{E}_k$, where

$$\mathcal{J}_k = \{i(k) : \; i(k) = (i_1, i_2, \dots, i_k), \; 1 \leq i_s \leq i_{s-1}, \; 1 \leq s \leq k, \; i_0 = 3\}, \quad k \geq 1,$$

and

$$\mathcal{E}_k = \{e_{i(k)} : \; e_{i(k)} = e_{i_1, i_2, \dots, i_k} = e_{i_1} + e_{i_2} + \dots + e_{i_k}, \; i(k) \in \mathcal{J}_k\}, \quad k \geq 1,$$

where $e_k = (\delta_{k,1}, \delta_{k,2}, \delta_{k,3})$, $1 \leq k \leq 3$, and $\delta_{i,j}$ is the Kronecker delta, such that $\varphi(i(k)) = e_{i(k)}$ for all $i(k) \in \mathcal{J}_k$. Also, it can be shown that the mapping φ is bijective.

Theorem 1. *Let $m_{0,0,k}$, where $k \geq 1$, be real constants satisfying*

$$0 < m_{0,0,k} \leq 1, \quad k \geq 1.$$

Then, the following are true:

(1) The branched continued fraction

$$1 - z_{1,0,0} - \cfrac{m_{0,0,1} z_{0,0,1}}{1 - (1 - m_{0,0,1}) z_{0,1,1} - \cfrac{m_{0,0,2}(1 - m_{0,0,1}) z_{0,0,2}}{1 - (1 - m_{0,0,2}) z_{1,0,2} - \cfrac{m_{0,0,3}(1 - m_{0,0,2}) z_{0,0,3}}{1 - \ddots}}}, \tag{7}$$

converges absolutely and uniformly for

$$|z_{1,0,2k}| \leq \frac{1}{2}, \quad |z_{0,1,2k+1}| \leq \frac{1}{2}, \quad |z_{0,0,k+1}| \leq \frac{1}{2}, \quad k \geq 1; \tag{8}$$

(2) The values of the branched continued fraction and of its approximants are in the closed domain

$$|w - 1| \leq 1. \tag{9}$$

Indeed, Theorem 1 (1) follows directly from Theorem 1 [19], with

$$m_{1,0,2r} = m_{0,1,2r+1} = 1, \quad z_{1,0,2r+1} = z_{0,1,2r} = 0, \quad r \geq 0.$$

Next, by setting

$$F_n^{(n)} = 1, \quad n \geq 1,$$

it is clear that

$$F_{2k-1}^{(2n)} = 1 - (1 - m_{0,0,2k-1}) z_{0,1,2k-1} - \frac{m_{0,0,2k}(1 - m_{0,0,2k-1}) z_{0,0,2k}}{F_{2k}^{(2n)}},$$

$$F_{2k-2}^{(2n)} = 1 - (1 - m_{0,0,2k-2}) z_{1,0,2k-2} - \frac{m_{0,0,2k-1}(1 - m_{0,0,2k-2}) z_{0,0,2k-1}}{F_{2k-1}^{(2n)}},$$

where $m_{0,0,0} = 0$ and

$$F_{2k-1}^{(2n+1)} = 1 - (1 - m_{0,0,2k-1}) z_{0,1,2k-1} - \frac{m_{0,0,2k}(1 - m_{0,0,2k-1}) z_{0,0,2k}}{F_{2k}^{(2n+1)}},$$

$$F_{2k}^{(2n+1)} = 1 - (1 - m_{0,0,2k}) z_{1,0,2k} - \frac{m_{0,0,2k+1}(1 - m_{0,0,2k}) z_{0,0,2k+1}}{F_{2k+1}^{(2n+1)}}$$

are valid for $n \geq 1$ and $1 \leq k \leq n$. Therefore, the nth approximants of Equation (7) can be written as follows:

$$f_n = 1 - z_{1,0,0} - \frac{m_{0,0,1} z_{0,0,1}}{F_1^{(n)}}.$$

In the proof of Theorem 1 [19], it is shown that

$$|F_1^{(n)}| \geq m_{0,0,1}, \quad n \geq 1, \tag{10}$$

and hence, according to Equations (8) and (10), any $n \geq 1$ yields

$$\begin{aligned} |f_n - 1| &\leq |z_{1,0,0}| + \frac{m_{0,0,1}|z_{0,0,1}|}{|F_1^{(n)}|} \\ &\leq \frac{1}{2} + \frac{1}{2} \\ &= 1, \end{aligned}$$

which proves Theorem 1 (2).

Note that this theorem is analogous to Theorem 11.1 in [20]. Also, the fact that the majorant method [17] (p. 51) and the formula for the difference of two approximants of the branched continued fraction [17] (p. 28) were used in the proof of Theorem 1 [19].

