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On Graphical Symmetric Spaces, Fixed-Point Theorems and the Existence of Positive Solution of Fractional Periodic Boundary Value Problems †

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† Dedicatedto the memory of Professor Hans-Peter A. Künzi. Professor Hans-Peter A. Künzi passed away in August 2020. Some of the ideas of this paper were discussed with him.

Abstract: The rationale of this work is to introduce the notion of graphical symmetric spaces and some fixed-point results are proved for $\mathcal{H}$-(ϑ, φ)-contractions in this setting. The idea of graphical symmetric spaces generalizes various spaces equipped with a function which characterizes the distance between two points of the space. Some topological properties of graphical symmetric spaces are discussed. Some fixed-point results for the mappings defined on graphical symmetric spaces are proved. The fixed-point results of this paper generalize and extend several fixed-point results in this new setting. The main results of this paper are applied to obtain the positive solutions of fractional periodic boundary value problems.

Keywords: graphical symmetric space; contraction; fixed point; periodic point; Caputo's derivative

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1. Introduction

In 1905, Fréchet [1] introduced the study of spaces equipped with a distance function by assigning a nonnegative value to each pair of arbitrary objects of a nonempty set. Hausdorff later named these spaces as metric spaces. In such spaces, the distance between two objects is characterized by a distance function or metric function which, apart from nonnegativity, also possess the identity of indiscernibles, the properties of symmetry and the triangular inequality. There are several generalizations of metric spaces and most of it are obtained by weakening or extending the above properties (see, for example, [2–7] and the references therein). In view of fixed-point theorems, the triangular inequality associated with metric function plays a crucial role when proving an iterative sequence to be convergent; thus, several authors have tried to find spaces in which the triangular inequality was introduced in a weaker or extended form in such a way that the existence of a fixed point still remained demonstrable. In symmetric spaces (see [5,7]) the triangular inequality of a metric function is dropped and to establish various properties and existence of a fixed point of contractive type mappings, various replacements of the triangular inequality are used.

Inspired by Shukla and Künzi [8] and Shukla et al. [6], in this paper we introduce a graphical structure on the set associated with the space and introduce the notion of graphical symmetric spaces. This notion extends, generalizes and improves several known generalized forms of metric spaces. Some fixed-point results in this new setting are also proved which generalize and extend several fixed-point results. For illustration and justification of concepts and claims, several examples are provided. An application of our main results in finding the positive solution of fractional periodic boundary value problems involving Caputo's fractional derivatives is presented.

2. Graphical Symmetric Spaces

Some basic notions and concepts are stated which are needed throughout the paper, some of them were initiated by Jachmyski [9], Shukla et al. [6] and Shukla and Künzi [8].

Let Υ be a nonempty set and consider a directed graph $\mathcal{Q}$, without parallel edges, such that the set $V(\mathcal{Q})$ of its vertices coincides with Υ, and the set of its edges $E(\mathcal{Q}) \subseteq \Upsilon \times \Upsilon$. Then, we say that Υ is endowed with the graph $\mathcal{Q} = (V(\mathcal{Q}), E(\mathcal{Q}))$. The conversion of the graph $\mathcal{Q}$ is denoted by $\mathcal{Q}^{-1}$, and it is defined by:

$$V(\mathcal{Q}^{-1}) = V(\mathcal{Q}) \text{ and } E(\mathcal{Q}^{-1}) = \{(\flat, v) \in \Upsilon \times \Upsilon \colon (v, \flat) \in E(\mathcal{Q})\}.$$

By $\widetilde{\mathcal{Q}}$, we denote the undirected graph obtained from $\mathcal{Q}$ by including all the edges of $\mathcal{Q}^{-1}$. More precisely, we define $V(\widetilde{\mathcal{Q}}) = V(\mathcal{Q})$ and $E(\widetilde{\mathcal{Q}}) = E(\mathcal{Q}) \cup E(\mathcal{Q}^{-1})$. If $\flat$ and v are vertices in a graph $\mathcal{Q}$, then a path in $\mathcal{Q}$ from $\flat$ to v of length $l \in \mathbb{N}$ is a sequence $\{\flat_i\}_{i=0}^{l}$ of $l+1$ vertices such that $\flat_0 = \flat, \flat_l = v$ and $(\flat_{i-1}, \flat_i) \in E(\mathcal{Q})$ for $i = 1, 2, \ldots, l$. A graph $\mathcal{Q}$ is called connected if there is a path between any two vertices. Moreover, two vertices $\flat$ and v of a directed graph are connected if there is a path from $\flat$ to v and a path from v to $\flat$. $\mathcal{Q}$ is weakly connected if, when treating all of its edges as being undirected, there is a path from every vertex to every other vertex. More precisely, $\mathcal{Q}$ is weakly connected if $\widetilde{\mathcal{Q}}$ is connected.

For a subgraph $\mathcal{H}$ of $\mathcal{Q}$, we define a set $\mathcal{C}_{\mathcal{H}}$ by:

$$\mathcal{C}_{\mathcal{H}} = \{(\flat, v) \in \Upsilon \times \Upsilon \colon \text{ there is a directed path from } \flat \text{ to } v \text{ in } \mathcal{H}\}.$$

Consider the relation $P = \{(\flat, v) \in \Upsilon \times \Upsilon \colon (\flat, v) \in \mathcal{C}_{\mathcal{H}}\}$. Then, P is called the relation of connectedness, and by the definition of a path in $\mathcal{H}$, it is clear that this relation is a transitive relation. A sequence $\{\flat_n\}$ in Υ is said to be $\mathcal{H}$-termwise-connected if $(\flat_n, \flat_{n+1}) \in \mathcal{C}_{\mathcal{H}}$ for all $n \in \mathbb{N}$. A vertex $\flat \in \Upsilon$ is called an isolated point of graph $\mathcal{H}$ if neither $(\flat, v)$ nor $(v, \flat)$ is in $\mathcal{C}_{\mathcal{H}}$ for all $v \in \Upsilon$. For a subset A of Υ, we denote $\Delta_A = \{(a, a) \colon a \in A\}$.

Throughout this paper, we assume that the graphs under consideration are directed, with nonempty sets of vertices and edges.

Definition 1. *Let Υ be a nonempty set endowed with a graph $\mathcal{Q}$ and $\sigma_{\mathcal{Q}} : \Upsilon \times \Upsilon \to \mathbb{R}$ be a function satisfying the following conditions: for all $\flat, v \in \Upsilon$ with $(\flat, v) \in \mathcal{C}_{\mathcal{Q}}$*

(GS1) $\sigma_{\mathcal{Q}}(\flat, v) \geq 0$;
(GS2) $\sigma_{\mathcal{Q}}(\flat, v) = 0$ *implies* $\flat = v$;
(GS3) $\sigma_{\mathcal{Q}}(\flat, v) = \sigma_{\mathcal{Q}}(v, \flat)$.

Then, the mapping $\sigma_{\mathcal{Q}}$ is called a graphical symmetry on Υ, and the pair $(\Upsilon, \sigma_{\mathcal{Q}})$ is called a graphical symmetric space.

