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Sustaining Quasi De-Sitter Inflation with Bulk Viscosity

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Abstract: The de-Sitter spacetime is a maximally symmetric Lorentzian manifold with constant positive scalar curvature that plays a fundamental role in modern cosmology. Here, we investigate bulk-viscosity-assisted quasi de-Sitter inflation, that is the period of accelerated expansion in the early universe during which $-\dot{H} \ll H^2$, with $H(t)$ being the Hubble expansion rate. We do so in the framework of a causal theory of relativistic hydrodynamics, which takes into account non-equilibrium effects associated with bulk viscosity, which may have been present as the early universe underwent an accelerated expansion. In this framework, the existence of a quasi de-Sitter universe emerges as a natural consequence of the presence of bulk viscosity, without requiring introducing additional scalar fields. As a result, the equation of state, determined by numerically solving the generalized momentum-conservation equation involving bulk viscosity pressure turns out to be time dependent. The transition timescale characterising its departure from an exact de-Sitter phase is intricately related to the magnitude of the bulk viscosity. We examine the properties of the new equation of state, as well as the transition timescale in the presence of bulk viscosity pressure. In addition, we construct a fluid description of inflation and demonstrate that, in the context of the causal formalism, it is equivalent to the scalar field theory of inflation. Our analysis also shows that the slow-roll conditions are realised in the bulk-viscosity-supported model of inflation. Finally, we examine the viability of our model by computing the inflationary observables, namely the spectral index and the tensor-to-scalar ratio of the curvature perturbations, and compare them with a number of different observations, finding good agreement in most cases.

Keywords: quasi de-Sitter inflation; bulk viscosity ; generalized Israel-Stewart theory

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1. Introduction

The inflationary paradigm plays a pivotal role in modern cosmology, which relies on the idea that the early universe underwent nearly exponential expansion within a very short interval of time [1,2]. While successfully providing explanations to all the drawbacks of the Big-Bang cosmology, inflation also presents natural explanations concerning the large-scale structure formation of the observable universe and the distribution of the galaxies [3–5]. The predictions of inflation associated with the temperature anisotropies measured in the cosmic microwave background radiation are well-supported by data sources, e.g., from the BICEP2 experiment [6], Wilkinson Microwave Anisotropy Probe (WMAP) [7–10], and Planck results [11,12].

The scalar field description is among the most-commonly adopted approaches for addressing inflation, where a scalar field, known as the "inflaton" field, and a specific potential are responsible for driving the accelerated expansion of the early universe [13]. In the last few decades, extensive studies have been carried out for addressing different facets of inflation using single scalar field models [14–16], multi-scalar field models [17,18], and

braneworld and string theory scenarios [19,20] in the framework of both Einstein's theory and alternative theories of gravity [21–23].

An alternative approach to the scalar field description is one in which inflation is governed by bulk viscosity pressure, providing a sufficient amount of negative pressure, which is essential for the accelerated expansion. In this approach, the dynamical equations are described by a relativistic theory of non-perfect cosmological fluids [24–26] (for recent works, we refer to [27–34] and the references therein). A well-developed and commonly employed relativistic theory of non-perfect fluids is due to Müller [35] and Israel and Stewart [36,37] and is normally referred to as the MIS theory. This phenomenological formulation, which is hyperbolic and stable, relies on the assumption that dissipative fluxes (in the form of bulk viscosity, shear viscosity, heat flux, and charge flow) in the system amounts to small departures from equilibrium [38,39] (see also [40] for an introduction). Consequently, the effects of these fluxes may be treated as perturbations to the equilibrium quantities.

Although the MIS theory is a valid relativistic hydrodynamical description of non-perfect fluids, it does not provide a reasonable description when applied to the epoch of the accelerated expansion of the early universe. This is because the basic assumptions behind the MIS theory (small departures away from equilibrium) are no longer respected in this epoch. More specifically, during inflation, the Strong Energy Condition (SEC) was violated, thus allowing bulk viscosity pressure to be comparable with the equilibrium pressure and energy density of the cosmological fluid. As a result, the departures from the equilibrium were large, instilling sufficiently large non-equilibrium effects, and the universe did not remain close to equilibrium during the inflationary period. Thus, the MIS formalism may fail to provide an accurate description of the cosmological evolution during inflation.

Due to the lack of a well-understood theory of non-equilibrium relativistic hydrodynamics valid during inflation, the out-of-equilibrium effects are accommodated in a phenomenological model first put forward by Maartens and Mendez [41], who proposed a generalisation of the MIS theory by taking into consideration the SEC. While preserving causality, the approach proposed in [41] introduces a new characteristic timescale that is non-vanishing when non-equilibrium effects are present. As a result, the modified evolution equation of the bulk viscosity pressure contains an additional coefficient responsible for the far-from-equilibrium effects. There are several important advantages of the formalism presented by Maartens and Mendez [41]. First, the MIS theory is correctly retrieved in the regime when the departures from the equilibrium are small. Second, the theory ensures the validity of the second law of thermodynamics; hence, the entropy production rate remains positive. Third, an upper bound on the maximum value of the bulk viscosity pressure exists and is compatible with the second law of thermodynamics.

Here, we extend the work of Maartens and Mendez [41] by relaxing the exact de-Sitter condition so as to assess whether a quasi de-Sitter expansion of the early universe, i.e., an inflationary scenario characterised by $-\dot{H} \ll H^2$, can be actually sustained by the bulk viscosity. To this scope, we introduce an *effective* equation of state (EOS), where the relation between the pressure and the energy density is no longer constant at all times, but becomes time dependent as a result of non-equilibrium effects and a time-dependent bulk viscous stress. However, as a new equilibrium is reached, the EOS stops evolving, and a new constant relation emerges between the pressure and the energy density. To quantify this behaviour, our model is described in terms of three parameters: the coefficient of the bulk viscosity $\bar{\zeta}$, the characteristic timescale of the non-equilibrium effects k, and the parameter relating the equilibrium pressure and energy density w_0 of the cosmological fluid.

In this way, we find that the quasi de-Sitter inflation is a natural outcome of the presence of a nonzero bulk viscosity and, hence, does not require the need for an inflaton field. In particular, the universe always exhibits departure from the exact de-Sitter universe, so that, as long as the EOS is time dependent, an exact de-Sitter universe cannot exist. The evolution of the transition timescale of the modified EOS from the exact de-Sitter solution is found to depend on the variation of each of the three aforementioned parameters of

the model. As an example, we find that this transition timescale during the quasi de-Sitter expansion is inversely proportional to the magnitude of the bulk viscosity. On the other hand, the timescale over which the non-equilibrium effects are present is larger than previously found, even though they do not have a significant impact in altering the qualitative behaviour of the equation of the state.

Furthermore, our analysis reveals that, as the universe evolves during inflation, the absolute magnitude of the bulk viscosity pressure diminishes before settling to a small and nearly constant value. We determined admissible regions of the parameters of the model under the quasi de Sitter conditions, and upon comparing with the exact de-Sitter case, we find that the admissible parameter region of one of the parameters (w_0; see below) is considerably larger than what has been computed so far. Finally, as a test of our model, we compute the inflationary observables, i.e., the spectral index and the tensor-to-scalar ratio of the curvature perturbations, and compare them with the observations. In this way, we find that, while our predictions are compatible with non-Planck-based observations, they are in tension based on the Planck data.

The paper is organised as follows. Section 2 provides an overview on the bulk-viscosity-driven inflation and is devoted to developing the mathematical background for the paper. In Section 3, the numerical results are presented, which were obtained by solving the generalized momentum-conservation equation in the presence of bulk viscous stress under quasi de-Sitter conditions. In Section 4, we discuss the implications of the numerical results and provide an estimate of the magnitude of the bulk viscosity coefficient compatible with the current results of the inflationary variables. Finally, the conclusions and outlook are reported in Section 5.

2. Bulk-Viscosity Driven Quasi De-Sitter Inflation: Mathematical Background

To study the inflationary scenario in the presence of the bulk viscosity, let us consider the homogeneous, isotropic, spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) metric as follows:

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j\,, \qquad (i = x, y, z) \tag{1}$$

where t is the cosmological time comoving with the expansion, which is the proper time measured by a free-falling observer and $a(t)$ is the scale-factor that describes the cosmological evolution of the universe. Hereafter, we will adopt units in which $8\,\pi G = c = 1$, with G being the gravitational constant and c the speed of light.