An important application of branched continued fractions is the representation of holomorphic functions by branched continued fractions, the elements of which are functions, particularly polynomials. And here, we need the following definition:

Definition 5. *A branched continued fraction whose elements are functions in a certain domain $D \subset \mathbb{C}^3$ converges uniformly on the set $E \subset D$ if its sequence of approximants converges uniformly on E. If this occurs for an arbitrary set E such that $\overline{E} \subset D$, (Here, $\overline{E}$ is the closure of the set E) then the branched continued fraction converges uniformly on every compact subset of D.*

3. Branched Continued Fractions and Analytic Continuation

In this section, we prove that the branched continued fraction in Equation (1) (as well as Equation (3)) converges in new symmetric domains and provides the analytic continuation in these domains. One of the key results here is Theorem 2. Its proof reveals a technique for extending the convergence, which is already known for a small domain, to a larger domain, and it uses some of the results for Theorem 6 [12]. It also shows that one of the reasons why parabolic regions are so important is that they form the basis of the cardioid domains. Note that here, the region is a domain together with all, part, or none of its boundary.

Theorem 2. *Let α_2, β_1, and γ_3 be complex constants which satisfy the conditions*

$$|u_k| + \mathrm{Re}(u_k) \leq pq(1-q), \quad k \geq 1, \tag{11}$$

where u_k, $k \geq 1$ are defined by Equation (2), $\gamma_3 \notin \{0, -1, -2, \ldots\}$, p is a positive number, and $0 < q < 1$. Then, the following are true:

(1) The branched continued fraction in Equation (1) converges uniformly on every compact subset of the domain

$$\Theta_{p,q}^{u,r} = \Omega_{p,q} \bigcup \Omega^{u,r}, \tag{12}$$

where

$$\Omega_{p,q} = \left\{ \mathbf{z} \in \mathbb{C}^3 : \ z_k \neq \left[\frac{q}{2}, +\infty\right), \ k = 1,2, \ |z_3| < \frac{1+\cos(\arg(z_3))}{2p} \right\} \tag{13}$$

and

$$\Omega^{u,r} = \left\{ \mathbf{z} \in \mathbb{C}^3 : \ |z_k| < \frac{1-r}{2}, \ k = 1,2, \ |z_3| < \frac{r(1-r)}{2u} \right\}, \tag{14}$$

where

$$u = \max_{k \in \mathbb{N}} \{|u_k|\}, \quad 0 < r < 1, \tag{15}$$

to a function $f(\mathbf{z})$ holomorphic in $\Theta_{p,q}^{u,r}$;

(2) The function $f(\mathbf{z})$ is an analytic continuation of the function on the left side of Equation (1) in Equation (12).

Proof. We show that (1) is valid in the domain in Equation (13), where for convenience we write the sets

$$z_k \neq \left[\frac{q}{2}, +\infty\right), \quad k = 1,2,$$

as

$$\mathrm{Re}(z_k e^{-(i/2)\arg(z_3)}) < \frac{q}{2} \cos\left(\frac{1}{2}\arg(z_3)\right), \quad k = 1,2.$$

Let

$$F_n^{(n)}(\mathbf{z}) = 1, \quad n \geq 1, \tag{16}$$

Then, it is clear that the recurrence relations

$$F_{2k-1}^{(2n)}(\mathbf{z}) = 1 - z_2 - \frac{u_{2k}z_3}{F_{2k}^{(2n)}(\mathbf{z})}, \quad F_{2k-2}^{(2n)}(\mathbf{z}) = 1 - z_1 - \frac{u_{2k-1}z_3}{F_{2k-1}^{(2n)}(\mathbf{z})}, \tag{17}$$

and

$$F_{2k-1}^{(2n+1)}(\mathbf{z}) = 1 - z_2 - \frac{u_{2k}z_3}{F_{2k}^{(2n+1)}(\mathbf{z})}, \quad F_{2k}^{(2n+1)}(\mathbf{z}) = 1 - z_1 - \frac{u_{2k+1}z_3}{F_{2k+1}^{(2n+1)}(\mathbf{z})},$$

are valid for $n \geq 1$ and $1 \leq k \leq n$.

Thus, the nth approximants of Equation (1) can be written as follows:

$$f_n(\mathbf{z}) = 1 - z_1 - \frac{u_1 z_3}{F_1^{(n)}(\mathbf{z})}. \tag{18}$$

Let us show that the approximants of the branched continued fraction in Equation (1) form a sequence of functions holomorphic in the domain in Equation (13). Since the numerator and denominator of each approximant are polynomials, they are the entire functions of three variables. And the quotient of two entire functions, where the denominator is not equal to zero, is a holomorphic function. Therefore, it suffices to show that

$$F_1^{(n)}(\mathbf{z}) \neq 0 \quad \text{for all} \quad n \geq 1 \quad \text{and} \quad \mathbf{z} \in \Omega_{p,q}. \tag{19}$$

Let n be an arbitrary natural number and $\mathbf{z}$ be an arbitrary fixed point in Equation (13). We set $\arg(z_3) = \psi$. Then the inequalities

$$\operatorname{Re}(F_{2k-1}^{(2n)}(\mathbf{z})e^{-i\psi/2}) > (1-q)\cos(\psi/2) \geq c > 0 \tag{20}$$

and

$$\operatorname{Re}(F_{2k-1}^{(2n+1)}(\mathbf{z})e^{-i\psi/2}) > (1-q)\cos(\psi/2) \geq c > 0 \tag{21}$$

are valid for $1 \leq k \leq n$.

