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Non-Lie Reduction Operators and Potential Transformations for a Special System with Applications in Plasma Physics

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Abstract: Non-Lie reduction operators, also known as nonclassical symmetries, are derived for special systems that appear in Plasma Physics. These operators are used to construct similarity mappings, which reduce the systems under study into systems of ordinary differential equations. Furthermore, potential equivalence transformations are presented. Based on these results, a number of exact solutions are constructed.

Keywords: system of diffusion equations; reduction operators; potential equivalence transformations; exact solutions

PACS: 02.30.Jr

MSC: 35A30; 35K55; 58J70

1. Introduction

The main application of Lie symmetries is the construction of mappings that reduce the number of independent variables in a system of partial differential equations. Bluman and Cole [1,2] derived a novel method for constructing reduction mappings that are not equivalent to the Lie ones, and they called them non-classical (symmetries) reductions. Later, these reductions were called conditional symmetries, Q-conditional symmetries and reduction operators by various authors [3–5]. Applications, further theory and more examples on non-classical symmetries can be found, for example, in [6–13]. The target of this approach is to derive operators which are not equivalent to Lie operators. Unlike the Lie method, where the determining system are linear partial differential equations, here the system consists nonlinear equations. This difference makes the nonclassical method more complicated to apply. There is no guarantee that this approach will lead to new operators different than the Lie ones, and this is a disadvantage of the method. In fact, not many results appear in the literature. More details and applications about this method can be found in Ref. [14]. Here, we also consider the notion of equivalence transformations, which play an important role in the theory of applications of Lie group to differential equations. The set of all equivalence transformations of a given family of differential equations forms a group which is called the equivalence group.

The system of nonlinear diffusion equations of the form

$$u_t = (u^m v^l u_x)_x, \;\; (uv)_t = \epsilon(u^{n+1} v^p v_x)_x + (u^m v^{l+1} u_x)_x \tag{1}$$

is used as a model to describe the consequence of nonlinearly coupled mass and heat diffusion in a plasma, which slowly diffuses in a strong magnetic field [15,16]. The dependent variables u and v represent the density and ionic temperature of the plasma, and the parameters m, n, l, p and ϵ are considered to be real numbers. Lie symmetry analysis of System (1) is presented in the recent works [17,18].

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Citation: Sophocleous, C. Non-Lie Reduction Operators and Potential Transformations for a Special System with Applications in Plasma Physics. *Symmetry* **2024**, *16*, 207. https://doi.org/10.3390/sym16020207

Academic Editors: Roman M. Cherniha and Eduard Marusic-Paloka

Received: 30 November 2023
Revised: 5 February 2024
Accepted: 7 February 2024
Published: 9 February 2024

In the case where $m = n$, $l = p$ and $\epsilon = 1$, System (1) can be written in the form

$$u_t = (u^{n-p} w^p u_x)_x, \;\; w_t = (u^{n-p} w^p w_x)_x, \tag{2}$$

where $w = uv$. In the case where $m = n$, $l = p$ and $\epsilon = -1$, System (1), after eliminating u_t in the second equation using the first, takes the form

$$u_t = (u^n v^p u_x)_x, \;\; v_t = -(u^n v^p v_x)_x. \tag{3}$$

We introduce the mapping $u = \phi_x$ and $w = \psi_x$, where ϕ and ψ are functions of x and t, to write System (2) in the potential form

$$\phi_t = \phi_x^{n-p} \psi_x^p \phi_{xx}, \;\; \psi_t = \phi_x^{n-p} \psi_x^p \psi_{xx}. \tag{4}$$

Similarly, we can write (3) in the potential form

$$\phi_t = \phi_x^n \psi_x^p \phi_{xx}, \;\; \psi_t = -\phi_x^n \psi_x^p \psi_{xx}. \tag{5}$$

In the present work, we consider the special case of System (4) with $n = -1$ and $p = 0$,

$$\phi_t = \frac{\phi_{xx}}{\phi_x}, \;\; \psi_t = \frac{\psi_{xx}}{\phi_x} \tag{6}$$

and the special case of System (5) with $n = -1$ and $p = 0$

$$\phi_t = \frac{\phi_{xx}}{\phi_x}, \;\; \psi_t = -\frac{\psi_{xx}}{\phi_x} \tag{7}$$

We derive non-Lie operators for the above two systems. The idea is similar to that of deriving potential (non-local) symmetries [19,20]. The notion of potential symmetries is applied when an equation can be written in a conserved form. In [7], two equivalent algorithms for finding nonclassical potential symmetries are presented. In the first one, the potential (auxiliary) systems are used, while in the second approach, the potential equation is used. Here, we used the second algorithm. Potential symmetries for System (1) can be found in [18].

We note that if $n = p = -1$ in (4), we obtain the symmetrical form of Equation (6) (interchange ϕ and ψ). Furthermore with $n = 0$ and $p = -1$ in (5), we obtain the system

$$\phi_t = \frac{\phi_{xx}}{\psi_x}, \;\; \psi_t = -\frac{\psi_{xx}}{\psi_x} \tag{8}$$

which is connected with (7) under the mapping $t \mapsto -t$, $x \mapsto x$, $\phi \mapsto \psi$, $\psi \mapsto \phi$.

In the next section, we derive equivalence transformations for the potential form of a general diffusion-type system. From the general result on potential equivalence transformations, we deduce interesting special cases. In Section 3, we derive non-Lie reduction operators for Systems (6) and (7). Finally, we construct exact solutions using the reduction operators and the equivalence transformations. Most of the calculations were performed with the assistance of the algebraic manipulation package REDUCE.

2. Potential Equivalence Transformations

We call two partial differential equations similar if they are connected by a point transformation. Such equations have similar sets of solutions, symmetries and other properties. It is important to derive such point transformations that link two equations from the same class of partial differential equations. We call these transformations form-preserving [21] or admissible [22]. When such transformations preserve the differential structure of the class, and may only change the arbitrary functions, they are called equivalence transformations, and they form a group. More on different kinds of equivalence groups and their applications can be found, for example, in reference [14].

We consider the system of the general class

$$u_t = [f(u,v)u_x]_x, \ \ v_t = [g(u,v)v_x]_x \tag{9}$$

and, introducing the potential variables ϕ and ψ, such that $u = \phi_x$ and $v = \psi_x$, we can write it as a system of four equations

$$\phi_x = u, \ \ \phi_t = f(u,v)u_x, \ \ \psi_x = v, \ \ \psi_t = g(u,v)v_x.$$

We eliminate the variables u and v to obtain the corresponding potential system

$$\phi_t = f(\phi_x, \psi_x)\phi_{xx}, \ \ \psi_t = g(\phi_x, \psi_x)\psi_{xx}. \tag{10}$$

We refer to equivalence transformations of System (10) as potential equivalence transformations of System (9). Examples of such transformations for diffusion-type equations are derived in [23,24]. Lie symmetries of (10) that induce potential symmetries of (9) can be found in [25]. Lie symmetries for System (9) are presented in [26].