To study inflation in the presence of the bulk viscous stress, let us consider the Eckart frame in which a non-perfect relativistic fluid is subjected to heat flux, bulk, and shear viscous stresses. Since the shear stress identically vanishes in the isotropic FLRW spacetime, the general form of the stress energy–momentum tensor of the cosmological fluid in terms of the heat flux q^μ and the bulk viscosity pressure Π is given by [40]

$$T^{\mu\nu} \;=\; eu^\mu u^\nu + (p + \Pi)h^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu\,, \tag{2}$$

where e and p are, respectively, the local equilibrium energy density and the fluid pressure, while $h^{\mu\nu} := g^{\mu\nu} + u^\mu u^\nu$ is the projection tensor orthogonal to the four-velocity u^μ and obviously satisfies the condition $h_{\mu\nu}u^\nu = 0$ [40]. As is customary in cosmology, we adopted a frame comoving with the fluid, so that the four-velocity of the fluid in such a frame is simply given by $u^\mu = (1, 0, 0, 0)$ and the four-acceleration $a^\mu := u^\beta \nabla_\beta u^\mu$ vanishes identically. The total effective pressure $p_{\rm eff}$ of the fluid in the presence of the bulk scalar stress Π is expressed as

$$p_{\rm eff} = p + \Pi\,. \tag{3}$$

In terms of the energy–momentum tensor, the energy density, the effective pressure, and the heat flux can be, respectively, expressed in the following way:

$$e = T_{\alpha\beta}u^{\alpha}u^{\beta}\,, \tag{4}$$

$$p_{\rm eff} = p + \Pi = \frac{1}{3}T_{\alpha\beta}h^{\alpha\beta}\,, \tag{5}$$

$$q^{\mu} = -K(\mathcal{D}^{\mu}_{\perp}T + Ta^{\mu})\,. \tag{6}$$

Here $\mathcal{D}^{\mu}_{\perp} = h^{\mu\alpha}\nabla_{\alpha}$ is the covariant derivative orthogonal to the four-velocity, K is the coefficient of the thermal conductivity, and T is the local temperature of the fluid. Equation (6) in the comoving frame ($a^{\mu} = 0$) of a homogeneous and isotropic universe (giving rise to a constant temperature) shows that the heat flux is trivially zero i.e., $u^{\mu}\nabla_{\mu}T = 0$ at the background level and will not be considered further here (see, however, [42] for an example in which the inflationary solutions of the early universe driven by heat flux are studied in the context of the MIS formulation). Using Equations (1) and (2), the Friedmann equations (i.e., the temporal and the spatial components of the Einstein equations) are given by

$$3H^2 = \kappa^2 e\,, \tag{7}$$

$$2\dot{H} + 3H^2 = -\kappa^2(p + \Pi)\,. \tag{8}$$

where $\kappa^2 := 8\pi G = 1$. The Hubble function is defined as

$$H(t) := \frac{\dot{a}}{a}\,, \tag{9}$$

where the "dot" will be employed hereafter to indicate a derivative with respect to the coordinate time t. From Equations (7) and (8) we obtain that

$$\dot{H} = -\frac{\kappa^2}{2}(e + p + \Pi)\,, \tag{10}$$

while the the four-divergence of the energy-momentum tensor leads to the energy-conservation equation

$$\dot{e} + 3H(e + p + \Pi) = 0\,. \tag{11}$$

Let us now consider that e and p are related by a barotropic EOS given by

$$p = w_0 e\,, \tag{12}$$

where w_0 is a constant, known as the EOS parameter, and expected to vary in the range $-1 \leq w_0 \leq 1$. The range of w_0 can be set from the validity/violation of a series of energy conditions, namely the Weak Energy Condition (WEC), Null Energy Condition (NEC), Dominant Energy Condition (DEC), and Strong Energy Condition (SEC). For an ideal cosmological fluid characterised by its energy density e and pressure p, the WEC predicts $e \geq 0$ and $e + p \geq 0$, which shows $w_0 \geq -1$. The NEC also gives rise to the condition $w_0 \geq -1$ and $e + p \geq 0$. The DEC shows that $e \geq \mid p \mid$, implying $w_0 \leq 1$. On the other hand, the SEC given by $e + 3p \geq 0$ implies $w_0 \geq -1/3$. The fact that the early universe underwent a phase of accelerated expansion results in the violation of the SEC, i.e., $e + 3p \leq 0$ or $w_0 \leq -1/3$. In particular, the exponential expansion of the universe is generated with the EOS $p = -e$ or $w_0 = -1$. Now, in the presence of the bulk viscosity, using Equations (7) and (8), Equation (8) leads to

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}[e + 3e\,w_0 + 3\Pi]\,. \tag{13}$$

The left-hand side of Equation (13) can also be expressed as

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = H^2(1 - \epsilon_{\rm H})\,, \qquad , \tag{14}$$

where $\epsilon_{\rm H} = -\dot{H}/H^2$ is also known as the slow-roll parameter (or first Hubble-flow). Since the accelerated expansion requires $\ddot{a} > 0$, during inflation the following conditions are always satisfied from Equations (13) and (14)

$$1 + 3w_0 + \frac{3\Pi}{e} < 0\,, \tag{15}$$

$$0 \leq \epsilon_{\rm H} < 1\,. \tag{16}$$

We note that $\dot{H} = 0$ corresponds to the "exact de-Sitter" expansion, for which $\epsilon_{\rm H} = 0$ and $a(t) \propto e^{H_0 t}$, where H_0 is a constant. The moment corresponding to $\epsilon_{\rm H} = 1$ is defined as the exit from inflation when $\ddot{a} = 0$ and the acceleration of the universe finally comes to a halt.

Because we are interested in the presence of the bulk viscosity pressure here, we introduce a new EOS, which we denote as the *effective* EOS, defined in the following way:

$$w_{\rm eff} := \frac{p_{\rm eff}}{e} \quad = \quad w_0 + \frac{\Pi}{e}\,. \tag{17}$$

Note also that $w_{\rm eff}$ is a time-dependent function, which, using Equations (10) and (17), can be re-expressed in terms of the Hubble function via an "effective EOS parameter":

$$w_{\rm eff} = -\left(1 + \frac{2}{3}\frac{\dot{H}}{H^2}\right). \tag{18}$$

Hence, so long as the condition

$$-\frac{2}{3}\left(\frac{\dot{H}}{H^2}\right) \ll 1\,, \tag{19}$$

is valid, $w_{\rm eff} \gtrsim -1$ remains true and the early universe undergoes a *nearly* exponential expansion supported by the bulk viscosity. On the other hand, in the absence of the bulk-viscosity pressure Equation (17) reduces to

$$w_{\rm eff} = w_0\,, \tag{20}$$

and becomes a constant. The universe is then filled with a perfect fluid and the corresponding pressure and the energy density are related by the EOS-parameter w_0.

2.1. Examining the Energy Conditions

In what follows, we analyse the energy conditions during the accelerated expansion of the universe in the presence of the bulk viscosity pressure. We start by recalling that the SEC is given by

$$T_{\alpha\beta}u^\alpha u^\beta - \frac{1}{2}Tu^\alpha u_\alpha \geq 0\,, \tag{21}$$

which can be alternatively expressed as

$$R_{\alpha\beta}u^\alpha u^\beta \geq 0\,. \tag{22}$$

In an FLRW spacetime, Equation (22) then reduces to

$$\frac{\ddot{a}}{a} \leq 0\,, \tag{23}$$

thus highlighting that the SEC must be violated to satisfy the condition $\ddot{a} > 0$ during the inflationary phase. On the other hand, the Weak Energy Condition (WEC) and the Null Energy Condition (NEC), respectively, give rise to the following conditions:

$$e \geq 0 \quad \Rightarrow \quad \frac{\dot{a}^2}{a^2} \geq 0\,, \tag{24}$$

$$e + p + \Pi \geq 0 \quad \Rightarrow \quad -\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \geq 0\,. \tag{25}$$

The WEC remains valid because the energy density of the viscous cosmological fluid is positive. Furthermore, since we will not consider inflationary models corresponding to $w_{\rm eff} < -1$, the NEC is also satisfied, and $\dot{H}$ remains negative during the quasi de-Sitter expansion.

Next, we recall that the relativistic theories of non-perfect fluids proposed by Eckart and Landau lead to dynamically unstable equilibrium states under linear perturbations and do not give rise to hyperbolic equations of motion that result in a violation of causality [38]. To counter these drawbacks, one of the best-developed and -studied approaches towards constructing a causal theory of the relativistic hydrodynamics of non-perfect fluids is the MIS formalism [36,37]. This approach makes use of the second-order gradients of the hydrodynamical variables and appropriately introduces relaxation time transport coefficients corresponding to all the dissipative quantities. Notwithstanding the complications brought about by this approach (see [43] and the references therein), the MIS formulation is able to remove all the drawbacks of the first-order theories of the relativistic hydrodynamics of non-perfect fluids proposed by Eckart and Landau [38,39]. We note that possible alternatives to the MIS formulation have recently been proposed addressing the existence of the hyperbolicity and causality of the hydrodynamical theory of relativistic non-perfect fluids [44,45]; while these formulations are interesting and deserve future attention, they will not be employed here, where we instead focus on the MIS formalism.

The most-crucial assumption of the MIS formalism is that it considers regimes that are near-equilibrium, that is where all the dissipative flux quantities are small compared to the equilibrium fluid variables. In the context of the present study, this implies that the bulk viscosity pressure Π must be smaller when compared to the equilibrium fluid pressure, i.e., $\Pi \ll p$, where the equilibrium cosmological fluid pressure is always non-negative, i.e., $p > 0$. For an accelerated expansion of the early universe, Equation (13), however, indicates that $e + 3(p + \Pi) < 0$, which, in turn, implies

$$-\Pi \quad > \quad p + \frac{1}{3}e\,. \tag{26}$$

Thus, the violation of the SEC leads to a scenario where Π is actually greater than the equilibrium fluid variables, and the validity of the MIS theory is, therefore, put to question for studying the inflationary regime in the presence of bulk viscosity.