Indeed, since $\mathbf{z}$ is an arbitrary fixed point in $\Omega_{p,q}$, then for its arbitrary neighborhood, there exists ε, $0 < \varepsilon \leq \pi/2$ such that

$$|\psi/2| \leq \pi/2 - \varepsilon,$$

and therefore

$$(1-q)\cos(\psi/2) \geq (1-q)\cos(\psi/2 - \varepsilon) = (1-q)\sin(\varepsilon) = c > 0.$$

Next, we prove the first inequality in Equation (20). From Equation (16), it is clear that Equation (20) is valid for $k = n$. Let the first inequality in Equation (20) hold for $k = s+1$, where $s + 1 \leq n$. Then, from Equation (17), one finds that

$$\begin{aligned}
F_{2s-1}^{(2n)}(\mathbf{z})e^{-i\psi/2} &= e^{-i\psi/2} - z_2 e^{-i\psi/2} - \frac{u_{2s} z_3 e^{-i\psi}}{F_{2s}^{(2n)}(\mathbf{z})e^{-i\psi/2}}, \\
F_{2s}^{(2n)}(\mathbf{z})e^{-i\psi/2} &= e^{-i\psi/2} - z_1 e^{-i\psi/2} - \frac{u_{2s+1} z_3 e^{-i\psi}}{F_{2s+1}^{(2n)}(\mathbf{z})e^{-i\psi/2}},
\end{aligned}$$

and hence, under Equations (11) and (13) and Corollary 2 [12], we have

$$\begin{aligned}
\operatorname{Re}(F_{2s}^{(2n)}(\mathbf{z})e^{-i\psi/2}) &= \operatorname{Re}(e^{-i\psi/2}) - \operatorname{Re}(z_1 e^{-i\psi/2}) - \operatorname{Re}\left(\frac{u_{2s+1} z_3 e^{-i\psi}}{F_{2s+1}^{(2n)}(\mathbf{z})e^{-i\psi/2}}\right) \\
&\geq \cos(\psi/2) - \operatorname{Re}(z_1 e^{-i\psi/2}) - |z_3| \frac{|u_{2s+1}| + \operatorname{Re}(u_{2s+1})}{2\operatorname{Re}(F_{2s+1}^{(2n)}(\mathbf{z})e^{-i\psi/2})} \\
&> \cos(\psi/2) - \frac{q\cos(\psi/2)}{2} - \frac{pq(1-q)}{2(1-q)\cos(\psi/2)} \frac{1+\cos(\psi)}{2p} \\
&= (1-q)\cos(\psi/2)
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Re}(F_{2s-1}^{(2n)}(\mathbf{z})e^{-i\psi/2}) &= \operatorname{Re}(e^{-i\psi/2}) - \operatorname{Re}(z_2 e^{-i\psi/2}) - \operatorname{Re}\left(\frac{u_{2s} z_3 e^{-i\psi}}{F_{2s}^{(2n)}(\mathbf{z})e^{-i\psi/2}}\right) \\
&\geq \cos(\psi/2) - \operatorname{Re}(z_2 e^{-i\psi/2}) - |z_3|\frac{|u_{2s}| + \operatorname{Re}(u_{2s})}{2\operatorname{Re}(F_{2s}^{(2n)}(\mathbf{z})e^{-i\psi/2})} \\
&> \cos(\psi/2) - \frac{q\cos(\psi/2)}{2} - \frac{pq(1-q)}{2(1-q)\cos(\psi/2)}\frac{1+\cos(\psi)}{2p} \\
&= (1-q)\cos(\psi/2).
\end{aligned}$$

Similarly, we obtain the first inequality in Equation (21). Thus, the inequalities in Equation (19) hold, and therefore, $\{f_n(\mathbf{z})\}$ is a sequence of functions holomorphic in the domain in Equation (13).