We have two methods for calculation of equivalence transformations: the direct, which was used first by Lie [27], and the Lie infinitesimal method, which was introduced by Ovsyannikov [28]. Here, we use the direct method. However, we only present the results without presenting any detailed analysis. We tabulate the results in Theorem 1.

Theorem 1. *System (10) admits the equivalence transformations*

$$\tilde{t} = \alpha_1 t + \alpha_2, \ \tilde{x} = \beta_1 x + \beta_2, \ \tilde{\phi} = \gamma_1 x + \gamma_2 \phi + \gamma_3, \ \tilde{\psi} = \delta_1 x + \delta_2 \psi + \delta_3, \tag{11}$$
$$\tilde{f} = \alpha^{-1}\beta_1^2 f, \ \ \tilde{g} = \alpha^{-1}\beta_1^2 g.$$

In the case where $g(\phi_x, \psi_x) = f(\phi_x, \psi_x)$, System (10) admits the equivalence transformations

$$\tilde{t} = \alpha_1 t + \alpha_2, \ \tilde{x} = \beta_1 x + \beta_2 \phi + \beta_3 \psi + \beta_4, \ \tilde{\phi} = \gamma_1 x + \gamma_2 \phi + \gamma_3 \psi + \gamma_4, \tag{12}$$
$$\tilde{\psi} = \delta_1 x + \delta_2 \phi + \delta_3 \psi + \delta_4, \ \tilde{f} = \alpha_1^{-1}(\beta_1 + \beta_2 \phi_x + \beta_3 \psi_x)^2 f.$$

The corresponding equivalence group for the nonlinear filtration equation $v_t = f(v_x)v_{xx}$ can be found in [29]. This book is an excellent source of references for the filtration equation, as well as for its physical applications.

It is important to give the special results for the two systems under study. These transformations will be used to construct exact solutions. From Theorem 1, we deduce that equivalence transformations for System (6) have the form

$$\tilde{t} = \beta_1 \gamma_1 t + \alpha_2, \ \tilde{x} = \beta_1 x + \beta_2, \ \tilde{\phi} = \gamma_1 \phi + \gamma_2, \ \tilde{\psi} = \delta_1 x + \delta_2 \phi + \delta_3 \psi + \delta_4$$

and

$$\tilde{t} = \beta_1 \gamma_1 t + \alpha_2, \ \tilde{x} = \beta_1 \phi + \beta_2, \ \tilde{\phi} = \gamma_1 x + \gamma_2, \ \tilde{\psi} = \delta_1 x + \delta_2 \phi + \delta_3 \psi + \delta_4.$$

We note that a special case is the hodograph transformation

$$\tilde{t} = t, \ \ \tilde{x} = \phi, \ \ \tilde{\phi} = x, \ \ \tilde{\psi} = \psi \tag{13}$$

which leaves (6) invariant. The corresponding result for the nonlinear fast diffusion equation $v_t = \frac{v_{xx}}{v_x}$ can be found in [29].

From Theorem 1, we deduce the equivalence transformations for (7) have the form

$$\tilde{t} = \beta_1 \gamma_1 t + \alpha_2, \ \tilde{x} = \beta_1 x + \beta_2, \ \tilde{\phi} = \gamma_1 \phi + \gamma_2, \ \tilde{\psi} = \delta_1 x + \delta_2 \psi + \delta_3.$$

Furthermore, we state some interesting special cases of the equivalence transformations (12). Initially, we write (6) in tilded variables

$$\tilde{\phi}_{\tilde{t}} = \frac{\tilde{\phi}_{\tilde{x}\tilde{x}}}{\tilde{\phi}_{\tilde{x}}}, \quad \tilde{\psi}_{\tilde{t}} = \frac{\tilde{\psi}_{\tilde{x}\tilde{x}}}{\tilde{\phi}_{\tilde{x}}}. \tag{14}$$

Equivalence transformation (12) maps (14) into equation

$$\phi_t = \frac{\phi_{xx}}{(\beta_1 + \beta_2\phi_x + \beta_3\psi_x)(\gamma_1 + \gamma_2\phi_x + \gamma_3\psi_x)}, \tag{15}$$

$$\psi_t = \frac{\psi_{xx}}{(\beta_1 + \beta_2\phi_x + \beta_3\psi_x)(\gamma_1 + \gamma_2\phi_x + \gamma_3\psi_x)}.$$

We fix certain constants to deduce some interesting mappings. Equation (14) is connected with

$$\phi_t = \frac{\phi_{xx}}{\mu_1\phi_x + \mu_2\psi_x + \mu_3}, \quad \psi_t = \frac{\psi_{xx}}{\mu_1\phi_x + \mu_2\psi_x + \mu_3} \tag{16}$$

under the transformation

$$\tilde{t} = \beta_1\gamma_1 t + \alpha_2, \;\; \tilde{x} = \beta_1 x + \beta_2, \;\; \tilde{\phi} = \gamma_1\mu_1\phi + \gamma_1\mu_2\psi + \gamma_1\mu_3 x + \gamma_2,$$

$$\tilde{\psi} = \delta_1 x + \delta_2\phi + \delta_3\psi + \delta_4$$

or the transformation

$$\tilde{t} = \beta_1\gamma_1 t + \alpha_2, \;\; \tilde{x} = \beta_1\mu_1\phi + \beta_1\mu_2\psi + \beta_1\mu_3 x + \beta_2, \;\; \tilde{\phi} = \gamma_1 x + \gamma_2,$$

$$\tilde{\psi} = \delta_1 x + \delta_2\phi + \delta_3\psi + \delta_4,$$

where $\beta_1\gamma_1(\delta_2\mu_1 - \delta_3\mu_2) \neq 0$. Equation (14) is connected with

$$\phi_t = \frac{\phi_{xx}}{\mu_1\phi_x^2 + \mu_2\phi_x + \mu_3}, \quad \psi_t = \frac{\psi_{xx}}{\mu_1\phi_x^2 + \mu_2\phi_x + \mu_3} \tag{17}$$

under the transformation

$$\tilde{t} = \beta_1\gamma_1 t + \alpha_2, \;\; \tilde{x} = \beta_1(\mu_2 - \gamma_2\mu_1)x + \beta_1\mu_1\phi + \beta_2, \;\; \tilde{\phi} = \gamma_1\phi + \gamma_1\gamma_2 x + \gamma_3,$$

$$\tilde{\psi} = \delta_1 x + \delta_2\phi + \delta_3\psi + \delta_4,$$

where $\mu_3 = (\mu_2 - \gamma_2\mu_1)\gamma_2$ and $\beta_1\gamma_1\delta_1(\mu_2 - 2\gamma_2\mu_1) \neq 0$. Equation (14) is connected with

$$\phi_t = \frac{\phi_{xx}}{\phi_x\psi_x}, \quad \psi_t = \frac{\psi_{xx}}{\phi_x\psi_x} \tag{18}$$

under the transformation

$$\tilde{t} = \beta_1\gamma_1 t + \alpha_2, \;\; \tilde{x} = \beta_1\psi + \beta_2, \;\; \tilde{\phi} = \gamma_1\phi + \gamma_2, \;\; \tilde{\psi} = \delta_1 x + \delta_2\phi + \delta_3\psi + \delta_4,$$

where $\beta_1\gamma_1\delta_1 \neq 0$. A special case is the hodograph transformation $\tilde{t} = t$, $\tilde{x} = \psi$, $\tilde{\phi} = \phi$, $\tilde{\psi} = x$.