Given these considerations, we adopt here the phenomenological approach prescribed by Maartens and Mendez [41], which not only incorporates the out-of-equilibrium effects by providing a minimalist modification to the MIS approach via the introduction of a characteristic scale τ_*, but also ensures that the MIS approach is recovered in the appropriate limit. In the context of the MIS formalism, the positivity of the entropy production leads to the following relations [42]:

$$\Pi = -\zeta\chi\,, \tag{27}$$

$$\chi = 3H + \frac{\tau_\Pi}{\zeta}\dot{\Pi} + \frac{\tau_\Pi}{2\zeta}\Pi\left[3H + \frac{\dot{\tau_\Pi}}{\tau_\Pi} - \frac{\dot{\zeta}}{\zeta} - \frac{\dot{T}}{T}\right], \tag{28}$$

where ζ and τ_Π are the bulk viscosity and the relaxation time coefficients, respectively (note that ζ has the dimensions of the inverse of mass). To include far-fro- equilibrium effects, the bulk viscosity pressure (27) is modified in a nonlinear manner as [41]

$$\Pi = -\frac{\zeta\chi}{1+\tau_*\chi}\,. \tag{29}$$

where $\tau_* \geq 0$ is the timescale of the onset of the non-equilibrium effects which become significant under the condition $\chi \gtrsim \tau_*^{-1}$. Clearly, the MIS approach is retrieved when χ is small, i.e., if $\chi \ll \tau_*^{-1}$. On the other hand, in the limit $\chi \to \infty$, the expression (29) ensures that $-\Pi$ does not become arbitrarily large but is bounded by the condition, $-\Pi \leq \zeta/\tau_*$. Additionally, the second law of thermodynamics for out-of-equilibrium systems implies that the entropy production should always be non-negative, namely

$$\nabla_\alpha S^\alpha \geq 0\,. \tag{30}$$

Since the MIS limit is recovered with $\tau_* = 0$, the four-divergence of the non-equilibrium entropy in that case is given by [41]

$$\nabla_\alpha S^\alpha = \frac{\Pi^2}{T\zeta} \geq 0\,, \tag{31}$$

which implies $\zeta > 0$. In other words, to guarantee that a smooth $\tau_* \to 0$ limit always exits, the condition $\zeta > 0$ needs to be considered.

Using now Equations (28) and (29), the momentum-conservation equation of the bulk-viscosity pressure becomes

$$\tau_\Pi\dot{\Pi}\left(1+\frac{\tau_*}{\zeta}\Pi\right)+\Pi(1+3H\tau_*) = -3\zeta H - \frac{1}{2}\tau_\Pi\Pi\left(3H+\frac{\dot{\tau_\Pi}}{\tau_\Pi}-\frac{\dot{\zeta}}{\zeta}-\frac{\dot{T}}{T}\right)\left(1+\frac{\tau_*}{\zeta}\Pi\right). \tag{32}$$

which recovers the MIS equivalent when $\tau_* = 0$.

Of course, the phenomenological prescription (29) does not provide a way to determine the timescale τ_*, which should in principle be estimated from microscopic considerations. Once again, we follow the prescription proposed by Maartens and Mendez [41], which simply relates the relaxation time to the characteristic timescale, i.e.,

$$\tau_* = k^2\tau_\Pi\,, \tag{33}$$

where k is a dimensionless proportionality constant. Let us now consider the ansatz that $\zeta \propto \sqrt{e}$, from which we can set

$$\zeta = \sqrt{3}\,\bar{\zeta}\,\frac{H}{\kappa}\,, \tag{34}$$

with $\bar{\zeta}$ a proportionality constant. Since ζ has the dimensions of an inverse of a mass, using Equation (34), the proportionality constant $\bar{\zeta}$ will have the same dimensions. In the absence of a heat flux and shear viscosity, the relation between the relaxation-time coefficient and ζ is given by [41]

$$\tau_\Pi = \frac{\zeta}{e(1+w_0)v_b^2}\,, \tag{35}$$

where v_b is the time-independent bulk viscosity (non-adiabatic) contribution to the sound speed V_s, namely, $V_s^2 := c_s^2 + v_b^2$, with c_s the adiabatic contribution defined as

$$c_s^2 := \frac{\partial p}{\partial e} = w_0\,, \tag{36}$$

where the last equality applies in the specific case of the barotropic EOS (12) (see Refs. [39,46] for a detailed discussion of the sound speed in the presence of non-adiabatic effects). Since

the sound speed cannot exceed unity (the speed of light), i.e., $w_0 + v_b^2 \leq 1$, the relaxation-time coefficient becomes

$$\tau_{\Pi} \geq \frac{\zeta}{e(1-w_0^2)} = \frac{\kappa\bar{\zeta}}{\sqrt{3}H(1-w_0^2)}\,, \tag{37}$$

where the second expression is obtained by substituting Equation (34) and Equation (7) in the first expression. Let us now consider the local temperature to be a simple function of the energy density, i.e., $T = T(e)$, and the integrability conditions of the second law of thermodynamics (Gibbs equation)

$$TdS = (p+e)d\left(\frac{1}{n}\right) + \frac{1}{n}de\,, \tag{38}$$

$$\frac{\partial^2 S}{\partial e \partial n} = \frac{\partial^2 S}{\partial n \partial e}\,, \tag{39}$$

then, from the above two conditions, one obtains,

$$T(e) \propto \exp\left(\frac{w_0}{w_0+1}\right). \tag{40}$$

Using the Friedmann Equation (10), we can express the bulk-viscosity pressure in terms of the Hubble function as

$$\Pi = -\left[2\dot{H} + 3H^2(1+w_0)\right]. \tag{41}$$

Substituting the expressions for τ_{Π}, $T(e)$, and v_b^2, together with Equations (34) and (36) in Equation (32), the Friedmann equation expressing the dynamics of the function $H = H(t)$ can be expressed as

$$\begin{aligned} 1 &+ \frac{2\left[\frac{2k^2}{3}\frac{\dot{H}}{H^2} + (w_0+1)\left(k^2+w_0-1\right)\right]\left[3(w_0+1)\frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3}\right]}{9\left(w_0^2-1\right)^2} \\ &+ \frac{\left(\frac{2\dot{H}}{3H^2} + w_0 + 1\right)(w_0^2 - 1 - \sqrt{3}k^2\bar{\zeta})}{\sqrt{3}\bar{\zeta}(w_0^2-1)} \\ &+ \frac{\left(\frac{2\dot{H}}{3H^2} + w_0 + 1\right)\left[w_0 + 1 - \frac{2\dot{H}}{3H^2}(2w_0+1)\right]\left[\frac{2k^2}{3}\frac{\dot{H}}{H^2} + (w_0+1)\left(k^2+w_0-1\right)\right]}{2(w_0-1)^2(w_0+1)^3} = 0\,. \end{aligned} \tag{42}$$

With an appropriate choice of the initial conditions and for a given choice of the free parameters, namely: $\bar{\zeta}, k$, and w_0, the evolution of the Hubble function $H(t)$ is determined from Equation (42), which, when substituted back into Equation (41), gives the evolution of the bulk viscosity pressure $\Pi(t)$.

An advantage of the formulation proposed in [41] is that the bulk viscosity pressure has an upper bound given by $-\Pi \leq \zeta/\tau_*$, which results in being fixed by the validity of the second law of thermodynamics. With this upper-bound constraint, the relation between k and v_b under the quasi de-Sitter condition on Hubble expansion is determined as follows:

$$k^2 \leq v_b^2\,, \qquad \Rightarrow \qquad k^2 = \tilde{\alpha}v_b^2\,, \qquad 0 \leq \tilde{\alpha} \leq 1 \tag{43}$$

$$v_b^2 \leq 1 - c_s^2 = 1 - w_0\,, \qquad \Rightarrow \qquad v_b^2 = \epsilon(1-w_0)\,, \qquad 0 \leq \epsilon \leq 1\,, \tag{44}$$

or, equivalently,

$$k^2 = \tilde{\alpha}v_b^2 = \alpha(1-w_0)\,, \tag{45}$$

where $\alpha := \tilde{\alpha}\epsilon$ is, again, a proportionality constant between zero and one. The non-equilibrium characteristic timescale τ_* of the non-equilibrium effects (33) is then given by

$$\tau_* = \kappa \frac{\alpha \bar{\zeta}}{H(1+w_0)}\,. \tag{46}$$

Hence, the non-equilibrium characteristic timescale can be completely parametrized by α for fixed values of $\bar{\zeta}$ and w_0.

2.2. Exact De-Sitter Expansion within the Generalised Causal Theory

For completeness, we briefly examine the *exact* de-Sitter inflationary scenario generated solely by a large bulk viscosity pressure, i.e., $\Pi \gg p$ in the generalised causal framework, which was first proposed by Maartens and Mendez [41]. The de-Sitter expansion corresponds to $H(t) = H_0 = \text{const.}$, with $H_0 > 0$ and $\dot{H}(t) = 0$ implying a constant EOS given by $w_{\text{eff}} = -1$. From Equations (42) and (45), one obtains

$$-27(w_0+1)^3\left(-\sqrt{3}(\alpha-1)\bar{\zeta} + 2w_0^2 + 2\sqrt{3}(\alpha-1)\bar{\zeta}w_0 - 2\right) = 0\,. \tag{47}$$

so that the bulk viscosity coefficient required to produce the de-Sitter expansion is given by

$$\bar{\zeta}_{\text{dS}} = \frac{2(w_0^2-1)}{\sqrt{3}(2w_0-1)(1-\alpha)}\,. \tag{48}$$

The variation of $\bar{\zeta}_{\text{dS}}$ as a function w_0 and α is shown in Figure 1. Since $\bar{\zeta}_{\text{dS}}$ has to be positive—so as not to violate the second law of thermodynamics—and finite, the permissible parameter space for w_0 becomes $-1 < w_0 < 0.5$. The solution corresponding to $w_0 = -1$ gives rise to a time-independent EOS depicting inflation driven by a perfect fluid, where the bulk viscosity does not have any role. Thus, the *exact* de-Sitter phase arises under two distinct situations: (a) if $w_0 = -1$, such that the accelerated expansion of the early universe is as a perfect fluid; (b) the accelerated expansion is driven by a large bulk viscosity pressure such that the coefficient of viscosity is given by Equation (48) [41]. In both cases, the effective EOS parameter is time independent and is given by $w_{\text{eff}} = -1$.