Remark 1. *The following are examples of some standard spaces which are a particular type of graphical symmetric spaces:*

- *If Υ is a nonempty set endowed with a graph $\mathcal{Q}$ such that $(\Upsilon, d_{\mathcal{Q}})$ is a graphical metric space (see, [6]), then $(\Upsilon, \sigma_{\mathcal{Q}})$ is a graphical symmetric space with the same graph $\mathcal{Q}$, where $\sigma_{\mathcal{Q}}(\flat, v) = d_{\mathcal{Q}}(\flat, v)$ for all $\flat, v \in \Upsilon$. Hence, every graphical metric space (thus every metric space) is also a graphical symmetric space.*
- *Every b-metric space (see, [3]) (Υ, b) is a graphical symmetric space $(\Upsilon, \sigma_{\mathcal{Q}})$ with the graph $\mathcal{Q}$ such that $E(\mathcal{Q}) = \Upsilon \times \Upsilon$, where $\sigma_{\mathcal{Q}}(\flat, v) = b(\flat, v)$ for all $\flat, v \in \Upsilon$. Similarly, one can see that every functional weighted metric space (see, [10]) is a graphical symmetric space.*
- *Every metriclike space (Υ, σ) (see, [2]) is a graphical symmetric space $(\Upsilon, \sigma_{\mathcal{Q}})$ with the graph $\mathcal{Q}$ such that $E(\mathcal{Q}) = \Upsilon \times \Upsilon$, where $\sigma_{\mathcal{Q}}(\flat, v) = \sigma(\flat, v)$ for all $\flat, v \in \Upsilon$. Hence, every partial metric space (see, [4]) is a graphical symmetric space.*
- *Every symmetric space (Υ, d) (see, [5,7]) is a graphical symmetric space $(\Upsilon, \sigma_{\mathcal{Q}})$ with the graph $\mathcal{Q}$ such that $E(\mathcal{Q}) = \Upsilon \times \Upsilon$, where $\sigma_{\mathcal{Q}}(\flat, v) = d(\flat, v)$ for all $\flat, v \in \Upsilon$.*

Example 1. *Let* $\Upsilon = \mathbb{R}$ *be endowed with the graph* $\mathcal{Q}$*, where* $E(\mathcal{Q}) = \{(\flat, \upsilon) \in \Upsilon \times \Upsilon \colon \flat = n\pi + \upsilon, n \in \mathbb{Z}\}$*. Then, the function* $\sigma_{\mathcal{Q}} \colon \Upsilon \times \Upsilon \to \mathbb{R}$ *is a graphical symmetry on* Υ*, where* $\sigma_{\mathcal{Q}}(\flat, \upsilon) = e^{-\flat\upsilon} + \sin(\flat - \upsilon)$ *for all* $\flat, \upsilon \in \Upsilon$*. Hence,* $(\Upsilon, \sigma_{\mathcal{Q}})$ *is a graphical symmetric space. Note that* $(\Upsilon, \sigma_{\mathcal{Q}})$ *does not fall under any class of spaces mentioned above.*

Example 2. *Let* $\Upsilon = [0, \infty)$*,* $\mathbb{Q}_{>0} = \mathbb{Q} \cap (0, \infty)$ *and* $\mathbb{Q}^c_{>0} = \Upsilon \setminus (\mathbb{Q}_{>0} \cup \{0\})$*. Let* Υ *be endowed with the graph* $\mathcal{Q}$*, where* $E(\mathcal{Q}) = \{(\flat, \upsilon) \in \Upsilon \times \Upsilon \colon (\flat, \upsilon) \in (\mathbb{Q}_{>0} \times \mathbb{Q}^c_{>0}) \cup (\mathbb{Q}^c_{>0} \times \mathbb{Q}_{>0})\}$*. Then, the function* $\sigma_{\mathcal{Q}} \colon \Upsilon \times \Upsilon \to \mathbb{R}$ *is a graphical symmetry on* Υ*, where*

$$\sigma_{\mathcal{Q}}(\flat, \upsilon) = \begin{cases} \flat\upsilon, & \text{if } (\flat, \upsilon) \in (\mathbb{Q}_{>0} \times \mathbb{Q}^c_{>0}) \cup (\mathbb{Q}^c_{>0} \times \mathbb{Q}_{>0}); \\ 2\flat + \upsilon, & \text{otherwise} \end{cases}$$

for all $\flat, \upsilon \in \Upsilon$*. Hence,* $(\Upsilon, \sigma_{\mathcal{Q}})$ *is a graphical symmetric space. Note that* $(\Upsilon, \sigma_{\mathcal{Q}})$ *does not fall under any class of spaces mentioned above.*

We next define some topological concepts in graphical symmetric spaces. As the graphs associated with the space are not necessarily undirected, we define the topology and concerned concepts in two ways.

Let $(\Upsilon, \sigma_{\mathcal{Q}})$ be a graphical symmetric space. For $\flat \in \Upsilon$ and $r > 0$, denote by $B^R_{\mathcal{Q}}(\flat, r)$ the right ball with center $\flat$ and radius r, where

$$B^R_{\mathcal{Q}}(\flat, r) = \{\upsilon \in \Upsilon \colon |\sigma_{\mathcal{Q}}(\flat, \upsilon) - \sigma_{\mathcal{Q}}(\flat, \flat)| < r, (\flat, \upsilon) \in \mathcal{C}_{\mathcal{Q}}\}.$$

Similarly, a left ball with center $\flat$ and radius r is denoted by $B^L_{\mathcal{Q}}(\flat, r)$, and

$$B^L_{\mathcal{Q}}(\flat, r) = \{\upsilon \in \Upsilon \colon |\sigma_{\mathcal{Q}}(\flat, \upsilon) - \sigma_{\mathcal{Q}}(\flat, \flat)| < r, (\upsilon, \flat) \in \mathcal{C}_{\mathcal{Q}}\}.$$

Remark 2. *It is obvious that if* $r_1 \leq r_2$*, then* $B^R_{\mathcal{Q}}(\flat, r_1) \subseteq B^R_{\mathcal{Q}}(\flat, r_2)$ *and* $B^L_{\mathcal{Q}}(\flat, r_1) \subseteq B^L_{\mathcal{Q}}(\flat, r_2)$ *.*

Consider the following two collections of subsets of Υ:

$$\tau_R = \{U \subset \Upsilon \colon \text{ for all } \flat \in U \text{ there exists } r > 0 \text{ such that } B^R_{\mathcal{Q}}(\flat, r) \subseteq U\}; \text{ and}$$

$$\tau_L = \{U \subset \Upsilon \colon \text{ for all } \flat \in U \text{ there exists } r > 0 \text{ such that } B^L_{\mathcal{Q}}(\flat, r) \subseteq U\}.$$

Remark 3. *We notice that a right ball (left ball) may be empty. In particular, if* $\flat \in \Upsilon$ *is an isolated point of* $\mathcal{Q}$*, then we have* $B^R_{\mathcal{Q}}(\flat, r) = B^L_{\mathcal{Q}}(\flat, r) = \emptyset$ *for all* $r > 0$*. In this case, the inclusion of any ball in a set becomes a trivial case. On the other hand, if* $\Delta_\Upsilon = \{(\flat, \flat) \colon \flat \in \Upsilon\} \subseteq E(\mathcal{Q})$*, then one can see that a right ball (left ball) is nonempty. In particular, if* $\Delta_\Upsilon \subseteq E(\mathcal{Q})$*, then each right ball (left ball) contains its center. Also, by using Remark 2, one can easily verify that the collections* τ_R *and* τ_L *are two topologies on* Υ *induced by* $\sigma_{\mathcal{Q}}$ *and called R-topology and L-topology, respectively. The members of* τ_R *and* τ_L *are called R-open and L-open sets, respectively. A subset* $A \subseteq \Upsilon$ *is called R-closed (respectively L-closed) if* $\Upsilon \setminus A$ *is R-open (respectively L-open).*