3. Non-Lie Operators

We construct non-Lie operators for the potential systems (6) and (7) which, originally, were used by Bluman and Cole [1], and they called them non-classical reductions. In order to obtain such reduction operators, we seek invariance of the differential equation in conjunction with its invariant surface condition. Precise and rigorous definitions for these reductions can be found in [8,14,30]. A number of examples of non-classical reductions for diffusion-type systems are given in the book [31], and in the recent works [32–36]. We can refer to the results derived in this section as potential non-Lie operators for the systems

$$u_t = (u^{-1}u_x)_x, \;\; w_t = (u^{-1}w_x)_x, \tag{19}$$

and

$$u_t = (u^{-1}u_x)_x, \;\; v_t = -(u^{-1}v_x)_x. \tag{20}$$

Here, we require invariance of System (6) (or System (7)) in conjunction with its invariance surface conditions,

$$\tau(t,x,\phi,\psi)\phi_t + \xi(t,x,\phi,\psi)\phi_x = \eta(t,x,\phi,\psi), \tag{21}$$
$$\tau(t,x,\phi,\psi)\psi_t + \xi(t,x,\phi,\psi)\psi_x = \mu(t,x,\phi,\psi)$$

under the infinitesimal transformations generated by the operator

$$\Gamma = \tau(t,x,\phi,\psi)\partial_t + \xi(t,x,\phi,\psi)\partial_x + \eta(t,x,\phi,\psi)\partial_\phi + \mu(t,x,\phi,\psi)\partial_\psi. \tag{22}$$

This invariance results in an overdetermined non-linear system of partial differential equations with unknown functions for the coefficients of the operator, $\tau(t,x,\phi,\psi)$, $\xi(t,x,\phi,\psi)$, $\eta(t,x,\phi,\psi)$ and $\mu(t,x,\phi,\psi)$. We require that $(\tau,\xi) \neq (0,0)$. Clearly, the non Lie reduction operator also has the form (22). In the case where $\tau \neq 0$, without loss of generality, we take $\tau = 1$. In the case where $\tau = 0$, we can take $\xi = 1$. Hence, two exclusive cases need to be considered. However, in the preset work, we only consider the case where $\tau = 1$.

The nonclassical method also unfolds Lie symmetries admitted by the system. It is essential to list only those reduction operators which are not equivalent to Lie symmetries. The symmetry Lie algebra for (6) is 10-dimensional, and is spanned by the operators

$$X_1 = \partial_t, \; X_2 = \partial_x, \; X_3 = \partial_\phi, \; X_4 = \partial_\psi, \; X_5 = t\partial_t + x\partial_x, \; X_6 = x\partial_x - \phi\partial_\phi,$$
$$X_7 = x\partial_\psi, \; X_8 = \psi\partial_\psi, \; X_9 = \phi\partial_\psi, \; X_{10} = (x\phi + 2t)\partial_\psi.$$

The symmetry Lie algebra for (7) is 8-dimensional, and is spanned by the above Lie operators $X_1 - X_8$. Also, the symmetry Lie algebra for System (19) is 7-dimensional, and is spanned by the operators

$$Y_1 = \partial_t, \; Y_2 = \partial_x, \; Y_3 = \partial_v, \; Y_4 = 2t\partial_t + x\partial_x, \; Y_5 = t\partial_t + u\partial_u, \; Y_6 = v\partial_v, \; Y_7 = u\partial_v$$

and System (20) admits the Lie symmetries $Y_1 - Y_6$.

For the nonclassical analysis, we use the corresponding results on the fast diffusion equation, which appear in [30]. Without presenting any detailed analysis, we give the non-Lie operators for Systems (6) and (7). We tabulate the determining equations that lead to the desired reduction operators in Appendix A.

3.1. Non-Lie Reduction Operators for System (6)

We present the non-Lie reduction operators (non-classical symmetries) admitted by the potential system (6). These results do not appear in the literature.

Case 1.1: $\Gamma_1 = \partial_t - \frac{2}{x}\partial_\phi + \frac{c}{x}\partial_\psi, \;\; \Gamma_2 = \partial_t - \frac{2}{x}\partial_\phi + x^2\partial_\psi.$

Case 1.2: We find that $\Gamma = \partial_t - 2\cot(x)\partial_\phi + \theta(x)\partial_\psi$, where the function $\theta(x)$ satisfies the differential equation $\theta''(x) - 2\csc^2(x)\theta(x) = 0$ which has the solution $\theta(x) = c(1 - x\cot(x)) + k\cot(x)$. Hence, we have two non-Lie operators

$$\Gamma_1 = \partial_t - 2\cot(x)\partial_\phi + c[1 - x\cot(x)]\partial_\psi, \;\; \Gamma_2 = \partial_t - 2\cot(x)\partial_\phi + k\cot(x)\partial_\psi.$$

Case 1.3: We have $\Gamma = \partial_t - 2\tanh(x)\partial_\phi + \theta(x)\partial_\psi$, where the function $\theta(x)$ satisfies the differential equation $\theta''(x) + 2\,\mathrm{sech}^2(x)\theta(x) = 0$. The solution is $\theta(x) = c(x\tanh(x) - 1) + k\tanh(x)$ and, therefore, we have two non-Lie operators

$$\Gamma_1 = \partial_t - 2\tanh(x)\partial_\phi + c[x\tanh(x) - 1]\partial_\psi, \;\; \Gamma_2 = \partial_t - 2\tanh(x)\partial_\phi + k\tanh(x)\partial_\psi.$$

Case 1.4: As in the previous case, we find the two non Lie operators:

$$\Gamma_1 = \partial_t - 2\coth(x)\partial_\phi + c(x\coth(x) - 1)\partial_\psi, \;\; \Gamma_2 = \partial_t - 2\coth(x)\partial_\phi + k\coth(x)\partial_\psi.$$

Case 2.1: $\Gamma_1 = \partial_t - \frac{2}{\phi+x}\partial_x - \frac{2}{\phi+x}\partial_\phi + \frac{c}{\phi+x}\partial_\psi, \;\; \Gamma_2 = \partial_t - \frac{2}{\phi+x}\partial_x - \frac{2}{\phi+x}\partial_\phi + (\phi+x)^2\partial_\psi.$

Case 2.2: Coefficient $\mu(t, x, \phi, \psi)$ in the operator Γ satisfies the same differential equation as in Case 1.2 with independent variable $\eta = x + \phi$. Hence, we have the two reduction operators:

$$\Gamma_1 = \partial_t - 2\cot(x+\phi)\partial_x - 2\cot(x+\phi)\partial_\phi + k\cot(x+\phi)\partial_\psi,$$
$$\Gamma_2 = \partial_t - 2\cot(x+\phi)\partial_x - 2\cot(x+\phi)\partial_\phi + c[1 - (x+\phi)\cot(x+\phi)]\partial_\psi.$$