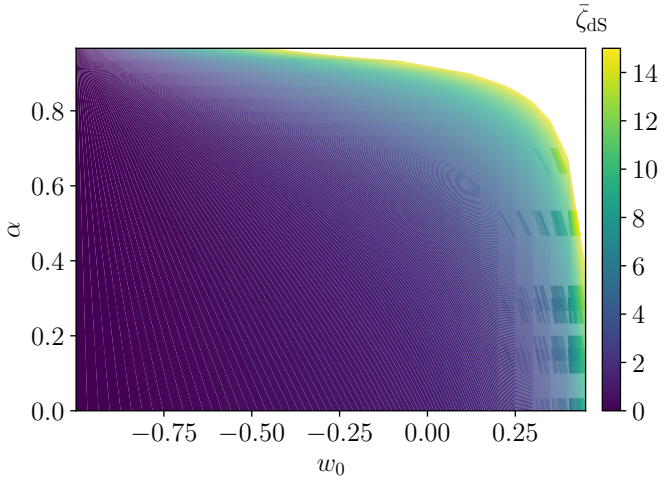


Figure 1. Behaviour of the maximum value of the bulk-viscosity coefficient $\bar{\zeta}_{\text{dS}}$ as a function of the two free parameters in the model, α and w_0 (see Equation (56)).

3. Numerical Solutions

In this Section, we investigate the emergence of a quasi de-Sitter phase supported by the bulk viscosity pressure within a causal MIS formulation. In this scope, we numerically solve the generalized momentum conservation Equation (32) and the effective EOS param-

eter (17) under the quasi de-Sitter condition so as to obtain the evolution of the Hubble function. More specifically, substituting Equation (45) into Equation (42) gives

$$\left(\frac{\ddot{H}}{H^3}+3(w_0+1)\frac{\dot{H}}{H^2}\right)\left(\frac{2\alpha}{3}\frac{\dot{H}}{H^2}+(\alpha-1)(w_0+1)\right)$$
$$+\frac{3}{2}(w_0^2-1)\left[\frac{\sqrt{3}}{\bar{\zeta}}\left(\sqrt{3}\alpha\bar{\zeta}+w_0+1\right)\left(\frac{2}{3}\frac{\dot{H}}{H^2}+w_0+1\right)-3(w_0+1)\right]+$$
$$\frac{9}{4(w_0+1)}\left(\frac{2}{3}\frac{\dot{H}}{H^2}+w_0+1\right)\left(w_0+1-\frac{2}{3}(2w_0+1)\frac{\dot{H}}{H^2}\right)\left(\frac{2\alpha}{3}\frac{\dot{H}}{H^2}+(\alpha-1)(w_0+1)\right)=0\,, \tag{49}$$

so that the Hubble function is determined by solving Equation (49) numerically for given values of α and w_0, under the quasi de-Sitter condition that $-\dot{H} \ll H^2$. Note that because $w_{\rm eff}$ is a time-dependent function under the quasi de-Sitter condition, we express Equation (49) in terms of a first-order ordinary differential equation in $w_{\rm eff}$ as follows

$$\frac{\dot{w}_{\rm eff}}{3H(t)}+\frac{(1+w_0+\sqrt{3}\,\alpha\,\bar{\zeta})(1-w_0^2)(w_{\rm eff}-w_0)}{\sqrt{3}\,\bar{\zeta}(1+w_0+\alpha(w_{\rm eff}-w_0))}+\frac{(1-w_0^2)(1+w_0)}{(1+w_0+\alpha(w_{\rm eff}-w_0))}$$
$$-\frac{(w_{\rm eff}-w_0)^2}{2(1+w_0)}=0\,, \tag{50}$$

where we have made use of the fact that

$$H(t)=\frac{2}{3}\left[\int(1+w_{\rm eff}(t))dt\right]^{-1}. \tag{51}$$

The evolution of $w_{\rm eff}$ is determined by numerically solving Equation (50), with initial condition $w_{\rm eff}(t_{\rm in}) \approx -1$ (i.e., $w_{\rm eff}-1=10^{-12}$) as expected if the universe is nearly de-Sitter at $t=t_{\rm in}$ and possesses a nonzero bulk viscosity pressure $\Pi(t_{\rm in}) \sim 3H_{\rm in}^2(w_{\rm eff}(t_{\rm in})-w_0)$, where $H_{\rm in}$ is the Hubble constant at $t=t_{\rm in}$.

Figure 2 shows the evolution of $H(t)$ under the quasi de-Sitter condition that $-\dot{H} \ll H^2$ for $\alpha=0.1$ and $w_0=-0.975$. Note that the Hubble function, which is initially set to take the value at which inflation is believed to have taken place, i.e., $H=10^{16}$ GeV $\approx 10^{-3}\,M_{\rm pl}$, remains essentially constant, but then, it starts to slowly fall off. During this period, as shown by Figure 2, the first Hubble slow-roll parameter remains small and satisfies the condition $\epsilon_{\rm H} \ll 1$. The behaviour of $w_{\rm eff}$ is shown in the various panels of Figure 3 for different values of the parameters $\bar{\zeta}$, α, and w_0.

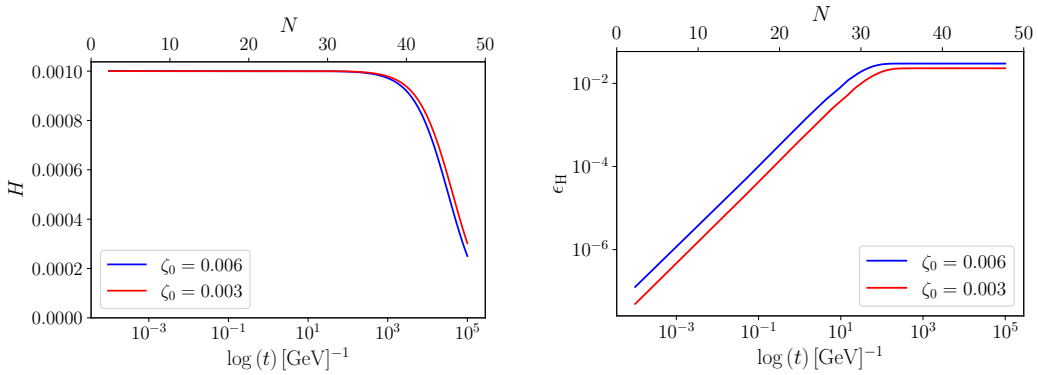


Figure 2. *Left panel:* Evolution of Hubble function $H(t)$ for for a representative choice of $\alpha=0.1$ and $w_0=-0.975$. *Right panel:* The same as in the left panel but for the slow-roll parameter (or first Hubble-flow) parameter. In both panels the top horizontal axes report time in terms of the number of e-foldings N.

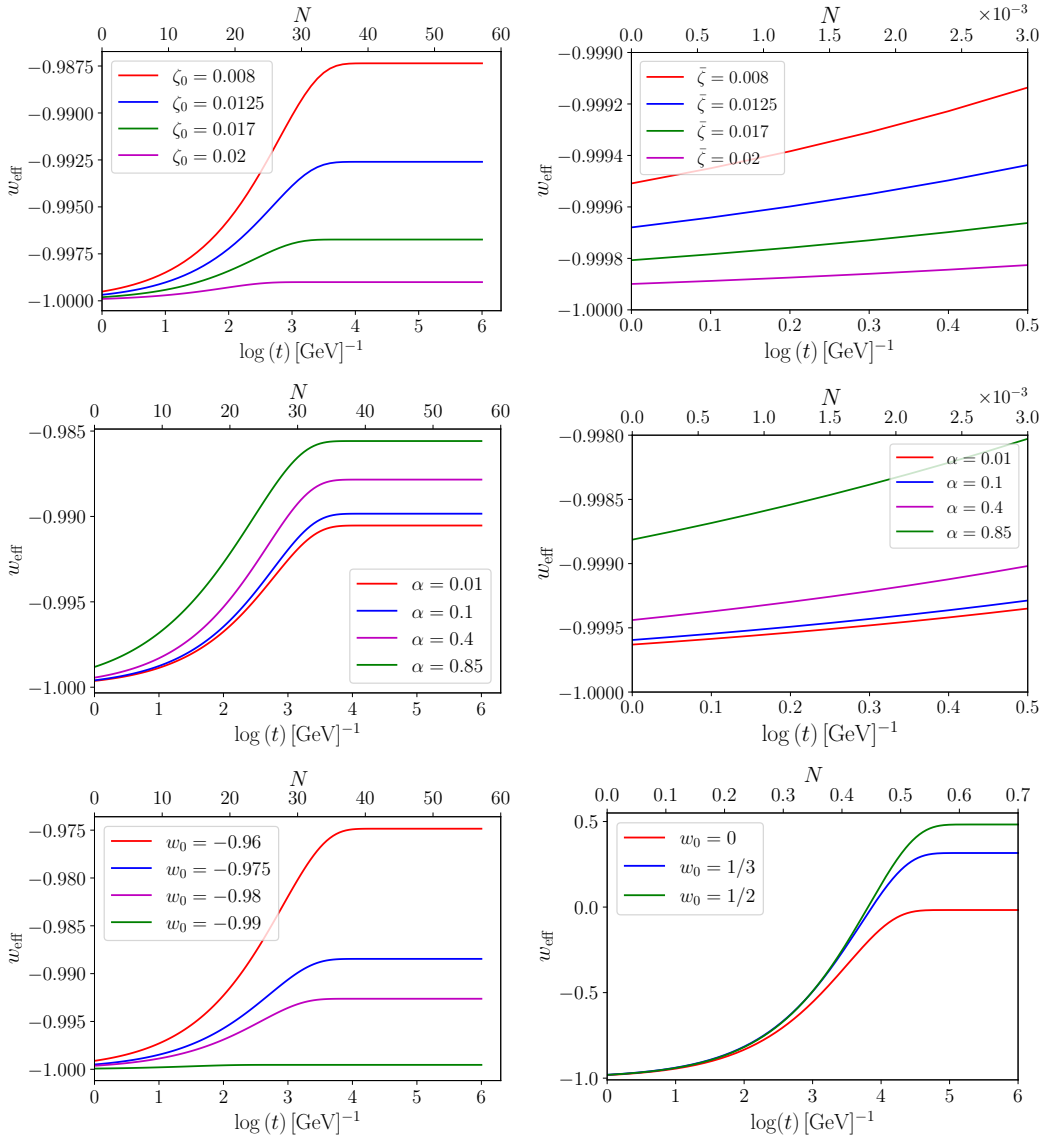


Figure 3. *Top row:* Evolution of $w_{\rm eff}$ for different values of $\bar{\zeta}$ for reference values of $\alpha = 0.1$ and $w_0 = -0.975$. The right panel simply shows a magnification of the early evolution. *Middle row:* the same as above but for different values of α while keeping fixed $w_0 = -0.975$ and $\bar{\zeta} = 0.01$. *Bottom row:* the same as above but for different values of w_0 while keeping fixed $\bar{\zeta} = 0.01$ and $\alpha = 0.3$. In all panels the top horizontal axes report time in terms of the number of e-foldings N.