Next, for an arbitrary compact subset Y of $\Omega_{p,q}$, there exists an open triple-disk

$$\Xi_R = \{\mathbf{z} \in \mathbb{C}^3 : \ |z_k| < R, \ 1 \leq k \leq 3\}, \quad R > 0, \tag{22}$$

such that $\mathrm{Y} \subset \Xi_R$, and hence, under Equations (18), (20), and (21), for any $n \geq 1$ and $\mathbf{z} \in \Omega_{p,q} \bigcap \Xi_R$, we have

$$\begin{aligned}
|f_n(\mathbf{z})| &\leq 1 + |z_1| + \frac{|u_1||z_3|}{\operatorname{Re}(F_1^{(n)}(\mathbf{z})e^{-i\psi/2})} \\
&< 1 + R + \frac{uR}{(1-q)\cos(\psi/2)} \\
&= C(\mathrm{Y}),
\end{aligned}$$

In other words, $\{f_n(\mathbf{z})\}$ is a sequence of functions uniformly bounded on every compact subset of Equation (13).

It is clear that for each L that satisfies the inequalities

$$0 < L < \min\left\{\frac{1-r}{2}, \frac{r(1-r)}{2u}, \frac{1}{p}, \frac{q}{2}\right\}$$

the domain

$$\Gamma_L = \{\mathbf{z} \in \mathbb{R}^3 : \ 0 < z_k < L, \ 1 \leq k \leq 3\}$$

is contained in Equation (13), particularly $\Gamma_{L/2} \subset \Omega_{p,q}$. Moreover, for any $\mathbf{z} \in \Gamma_L, \Gamma_L \subset \Omega_{p,q}$, using Equation (15), one finds that

$$|z_1| < \frac{1-r}{2}, \quad |z_2| < \frac{1-r}{2}, \quad |u_k z_3| < \frac{r(1-r)}{2}, \quad k \geq 1,$$

In other words, the elements of Equation (1) satisfy Theorem 1 with $m_{0,0,k} = r$, where $k \geq 1$. Thus, under Theorem 1 (1), the branched continued fraction in Equation (1) converges in Γ_L, $\Gamma_L \subset \Omega_{p,q}$. An application of Theorem 5 [12] then yields the uniform convergence of Equation (1) to a holomorphic function on every compact subset of Equation (13).

Therefore, to prove (1), it suffices to show that this assertion is also valid in the domain in Equation (14). An application of Theorem 1 (1) with $m_{0,0,k} = r$, where $k \geq 1$, shows that the branched continued fraction in Equation (1) converges for all $\mathbf{z} \in \Omega^{u,r}$. Theorem 1 (2) implies that the approximants of Equation (1) all lie in the closed domain in Equation (9) if $\mathbf{z} \in \Omega^{u,r}$. Hence, under Theorem 5 [12], the convergence of the branched continued fraction in Equation (1) is uniform on every compact subset of the domain in Equation (14).

The proof of (2) is analogous to the proof of Theorem 6 (2) [12], and hence it was omitted. Note that from the proof of Theorem 6 (2) [12], it follows that the branched continued fraction in Equation (1) corresponds at $\mathbf{z} = \mathbf{0}$ to the function on the left side of Equation (1). Since, as was proved above, the sequence of its approximants converges uniformly on each compact subset of some neighborhood of the origin to a function $f(\mathbf{z})$ holomorphic in this neighborhood, one can apply the PC method (see [12]) and easily prove that the function $f(\mathbf{z})$ to which the branched continued fraction in Equation (1) converges on the domain in Equation (12) is an analytic continuation of the function on the left side of Equation (1) in this domain. □

Corollary 1. *Let α_2 and γ_3 be complex constants satisfying Equation (11), where*

$$u_{2k-1} = \frac{(\alpha_2+k-1)(\gamma_3+k-2)}{(\gamma_3+2k-3)(\gamma_3+2k-2)}, \quad u_{2k} = \frac{k(\gamma_3+k-1-\alpha_2)}{(\gamma_3+2k-2)(\gamma_3+2k-1)}, \quad k \geq 1, \tag{23}$$

p is a positive number, $\gamma_3 \notin \{1, 0, -1, -2, \ldots\}$, and $0 < q < 1$. Then, we have

$$\cfrac{1}{1-z_1-\cfrac{u_1z_3}{1-z_2-\cfrac{u_2z_3}{1-z_1-\cfrac{u_3z_3}{1-z_2-\cfrac{u_4z_3}{1-\ddots}}}}} \tag{24}$$

converging uniformly on every compact subset of Equation (12) to a function $f(\mathbf{z})$ holomorphic in $\Theta_{p,q}^{u,r}$, and $f(\mathbf{z})$ is an analytic continuation of the function

$$F_K(\alpha_1, \alpha_2, 1, \beta_2; \alpha_1, \beta_2, \gamma_3; \mathbf{z}) \tag{25}$$

in the domain in Equation (12).

Theorem 3 is symmetric to Theorem 2, and thus it can be proven in much the same way as in Theorem 2.