Remark 4. *In a graphical symmetric space, a right ball (left ball) is not necessarily an open set in the topology* τ_R *(*τ_L*), and this fact is independent of the graphical structure associated with the space, that is, it is true even if we take the associated graph as the universal graph. For instance, let* $f \colon (0, \infty) \to (0, \infty)$ *be a one-to-one function and* $\Upsilon = \{-1, 0, f(r) \colon r \in (0, \infty)\}$*. Let* Υ *be endowed with the graph* $\mathcal{Q}$*, where* $E(\mathcal{Q}) = \Upsilon \times \Upsilon$ *(the universal graph). Define* $\sigma_{\mathcal{Q}} \colon \Upsilon \times \Upsilon \to \mathbb{R}$ *by:*

$$\sigma_{\mathcal{Q}}(\flat, \flat) = 0, \sigma_{\mathcal{Q}}(\flat, \upsilon) = \sigma_{\mathcal{Q}}(\upsilon, \flat) \text{ for all } \flat, \upsilon \in \Upsilon;$$
$$\sigma_{\mathcal{Q}}(-1, 0) = \frac{1}{2}, \sigma_{\mathcal{Q}}(0, f(r)) = \frac{r}{2};$$
$$\sigma_{\mathcal{Q}}(-1, f(r)) = 1 \text{ for all } r \in (0, \infty).$$

Then, it is easy to see that $(\Upsilon, \sigma_{\mathcal{Q}})$ is a graphical symmetric space. Now, consider the right ball with center -1 and radius 1, that is, $B^{R}_{\mathcal{Q}}(-1,1)$. Then, since $|\sigma_{\mathcal{Q}}(-1,0) - \sigma_{\mathcal{Q}}(-1,-1)| = \frac{1}{2} < 1$ and $(-1,0) \in \mathcal{C}_{\mathcal{Q}}$, $0 \in B^{R}_{\mathcal{Q}}(-1,1)$. But note that for every $r > 0$, we have $f(r) \in B^{R}_{\mathcal{Q}}(0,r)$ and $f(r) \notin B^{R}_{\mathcal{Q}}(-1,1)$. Hence, $B^{R}_{\mathcal{Q}}(0,r) \nsubseteq B^{R}_{\mathcal{Q}}(-1,1)$ for all $r > 0$. This shows that $B^{R}_{\mathcal{Q}}(-1,1)$ is not an R-open set. Similarly, it can be shown that $B^{L}_{\mathcal{Q}}(-1,1)$ is not an L-open set.

Remark 5. *In general, the topologies τ_R and τ_L are not $\top_1$. For instance, let $\Upsilon = \mathbb{R}$ be endowed with the graph $\mathcal{Q}$, where $E(\mathcal{Q}) = \Upsilon \times \Upsilon$. Define $\sigma_{\mathcal{Q}} \colon \Upsilon \times \Upsilon \to \mathbb{R}$ by:*

$$\sigma_{\mathcal{Q}}(\flat, \upsilon) = \sigma_{\mathcal{Q}}(\upsilon, \flat) \text{ for all } \flat, \upsilon \in \Upsilon;$$
$$\sigma_{\mathcal{Q}}(1,2) = \sigma_{\mathcal{Q}}(1,1) = 1;$$
$$\sigma_{\mathcal{Q}}(\flat, \upsilon) = 2 \text{ in all other cases.}$$

Then, it is easy to see that $(\Upsilon, \sigma_{\mathcal{Q}})$ is a graphical symmetric space. Note that the singleton $\{2\}$ is neither R-closed nor L-closed, because $1 \in \Upsilon \setminus \{2\}$ and $2 \in B^{R}_{\mathcal{Q}}(1,r)$ ($2 \in B^{L}_{\mathcal{Q}}(1,r)$) for all $r > 0$.

Theorem 1. *Suppose at least one of the following conditions is satisfied:*

(A) $\sigma_{\mathcal{Q}}(\flat, \flat) = 0$ *for all* $\flat \in \Upsilon$;
(B) $\sigma_{\mathcal{Q}}(\flat, \upsilon) \neq \sigma_{\mathcal{Q}}(\upsilon, \upsilon)$ *for all* $\flat, \upsilon \in \Upsilon$ *with* $\flat \neq \upsilon$.

Then, the topologies τ_R and τ_L are $\top_1$.

Proof. Let $\flat \in \Upsilon$. We show that the singleton $\{\flat\}$ is R-closed, that is, the set $\Upsilon \setminus \{\flat\}$ is an R-open set. If $\Upsilon \setminus \{\flat\} = \emptyset$, we are done. If $\upsilon \in \Upsilon \setminus \{\flat\}$, we show that there exists $r > 0$ such that $\flat \notin B^{R}_{\mathcal{Q}}(\upsilon, r)$, that is, $B^{R}_{\mathcal{Q}}(\upsilon, r) \subseteq \Upsilon \setminus \{\flat\}$. We consider the following cases:

(I) If $(\upsilon, \flat) \notin \mathcal{C}_{\mathcal{Q}}$, we have $\flat \notin B^{R}_{\mathcal{Q}}(\upsilon, r)$ for all $r > 0$.
(II) If $(\upsilon, \flat) \in \mathcal{C}_{\mathcal{Q}}$, by (GS1) and (GS2) we have $\sigma_{\mathcal{Q}}(\upsilon, \flat) = \sigma_{\mathcal{Q}}(\flat, \upsilon) > 0$ (since $\upsilon \neq \flat$). Let $r = |\sigma_{\mathcal{Q}}(\flat, \upsilon) - \sigma_{\mathcal{Q}}(\upsilon, \upsilon)|$; then, since at least one of the conditions (A) and (B) is satisfied, we must have $r > 0$ and $B^{R}_{\mathcal{Q}}(\upsilon, r) \subseteq \Upsilon \setminus \{\flat\}$.

Therefore, in each case, we can find $r > 0$ such that $B^{R}_{\mathcal{Q}}(\upsilon, r) \subseteq \Upsilon \setminus \{\flat\}$; thus, $\Upsilon \setminus \{\flat\}$ is R-open. Similarly, one can show that $\Upsilon \setminus \{\flat\}$ is L-open. □

Remark 6. *It is well known that the R-topology and L-topology induced by a graphical metric are not necessarily $\top_2$ (Hausdorff) (see, Theorem 3.4 of [8]). Also, every graphical metric space is a graphical symmetric space (with the same graph) and it is easy to see that the R-topology (L-topology) induced by a graphical metric is the same as the R-topology (L-topology) induced by the corresponding graphical symmetry. Hence, we can say that the topologies induced by a graphical symmetry are not necessarily $\top_2$. Similarly, in a graphical symmetric space, the R-open and L-open balls with the same center and radius are not necessarily the same, as shown by Example 3.1 of [8].*

Definition 2. *Let $(\Upsilon, \sigma_{\mathcal{Q}})$ be a graphical symmetric space and $\{\flat_n\}$ be a sequence in Υ. Then, the sequence $\{\flat_n\}$ is called $\sigma_{\mathcal{Q}}$-convergent to $\flat \in \Upsilon$, and $\flat$ is called a $\sigma_{\mathcal{Q}}$-limit of $\{\flat_n\}$ if $\lim_{n\to\infty} \sigma_{\mathcal{Q}}(\flat_n, \flat) = \sigma_{\mathcal{Q}}(\flat, \flat)$. The sequence $\{\flat_n\}$ is called τ_R-convergent to $\flat \in \Upsilon$ (or convergent to $\flat$ with respect to τ_R, or simply R-convergent) if for every given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\flat_n \in B^{R}_{\mathcal{Q}}(\flat, \varepsilon)$ for all $n > n_0$. The τ_L-convergence (or convergence with respect to τ_L, or L-convergence) is defined in a similar manner.*

Remark 7. *It is clear from the above definition that if a sequence is R-convergent to some point $\flat$, then it is $\sigma_{\mathcal{Q}}$-convergent to the same point $\flat$. The next example shows that the converse of this fact is not true; also, this example shows that a sequence in a graphical symmetric space may $\sigma_{\mathcal{Q}}$-converge (τ_R-converge) to more than one limit. A similar conclusion holds for L-convergence as well.*