In essence, all variations of the parameters show that, through the time-evolving bulk viscosity pressure $\Pi(t)$, the effective EOS parameter $w_{\rm eff}$ evolves with time and, finally, attains a nearly constant value at a late time value during the quasi de-Sitter inflationary period. More specifically, the effect of varying $\bar{\zeta}$ on the effective EOS parameter is shown in the top row of Figure 3, where it is found that $w_{\rm eff}$ departs from the exact de-Sitter value as $\bar{\zeta}$ is slowly decreased from its maximum value $\bar{\zeta}_{\rm dS}$ ($\bar{\zeta}_{\rm dS} = 0.0214$ for $\alpha = 0.1$ and $w_0 = -0.975$). As a result, the more the magnitude of $\bar{\zeta}$ is decreased, the greater the departure from the exact de-Sitter evolution. Similarly, the influence of the variation of α on $w_{\rm eff}$ is presented in the middle row of Figure 3, where it can be observed that, as α is increased, $w_{\rm eff}$ deviates swiftly from the de-Sitter phase, implying that an increase of the non-equilibrium characteristic timescale τ_* leads to an increasing large departure from the exact de-Sitter state. Stated differently, if non-equilibrium effects are present for a sufficiently long time, the universe cannot remain close to an exact de-Sitter state. Finally, the bottom row of Figure 3 reports the behaviour of $w_{\rm eff}$ due to the variations of

the parameter w_0 while keeping α and $\bar{\zeta}$ fixed. Since the value $w_0 = -1$ is not allowed, decreasing $|w_0|$ leads to departures of $w_{\rm eff}$ from the exact de-Sitter phase for finite and positive values of $\bar{\zeta}$ and α.

It is worth noting that the profile of $w_{\rm eff}$ is a feature that is purely a consequence of bulk viscous effects subject to quasi de-Sitter conditions. In this regard, an insightful example is the evolution when $w_0 = 0$ (see the bottom-right panel of Figure 3). In this case, in fact, the evolution of $w_{\rm eff}$ depends only on $\bar{\zeta}$ and α. The effective EOS parameter exhibits a departure from the exact de-Sitter phase, which is qualitatively very similar to the cases when $w_0 \neq 0$. Furthermore, the late-time value of $w_{\rm eff}$ increases systematically when decreasing $\bar{\zeta}$, and it increases for larger values of α and w_0. Finally, we note that, during the quasi de-Sitter expansion, the absolute magnitude of the bulk viscosity pressure can be estimated using the quantity $|w_{\rm eff} - w_0|$, which is reported in Figure 4. As is in part to be expected, the bulk viscosity pressure decreases as $\bar{\zeta}$ decreases from $\bar{\zeta}_{\rm dS}$ and becomes zero in the limit in which $\bar{\zeta} \to 0$. All in all, the results presented in Figure 3 clearly indicate that the effective EOS parameter evolves following a unique functional behaviour, namely, from an almost constant and small initial value to an almost constant and large final value. However, the details on the transition between these two states is a function of the three parameters involved in the evolution: $\bar{\zeta}$, α, and w_0.

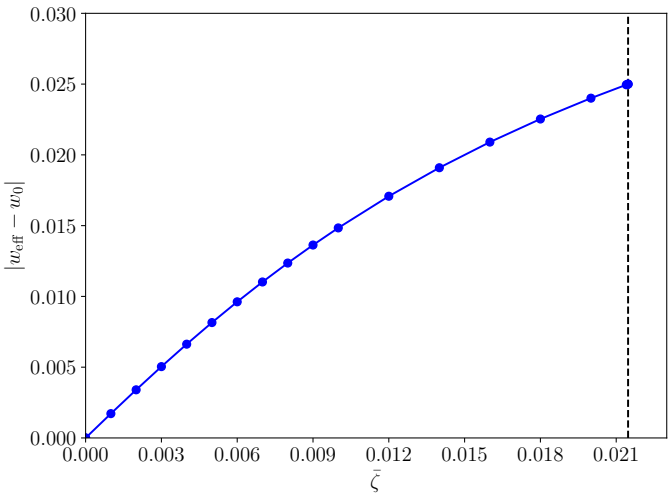


Figure 4. Variation of the absolute magnitude of the bulk-viscosity pressure $|w_{\rm eff} - w_0|$ as a function of the bulk-viscosity coefficient $\bar{\zeta}$ for reference values of $\alpha = 0.1$ and $w_0 = -0.975$; the vertical dotted line marks the maximum value $\bar{\zeta}_{\rm dS} = 0.02147$.

4. Results and Interpretation

In what follows, we summarise the bulk of the results obtained in our numerical investigation and provide a global interpretation of the various behaviours encountered. We start by considering what are the ranges in which the parameters of our approach, namely w_0, $\bar{\zeta}$, and α, can vary.

4.1. Admissible Regions of the Parameters

We first note that we have restricted our study to the values of the EOS parameter such that $-1 \leq w_0 \leq 1$, so that we do not consider the case $w_0 < -1$ for which the scenario of phantom inflation is often considered [47–49]. Clearly, using Equation (49) with $w_0 = -1$ yields $\dot{H} = 0$, that is the standard and non-viscous *exact* de-Sitter universe (note that, in this case, $\bar{\zeta}_{\rm dS} = 0$, as expressed by Equation (48)). On the other hand, when $w_0 = 1$, Equation (49) gives rise to $\dot{H} \neq 0$, so a *quasi* de-Sitter phase can in principle exist in this limit provided the conditions $\dot{H} < 0$ and $-\dot{H}/H^2 \ll 1$ are valid. What remains problematic, however, is that, in this case, the relaxation timescale τ_{Π} diverges; for this reason, we will not consider the value $w_0 = 1$ as admissible in our analysis. Another potentially

problematic value for the EOS parameter is $w_0 = 1/2$, which results in a diverging $\bar{\zeta}_{\rm dS}$ [41]; however, so long as the quasi de-Sitter conditions remain in place, i.e., $\dot{H} < 0$ and $-\dot{H}/H^2 \ll 1$, the issue of the divergence of $\bar{\zeta}_{\rm dS}$ at $w_0 = 1/2$ does not arise. This is shown in the bottom row of Figure 3, where we observe that $w_{\rm eff}$ is finite and well-behaved over the entire interval $-1 < w_0 < 1$ for different magnitudes of $\bar{\zeta}$.

Let us now consider the set of possible solutions starting from the inviscid limit ($\bar{\zeta} = 0$), which leads to $w_{\rm eff} = w_0$ i.e., a constant effective EOS parameter. Under these conditions, Equation (50) reduces to

$$\frac{3\sqrt{3}}{2}(w_0^2 - 1)(1 + w_0)\left(\frac{2}{3}\frac{\dot{H}}{H^2} + 1 + w_0\right) = 0\,, \tag{52}$$

whose possible solutions are:

(a) $w_0 = 1$: this is the perfect-fluid cosmological solution with an ultra-stiff EOS [40].
(b) $w_0 = -1$: this corresponds to the exact de-Sitter solution.
(c) $\frac{2}{3}\frac{\dot{H}}{H^2} + 1 + w_0 = 0$: the corresponding solution is given by

$$H(t) = \frac{2}{3t(w_0 + 1) - 2c_1}\,, \tag{53}$$

where the integration constant may be chosen to be $c_1 = 0$ or $c_1 < 0$ to keep $H(t)$ finite. More specifically, if $c_1 = 0$, the scale-factor is then given by

$$a(t) \propto [3t(1 + w_0)]^{2/3(w_0+1)}\,. \tag{54}$$

Equations (53) and (54) thus provide straightforward expressions for the evolution of the Hubble parameter and of the scale-factor for either pressureless matter ($w_0 = 0$) or for a radiation-dominated ($w_0 = 1/3$) universe.