Theorem 3. *Let α_2, β_1, and γ_3 be complex constants which satisfy the conditions*

$$|v_k| + \mathrm{Re}(v_k) \leq pq(1-q), \quad k \geq 1, \tag{26}$$

where v_k with $k \geq 1$ are defined by Equation (4), $\gamma_3 \notin \{0, -1, -2, \ldots\}$, p is a positive number, and $0 < q < 1$. Then, thet following are true:

(1) *The branched continued fraction in Equation (3) converges uniformly on every compact subset of the domain*

$$\Theta_{p,q}^{v,r} = \Omega_{p,q} \bigcup \Omega^{v,r}, \tag{27}$$

where $\Omega_{p,q}$ is defined by Equation (13) and

$$\Omega^{v,r} = \left\{ \mathbf{z} \in \mathbb{C}^3 : \; |z_k| < \frac{1-r}{2}, \; k = 1, 2, \; |z_3| < \frac{r(1-r)}{2v} \right\},$$

where

$$v = \max_{k \in \mathbb{N}} \{|v_k|\}, \quad 0 < r < 1,$$

for a function $f(\mathbf{z})$ holomorphic in $\Theta_{p,q}^{v,r}$;

(2) *The function $f(\mathbf{z})$ is an analytic continuation of the function on the left side of Equation (3) in Equation (27).*

Corollary 2. *Let β_1, and γ_3 be complex constants satisfying Equation (26), where*

$$v_{2k-1} = \frac{(\beta_1+k-1)(\gamma_3+k-2)}{(\gamma_3+2k-3)(\gamma_3+2k-2)}, \quad v_{2k} = \frac{k(\gamma_3+k-1-\beta_1)}{(\gamma_3+2k-2)(\gamma_3+2k-1)}, \quad k \geq 1, \tag{28}$$

p is a positive number, $\gamma_3 \notin \{1, 0, -1, -2, \ldots\}$, and $0 < q < 1$. Then, we have

$$\cfrac{1}{1 - z_2 - \cfrac{v_1 z_3}{1 - z_1 - \cfrac{v_2 z_3}{1 - z_2 - \cfrac{v_3 z_3}{1 - z_1 - \cfrac{v_4 z_3}{1 - \ddots}}}}}, \tag{29}$$

converging uniformly on every compact subset of Equation (27) to a function $f(\mathbf{z})$ holomorphic in $\Theta_{p,q}^{v,r}$, and $f(\mathbf{z})$ is an analytic continuation of the function

$$F_K(\alpha_1, 1, \beta_1, \beta_2; \alpha_1, \beta_2, \gamma_3; \mathbf{z}) \tag{30}$$

in the domain in Equation (27).

An application of Theorem 2 follows:

Theorem 4. *Let α_2, β_1, and γ_3 be real constants such that*

$$-u \leq u_k < 0, \quad k \geq 1, \tag{31}$$

where u is a positive number and u_k, where $k \geq 1$, are defined by Equation (2). Then, the following are true:

(1) The branched continued fraction in Equation (1) converges uniformly on every compact subset of the domain

$$\Theta_u = \left\{ \mathbf{z} \in \mathbb{C}^3 : \ z_k \notin \left[\frac{1}{2}, +\infty\right), \ k = 1, 2, \ z_3 \notin \left(-\infty, -\frac{1}{8u}\right] \right\} \tag{32}$$

to a function $f(\mathbf{z})$ holomorphic in Θ_u;

(2) The function $f(\mathbf{z})$ is an analytic continuation of the function on the left side of Equation (1) in Equation (32).

Proof. If $u_k < 0$ for $k \geq 1$, then the conditions in Equation (11) hold for all $p > 0$ and $0 < q < 1$. Let Υ be an arbitrary compact set contained in Equation (32). Then, $\Upsilon \subseteq \Theta_{p,q}^{u,r} \subseteq \Theta_u$ for some p which is sufficiently small and q sufficiently close to one, whose $\Theta_{p,q}^{u,r}$ is the domain in Equation (12). Theorem 3 is thus an immediate consequence of Theorem 2. □

Note that Equation (32) is the Cartesian product of two planes cut along the real axis from $1/2$ to $+\infty$ and one plane cut along the real axis from $-1/(8u)$ to $-\infty$, where u is a positive number satisfying Equation (31).

Corollary 3. *Let α_2 and γ_3 be real constants satisfying Equation (31), where u_k, $k \geq 1$ are defined by Equaton (23) and u is a positive number. Then, Equation (24) converges uniformly on every compact subset of Equation (32) to a function $f(\mathbf{z})$ holomorphic in Θ_u, and $f(\mathbf{z})$ is an analytic continuation of Equation (25) in Equation (32).*

An application of Theorem 3 is Theorem 5 below, which can be proven in much the same way as Theorem 4.