Case 2.3: Using the results in Case 1.3, we derive the two reduction operators

$$\Gamma_1 = \partial_t - 2\tanh(x+\phi)\partial_x - 2\tanh(x+\phi)\partial_\phi + k\tanh(x+\phi)\partial_\psi,$$

$$\Gamma_2 = \partial_t - 2\tanh(x+\phi)\partial_x - 2\tanh(x+\phi)\partial_\phi + c[(x+\phi)\tanh(x+\phi) - 1]\partial_\psi.$$

Case 2.4: We replace tanh by coth in Case 2.3 to obtain

$$\Gamma_1 = \partial_t - 2\coth(x+\phi)\partial_x - 2\coth(x+\phi)\partial_\phi + k\coth(x+\phi)\partial_\psi,$$

$$\Gamma_2 = \partial_t - 2\coth(x+\phi)\partial_x - 2\coth(x+\phi)\partial_\phi + c[(x+\phi)\coth(x+\phi) - 1]\partial_\psi.$$

Case 3: Finally, we have the reduction operator

$$\Gamma = \partial_t + \frac{2x}{2t - \phi x}\partial_x + \frac{2\phi}{2t - \phi x}\partial_\phi + \frac{2\psi}{2t - \phi x}\partial_\psi.$$

In the next section, using these operators, we construct reduction mappings with the target to solve the reduced systems.

3.2. Non-Lie Reduction Operators for System (7)

Similarly, we derive the non-Lie operators for the potential system (7), which are all new results. We have the following operators:

Case 1.1: $\Gamma_1 = \partial_t - \frac{2}{x}\partial_\phi + c\sqrt{x}\cos\left(\frac{\sqrt{7}\log(x)}{2}\right)\partial_\psi, \;\; \Gamma_2 = \partial_t - \frac{2}{x}\partial_\phi + c\sqrt{x}\sin\left(\frac{\sqrt{7}\log(x)}{2}\right)\partial_\psi.$

Case 1.2: $\Gamma = \partial_t - 2\cot(x)\partial_\phi + \theta(x)\partial_\psi$, where the function $\theta(x)$ satisfies the differential equation

$$\theta''(x) + 2\csc^2(x)\theta(x) = 0. \tag{23}$$

Case 1.3: $\Gamma = \partial_t - 2\tanh(x)\partial_\phi + \theta(x)\partial_\psi$, where the function $\theta(x)$ satisfies the differential equation

$$\theta''(x) - 2\,\mathrm{sech}^2(x)\theta(x) = 0. \tag{24}$$

Case 1.4: $\Gamma = \partial_t - 2\coth(x)\partial_\phi + \theta(x)\partial_\psi$, where the function $\theta(x)$ satisfies the differential equation

$$\theta''(x) + 2\,\mathrm{csch}^2(x)\theta(x) = 0. \tag{25}$$

Differential Equations (23)–(25) can be solved in terms of the Legendre functions.

4. Exact Solutions

Similarly to the case of Lie symmetries, the operators derived in the previous section can be used to construct reduction mappings that transform System (6) into a system with ordinary differential equations. The reductions are obtained by solving the invariant surface conditions (21). In certain cases, the reduced systems can be solved and, with the use of the mappings, we obtain exact solutions for the original system. We consider the cases of the previous section. Since the hodograph transformation (13) leaves System (6) invariant, it can be used to construct new solutions from known ones.

4.1. Exact Solutions for System (19)

We separately consider each case derived in Section 3.1. It appears that the exact solutions obtained using non-Lie reductions are new. However, we need to emphasize that, since the corresponding search for exact solutions using Lie symmetries does not appear in the literature, it is possible certain solutions may be derived from Lie reductions. For example, the exact solutions of case 1.1 can be obtained using the Lie symmetries admitted by System (19).

Case 1.1: The operator Γ_1 provides the similarity mapping $\phi(t,x) = -\frac{2t}{x} + F(x)$, $\psi(t,x) = \frac{ct}{x} + G(x)$ that reduces (6) into the system

$$xF''(x) + 2F'(x) = 0, \ \ xG''(x) - cF'(x) = 0.$$

Solving this system, we obtain the simple exact solution

$$\phi(t,x) = \frac{c_1 + c_2 x - 2t}{x}, \ \ \psi(t,x) = \frac{2c_3 x^2 + 2c_4 x + 2ct - cc_1}{2x}$$

of System (6). Using the hodograph transformation (13) and the above solution, we obtain the same solution (with different constants). The corresponding solution for System (19) is

$$u(t,x) = \frac{2t - c_1}{x^2}, \ \ w(t,x) = \frac{2c_3 x^2 - 2ct + c_1 c}{2x^2}.$$

Similarly, the second operator Γ_2 leads to the solution

$$\phi(t,x) = \frac{c_1 + c_2 x - 2t}{x}, \ \ \psi(t,x) = \frac{1}{2}\left(-c_1 x^2 + 2c_3 x + 2c_4 + 2tx^2\right).$$

We use the hodograph transformation (13) to obtain the solution

$$\phi(t,x) = \frac{c_1 - 2t}{x - c_2},$$

$$\psi(t,x) = \frac{2c_4 x^2 + 2(c_1 c_3 - 2c_2 c_4 - 2c_3 t)x + 8t^3 - 12c_1 t^2 + 2(3c_1^2 + 2c_2 c_3)t + 2c_2^2 c_4 - c_1^3 - 2c_1 c_2 c_3}{2(x - c_2)^2}$$

The corresponding solutions for System (19) are

$$u(t,x) = \frac{2t - c_1}{x^2}, \ \ w(t,x) = 2tx - c_1 x + c_3$$

and

$$u(t,x) = \frac{2t - c_1}{(x - c_2)^2}, \ \ w(t,x) = \frac{2c_3 tx - c_1 c_3 x - 8t^3 + 12c_1 t^2 - (6c_1^2 + 2c_2 c_3)t + c_1^3 + c_1 c_2 c_3}{(x - c_2)^3}.$$

Case 1.2: From the operator Γ_1 we obtain the reduction $\phi(t,x) = -2t\cot(x) + F(x)$, $\psi(t,x) = ct(1 - x\cot(x)) + G(x)$, and the reduced system has the form

$$F''(x) + 2\cot(x)F'(x) = 0, \;\; G''(x) - ct(1 - x\cot(x))F'(x) = 0.$$

Solving the above system, we find the solution

$$\phi(t,x) = -(c_1 + 2t)\cot(x) + c_2, \;\; \psi(t,x) = \frac{1}{2}c(c_1 + 2t)(1 - x\cot(x)) + c_3 x + c_4$$

of System (6). Similarly, the reduction operator Γ_2 leads to the solution

$$\phi(t,x) = -(c_1 + 2t)\cot(x) + c_2, \;\; \psi(t,x) = \frac{1}{2}k(c_1 + 2t)\cot(x) + c_3 x + c_4.$$