Example 3. *Consider the graphical symmetric space* $(\Upsilon, \sigma_{\mathcal{Q}})$*, where* $\Upsilon = \mathbb{R}$ *is endowed with the graph* $\mathcal{Q}$ *with* $E(\mathcal{Q}) = \{(\flat, v) \in \Upsilon \times \Upsilon \colon \flat, v \in \mathbb{Q}_{\neq 0}, \flat \geq v\}$*, and* $\sigma_{\mathcal{Q}} \colon \Upsilon \times \Upsilon \to \mathbb{R}$ *is given by:*

$$\sigma_{\mathcal{Q}}(\flat, v) = \begin{cases} 0 & \text{if } \flat = v; \\ |\flat v|, & \text{if } \flat, v \in \mathbb{Q}_{\neq 0} \text{ or } \flat \in \mathbb{Q}_{\neq 0}, v \in \mathbb{Q}^c_{\neq 0}; \\ 2\flat + v, & \text{otherwise} \end{cases}$$

for all $\flat, v \in \Upsilon$*, where* $\mathbb{Q}_{\neq 0} = \mathbb{Q} \cap \mathbb{R} \setminus \{0\}$, $\mathbb{Q}^c_{\neq 0} = \mathbb{R} \setminus (\mathbb{Q}_{\neq 0} \cup \{0\})$. *Consider the sequence* $\{\flat_n\}$ *in* Υ*, where* $\flat_n = \dfrac{1}{n}$ *for all* $n \in \mathbb{N}$*. Then, it is clear that* $\lim_{n\to\infty} \sigma_{\mathcal{Q}}(\flat_n, \flat) = 0$ *for all* $\flat \in \mathbb{R}$*; hence, the sequence* $\{\flat_n\}$ *is* $\sigma_{\mathcal{Q}}$*-convergent to each* $\flat \in \mathbb{R}$*. On the other hand, if* $\flat \in \mathbb{Q}^c_{\neq 0}$*, then* $\{\flat_n\}$ *is* $\sigma_{\mathcal{Q}}$*-convergent to* $\flat$*, but note that for every* $\flat \in \mathbb{Q}^c_{\neq 0}$*, we have* $\flat_n \notin B^R_{\mathcal{Q}}(\flat, \varepsilon)$ *for all* $\varepsilon > 0$ *and for all* $n \in \mathbb{N}$*. Hence, we conclude that the sequence* $\{\flat_n\}$ *is not R-convergent to* $\flat$ *for all* $\flat \in \mathbb{Q}^c_{\neq 0}$*. Also, if* $\flat \in \mathbb{Q}_{\neq 0}$*, then it is easy to see that* $\{\flat_n\}$ *is R-convergent to* $\flat$*. Thus, a sequence in graphical symmetric spaces may be* $\sigma_{\mathcal{Q}}$*-convergent (R-convergent) to more than one limit.*

Definition 3. *Let* $(\Upsilon, \sigma_{\mathcal{Q}})$ *be a graphical symmetric space and* $\{\flat_n\}$ *be a sequence in* Υ*. Then, the sequence* $\{\flat_n\}$ *is called* $\sigma_{\mathcal{Q}}$*-Cauchy if* $\lim_{n\to\infty} \sigma_{\mathcal{Q}}(\flat_n, \flat_m)$ *exists.*

As there are three ways to define convergence in graphical symmetric spaces, we introduce three types of completeness of symmetric spaces:

Definition 4. *Let* $(\Upsilon, \sigma_{\mathcal{Q}})$ *be a graphical symmetric space and* $\{\flat_n\}$ *be a sequence in* Υ*. Then,* $(\Upsilon, \sigma_{\mathcal{Q}})$ *is called* $\sigma_{\mathcal{Q}}$*-complete if every* $\sigma_{\mathcal{Q}}$*-Cauchy sequence* $\{\flat_n\}$ *in* Υ *is* $\sigma_{\mathcal{Q}}$*-convergent to* $\flat \in \Upsilon$ *such that* $\lim_{n\to\infty} \sigma_{\mathcal{Q}}(\flat_n, \flat_m) = \lim_{n\to\infty} \sigma_{\mathcal{Q}}(\flat_n, \flat) = \sigma_{\mathcal{Q}}(\flat, \flat)$*. The space* $(\Upsilon, \sigma_{\mathcal{Q}})$ *is called R-complete (respectively, L-complete) if every* $\sigma_{\mathcal{Q}}$*-Cauchy sequence* $\{\flat_n\}$ *in* Υ *is R-convergent (respectively, L-convergent) to* $\flat \in \Upsilon$ *such that* $\lim_{n\to\infty} \sigma_{\mathcal{Q}}(\flat_n, \flat_m) = \sigma_{\mathcal{Q}}(\flat, \flat)$*.*

Remark 8. *From the above definition and Remark 7 it is clear that R-completeness and L-completeness implies* $\sigma_{\mathcal{Q}}$*-completeness, but the converse of this fact is not necessarily true (see the next example). Hence, R-completeness and L-completeness are stronger notions than the notion of* $\sigma_{\mathcal{Q}}$*-completeness.*

Example 4. *Consider the graphical symmetric space* $(\Upsilon, \sigma_{\mathcal{Q}})$*, where* $\Upsilon = [0, \infty)$ *is endowed with the graph* $\mathcal{Q}$ *with* $E(\mathcal{Q}) = \Delta_\Upsilon \cup \{(\flat, v) \in \Upsilon \times \Upsilon \colon 0 < \flat < v\}$*, and* $\sigma_{\mathcal{Q}} \colon \Upsilon \times \Upsilon \to \mathbb{R}$ *is given by* $\sigma_{\mathcal{Q}}(\flat, v) = \flat + v$ *for all* $\flat, v \in \Upsilon$*. Then, it is easy to see that* $(\Upsilon, \sigma_{\mathcal{Q}})$ *is* $\sigma_{\mathcal{Q}}$*-complete. Consider the sequence* $\{\flat_n\}$ *in* Υ*, where* $\flat_n = \dfrac{1}{n}$ *for all* $n \in \mathbb{N}$*. Then, it is easy to see that this sequence is a* $\sigma_{\mathcal{Q}}$*-Cauchy sequence in* Υ*, but there exists no* $\flat \in \Upsilon$ *for which we can find* $n_0 \in \mathbb{N}$ *such that* $\flat_n \in B^R_{\mathcal{Q}}(\flat, \varepsilon)$ *for all* $n > n_0$ *or* $\flat_n \in B^L_{\mathcal{Q}}(\flat, \varepsilon)$ *for all* $n > n_0$*. Therefore,* $(\Upsilon, \sigma_{\mathcal{Q}})$ *is neither R-complete nor L-complete.*

Inspired by Shukla and Künzi [8] and Shukla et al. [11], we introduce the following versions of Cauchy sequences and completeness of graphical symmetric spaces.