To determine the maximum allowed value of $\bar{\zeta}$, Equation (50) is re-expressed in terms of $\epsilon_{\rm H}$ (which lies in the interval $0 \leq \epsilon_{\rm H} < 1$). Substituting $w_{\rm eff} = -1 + (2/3)\epsilon_{\rm H}$ in Equation (50) one arrives at

$$\frac{\dot{\epsilon}_{\rm H}}{3H(1 - w_0^2)} + \frac{(1 + w_0 + \sqrt{3}\alpha\bar{\zeta})(-1 + \frac{2}{3}\epsilon_{\rm H} - w_0)}{\sqrt{3}\bar{\zeta}[1 + w_0 + \alpha(-1 + \frac{2}{3}\epsilon_{\rm H} - w_0)]} + \frac{(1 + w_0)}{[1 + w_0 + \alpha(-1 + \frac{2}{3}\epsilon_{\rm H} - w_0)]} - \frac{(1 - \frac{2}{3}\epsilon_{\rm H} + w_0)^2}{2(1 + w_0)(1 - w_0^2)} = 0\,. \tag{55}$$

The maximum value of the bulk viscosity $\bar{\zeta}$ required for sustaining an accelerated expansion $\bar{\zeta}_{\rm max}$ is obtained when considering $\epsilon_{\rm H} = 0$ and $\dot{\epsilon_{\rm H}} \approx 0$ and is found to be

$$\bar{\zeta}_{\rm max} = \frac{2(w_0^2 - 1)}{\sqrt{3}(1 - \alpha)(2w_0 - 1)} = \bar{\zeta}_{\rm dS}\,, \tag{56}$$

which exactly matches with Equation (48) and equals to the magnitude of the bulk viscosity that would give rise to an exact de-Sitter expansion.

As a consistency check, we can set $\bar{\zeta} = \bar{\zeta}_{\rm max}$ in Equation (50) to obtain

$$(w_{\rm eff} + 1)\Big[w_{\rm eff}(-\alpha - 3\alpha w_0 + w_0 + 1) + \alpha w_{\rm eff}^2 + (2\alpha + 1)w_0^2 + 2w_0^3 + (\alpha - 3)w_0 - 2\Big] = 0\,, \tag{57}$$

which has three real roots one of which is $w_{\rm eff} = -1$. The other two roots are

$$w_{\rm eff} = \frac{-\alpha + (w_0 + 1)\sqrt{\alpha^2 + \alpha(6 - 8w_0) + 1} - 3\alpha w_0 + w_0 + 1}{2\alpha}\,, \tag{58}$$

and

$$w_{\text{eff}} = \frac{\alpha + (w_0+1)\sqrt{\alpha^2+\alpha(6-8w_0)+1} + 3\alpha w_0 - w_0 - 1}{2\alpha}, \tag{59}$$

and are not relevant here, since for Equation (58) $w_{\text{eff}} < -1$ in the range $0 < \alpha < 1$, while for Equation (59) $w_{\text{eff}} > 1$ in the range $-0.5 < w_0 < 1$ and for all values of α. As a result, $w_{\text{eff}} = -1$ is the only possible solution of Equation (50) during the exact de-Sitter phase when $\bar{\zeta} = \bar{\zeta}_{\max}$.

In order to represent the effects of the variations of w_0, $\bar{\zeta}$ and α on the solution obtained during the quasi de-Sitter inflationary period, we present in Figure 5 with colour maps the values of $-\dot{H}/H^2$ obtained when varying either $\bar{\zeta}$ or α while keeping one of these quantities fixed. In each case, we considered the admissible ranges of the parameters $\alpha, \bar{\zeta}$, and w_0 and found that quasi de-Sitter solutions exist over the entire interval $-1 < w_0 < 1$ of w_0. However, the top panels of Figure 5 also show that significant deviations from the quasi de-Sitter evolutions can take place if $\bar{\zeta}$ is kept small. More specifically, while for large negative values of w_0 (i.e., $w_0 \lesssim -0.81$), a de-Sitter evolution is possible even for large values of the bulk viscosity, as the EOS parameter increases, a substantial amount of bulk viscosity is necessary to ensure a de-Sitter evolution. In particular, setting as a critical value $-\dot{H}/H^2 = 0.00016$, we found the a de-Sitter evolution is possible only for $\bar{\zeta} \gtrsim 10^{-4}$ (see left panel of Figure 5). Interestingly, for $w_0 \gtrsim -0.25$, the amount of bulk viscosity needed to ensure a de-Sitter evolution is essentially constant a corresponds to $\bar{\zeta} \sim 2 \times 10^{-3}$. Stated differently, while a quasi de-Sitter evolution is possible for any value of w_0, if the bulk viscosity is large, the latter is small, only a delicate balance is required between w_0 and $\bar{\zeta}$. Note also that the considerations made above depend – although not sensitively – on the value of α, as can be appreciated when comparing the left and right top panels of Figure 5. More specifically, the regions of quasi de-Sitter evolution tend to be slightly reduced as α is increased.

The lower-left panel of Figure 5, on the other hand, shows that the impact of the α parameter is small so long as $w_0 \sim -1$ and the deviation from the quasi de-Sitter phase remains small over the entire range of α. For instance, a slight deviation from the quasi de-Sitter phase takes place only for the bulk viscosity coefficient around $\bar{\zeta} \lesssim 10^{-3}$, with the deviation becoming comparatively larger as α becomes close to unity. However, for bulk viscosities $\bar{\zeta} \gtrsim 10^{-3}$, a quasi de-Sitter phase always exists independently of the value of α. Finally, the lower-right panel of Figure 5 reports the deviation from the exact de-Sitter phase when considering a fixed bulk viscosity coefficient $\bar{\zeta} = 0.001$ and varying the two coefficients w_0 and α. In this case, it is clear that the quasi de-Sitter expansion takes place for all values of w_0 so long as $\alpha \lesssim 0.9$. However, when $\alpha \gtrsim 0.9$, deviations from the de-Sitter phasebecome large for most of the range of w_0 and, in particular, if $w_0 \geq -0.75$.

Given that w_{eff} does not remain constant during the quasi de-Sitter inflation, but increases till reaching a new stationary value (see Figure 3), it is reasonable to define a timescale τ that characterises how rapid the transition is from the exact de-Sitter phase to the new quasi de-Sitter inflation. We use the behaviour shown in Figure 3 to express the evolution of w_{eff} as

$$\tilde{w}_{\text{eff}}(x) = -1 + \frac{A}{1 + Be^{-x/\tau}}, \tag{60}$$

where $x := \log_{10} t$ and the dimensionless constants A, B depend on the bulk viscosity. Using a best-fit approach and the numerical results presented in Figure 3, it is possible to compute both τ and A, B. Unsurprisingly, and just like w_{eff}, also τ depends on $\bar{\zeta}$, α, and w_0, so that, for α and w_0 fixed, τ simply decreases as $\bar{\zeta}$ is increased up to $\bar{\zeta}_{\text{dS}}$. Indeed, for $w_0 \to -1$, a considerable time is needed for w_{eff} to deviate from the exact de-Sitter phase (see the bottom-left panel in Figure 3). This behaviour of w_{eff} is reflected in τ as well, which increases only very slowly even if $\bar{\zeta} \ll \bar{\zeta}_{\text{dS}}$.

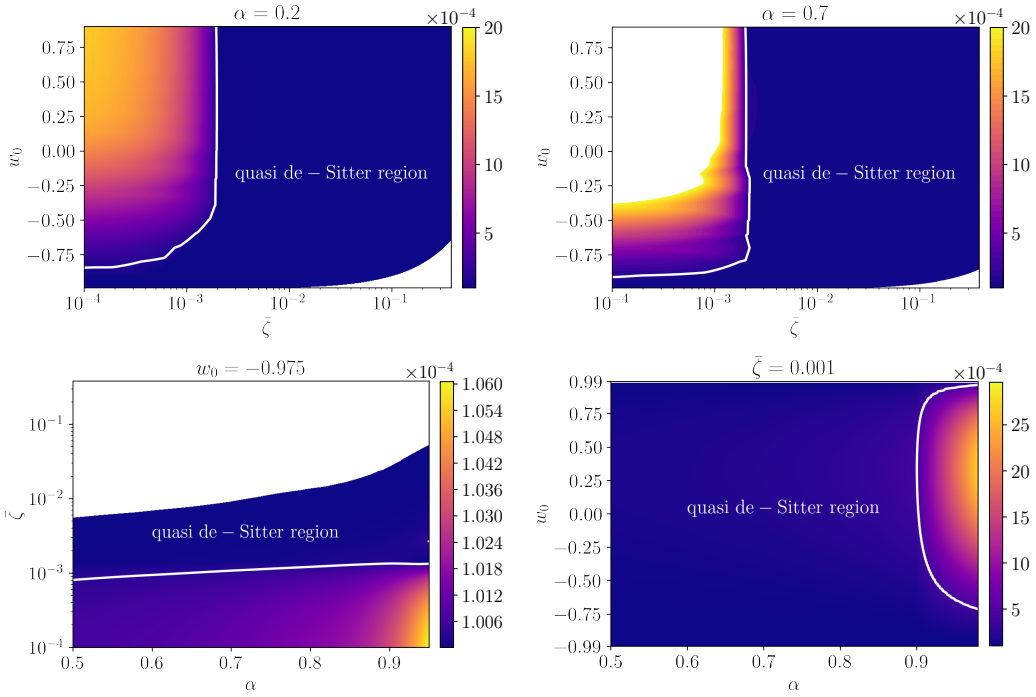


Figure 5. Colour maps of the measure of the deviation from the quasi de-Sitter condition via the quantity $-\dot{H}/H^2$ ($-\dot{H}/H^2 \ll 1$ corresponding to a quasi de-Sitter expansion; the white line marks the critical value $-\dot{H}/H^2 = 0.00016$. The maps are obtained by solving Equation (49) for different magnitudes of the parameters $\bar{\zeta}$, w_0, and α, with the first row referring to scenarios in which $\alpha = 0.2$ (left) and $\alpha = 0.7$ (right). The bottom row is the same as the top one but when considering constant values of $w_0 = -0.975$ (left) and $\bar{\zeta} = 0.001$ (right), respectively.