Theorem 5. *Let α_2, β_1, and γ_3 be real constants such that*

$$-v \leq v_k < 0, \quad k \geq 1, \tag{33}$$

where v is a positive number and v_k, $k \geq 1$ are defined by Equation (4). Then, the following are true:

(1) The branched continued fraction in Equation (3) converges uniformly on every compact subset of the domain

$$\Theta_v = \left\{ \mathbf{z} \in \mathbb{C}^3 : \ z_k \notin \left[\frac{1}{2}, +\infty\right), \ k = 1, 2, \ z_3 \notin \left(-\infty, -\frac{1}{8v}\right] \right\} \tag{34}$$

to function $f(\mathbf{z})$, which is holomorphic in Θ_v;

(2) The function $f(\mathbf{z})$ is an analytic continuation of the function on the left side of Equation (3) in Equation (34).

Corollary 4. *Let β_1, and γ_3 be real constants satisfying Equation (33), where v_k, $k \geq 1$ are defined by Equation (28) and v is a positive number. Then, Equation (29) converges uniformly on every compact subset of Equation (34) to a function $f(\mathbf{z})$, which is holomorphic in Θ_v, and $f(\mathbf{z})$ is an analytic continuation of Equation (30) in (34).*

The following result gives the analytical extension domain that is the Cartesian product of two planes cut along the real axis from $(1 - r)/2$ to $+\infty$ and one plane cut along the real axis from $1/(8u)$ to ∞, where $0 < r \leq 1/2$ and u is a positive number that satisfies the following conditions (Equation (35)):

Theorem 6. *Let α_2, β_1, and γ_3 be real constants which satisfy the conditions*

$$0 < u_k \leq u, \quad k \geq 1, \tag{35}$$

where u_k, $k \geq 1$ are defined by Equation (2) and u is a positive number. Then, the following are true:

(1) The branched continued fraction in Equation (1) converges uniformly on every compact subset of the domain

$$\Theta_{u,r} = \left\{ \mathbf{z} \in \mathbb{C}^3 : \ z_k \notin \left[\frac{1-r}{2}, +\infty\right), \ k = 1, 2, \ z_3 \notin \left[\frac{1}{8u}, +\infty\right) \right\}, \tag{36}$$

where $0 < r \leq 1/2$ for function $f(\mathbf{z})$, which is holomorphic in $\Theta_{u,r}$;

(2) The function $f(\mathbf{z})$ is an analytic continuation of the function on the left side of Equation (1) in Equation (36).

Proof. First of all, for convenience, we write the domain in Equation (36) as

$$\Theta_{u,r} = \bigcup_{-\pi/2 < \psi < \pi/2} \Omega_{u,\psi} \bigcup \Omega^{u,r},$$

where

$$\Omega_{u,\psi} = \left\{ \mathbf{z} \in \mathbb{C}^3 : \ \mathrm{Re}(z_2 e^{-i\psi}) < \frac{\cos(\psi)}{4}, \ k = 1, 2, \ |z_1| + \mathrm{Re}(z_3 e^{-2i\psi}) < \frac{\cos^2(\psi)}{4u} \right\} \tag{37}$$

and $\Omega^{u,r}$ is defined by Equation (14) with $0 < r \leq 1/2$.

As in the proof of Theorem 2, we show that (1) is valid in the domain

$$\bigcup_{-\pi/2 < \psi < \pi/2} \Omega_{u,\psi}. \tag{38}$$

Let n be an arbitrary natural number, ψ be an arbitrary real value from $(-\pi/2, \pi/2)$, and $\mathbf{z}$ be an arbitrary fixed point in Equation (37). Then, we have

$$\mathrm{Re}(F_{2k-1}^{(2n)}(\mathbf{z})e^{-i\psi}) > \frac{\cos(\psi)}{2} > 0 \tag{39}$$

and

$$\mathrm{Re}(F_{2k-1}^{(2n+1)}(\mathbf{z})e^{-i\psi}) > \frac{\cos(\psi)}{2} > 0 \tag{40}$$

being valid for $1 \leq k \leq n$.

Let us prove Equation (39). From Equation (16), it is clear that the inequalities in Equation (39) are valid for $k = n$. Assuming through the induction that Equation (39) holds for $k = s + 1$, where $s + 1 \leq n$, then from Equation (17), one obtains

$$\begin{aligned}
F_{2s-1}^{(2n)}(\mathbf{z})e^{-i\psi} &= e^{-i\psi} - z_2 e^{-i\psi} - \frac{u_{2s} z_3 e^{-2i\psi}}{F_{2s}^{(2n)}(\mathbf{z})e^{-i\psi}}, \\
F_{2s}^{(2n)}(\mathbf{z})e^{-i\psi} &= e^{-i\psi} - z_1 e^{-i\psi} - \frac{u_{2s+1} z_3 e^{-2i\psi}}{F_{2s+1}^{(2n)}(\mathbf{z})e^{-i\psi}}.
\end{aligned}$$