Definition 5. *Let* $(\Upsilon, \sigma_{\mathcal{Q}})$ *be a graphical symmetric space and* $\{\flat_n\}$ *be a sequence in* Υ*. Then, the sequence* $\{\flat_n\}$ *is called 0-*$\sigma_{\mathcal{Q}}$*-Cauchy if* $\lim_{n\to\infty} \sigma_{\mathcal{Q}}(\flat_n, \flat_m) = 0$*. The space* $(\Upsilon, \sigma_{\mathcal{Q}})$ *is called 0-*$\sigma_{\mathcal{Q}}$*-complete if every 0-*$\sigma_{\mathcal{Q}}$*-Cauchy sequence in* Υ *is* $\sigma_{\mathcal{Q}}$*-convergent to some* $\flat \in \Upsilon$ *such that* $\sigma_{\mathcal{Q}}(\flat, \flat) = 0$*. If* $\mathcal{H}$ *is a graph such that* $V(\mathcal{H}) = \Upsilon$*, then the space* $(\Upsilon, \sigma_{\mathcal{Q}})$ *is called* $\mathcal{H}$*-*$\sigma_{\mathcal{Q}}$*-complete if every* $\mathcal{H}$*-termwise-connected* $\sigma_{\mathcal{Q}}$*-Cauchy sequence* $\{\flat_n\}$ *in* Υ *is* $\sigma_{\mathcal{Q}}$*-convergent to* $\flat \in \Upsilon$ *such that* $\lim_{n\to\infty} \sigma_{\mathcal{Q}}(\flat_n, \flat_m) = \lim_{n\to\infty} \sigma_{\mathcal{Q}}(\flat_n, \flat) = \sigma_{\mathcal{Q}}(\flat, \flat)$*. While* $(\Upsilon, \sigma_{\mathcal{Q}})$ *is called 0-*$\mathcal{H}$*-*$\sigma_{\mathcal{Q}}$*-complete if every* $\mathcal{H}$*-termwise-connected 0-*$\sigma_{\mathcal{Q}}$*-Cauchy sequence in* Υ *is* $\sigma_{\mathcal{Q}}$*-convergent to some* $\flat \in \Upsilon$ *such that* $\sigma_{\mathcal{Q}}(\flat, \flat) = 0$*.*

It is not hard to see that every $\sigma_{\mathcal{Q}}$-complete space is 0-$\sigma_{\mathcal{Q}}$-complete and every 0-$\sigma_{\mathcal{Q}}$-complete space is 0-$\mathcal{H}$-$\sigma_{\mathcal{Q}}$-complete. The following example shows that the converse assertions of these facts do not hold, so the above notions of completeness are even weaker than $\sigma_{\mathcal{Q}}$-completeness, and the notion of 0-$\mathcal{H}$-$\sigma_{\mathcal{Q}}$-completeness is the most general among the other notions mentioned above.

Example 5.

(A) *Consider* $\Upsilon = [0,\infty) \cap \mathbb{Q}$ *endowed with the graph* $\mathcal{Q}$*, where* $E(\mathcal{Q}) = \Upsilon \times \Upsilon$*. Then, the function* $\sigma_{\mathcal{Q}} \colon \Upsilon \times \Upsilon \to \mathbb{R}$ *is a graphical symmetry on* Υ*, where* $\sigma_{\mathcal{Q}}(\flat, v) = \max\{\flat, v\}$ *for all* $\flat, v \in \Upsilon$*, and* $(\Upsilon, \sigma_{\mathcal{Q}})$ *is 0-*$\sigma_{\mathcal{Q}}$*-complete but not* $\sigma_{\mathcal{Q}}$*-complete.*

(B) *Consider* $\Upsilon = \mathbb{R}$ *endowed with graph* $\mathcal{Q}$ *and* $\mathcal{H}$*, where* $E(\mathcal{Q}) = E(\mathcal{H}) = \{(\flat, v) \in \mathbb{Q} \times \mathbb{Q} \colon 0 \leq v < \flat \leq 1\}$*. Then, the function* $\sigma_{\mathcal{Q}} \colon \Upsilon \times \Upsilon \to \mathbb{R}$ *is a graphical symmetry on* Υ*, where*

$$\sigma_{\mathcal{Q}}(\flat, v) = \begin{cases} 1 - \dfrac{\flat + v}{2}, & \text{if } \flat \neq v \text{ and } \flat, v \notin \mathbb{Q}; \\ \max\{\flat, v\}, & \text{otherwise.} \end{cases}$$

Note that $(\Upsilon, \sigma_{\mathcal{Q}})$ *is 0-*$\mathcal{H}$*-*$\sigma_{\mathcal{Q}}$*-complete but not 0-*$\sigma_{\mathcal{Q}}$*-complete.*

In the next section, we prove some fixed-point results for a class of mappings defined on graphical symmetric spaces.

3. Fixed Point Theorems

We first state some definitions which are needed in the sequel.

Let $\vartheta \colon [0,\infty) \to [0,\infty)$ be a function and $(\Upsilon, \sigma_{\mathcal{Q}})$ be a graphical symmetric space. In all subsequent discussions, we assume that $\mathcal{H}$ is a subgraph of $\mathcal{Q}$ with $\Delta_{\mathcal{D}_{\mathcal{H}}} \subseteq \mathcal{C}_{\mathcal{H}}$, where $\mathcal{D}_{\mathcal{H}} = \left\{\flat \in \Upsilon \colon (\flat, v) \in \widetilde{\mathcal{C}_{\mathcal{H}}} \text{ for some } v \in \Upsilon\right\}$. A subset A of Υ is called $\mathcal{H}$-bounded with respect to ϑ, if $\vartheta(\sigma_{\mathcal{Q}}(\flat, v)) \leq K$ for all $\flat, v \in A$ with $(\flat, v) \in \mathcal{C}_{\mathcal{H}}$, where $K \in [0,\infty)$ is a fixed number.

By Ψ, we denote the class of the functions $\vartheta \colon [0,\infty) \to [0,\infty)$ such that for every sequence of nonnegative numbers $\{a_{n,m}\}_{n,m \in \mathbb{N}}$, we have $\lim_{n,m\to\infty} \vartheta(a_{n,m}) = 0$ if and only if $\lim_{n,m\to\infty} a_{n,m} = 0$.

Definition 6. *Let* $(\Upsilon, \sigma_{\mathcal{Q}})$ *be a graphical symmetric space,* $\top \colon \Upsilon \to \Upsilon$ *a mapping and* $\mathcal{H}$ *be a subgraph of* $\mathcal{Q}$*. Then, the mapping* $\top$ *is called a graphical* $\mathcal{H}$*-*(ϑ, φ)*-contraction if there exist* $\vartheta, \varphi \in \Psi$ *such that:*

$$\vartheta(\sigma_{\mathcal{Q}}(\top\flat, \top v)) \leq \vartheta(\sigma_{\mathcal{Q}}(\flat, v)) - \varphi(\sigma_{\mathcal{Q}}(\flat, v)), \text{ for all } \flat, v \in \Upsilon \text{ with } (\flat, v) \in \mathcal{C}_{\mathcal{H}}.$$

If $\flat_0 \in \Upsilon$*, the iterative sequence* $\{\flat_n\}$*, where* $\flat_n = \top^n \flat_0$ *for all* $n \in \mathbb{N}$*, is called the* $\top$*-Picard sequence with initial value* $\flat_0$*. The mapping* $\top$ *is called* $\mathcal{H}$*-edge-preserving if* $(\flat, v) \in E(\mathcal{H})$ *implies* $(\top\flat, \top v) \in E(\mathcal{H})$ *for all* $\flat, v \in \Upsilon$*.*

Theorem 2. *Let* $(\Upsilon, \sigma_{\mathcal{Q}})$ *be a 0-*$\mathcal{H}$*-*$\sigma_{\mathcal{Q}}$*-complete graphical symmetric space and* $\top \colon \Upsilon \to \Upsilon$ *be an* $\mathcal{H}$*-edge-preserving graphical* $\mathcal{H}$*-*(ϑ, φ)*-contraction. Suppose that the following conditions hold:*

(A) *There exists* $\flat_0 \in \Upsilon$ *such that the orbit of* $\top$ *with initial value* $\flat_0$*, that is,* $O(\top, \flat_0) = \{\top^r \flat_0 \colon r = 0, 1, \ldots\}$*, is* $\mathcal{H}$*-bounded with respect to both* ϑ *and* φ*, and* $(\flat_0, \top\flat_0) \in \mathcal{C}_{\mathcal{H}}$*;*