Using the above solutions and transformation (13), we derive two new solutions:

$$\phi(t,x) = \cot^{-1}\left(\frac{c_2 - x}{c_1 + 2t}\right), \;\; \psi(t,x) = \frac{1}{2}c\left[c_1 + 2t - (c_2 - c_3 - x)\cot^{-1}\left(\frac{c_2 - x}{c_1 + 2t}\right)\right] + c_4,$$

$$\phi(t,x) = \cot^{-1}\left(\frac{c_2 - x}{c_1 + 2t}\right), \;\; \psi(t,x) = \frac{1}{2}k(c_2 - x) + c_3\cot^{-1}\left(\frac{c_2 - x}{c_1 + 2t}\right) + c_4.$$

Here, the upper index -1 denotes inverse function. The corresponding solutions of System (19) have the form

$$u(t,x) = (2t + c_1)\csc^2(x), \;\; w(t,x) = \frac{1}{2}[c(2t + c_1)(x\csc^2(x) - \cot(x)) + 2c_3],$$

$$u(t,x) = (2t + c_1)\csc^2(x), \;\; w(t,x) = -\frac{1}{2}k(2t + c_1)\csc^2(x) + c_3,$$

$$u(t,x) = \frac{2t + c_1}{(x - c_2)^2 + (2t + c_1)^2}, \;\; w(t,x) = \frac{c}{2}\left[\cot^{-1}\left(\frac{c_2 - x}{c_1 + 2t}\right) + \frac{(2t + c_1)(x + c_3 - c_2)}{(x - c_2)^2 + (2t + c_1)^2}\right],$$

$$u(t,x) = \frac{2t + c_1}{(x - c_2)^2 + (2t + c_1)^2}, \;\; w(t,x) = -\frac{k}{2} - \frac{2c_3(2t + c_1)}{(x - c_2)^2 + (2t + c_1)^2}.$$

Case 1.3: Here, the reductions are similar to the previous case. Using the operators Γ_1 and Γ_2, we find the solutions of System (6)

$$\phi(t,x) = (c_1 - 2t)\tanh(x) + c_2, \;\; \psi(t,x) = \frac{1}{2}k(2t - c_1)\tanh(x) + c_3 x + c_4$$

and

$$\phi(t,x) = (c_1 - 2t)\tanh(x) + c_2, \;\; \psi(t,x) = \frac{1}{2}c(2t - c_1)(x\tanh(x) - 1) + c_3 x + c_4.$$

Hodograph transformation (13) produces the new solutions

$$\phi(t,x) = \tanh^{-1}\left(\frac{x - c_2}{c_1 - 2t}\right), \;\; \psi(t,x) = \frac{1}{2}k(c_2 - x) + c_3\tanh^{-1}\left(\frac{x - c_2}{c_1 - 2t}\right) + c_4$$

and

$$\phi(t,x) = \tanh^{-1}\left(\frac{x - c_2}{c_1 - 2t}\right), \;\; \psi(t,x) = \frac{1}{2}c\left[(c_2 + c_3 - x)\tanh^{-1}\left(\frac{x - c_2}{c_1 - 2t}\right) + c_1 - 2t\right] + c_4.$$

The corresponding solutions of System (19) have the form

$$u(t,x) = (c_1 - 2t)\mathrm{sech}^2(x), \;\; w(t,x) = \frac{1}{2}(2t - c_1)\mathrm{sech}^2(x) + c_3,$$

$$u(t,x) = (c_1 - 2t)\mathrm{sech}^2(x), \;\; w(t,x) = \frac{1}{2}[c(2t - c_1)(x\mathrm{sech}^2(x) + \tanh(x)) + 2c_3],$$

$$u(t,x) = \frac{2t - c_1}{(x - c_2)^2 - (2t - c_1)^2}, \;\; w(t,x) = -\frac{k}{2} + \frac{c_2(2t - c_1)}{(x - c_2)^2 - (2t - c_1)^2},$$

$$u(t,x) = \frac{2t - c_1}{(x - c_2)^2 - (2t - c_1)^2}, \;\; w(t,x) = \frac{c}{2}\left[\tanh^{-1}\left(\frac{x - c_2}{2t - c_1}\right) + \frac{(2t - c_1)(c_3 + c_2 - x)}{(x - c_2)^2 - (2t - c_1)^2}\right].$$

Case 1.4: The results are obtained by replacing tanh with coth in case 1.3.

In case 2, we only present the solutions of System (6). The corresponding solutions of System (19), $u = \phi_x(t,x)$, $w = \psi_x(t,x)$, are very lengthy, and we shall not tabulate them.

Case 2.1: The reduction operator Γ_1 leads to the same solutions as in the case 1.1, with different constants. From Γ_2, we obtain the reduction mapping

$$\phi(t,x) = x + F(\omega), \;\; \psi(t,x) = -\frac{1}{16}(\phi + x)^4 + G(\omega), \;\; \omega = x\phi(t,x) + 2t$$

which reduces (6) to the system

$$F''(\omega) = 0, \;\; \frac{3}{4}\left[F(\omega)F'(\omega) + 2\right]^2 - G''(\omega) = 0.$$

Solving this reduced system, we find the solution

$$\begin{aligned}
&\phi(t,x) = \frac{x + 2c_1 t + c_2}{1 - c_1 x},\\
&\psi(t,x) = -\frac{1}{16}(\phi + x)^4 + \frac{(x\phi + 2t)^2}{16}\left[c_1^2(x\phi + 2t) + 2(c_1 c_2 + 2)\right]^2\\
&+\frac{1}{8}(c_1 c_2 + 2)^2(x\phi + 2t)^2 + c_3(x\phi + 2t) + c_4.
\end{aligned}$$

We point out that the hodograph transformation (13) does not provide a new solution.

Case 2.2: We use the reduction operator Γ_2 to produce the following two solutions:

$$\phi(t,x) = -\tan^{-1}[\tanh(2t)\cot(x)],$$

$$\psi(t,x) = \frac{c}{4}\log\left[\frac{\cos(\phi + x)}{\cos(\phi - x)}\right] + \frac{c}{2}x\phi + c_1\phi + c_2 x$$

and

$$\phi(t,x) = \tan^{-1}\left[\frac{1 - \tanh(2t) + (1 + \tanh(2t))\tan(x)}{1 + \tanh(2t) + (1 - \tanh(2t))\tan(x)}\right],$$

$$\psi(t,x) = \frac{c}{4}\log\left[\frac{\cos(\phi + x)}{\sin(\phi - x)}\right] + \frac{c}{2}x\phi + c_1\phi + c_2 x.$$