Then, under Equations (35) and (37), and Corollary 2 [12], we have

$$\begin{aligned}
\mathrm{Re}(F_{2s}^{(2n)}(\mathbf{z})e^{-i\psi}) &= \mathrm{Re}(e^{-i\psi}) - \mathrm{Re}(z_1 e^{-i\psi}) - \mathrm{Re}\left(\frac{u_{2s+1} z_3 e^{-2i\psi}}{F_{2s+1}^{(2n)}(\mathbf{z})e^{-i\psi}}\right) \\
&\geq \cos(\psi) - \mathrm{Re}(z_1 e^{-i\psi}) - |u_{2s+1}| \frac{|z_3| + \mathrm{Re}(z_3 e^{-2i\psi})}{2\,\mathrm{Re}(F_{2s+1}^{(2n)}(\mathbf{z})e^{-i\psi})} \\
&> \cos(\psi) - \frac{\cos(\psi)}{4} - \frac{\cos(\psi)}{4} \\
&= \frac{\cos(\psi)}{2}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Re}(F_{2s-1}^{(2n)}(\mathbf{z})e^{-i\psi})\, &\mathrm{Re}(e^{-i\psi}) - \mathrm{Re}(z_2 e^{-i\psi}) - \mathrm{Re}\left(\frac{u_{2s} z_3 e^{-2i\psi}}{F_{2s}^{(2n)}(\mathbf{z})e^{-i\psi}}\right) \\
&\geq \cos(\psi) - \mathrm{Re}(z_2 e^{-i\psi}) - |u_{2s}| \frac{|z_3| + \mathrm{Re}(z_3 e^{-2i\psi})}{2\,\mathrm{Re}(F_{2s}^{(2n)}(\mathbf{z})e^{-i\psi})} \\
&> \cos(\psi) - \frac{\cos(\psi)}{4} - \frac{\cos(\psi)}{4} \\
&= \frac{\cos(\psi)}{2}.
\end{aligned}$$

In the same way, we obtain the inequalities in Equation (40). Thus, the inequalities

$$F_1^{(n)}(\mathbf{z}) \neq 0 \quad \text{for all} \quad n \geq 1 \quad \text{and} \quad \mathbf{z} \in \Omega_{u,\psi}.$$

hold, and therefore, $\{f_n(\mathbf{z})\}$ is a sequence of functions holomorphic in the domain in Equation (37).

For an arbitrary compact subset Υ of Equation (38), there exists an open triple-disk (Equation (22)) such that $\Upsilon \subset \Xi_R$. We cover Υ with domains of the form

$$\Omega_{u\,\psi,R} = \bigcup_{-\pi/2<\psi<\pi/2} \Omega_{u,\psi} \bigcap \Xi_R,$$

and choose from this cover a finite subcover

$$\Omega_{u\,\psi_1,R}, \Omega_{u\,\psi_2,R}, \ldots, \Omega_{u\,\psi_k,R}.$$

Under Equations (18), (39), and (40), for any $n \geq 1$, $s \in \{1, 2, \ldots, k\}$, and $\mathbf{z} \in \Omega_{u\,\psi_s,R}$, we have

$$\begin{aligned} |f_n(\mathbf{z})| &\leq 1 + |z_1| + \frac{|u_1||z_3|}{\operatorname{Re}(F_1^{(n)}(\mathbf{z})e^{-i\psi})} \\ &< 1 + R + \frac{2uR}{\cos(\psi)} \\ &= C(\Omega_{u\,\psi_s,R}). \end{aligned}$$

We set

$$C(\mathrm{Y}) = \max_{s \in \{1,2,\ldots,k\}} C(\Omega_{u\,\psi_s,R}).$$

Then, for any $n \geq 1$ and $\mathbf{z} \in \mathrm{Y}$, we have

$$|f_n(\mathbf{z})| \leq C(\mathrm{Y}),$$

In other words, $\{f_n(\mathbf{z})\}$ is a sequence of functions uniformly bounded on every compact subset of Equation (38).

It is clear that for each L such that

$$0 < L < \min\left\{\frac{1}{4}, \frac{1}{8u}\right\}$$

the domain

$$\Delta_L = \{\mathbf{z} \in \mathbb{R}^3 : \; -L < z_k < 0, \; 1 \leq k \leq 3\}$$

is contained in Equation (38) (e.g., $\Delta_{L/2}$). Taking into account Equation (35), for any $\mathbf{z} \in \Delta_L$, where Δ_L is contained in Equation (38), one can find that

$$|z_1| < \frac{1}{4}, \quad |z_2| < \frac{1}{4}, \quad |u_k z_3| < \frac{1}{8}, \quad k \geq 1,$$

In other words, the elements of Equation (1) satisfy Theorem 1, with $m_{0,0,k} = 1/2$ and $k \geq 1$. Thus, according to Theorem 1 (1), the branched continued fraction in Equation (1) converges in Δ_L, where Δ_L is contained in Equation (38). It follows from Theorem 5 [12] that the convergence is uniform on compact subsets of Equation (38) to a holomorphic function in this domain.