(B) *If an* $\mathcal{H}$*-termwise-connected* $\top$*-Picard sequence* $\{z_n\}$ *is* $\sigma_{\mathcal{Q}}$*-convergent to some point in* Υ*, then there exist a* $\sigma_{\mathcal{Q}}$*-limit* $z \in \Upsilon$ *of* $\{z_n\}$ *and* $n_0 \in \mathbb{N}$ *such that* $(z_n, z) \in \mathcal{C}_{\mathcal{H}}$ *for all* $n > n_0$ *or* $(z, z_n) \in \mathcal{C}_{\mathcal{H}}$ *for all* $n > n_0$*.*

Then, there exists $u \in \Upsilon$ *such that the* $\top$*-Picard sequence* $\{\flat_n\}$ *with initial value* $\flat_0 \in \Upsilon$ *is* $\mathcal{H}$*-termwise-connected and* $\sigma_{\mathcal{Q}}$*-convergent to* $\top^r u$ *with* $\sigma_{\mathcal{Q}}(\top^r u, \top^r u) = 0$ *for all* $r \in \mathbb{N} \cup \{0\}$*.*

Proof. Let $\flat_0 \in Y$ be such that $O(\top, \flat_0)$ is $\mathcal{H}$-bounded with respect to both ϑ and φ, and $(\flat_0, \top\flat_0) \in \mathcal{C}_{\mathcal{H}}$. We first show that the $\top$-Picard sequence $\{\flat_n\}$ with initial value $\flat_0$ is a 0-$\sigma_{\mathcal{Q}}$-Cauchy sequence.

Since $(\flat_0, \top\flat_0) \in \mathcal{C}_{\mathcal{H}}$, we have $(\flat_0, \flat_1) \in \mathcal{C}_{\mathcal{H}}$. As $\top$ is $\mathcal{H}$-edge-preserving, we have $(\top\flat_0, \top\flat_1) \in \mathcal{C}_{\mathcal{H}}$, that is, $(\flat_1, \flat_2) \in \mathcal{C}_{\mathcal{H}}$. A repetition of this argument leads us to the conclusion: $(\flat_{n-1}, \flat_n) \in \mathcal{C}_{\mathcal{H}}$ for all $n \in \mathbb{N}$. Since the relation of connectedness is a transitive relation, we have $(\flat_n, \flat_m) \in \mathcal{C}_{\mathcal{H}}$ for all $n, m \in \mathbb{N}$ with $m > n$. As $\top$ is a graphical $\mathcal{H}$-(ϑ, φ)-contraction, for every $m > n$, we have

$$\begin{aligned} \vartheta(\sigma_{\mathcal{Q}}(\flat_{n+1}, \flat_{m+1})) &= \vartheta(\sigma_{\mathcal{Q}}(\top\flat_n, \top\flat_m)) \\ &\leq \vartheta(\sigma_{\mathcal{Q}}(\flat_n, \flat_m)) - \varphi(\sigma_{\mathcal{Q}}(\flat_n, \flat_m)). \end{aligned}$$

This gives

$$\vartheta(\sigma_{\mathcal{Q}}(\flat_{n+1}, \flat_{m+1})) + \varphi(\sigma_{\mathcal{Q}}(\flat_n, \flat_m)) \leq \vartheta(\sigma_{\mathcal{Q}}(\flat_n, \flat_m)). \tag{1}$$

Since the condition (A) is satisfied, for each $n \in \mathbb{N}$, two sets of nonnegative numbers: $\{\vartheta(\sigma_{\mathcal{Q}}(\flat_{n-1}, \flat_{m-1}))\colon n, m \in \mathbb{N}, m > n\}$ and $\{\varphi(\sigma_{\mathcal{Q}}(\flat_{n-1}, \flat_{m-1}))\colon n, m \in \mathbb{N}, m > n\}$ are bounded. Hence, for each $n \in \mathbb{N}$, the following numbers must exist:

$$s_n = \sup_{m>n} \vartheta(\sigma_{\mathcal{Q}}(\flat_n, \flat_m)); \text{ and } p_n = \sup_{m>n} \varphi(\sigma_{\mathcal{Q}}(\flat_n, \flat_m)).$$

As $\sup_{m>n} \vartheta(\sigma_{\mathcal{Q}}(\flat_{n+1}, \flat_{m+1})) = \sup_{m>n+1} \vartheta(\sigma_{\mathcal{Q}}(\flat_{n+1}, \flat_m))$, taking the supremum over $m(> n)$ in (1), we obtain

$$\sup_{m>n+1} \vartheta(\sigma_{\mathcal{Q}}(\flat_{n+1}, \flat_m)) + \sup_{m>n} \varphi(\sigma_{\mathcal{Q}}(\flat_n, \flat_m)) \leq \sup_{m>n} \vartheta(\sigma_{\mathcal{Q}}(\flat_n, \flat_m)).$$

Hence, by definitions of s_n and p_n we obtain:

$$s_{n+1} + p_n \leq s_n \text{ for all } n \in \mathbb{N}. \tag{2}$$

Since $\vartheta, \varphi \in \Psi$, we have $0 \leq s_n, p_n$; so, by (2), the sequence $\{s_n\}$ is a nonincreasing sequence of real numbers which is bounded below as well. Hence, it must be convergent to some nonnegative number s, that is,

$$\lim_{n\to\infty} s_n = s.$$

By (2), we have

$$0 \leq p_n \leq s_n - s_{n+1} \text{ for all } n \in \mathbb{N},$$

which, with the fact that $\lim_{n\to\infty} s_n = s$, gives

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} \sup_{m>n} \varphi(\sigma_{\mathcal{Q}}(\flat_n, \flat_m)) = 0.$$

As $0 \leq \varphi(\sigma_{\mathcal{Q}}(\flat_n, \flat_m)) \leq \sup_{m>n} \varphi(\sigma_{\mathcal{Q}}(\flat_n, \flat_m))$ for all $m > n$, the above equality yields

$$\lim_{n,m\to\infty} \varphi(\sigma_{\mathcal{Q}}(\flat_n, \flat_m)) = 0.$$

As $\varphi \in \Psi$, the above equality implies that

$$\lim_{n,m\to\infty} \sigma_{\mathcal{Q}}(\flat_n, \flat_m) = 0.$$

Thus, we have showed that the $\top$-Picard sequence $\{\flat_n\}$ with initial value $\flat_0$ is an $\mathcal{H}$-termwise-connected 0-$\sigma_{\mathcal{Q}}$-Cauchy sequence.

By the 0-$\mathcal{H}$-$\sigma_{\mathcal{Q}}$-completeness of Y, the sequence $\{\flat_n\}$ is $\sigma_{\mathcal{Q}}$-convergent to some point in Y, and by condition (B), there exist a $\sigma_{\mathcal{Q}}$-limit $u \in$ Y of $\{\flat_n\}$ and $n_0 \in \mathbb{N}$ such that

$$\lim_{n,m\to\infty} \sigma_{\mathcal{Q}}(\flat_n, \flat_m) = \lim_{n\to\infty} \sigma_{\mathcal{Q}}(\flat_n, u) = \sigma_{\mathcal{Q}}(u, u) = 0$$

and $(\flat_n, u) \in \mathcal{C}_{\mathcal{H}}$ for all $n > n_0$ or $(u, \flat_n) \in \mathcal{C}_{\mathcal{H}}$ for all $n > n_0$.