Case 2.3: Similarly, the reduction operator Γ_2 produces the solutions

$$\phi(t,x) = -\tanh^{-1}[\tanh(x)\coth(2t)],$$

$$\psi(t,x) = \frac{c}{4}\log\left[\frac{\sinh(\phi + x)}{\sinh(\phi - x)}\right] - \frac{c}{2}x\phi + c_1\phi + c_2 x$$

and

$$\phi(t,x) = \tanh^{-1}\left[\frac{1 - \tanh(2t) - (1 + \tanh(2t))\tanh(x)}{1 + \tanh(2t) + (1 - \tanh(2t))\tanh(x)}\right],$$

$$\psi(t,x) = \frac{c}{4}\log\left[\frac{\sinh(\phi + x)}{\cosh(\phi - x)}\right] - \frac{c}{2}x\phi + c_1\phi + c_2 x.$$

Case 2.4: As before, the reduction operator Γ_2 leads to the exact solutions

$$\phi(t,x) = \coth^{-1}[\tanh(x)\coth(2t)],$$
$$\psi(t,x) = \frac{c}{4}\log\left[\frac{\cosh(\phi+x)}{\cosh(\phi-x)}\right] + \frac{c}{2}x\phi + c_1\phi + c_2 x$$

and

$$\phi(t,x) = -\coth^{-1}\left[\frac{1-\tanh(2t)-(1+\tanh(2t))\tanh(x)}{1+\tanh(2t)+(1-\tanh(2t))\tanh(x)}\right],$$
$$\psi(t,x) = \frac{c}{4}\log\left[\frac{\cosh(\phi-x)}{\sinh(\phi+x)}\right] - \frac{c}{2}x\phi + c_1\phi + c_2 x.$$

Case 3. The solutions are similar to case 1.1.

As we pointed out earlier, some exact solutions can be derived using Lie reductions. Based on the results for the fast diffusion equation $u_t = (u^{-1}u_x)_x$ [24,30,37–39], we deduce that the solutions of case 1.1 can be obtained using Lie reductions.

4.2. Exact Solutions for System (20)

We use the reduction operators of Section 3.2 to derive similarity mappings for System (7). Solving the reduced systems, we obtain exact solutions for System (7) and, consequently, $u(t,x) = \phi_x$ and $v(t,x) = \psi_x$ are the corresponding solutions of the original system (20). We list the solutions of System (7). Not all solutions obtained appear in the literature.

Case 1.1: Reduction operator Γ_1 provides the solution

$$\phi(t,x) = \frac{c_1 + c_2 x - 2t}{x},$$
$$\psi(t,x) = ct\sqrt{x}\cos\left(\frac{\sqrt{7}\log(x)}{2}\right) - \frac{1}{2}cc_1\sqrt{x}\cos\left(\frac{\sqrt{7}\log(x)}{2}\right) + c_3 x + c_4$$

and Γ_2 produces the solution

$$\phi(t,x) = \frac{c_1 + c_2 x - 2t}{x},$$
$$\psi(t,x) = ct\sqrt{x}\sin\left(\frac{\sqrt{7}\log(x)}{2}\right) - \frac{1}{2}cc_1\sqrt{x}\sin\left(\frac{\sqrt{7}\log(x)}{2}\right) + c_3 x + c_4.$$

Case 1.2: Using the reduction operator Γ, we find the solution

$$\phi(t,x) = (c_1 - 2t)\cot(x), \quad \psi(t,x) = t\theta(x) + G(x),$$

where $\theta(x)$ satisfies Equation (23) and $G''(x) = c_1\theta(x)\csc^2(x)$.

Case 1.3: Similarly, we have the solution

$$\phi(t,x) = (c_1 - 2t)\tanh(x), \quad \psi(t,x) = t\theta(x) + G(x),$$

where $\theta(x)$ satisfies Equation (24) and $G''(x) = -c_1\theta(x)\mathrm{sech}^2(x)$.

Case 1.4: Finally, we find the solution

$$\phi(t,x) = (c_1 - 2t)\coth(x), \quad \psi(t,x) = t\theta(x) + G(x),$$

where $\theta(x)$ satisfies Equation (25) and $G''(x) = c_1\theta(x)\mathrm{csch}^2(x)$.

5. Final Remarks

Group analysis of differential equations provides systematic methods for deducing exact solutions of nonlinear general partial differential equations. Such a method was introduced by Bluman and Cole [1], and they called it non-classical. The target is to derive similarity reductions that are not equivalent to Lie reductions. This method was used in this article to derive non-Lie operators for (6) and (7), which are potential systems of (19) and (20), respectively. The obtained results are employed to construct exact solutions for the corresponding systems. Motivated by the present work, one can search for non-Lie operators directly for Systems (19) and (20). The related results for the single equation, the fast diffusion equation $u_t = (u^{-1}u_x)_x$, can be found in [39]. A more difficult task is to search for such operators for the general system (1), or for its special cases (2) and (3), or for their potential forms. Furthermore, as pointed out earlier, similarity solutions that result from Lie symmetries for the special systems (19) and (20) do not appear in the literature, and therefore this can be the subject of a further study. For example, the Lie symmetry $X_5 = t\partial_t + x\partial_x$ admitted by System (6) produces the solutions $(\phi, \psi) = \{(-\frac{2t}{x}, -\frac{2t}{x}),\ (2\tan^{-1}\frac{x}{t}, \tan^{-1}\frac{x}{t}),\ (-2\coth^{-1}\frac{x}{t}, -\coth^{-1}\frac{x}{t})\}$. Consequently, we have the solutions of System (19) $(u, w) = \{(\frac{2t}{x^2}, \frac{2t}{x^2}),\ (\frac{2t}{x^2+t^2}, \frac{t}{x^2+t^2}),\ (\frac{2t}{x^2-t^2}, \frac{t}{x^2-t^2})\}$.

An equivalence transformation for a class of partial differential equations has the property that it transforms any member of the class to an equation, which is also a member of the class. This is useful, for example, when converting equations to a known form on which an established theory can be called. Here, we have derived the equivalence group for a potential form of the system under study. This potential form is obtained using the conservation laws admitted by the system. The complete list of conservation laws for the general class of systems (9) and the corresponding potential systems can be found in [25]. One new task is to construct the equivalence group for these potential systems. Furthermore, these potential systems lead to additional transformations. We present one example. Introducing the potential variables ϕ and ψ, System (19) can be written as a system with four equations,

$$\phi_x = u,\ \ \phi_t = \frac{u_x}{u},\ \ \psi_x = w,\ \ \psi_t = \frac{w_x}{u}.$$

This system remains invariant under the mapping

$$t \mapsto t,\ \ x \mapsto \phi,\ \ u \mapsto \frac{1}{u},\ \ w \mapsto \frac{w}{u},\ \ \phi \mapsto x,\ \ \psi \mapsto \psi$$

which leads to the transformation

$$dt \mapsto dt,\ \ dx \mapsto u dx + \frac{u_x}{u} dt,\ \ u \mapsto \frac{1}{u},\ \ w \mapsto \frac{w}{u}$$

that leaves System (19) invariant. The above mapping can be written in an integrated form [40].

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Acknowledgments: We thank the anonymous referees for their suggestions, which improved the manuscript.