The fact that (1) is also valid in the domain in Equation (14) with $0 < r \leq 1/2$ can be proven in much the same way as in the proof of Theorem 2 (1). The proof of (2) is analogous to the proof of Theorem 2 (2) and Theorem 6 (2) [12], and hence it was omitted. □

Corollary 5. *Let α_2 and γ_3 be real constants satisfying Equation (35), where u_k, $k \geq 1$ are defined by Equation (23) and u is a positive number. Then, Equation (24) converges uniformly on every compact subset of Equation (36) to a function $f(\mathbf{z})$ holomorphic in $\Theta_{u,r}$, and $f(\mathbf{z})$ is an analytic continuation of Equation (25) in this domain.*

Finally, we have the following theorem, which is symmetric to Theorem 6:

Theorem 7. *Let α_2, β_1, and γ_3 be complex constants which satisfy the conditions*

$$0 < v_k \leq v, \quad k \geq 1, \tag{41}$$

where v_k, $k \geq 1$ are defined by Equation (4) and v is a positive number. Then, the following are true:

(1) *The branched continued fraction in Equation (3) converges uniformly on every compact subset of the domain*

$$\Theta_{v,r} = \left\{ \mathbf{z} \in \mathbb{C}^3 : \ z_k \notin \left[\frac{1-r}{2}, +\infty\right), \ k = 1,2, \ z_3 \notin \left[\frac{1}{8v}, +\infty\right) \right\}, \tag{42}$$

where $0 < r \leq 1/2$ *for the function* $f(\mathbf{z})$*, which is holomorphic in* $\Theta_{v,r}$*;*

(2) *The function* $f(\mathbf{z})$ *is an analytic continuation of the function on the left side of Equation (3) in Equation (42).*

Corollary 6. *Let* β_1 *and* γ_3 *be real constants satisfying Equation (41), where* v_k*,* $k \geq 1$ *are defined by Equation (28) and* v *is a positive number. Then, Equation (29) converges uniformly on every compact subset of Equation (42) to a function* $f(\mathbf{z})$*, which is holomorphic in* $\Theta_{v,r}$*, and* $f(\mathbf{z})$ *is an analytic continuation of Equation (30) in this domain.*

When comparing the domain of the analytical continuation in Equation (5) and Equation (36) (or Equation (42)), we note that they are different under the same conditions for the parameters of Lauricella–Saran's hypergeometric functions F_K.

4. Discussions and Conclusions

We considered the representation and extension of the analytic functions of three variables by a special family of functions: branched continued fractions. The main results were new symmetric domains of analytical continuation for Lauricella–Saran's hypergeometric functions F_K with certain conditions for real and complex parameters, which were established using their branched continued fraction representations. In particular, in the case of real parameters, we obtained the Cartesian product of two planes cut along the real axis from $1/2$ to $+\infty$ and one plane cut along the real axis from $-1/(8u)$ to $-\infty$, where u is a positive number. To prove the above, we used a technique that extends the convergence of branched continued fractions, which is already known for a small domain, to a larger domain, as well as the PC method (see [12]) to prove that they were also domains of analytical continuation. In fact, in pairs, they were proven to be effective tools and could therefore be applied to Lauricella–Saran's other functions. However, we could not establish such domains of analytical continuation for Lauricella–Saran's hypergeometric functions F_K with arbitrary admissible real or complex parameters. Unfortunately, the well-developed methods for investigating the convergence of continued fractions are generally not carried over to their multidimensional generalization of branched continued fractions. Therefore, there is a need to develop new methods that would provide effective convergence criteria in both the partial and general cases.

The branched continued fraction, being a generalization of the continued fraction, is interesting in itself because it has good approximate properties, such as wide regions of convergence and numerical stability. Therefore, further investigations can be continued in various directions. First, one can try to extend the domains of convergence of the branched continued fraction expansions with real and complex coefficients in their elements using parabolic regions of convergence and angular domains, which can be found in [21–24], respectively. To render branchedcontinued fractions as more useful in computation, it is necessary to know more about their rate of convergence and numerical stability. Therefore, truncation error analysis and the computational stability of the branched continued fraction expansions are other directions. These are interesting and somewhat new directions, and there are not many results here (see [25–30]).

The numerical experiments in [12,31–35] have shown that branched continued fraction expansions provide a useful tool for representing and computing the values of analytic functions. Therefore, a no less interesting and essential direction of research is the application of branched continued fractions to compute special functions, including Lauricella–Saran's hypergeometric functions, which naturally arise in various problems in various fields of science, especially physics (see, for example, [4–6,8,36] and also [37] (Part 5), [38] (Chapters 7 and 8), [39], and [40] (Chapters 5)).

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