By mathematical induction, we show that $\{\flat_n\}$ is $\sigma_{\mathcal{Q}}$-convergent to $\top^r u$ with $\sigma_{\mathcal{Q}}(\top^r u, \top^r u) = 0$ for all $r \in \mathbb{N}$. Suppose that $(\flat_n, u) \in \mathcal{C}_{\mathcal{H}}$ for all $n > n_0$ (the proof for the second case is same). Then, as $\top$ is a graphical $\mathcal{H}$-(ϑ, φ)-contraction, we have

$$\begin{array}{rcl} \vartheta(\sigma_{\mathcal{Q}}(\flat_{n+1}, \top u)) & = & \vartheta(\sigma_{\mathcal{Q}}(\top \flat_n, \top u)) \\ & \leq & \vartheta(\sigma_{\mathcal{Q}}(\flat_n, u)) - \varphi(\sigma_{\mathcal{Q}}(\flat_n, u)). \end{array}$$

Since u is a $\sigma_{\mathcal{Q}}$-limit of the sequence $\{\flat_n\}$, we have $\lim_{n\to\infty} \sigma_{\mathcal{Q}}(\flat_n, u) = \sigma_{\mathcal{Q}}(u, u) = 0$, and since $\vartheta, \varphi \in \Psi$, the real sequences $\{\vartheta(\sigma_{\mathcal{Q}}(\flat_n, u))\}$ and $\{\varphi(\sigma_{\mathcal{Q}}(\flat_n, u))\}$ of nonnegative real numbers must converge to zero. Therefore, the above inequality yields

$$\lim_{n\to\infty} \sigma_{\mathcal{Q}}(\flat_{n+1}, \top u) = 0.$$

As $(\flat_n, u) \in \mathcal{C}_{\mathcal{H}}$ for all $n > n_0$ and $\top$ is $\mathcal{H}$-edge-preserving, we have $(\flat_{n+1}, \top u) = (\top \flat_n, \top u) \in \mathcal{C}_{\mathcal{H}}$ for all $n > n_0$, so $\top u \in \mathcal{D}_{\mathcal{H}}$. Also, as $\Delta_{\mathcal{D}_{\mathcal{H}}} \subseteq \mathcal{C}_{\mathcal{H}}$ and $\top$ is a graphical $\mathcal{H}$-(ϑ, φ)-contraction, we have:

$$\begin{array}{rcl} \vartheta(\sigma_{\mathcal{Q}}(\top u, \top u)) & \leq & \vartheta(\sigma_{\mathcal{Q}}(u, u)) - \varphi(\sigma_{\mathcal{Q}}(u, u)) \\ & = & \vartheta(0) - \varphi(0) \\ & = & 0 \end{array}$$

and as $\vartheta \in \Psi$, the above inequality implies that $\sigma_{\mathcal{Q}}(\top u, \top u) = 0$.

As an induction hypothesis, suppose that $\{\flat_n\}$ is $\sigma_{\mathcal{Q}}$-convergent to $\top^k u$ and $\sigma_{\mathcal{Q}}(\top^k u, \top^k u) = 0$ for some $k \in \mathbb{N}$. Since $(\flat_n, u) \in \mathcal{C}_{\mathcal{H}}$ for all $n > n_0$ and $\top$ is $\mathcal{H}$-edge-preserving, we must have $(\flat_n, \top^k u) \in \mathcal{C}_{\mathcal{H}}$ for all $n > n_0 + k$. Hence, for $n > n_0 + k$, we have

$$\begin{array}{rcl} \vartheta(\sigma_{\mathcal{Q}}(\flat_{n+1}, \top^{k+1} u)) & = & \vartheta(\sigma_{\mathcal{Q}}(\top \flat_n, \top \top^k u)) \\ & \leq & \vartheta(\sigma_{\mathcal{Q}}(\flat_n, \top^k u)) - \varphi(\sigma_{\mathcal{Q}}(\flat_n, \top^k u)). \end{array}$$

Since $\top^k u$ is a $\sigma_{\mathcal{Q}}$-limit of the sequence $\{\flat_n\}$, we have $\lim_{n\to\infty} \sigma_{\mathcal{Q}}(\flat_n, \top^k u) = \sigma_{\mathcal{Q}}(\top^k u, \top^k u) = 0$, and using arguments similar to the previous case, the above inequality yields

$$\lim_{n\to\infty} \sigma_{\mathcal{Q}}(\flat_{n+1}, \top^{k+1} u) = 0.$$

As $(\flat_n, \top^k u) \in \mathcal{C}_{\mathcal{H}}$ for all $n > n_0 + k$ and $\top$ is $\mathcal{H}$-edge-preserving, $\top^{k+1} u \in \mathcal{D}_{\mathcal{H}}$. Also, as $\Delta_{\mathcal{D}_{\mathcal{H}}} \subseteq \mathcal{C}_{\mathcal{H}}$ and $\top$ is a graphical $\mathcal{H}$-(ϑ, φ)-contraction, we have:

$$\begin{array}{rcl} \vartheta(\sigma_{\mathcal{Q}}(\top^{k+1} u, \top^{k+1} u)) & \leq & \vartheta(\sigma_{\mathcal{Q}}(\top^k u, \top^k u)) - \varphi(\sigma_{\mathcal{Q}}(\top^k u, \top^k u)) \\ & = & \vartheta(0) - \varphi(0) \\ & = & 0 \end{array}$$

and as $\vartheta \in \Psi$, the above inequality implies that $\sigma_{\mathcal{Q}}(\top^{k+1} u, \top^{k+1} u) = 0$. This completes the induction.

Thus, the $\top$-Picard sequence $\{\flat_n\}$ with initial value $\flat_0 \in$ Y is $\mathcal{H}$-termwise-connected and $\sigma_{\mathcal{Q}}$-convergent to $\top^r u$ with $\sigma_{\mathcal{Q}}(\top^r u, \top^r u) = 0$ for all $r \in \mathbb{N} \cup \{0\}$. $\square$

Example 6. *Consider the set* $\Upsilon = \mathbb{R}$ *equipped with the graphs* $\mathcal{Q}, \mathcal{H}$*, where* $E(\mathcal{Q}) = E(\mathcal{H}) = \{(\flat, v) \in \Upsilon \times \Upsilon : 0 < v \leq \flat \leq 1\}$*, and consider the function* $\sigma_{\mathcal{Q}} : \Upsilon \times \Upsilon \to \mathbb{R}$ *defined by:*

$$\sigma_{\mathcal{Q}}(\flat, v) = \begin{cases} 0, & \text{if } \flat = v; \\ \sin^{-1}(\flat v), & \text{if } \flat \neq v; \end{cases}$$

Then, $\sigma_{\mathcal{Q}}$ *is a graphical symmetry on* Υ*, and* $(\Upsilon, \sigma_{\mathcal{Q}})$ *is a 0-*$\mathcal{H}$*-*$\sigma_{\mathcal{Q}}$*-complete graphical symmetric space. Let* $\top : \Upsilon \to \Upsilon$ *be a mapping defined by* $\top\flat = \sin\flat$ *for all* $\flat \in \Upsilon \setminus \{0\}$*,* $\top 0 = 1$*, and let* $\vartheta, \varphi : [0, \infty) \to [0, \infty)$ *be two functions defined by* $\vartheta(t) = t$ *for all* $t \in [0, \infty)$ *and*

$$\varphi(t) = \begin{cases} t - \sin(t), & \text{if } t \in [0, \pi/2]; \\ \frac{t}{2}, & \text{otherwise.} \end{cases}$$