Conflicts of Interest: The author declares no conflicts of interest.

Appendix A

We apply the second extension $\Gamma^{(2)}$ of Operator (22) (with $\tau = 1$) on System (6). We eliminate ϕ_{xx} and ψ_{xx} from (6), and $\phi_t,\ \psi_t,\ \phi_{xt},\ \phi_{tt},\ \psi_{xt},\ \psi_{tt}$ from the invariant surfaces (21), to obtain two identities: $E_1(t, x, \phi, \psi, \xi, \eta, \mu, \phi_x, \psi_x) = 0$ and $E_2(t, x, \phi, \psi, \xi, \eta, \mu, \phi_x, \psi_x) = 0$. E_1 and E_2 are polynomials in the derivatives ϕ_x and ψ_x. We take the coefficients of powers of ϕ_x and ψ_x in $E_1 = 0$ to find the following system of nonlinear determining equations:

$$\begin{aligned}
&\xi_{\phi\phi} - \xi\xi_\phi = 0,\ 2\xi_{\phi\psi} - \xi\xi_\psi = 0,\ 2\xi_{x\phi} + \eta\xi_\phi - \eta_{\phi\phi} - \xi\eta_\phi - \xi_t - \xi\xi_x = 0, \\
&\xi_{\psi\psi} = 0,\ 2\xi_{x\psi} + \eta\xi_\psi - 2\eta_{\phi\psi} - \xi\eta_\psi = 0,\ \eta_{\psi\psi} = 0, \\
&\eta_t - 2\eta_{x\phi} + \eta\eta_\phi - \xi\eta_x + \xi_{xx} + \eta\xi_x = 0,\ 2\eta_{x\psi} - \eta\eta_\psi = 0,\ \eta_{xx} - \eta\eta_x = 0,
\end{aligned} \tag{A1}$$

and the corresponding coefficients in $E_2 = 0$ give the nonlinear equations

$$\begin{aligned}
&\xi_{\psi\psi} = 0,\ 2\xi_{\phi\psi} - \xi\xi_\psi = 0,\ 2\xi_{x\psi} - \xi\eta_\psi - \mu_{\psi\psi} = 0,\ \xi_{\phi\phi} - \xi\xi_\phi = 0, \\
&2\xi_{x\phi} + \mu\xi_\psi - \xi\eta_\phi - 2\mu_{\phi\psi} - \xi_t - \xi\xi_x = 0,\ \mu\eta_\psi - \xi\eta_x - 2\mu_{x\psi} + \xi_{xx} = 0, \\
&\mu_{\phi\phi} - \mu\xi_\phi = 0,\ \mu\eta_\phi + \mu_t - 2\mu_{x\phi} + \mu\xi_x = 0,\ \mu\eta_x - \mu_{xx} = 0.
\end{aligned} \tag{A2}$$

The solution of the determining system (A1) and (A2) leads to the desired reduction operators.

Similarly, we find the following determining equations for System (7):

$$\begin{aligned}
&\xi_{\phi\phi} - \xi\xi_\phi = 0,\ 2\phi\xi_{\phi\psi} + \xi_\psi = 0,\ \xi_{\psi\psi} = 0,\ \eta_{\psi\psi} = 0, \\
&2\xi_{x\phi} + \eta\xi_\phi - \eta_{\phi\phi} - \xi\eta_\phi - \xi_t - \mu\xi_\psi - \xi\xi_x = 0, \\
&2\phi\xi_{x\psi} + \phi\eta\xi_\psi - 2\phi\eta_{\phi\psi} - 2\phi\xi\eta_\psi - \eta_\psi - \phi^2\xi\xi_\psi = 0, \\
&\eta_t - 2\eta_{x\phi} + \eta\eta_\phi + \mu\eta_\psi - \xi\eta_x + \phi\mu\xi_\psi + \xi_{xx} + \eta\xi_x = 0, \\
&2\eta_{x\psi} - \eta\eta_\psi - \phi\xi\eta_\psi = 0,\ \eta_{xx} + \phi\mu\eta_\psi - \eta\eta_x = 0
\end{aligned} \tag{A3}$$

and

$$\begin{aligned}
&\xi_{\psi\psi} = 0,\ 2\phi\xi_{\phi\psi} + \xi_\psi = 0,\ \eta_\psi - \phi\mu_{\psi\psi} + 2\phi\xi_{x\psi} - 2\phi^2\xi\xi_\psi = 0, \\
&\xi_{\phi\phi} - \xi\xi_\phi = 0,\ \phi\eta_\phi - 2\phi^2\mu_{\phi\psi} + 2\phi^2\xi_{x\phi} + \phi^2\eta\xi_\phi - \phi^3\xi\xi_\phi - \eta = 0, \\
&\eta_x - 2\phi\mu_{\phi\psi} - \phi^2\xi_t - \phi^2\eta\xi_\phi + 2\phi^2\mu\xi_\psi + \phi\xi_{xx} - 2\phi^2\xi\xi_x - \phi\xi\eta = 0, \\
&\phi\mu_{\phi\phi} - \phi\xi\mu_\phi - \mu_\phi = 0,\ 2\phi\mu_{x\phi} + \phi\eta\mu_\phi + \phi^2\xi\mu_\phi - \mu_x - 2\phi^2\mu\xi_\phi = 0, \\
&2\phi\mu_{x\phi} + \phi\eta\mu_\phi + \phi^2\xi\eta_\phi - \mu_x - 2\phi^2\mu\xi_\phi = 0.
\end{aligned} \tag{A4}$$

The solution for the determining system (A3) and (A4) produces the results of Section 3.2.

References

1. Bluman, G.W.; Cole, J.D. The general similarity solution of the heat equation. *J. Math. Mech.* **1969**, *18*, 1025–1042.
2. Bluman, G.W.; Cole, J.D. *Similarity Methods for Differential Equations*; Springer: New York, NY, USA; Heidelberg, Germany, 1974; Volume 13.
3. Fushchich, W.I.; Serov N.I. Conditional invariance and exact solutions of a nonlinear acoustics equation. *Dokl. Akad. Nauk Ukrain. SSR A* **1988**, *10*, 27–31. (In Russian)
4. Fushchich, W.I.; Shtelen, W.M.; Serov, M.I.; Popovych, R.O. Q-conditional symmetry of the linear heat equation. *Proc. Acad. Sci. Ukr.* **1992**, *12*, 28–33.
5. Levi, D.; Winternitz, P. Non-classical symmetry reduction: Example of the Boussinesq equation. *J. Phys. A Math. Gen.* **1989**, *22*, 2915–2924. [CrossRef]
6. Cherniha, R.; Serov, M.; Pliukhin, O. *Nonlinear Reaction-Diffusion-Convection Equations*; CRC Press: Boca Raton, FL, USA, 2018.
7. Bluman, G.W.; Yan, Z. Nonclassical potential solutions of partial differential equations. *Eur. J. Appl. Math.* **2005**, *16*, 239–261. [CrossRef]
8. Fushchych, W.I.; Tsyfra, I.M. On a reduction and solutions of the nonlinear wave equations with broken symmetry. *J. Phys. A Math. Gen.* **1987**, *20*, L45–L48. [CrossRef]
9. Fushchych, W.I.; Zhdanov, R.Z. Conditional symmetry and reduction of partial differential equations. *Ukr. Mat. Zhurnal* **1992**, *44*, 970–982; Translated in *Ukr. Math. J.* **1992**, *44*, 875–886. (In Ukrainian) [CrossRef]
10. Arrigo, D.J.; Hill, J.M.; Broadbridge, P. Nonclassical symmetry reductions of the linear diffusion equation with a nonlinear source. *IMA J. Appl. Math.* **1994**, *52*, 1–24. [CrossRef]
11. Arrigo, D.J.; Hill, J.M. Nonclassical symmetries for nonlinear diffusion and absorption. *Stud. Appl. Math.* **1995**, *94*, 21–39. [CrossRef]