4.2. Estimation of the Model Parameters

We next turn our attention to estimating the allowed values of these parameters by using constraints from the primordial power spectrum generated by curvature perturbations during the inflationary period. Such an analysis will not only enable us to estimate realistic values of the bulk-viscosity coefficient and characteristic timescales, but will also allow us to examine the observational viability of the quasi de-Sitter model of inflation supported by bulk viscosity. A straightforward procedure in this direction involves the evaluation of the inflationary variables, e.g., the spectral index of scalar curvature perturbations n_s and the tensor-to-scalar ratio r, in terms of the model parameters $\bar{\zeta}$, α, and w_0, as well as the comparison with the experimentally observed values of n_s and r. A possible way to do this is to construct the fluid description of inflation in the context of the generalised causal theory following the procedure suggested by Bamba and Odintsov [50], who introduced an EOS which included the bulk viscosity, but did not consider a causal theory of hydrodynamics.

Typically, the inflationary variables are expressed in terms of slow-roll parameters which in turn are represented in terms of the potential of a scalar field driving inflation and their respective derivatives. The equivalence between the fluid description of inflation and the scalar-field description of inflation can be established provided the EOS of the bulk viscous fluid and that of the equation of the state followed by the pressure and energy density of a scalar field are the same [50] (A similar strategy can be found in [51] which addresses bulk viscosity driven inflation using MIS theory by taking into account of particle production.). Hence, we will first elucidate the equivalence between the two descriptions.

We recall that the Friedmann equations when expressed in terms of the e-folding number N are given by

$$3(H(N))^2 = e(N), \tag{61}$$
$$2H(N)H'(N) + 3(H(N))^2 = -p_{\text{eff}}(N), \tag{62}$$

where N is defined as $N := \int_{t_i}^{t_f} H dt$, so that the Hubble function becomes $H = H(N)$. Here, t_i and t_f are respectively the times corresponding to the beginning and end of inflation, while the index $'$ is used to indicate the derivative with respect to N. From Equation (61) and Equation (62), we then obtain

$$\frac{1}{2}[e(N) - p_{\text{eff}}(N)] = H'(N)H(N) + 3(H(N))^2. \tag{63}$$

On the other hand, in the single scalar-field description of inflation, the action is given by

$$S := \int d^4x \sqrt{-g}\left(\frac{R}{2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right), \tag{64}$$

where $\phi = \phi(t)$ is also known as the inflaton field and $V(\phi)$ is the inflaton potential and $\sqrt{-g}$ is the determinant of the metric. The components of the Einstein equations are

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) = e_\phi, \tag{65}$$
$$-(2\dot{H} + 3H^2) = \frac{1}{2}\dot{\phi}^2 - V(\phi) = p_\phi, \tag{66}$$

where e_ϕ and p_ϕ are the energy density and the pressure of the scalar field, respectively. Combining these two equations one obtains that

$$\dot{\phi}^2 = -2\dot{H}. \tag{67}$$

The equivalence between two descriptions of inflation is obtained once the scalar field ϕ and the cosmological time t are both rescaled by an auxiliary scalar field Φ, thus implying $\phi = \phi(\Phi)$ and $t = t(\Phi)$, and when Φ identified with the e-folding number N. Under these conditions, Equation (65) and Equation (66) can be rewritten as,

$$3H^2(N) = \frac{1}{2}H^2(N)\left(\frac{d\phi}{d\Phi}\right)^2\Bigg|_{\Phi=N} + V(\phi(\Phi))|_{\Phi=N}, \tag{68}$$
$$-2H(N)H'(N) = \frac{1}{2}H^2(N)\left(\frac{d\phi}{d\Phi}\right)^2\Bigg|_{\Phi=N} - V(\phi(\Phi))|_{\Phi=N}. \tag{69}$$

Using Equations (65)–(69), the scalar-field potential is given by

$$V(N) = H'(N)H(N) + 3H^2(N) = \frac{1}{2}(e_\phi - p_\phi). \tag{70}$$

Since the bulk viscous fluid and the scalar field possess the same EOS, together with Equation (63) the potential can be expressed as

$$V(N) = H'(N)H(N) + 3H^2(N) = \frac{1}{2}[e(N) - p_{\text{eff}}(N)]. \tag{71}$$

and Equation (50) can be re-written in the following way in terms of N, namely,

$$\tau_{\Pi}\frac{d\Pi(N)}{dN}\left(1+k^2\frac{\tau_{\Pi}}{\zeta(N)}\Pi(N)\right)+\frac{\Pi(N)}{H(N)}\left(1+3k^2H(N)\tau_{\Pi}\right)$$
$$=-3\zeta(N)-\frac{1}{2}\tau_{\Pi}\Pi(N)\left[3+\frac{1}{\tau_{\Pi}}\frac{d\tau_{\Pi}}{dN}-\frac{1}{\zeta}\frac{d\zeta}{dN}-\frac{1}{T}\frac{dT}{dN}\right]\left(1+k^2\frac{\tau_{\Pi}}{\zeta(N)}\Pi(N)\right), \quad (72)$$

where the bulk viscosity is expressed as

$$\Pi(N) \;=\; -2H'(N)H(N)+3(1+w_0)H^2(N)\,. \quad (73)$$

Similarly, the relaxation time coefficient, the temperature and the bulk viscosity can also be expressed in terms of the e-folding number. Using the potential, it is then possible to determine the slow-roll parameters $\epsilon_{\rm V}(N)$ and $\eta_{\rm V}(N)$ purely in terms of $H(N)$, $H'(N)$ and $H''(N)$ as [52]

$$\epsilon_{\rm V}(N)=-\frac{H}{4H'}\left[\frac{6H'/H+H''/H+(H'/H)^2}{3+H'/H}\right]^2, \quad (74)$$

and

$$\eta_{\rm V}(N)=-\frac{1}{2}\left[\frac{9H'/H+3H''/H+\frac{1}{2}(H'/H)^2-\frac{1}{2}(H''/H')^2+3(H''/H')+H'''/H'}{(3+H'/H)}\right]. \quad (75)$$

Typically, the energy scale of inflation and the e-folding number are respectively taken to be $H_{in}\simeq 10^{-3}\,M_{\rm Pl}$, where $M_{\rm Pl}$ is the Planck mass, and $N\simeq 50$. The energy scale of inflation becomes $H_{in}=10^{-3}$. Solving Equation (72) numerically under the condition that $-H'(N)\ll H(N)$ (or, equivalently, that $\dot{\phi}^2\ll V$ in a scalar-field description), we have found that $\epsilon_{\rm V}(N)$ and $\eta_{\rm V}(N)$ remain constant and smaller than unity when N is taken to vary in the interval $N\in[0,50]$ and α, w_0 and $\bar{\zeta}$ are varied appropriately (see the discussion in Sec. 4.3). This suggests that slow-roll conditions are indeed obeyed during the quasi de-Sitter inflationary phase described by a generalised causal theory. Consequently, the inflationary observables n_s and r that characterise the perturbation spectra can then be related to the potential slow-roll parameters $\epsilon_{\rm V}(N)$ and $\eta_{\rm V}(N)$ as follows (see, e.g., [53])

$$n_s \;=\; 1-6\epsilon_{\rm V}+2\eta_{\rm V}\,, \quad (76)$$
$$r \;=\; 16\epsilon_{\rm V}\,. \quad (77)$$

We note that, while for exact de-Sitter expansion, the de-Sitter symmetries predict $n_s=1$ and $r=0$, observations from the Cosmic Microwave Background (CMB) strongly suggest a small but non-zero deviation of n_s, thus suggesting a scale dependence of the density fluctuations and indicating a nearly de-Sitter expansion, as well as a small, but nonzero value for r.

The behaviour of n_s and r as predicted by our model is illustrated in Figure 6, where we evaluated these quantities using Eqs. (76) and (77) for different values of w_0, α, and $\bar{\zeta}$ when they are varied within the corresponding allowed ranges. Note that the results show an essentially linear behaviour and that each point of a specific line corresponds to a unique value of $\bar{\zeta}$ for fixed values of α and w_0, increasing from left to right as n_s is increased. Note also that, while each individual line represents a unique solution for fixed values of α and w_0, the actual variations in the spectral index are rather small, and this explains the expanded scale shown in Figure 6. In turn, this implies that the effective space of allowed values for n_s and r is essentially one-dimensional in our model and that, in the limit $n_s\to 1$, the tensor-to-scalar ratio is also vanishingly small, i.e., $r\to 0$.

4.3. Comparison with Observations

In order to examine the compatibility of our model with current observational data, we considered three datasets obtained on observations from the Planck satellite and from other, non-Planck-based observations. More specifically, we considered the observational constraints on n_s and r based on the Planck2018+ Keck-Array BK15 ("Planck+BK15") data sets [54–57], from the Atacama Cosmology Telescope DR4 likelihood combined with the WMAP satellite data set ("ACTPol+WMAP") [58,59], and from the South-Pole Telescope polarization measurements ("SPT3G+WMAP") [58,60]. The comparison of our model with these three observational datasets was carried out by considering two reference viscous cosmology datasets, namely $w_0 = -0.98, \alpha = 0.1$ and $w_0 = -0.99, \alpha = 0.1$, which are shown in the three panels of Figure 7 (given the small variance, they appear as solid straight lines in the panels). Also reported are the constraints from the various datasets, namely ACTPol+WMAP, SPT3G+WMAP, and Planck+BK15, from left to right. The comparison in Figure 7 shows that the proposed model of inflation is clearly compatible with both the ACTPol+WMAP and the SPT3G+WMAP results, since the straight lines generated by the two reference viscous cosmology datasets pass through the shaded regions in the (n_s, r) plane.