12. Nucci, M.C. Nonclassical symmetries and Bäcklund transformations. *J. Math. Anal. Appl.* **1993**, *178*, 294–300. [CrossRef]
13. Nucci, M.C.; Ames, W.F. Classical and nonclassical symmetries for the Helmholtz equation. *J. Math. Anal. Appl.* **1993**, *178*, 584–591. [CrossRef]
14. Vaneeva, O.O.; Popovych, R.O.; Sophocleous, C. Enhanced group analysis and exact solutions of variable coefficient semilinear diffusion equations with a power source. *Acta Appl. Math.* **2009**, *106*, 1–46. [CrossRef]
15. Rosenau, P.; Hyman, J.M. Analysis of nonlinear mass and energy diffusion. *Phys. Rev. A* **1985**, *32*, 2370–2373. [CrossRef] [PubMed]
16. Rosenau, P.; Hyman, J.M. Plasma diffusion across a magnetic field. *Phys. D* **1986**, *20*, 444–446. [CrossRef]
17. Charalambous, K.; Sophocleous, C. Lie symmetries of a system arising in Plasma Physics. *Math. Methods Appl. Sci.* **2018**, *41*, 1331–1343. [CrossRef]
18. Charalambous, K.; Sophocleous, C. Special transformation properties for certain equations with applications in Plasma Physics. *Math. Methods Appl. Sci.* **2021**, *44*, 14776–14790. [CrossRef]
19. Bluman, G.W.; Reid, G.J.; Kumei, S. New classes of symmetries for partial differential equations. *J. Math. Phys.* **1988**, *29*, 806–811. [CrossRef]
20. Bluman, G.W.; Kumei, S. *Symmetries and Differential Equations*; Springer: New York, NY, USA, 1989.
21. Kingston, J.G.; Sophocleous, C. On Form-Preserving Point Transformations of Partial Differential Equations. *J. Phys. A Math. Gen.* **1998**, *31*, 1597–1619. [CrossRef]
22. Popovych, R.; Kunzinger, M.; Eshraghi, H. Admissible Transformations and Normalized Classes of Nonlinear Schrodinger Equations. *Acta Appl. Math.* **2010**, *109*, 315–359. [CrossRef]
23. Sophocleous, C. Continuous and discrete transformations of a one-dimensional porous medium equation. *J. Nonlinear Math. Phys.* **1999**, *6*, 355–364. [CrossRef]
24. Popovych, R.O.; Ivanova, N.M. Potential equivalence transformations for nonlinear diffusion-convection equations. *J. Phys. A Math. Gen.* **2005**, *38*, 3145–3155. [CrossRef]
25. Ivanova, N.M.; Sophocleous, C. Conservation laws and potential symmetries of systems of diffusion equations. *J. Phys. A Math. Theor.* **2008**, *41*, 235201. [CrossRef]
26. Kontogiorgis, S.; Sophocleous, C. Group classification of systems of diffusion equations. *Math. Methods Appl. Sci.* **2017**, *40*, 1746–1756. [CrossRef]
27. Lie, S. Klassifikation und Integration von gewohnlichen Differentialgleichungen zwischen x, y, die eine Gruppe von Transformationen gestatten I, II. *Math. Ann.* **1888**, *32*, 213–281. [CrossRef]
28. Ovsiannikov, L.V. *Group Analysis of Differential Equations*; Academic Press: New York, NY, USA, 1982.
29. Ibragimov, N.H. *Symmetries, Exact Solutions and Conservation Laws, Lie Group Analysis of Differential Equations*; Chemical Rubber Company: Boca Raton, FL, USA, 1994; Volume 1.
30. Popovych, R.O.; Vaneeva, O.O.; Ivanova, N.M. Potential nonclassical symmetries and solutions of fast diffusion equation. *Phys. Lett. A* **2007**, *362*, 166–173. [CrossRef]
31. Cherniha, R.; Davydovych, V. *Nonlinear Reaction-Diffusion Systems*; Lecture Notes in Math, 2196; Springer: Cham, Switzerland, 2017.
32. Sil, S.; Sekhar, T.R. Nonclassical symmetry analysis, conservation laws of one-dimensional macroscopic production model and evolution of nonlinear waves. *J. Math. Anal. Appl.* **2021**, *497*, 124847. [CrossRef]
33. Sil, S.; Sekhar, T.R. Nonclassical potential symmetry analysis and exact solutions for a thin film model of a perfectly soluble anti-surfactant solution. *Appl. Math. Comput.* **2023**, *440*, 127660. [CrossRef]
34. Cherniha, R.; Davydovych, V. Conditional symmetries and exact solutions of a nonlinear three-component reaction-diffusion model. *Eur. J. Appl. Math.* **2021**, *32*, 280–300. [CrossRef]
35. Cherniha, R.; Davydovych, V.; King, J.R. The Shigesada-Kawasaki-Teramoto model: Conditional symmetries, exact solutions and their properties. *Commun. Nonlinear Sci. Numer. Simul.* **2023**, *124*, 107313. [CrossRef]
36. Broadbridge, P.; Cherniha, R.; Goard, J.M. Exact nonclassical symmetry solutions of Lotka Volterra-type population systems. *Eur. J. Appl. Math.* **2023**, *34*, 998–1016. [CrossRef]
37. Polyanin, A.D.; Zaitsev, V.F. *Hanbook of Nonlinear Partial Differential Equations*; Chapman & Hall/CRC: Boca Raton, FL, USA, 2004.
38. Popovych, R.O.; Ivanova, N.M. New results on group classification of nonlinear diffusion-convection equations. *J. Phys. A Math. Gen.* **2004**, *37*, 7547–7565. [CrossRef]
39. Gandarias, M.L. New symmetries for a model of fast diffusion. *Phys. Lett. A* **2001**, *286*, 153–160. [CrossRef]
40. Bluman, G.W.; Kumei, S. On the remarkable nonlinear diffusion equation $(\partial/\partial x)[a(u+b)^{-2}(\partial u/\partial x)] - (\partial u/\partial t) = 0$. *J. Math. Phys.* **1980**, *21*, 1019–1023. [CrossRef]