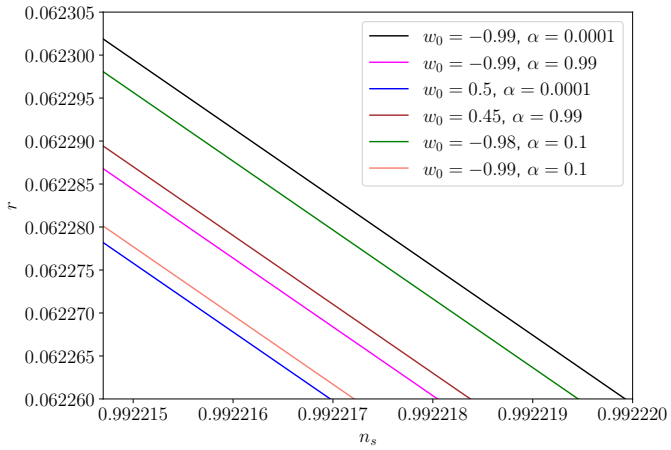


Figure 6. Behaviour of the spectral index n_s and of the tensor-to-scalar ratio r as predicted by our model. These quantities are evaluated using Equations (76)–(77) for different values of w_0, α and $\bar{\zeta}$ when they are varied within the corresponding allowed ranges.

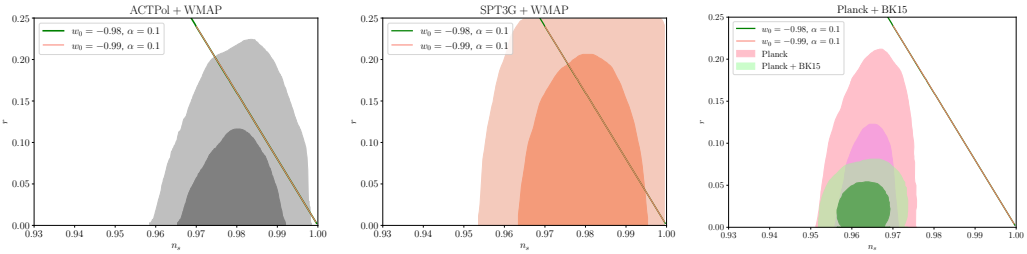


Figure 7. Comparison of bulk-viscous model of inflation (straight lines) with the marginalised contours of Planck+BK15, ACTPol+WMAP and SPT3G+WMAP results. The contours are obtained for n_s, r using marginalised joint 68% and 95% confidence-level regions from the observational data.

Furthermore, we note that when $w_0 \to -1$, it is possible (and straightforward) to select values of $\bar{\zeta}$ and α within their allowed ranges of variation such that n_s and r match the observational constraints. On the other hand, when $w_0 > 0$, the bulk-viscosity coefficient needed is very large (typically $\mathcal{O}(1) - \mathcal{O}(10)$) in order to produce compatible values of n_s and r (When considering $w_0 > 0.5$, the values of n_s and r obtained easily fall out of the range allowed by observations. For example, for $w_0 = 0.52$, we obtain $n_s = 0.9061$ and $r = 0.7455$.). Note also that since Figure 7 shows that the observations from ACTPol+WMAP predict

a lower value of r when compared to the the SPT3G+WMAP data, the magnitude of the bulk-viscosity coefficient required to produce compatible values of n_s and r will be different for the observational datasets. For instance, with $w_0 = -0.98, \alpha = 0.1$, a bulk viscosity coefficient $\bar{\zeta} \sim 7 \times 10^{-3}$ ($\bar{\zeta} \sim 6 \times 10^{-3}$) is necessary to produce values of n_s and r compatible with the constraints from the ACTPol+WMAP (SPT3G+WMAP) observations.

Finally, it should be remarked that, although our model is supported by the ACTPol+WMAP and SPT3G+WMAP results, it is incompatible with the current Planck+BK15 datasets (third panel of Figure 7) for all the values of α and w_0 that are allowed. In particular, the Planck+BK15 observations seem to predict values that are systematically smaller than that obtained in our model for all of the values that the spectral index is allowed to take.

It is worthwhile to note that our results are indeed in agreement with those presented in Refs. [61,62], where a higher value of n_s was also found and, thus, in accord with the constraints from ACTPol+WMAP and SPT3G+WMAP. In this sense, unless the differences are due to yet-undetected systematic differences among the observations, the bulk viscosity model of inflation suggests a possible and slight modification of the ΛCDM model scenario [63]. On the other hand, the discrepancy in the value of n_s may also be related to the simplified phenomenological model adopted in this work for incorporating out-of-equilibrium effects during the inflationary period of the early universe. The inclusion of non-equilibrium effects when gradients are large, as performed, for example, in [64], may reduce the discrepancy with the Planck+BK15 data, and we will address this possibility in future work.

A few remarks are worth making before concluding. First, several works, for example [34,41,50,65], have discussed the employment of dissipative hydrodynamics with a bulk viscosity to model the inflationary phase of the universe's evolution (the latter is the only phase of the cosmological evolution we are interested in). In this respect, our approach is not novel and has been employed by several authors and validated in the literature. Obviously, such a description is valid if the local mean-free-path of the particle collisions is much smaller than the scale of the system (this is known as the Knudsen number criterion [40]), which is obviously one of our background assumptions. In addition, hydrodynamics has been shown to offer a correct quantitative description of systems that are not close to local equilibrium, for instance in heavy-ion collision physics, e.g., [66,67], or even when the mean-free-path is comparable with the scale of the system, e.g., [68,69]. All of these examples indicate that, as an effective theory, hydrodynamics has a validity that is far larger than what one would expect by simply considering the Knudsen number criterion. Second, while our approach would not be able to describe a post-inflationary phase such as reheating, this is not what we are dealing with in our manuscript, which concentrates only on the evolution of the universe where a hydrodynamical description is possible and, indeed, has been employed by numerous authors before us. Finally, while it would be extremely interesting to match the end of the inflationary phase with one where the particle density is so small so as to require the use of a kinetic theory approach, this is well beyond the scope of our paper and may be addressed in future work.

5. Conclusions and Outlook

There is a widespread consensus that the early universe underwent a phase of quasi de-Sitter expansion, and this is normally modelled by means of a suitable scalar field and of an associated potential. We have presented an alternative modelling of the inflationary expansion that is not based on a scalar field, but that involves, instead, a bulk viscous cosmological fluid to sustain a quasi de-Sitter expansion in the early universe. Our model is set in the framework of the generalised causal theory of hydrodynamics, and by taking into account out-of-equilibrium effects, it reveals that, if the cosmological fluid possessed a nonzero bulk viscosity, then a quasi de-Sitter inflation would arise naturally in this scenario without invoking additional fields and without assuming an EOS of the type $p = w_0\, e = -e$ relating the equilibrium pressure and the energy density. Hence, a quasi de-Sitter inflation can be realised purely with the help of the bulk viscosity and by taking into consideration

the non-equilibrium effects that inevitably arise due to the violation of the SEC during the accelerated expansion of the early universe.

The model of inflation presented here has several interesting features. First, while the bulk viscosity provides the necessary negative pressure required for the accelerated expansion, as a matter of course, the effective EOS—expressed in terms of the ratio between the effective pressure and the energy density, $w_{\rm eff}$, and that embodies the bulk viscous and non-equilibrium effects—becomes a time-dependent function describing the time evolution of the inflationary phase. Second, the evolution of $w_{\rm eff}$ follows a rather simple and unique behaviour that reflects a smooth transition from the exact de-Sitter phase and to a subsequent levelling off to a constant value at later times. While this behaviour is obtained by numerically solving the generalized momentum-conservation equation, the functional behaviour of $w_{\rm eff}$ can be assimilated as a simple Logistic function, and hence, the associated timescale for the transition from exact de-Sitter to the new quasi de-Sitter phase can be estimated accurately. This timescale is a simple function of the three parameters of the systems ($\bar{\zeta}$, α, and w_0), and unsurprisingly, it decreases as the magnitude of the bulk viscosity coefficient is increased up to the exact de-Sitter value. Third, in contrast with the standard inflationary scenario where $-1 < w_0 < 1/2$, our bulk viscous model allows for a larger range of values, namely $-1 < w_0 < 1$, and is well-behaved for $w_0 = 0$. Finally, the equivalence between the non-perfect fluid description of inflation presented here with the scalar field theory of inflation is maintained also when considering the observational constraints, in particular when expressing the constraints coming from Planck2018 [12], Keck-Array BK15, ACTPol+WMAP, and SPT3G+WMAP [61], in terms of the standard inflationary variables like the spectral index of scalar density perturbations n_s and the tensor-to-scalar ratio r. We find that these constraints can be easily satisfied by suitable choices of $\bar{\zeta}$, α, and w_0 for the ACTPol+WMAP and SPT3G+WMAP datasets, while tensions appear in the case of the Planck+BK15 datasets.

The work presented here can be extended and improved in a number of directions, for example by investigating the possible origin of bulk viscosity in the early universe and relating it to the mechanism of particle production [25]. Similarly, it would be interesting to investigate the precise mechanism giving rise to the exit from the quasi de-Sitter inflationary phase, and this could lead to conducive conditions for reheating. Similarly, the study of warm inflation recently suggested in Ref. [70] could also be re-analysed when framed in the presence of the bulk viscosity and non-equilibrium effects and, finally, before a kinetic theory approach needs to be employed when the the hydrodynamic description used ceases to be valid. Some steps in this direction have already been taken, and the reheating process of the universe in the presence of the bulk viscosity has been addressed by several authors, for example within the warm inflation scenario [51,71,72]. Following their approach, it would be interesting to study the reheating process in the contest of the generalised MIS theory. We leave all of these investigations to future works.

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