Then, it is easy to see that $\top$ *is a graphical* $\mathcal{H}$*-*(ϑ, φ)*-contraction. Since* $\top$ *is an increasing function on* $(0, 1]$*,* $\top$ *is* $\mathcal{H}$*-edge-preserving. For any* $\flat_0 \in (0, 1]$ *the orbit* $O(\top, \flat_0)$ *is* $\mathcal{H}$*-bounded with respect to both* ϑ *and* φ *and* $(\flat_0, \top\flat_0) \in \mathcal{C}_{\mathcal{H}}$*. Also, if an* $\mathcal{H}$*-termwise-connected* $\top$*-Picard sequence* $\{\flat_n\}$ *is* $\sigma_{\mathcal{Q}}$*-convergent to some point in* Υ*, then there exist a limit* $z \in \Upsilon$ *of* $\{\flat_n\}$ *and* $n_0 \in \mathbb{N}$ *such that* $(z, \flat_n) \in \mathcal{C}_{\mathcal{H}}$ *for all* $n > n_0$ *(for example,* $z = 1$ *with* $n_0 = 1$*). Hence, all the conditions of Theorem 2 are satisfied, but* $\top$ *has no fixed point.*

Remark 9. *The above example shows that the conditions used in Theorem 2 are sufficient to ensure the convergence of a particular* $\mathcal{H}$*-termwise-connected* $\top$*-Picard sequence, where* $\top$ *is an* $\mathcal{H}$*-edge-preserving graphical* $\mathcal{H}$*-*(ϑ, φ)*-contraction. But the conditions of the above theorem are insufficient to ensure the existence of a fixed point of* $\top$*, so we introduce the following property:*

Definition 7 (Property (S))**.** *Let* $(\Upsilon, \sigma_{\mathcal{Q}})$ *be a graphical symmetric space and* $\top : \Upsilon \to \Upsilon$ *be a mapping. Then, the quadruple* $(\Upsilon, \sigma_{\mathcal{Q}}, \mathcal{H}, \top)$ *possesses the property (S) if:*

$$\text{whenever an } \mathcal{H}\text{-termwise-connected } \top\text{-Picard sequence } \{\flat_n\} \text{ has two } \sigma_{\mathcal{Q}}\text{-limits } u \text{ and } v, \\ \text{where } u \in \Upsilon, v \in \top(\Upsilon), \text{ then } u = v. \tag{3}$$

Remark 10. *Note that in Example 6, all the conditions of Theorem 2 are satisfied, but the quadruple* $(\Upsilon, \sigma_{\mathcal{Q}}, \mathcal{H}, \top)$ *does not possess the property (S), for example, consider the* $\mathcal{H}$*-termwise-connected* $\top$*-Picard sequence* $\{\flat_n\}$*, where* $\flat_0 = 1$*; then, this sequence has two* $\sigma_{\mathcal{Q}}$*-limits* $\flat(= 1) \in \Upsilon$ *and* $v(= \sin 1) \in \top(\Upsilon)$*, but* $\sin 1 \neq 1$*. This fact shows the significance of the property (S).*

Theorem 3. *Let* $(\Upsilon, \sigma_{\mathcal{Q}})$ *be a 0-*$\mathcal{H}$*-*$\sigma_{\mathcal{Q}}$*-complete graphical symmetric space and* $\top : \Upsilon \to \Upsilon$ *be an* $\mathcal{H}$*-edge-preserving graphical* $\mathcal{H}$*-*(ϑ, φ)*-contraction. Suppose that all the conditions of Theorem 2 are satisfied and the quadruple* $(\Upsilon, \sigma_{\mathcal{Q}}, \mathcal{H}, \top)$ *possesses the property (S), then* $\top$ *has a fixed point in* Υ*.*

Proof. Theorem 2 shows that the $\top$-Picard sequence $\{\flat_n\}$ with initial value $\flat_0$ is $\sigma_{\mathcal{Q}}$-convergent to $\top^k u$ for all $k \in \mathbb{N} \cup \{0\}$. Also, by the proof of Theorem 2, it is clear that this $\top$-Picard sequence is $\mathcal{H}$-termwise-connected. As $\top^0 u = u \in \Upsilon$, $\top u \in \top(\Upsilon)$ and the quadruple $(\Upsilon, \sigma_{\mathcal{Q}}, \mathcal{H}, \top)$ possesses the property (S), we must have $\top u = u$, that is, u is the fixed point of $\top$. □

Example 7. *Consider the set* $\Upsilon = \mathbb{R}$ *equipped with the graphs* $\mathcal{Q}, \mathcal{H}$*, where* $E(\mathcal{Q}) = E(\mathcal{H}) = [0, 1] \times [0, 1]$*, and consider the function* $\sigma_{\mathcal{Q}} : \Upsilon \times \Upsilon \to \mathbb{R}$ *defined by:*

$$\sigma_{\mathcal{Q}}(\flat, v) = \begin{cases} 0, & \text{if } \flat = v \in [0, 1]; \\ \max\{\flat, v\}, & \text{if } \flat, v \in [0, 1], \flat \neq v; \\ \flat - v, & \text{otherwise.} \end{cases}$$

Then, $\sigma_{\mathcal{Q}}$ is a graphical symmetry on Υ, and $(\Upsilon, \sigma_{\mathcal{Q}})$ is a 0-$\mathcal{H}$-$\sigma_{\mathcal{Q}}$-complete graphical symmetric space. Let $\top\colon \Upsilon \to \Upsilon$ be a mapping defined by $\top\flat = 0$ for all $\flat \in [0,1]$ and $\top\flat = \flat$ if $\flat \in \mathbb{R} \setminus [0,1]$. Consider the functions $\vartheta, \varphi\colon [0,\infty) \to [0,\infty)$ defined by $\vartheta(t) = \varphi(t) = t$ for all $t \in [0,\infty)$. Then, it is easy to see that $\top$ is a graphical $\mathcal{H}$-(ϑ,φ)-contraction which is $\mathcal{H}$-edge-preserving. For $\flat_0 = 0$ the orbit $O(\top, \flat_0)$ is $\mathcal{H}$-bounded with respect to both ϑ and φ, and $(\flat_0, \top\flat_0) = (0,0) \in \mathcal{C}_{\mathcal{H}}$. Also, if an $\mathcal{H}$-termwise-connected $\top$-Picard sequence $\{\flat_n\}$ is $\sigma_{\mathcal{Q}}$-convergent to some point $z \in \Upsilon$, then $z = 0$ and $(0, \flat_n) \in \mathcal{C}_{\mathcal{H}}$ for all $n \geq 1$. Hence, all the conditions of Theorem 2 are satisfied. Note that an $\mathcal{H}$-termwise-connected $\top$-Picard sequence in Υ must be a constant sequence with every term of it equal to zero, and if it has a $\sigma_{\mathcal{Q}}$-limit $u \in \Upsilon$, then we must have $u = 0$. Hence, the quadruple $(\Upsilon, \sigma_{\mathcal{Q}}, \mathcal{H}, \top)$ possesses the property (S). Hence, by Theorem 2, the mapping $\top$ must have a fixed point in Υ. Note that all the points of the set $(-\infty, 0] \cup (1,\infty)$ are the fixed points of $\top$.

The above example shows that the conditions used in Theorem 3 can only ensure the existence of the fixed point of an $\mathcal{H}$-edge-preserving graphical $\mathcal{H}$-(ϑ,φ)-contraction, but not the uniqueness of the fixed point.

By $\mathcal{F}(\top)$, we denote the set of all fixed points of a mapping $\top\colon \Upsilon \to \Upsilon$. For any subgraph $\mathcal{H}$ of $\mathcal{Q}$ and for a fixed number $k \in \mathbb{N}$, the set $\{z \in \Upsilon\colon (z, \top^k z) \in \mathcal{C}_{\mathcal{H}}\}$ is denoted by ${}_k\Upsilon^{\mathcal{H}}_{\top}$, and $(z, \top^k z)$ is called the corresponding edge of $z \in {}_k\Upsilon^{\mathcal{H}}_{\top}$. A subset A of Υ is said to be $\mathcal{H}$-connected if for every distinct $\flat, v \in A$, we have $(\flat, v) \in \mathcal{C}_{\mathcal{H}}$ or $(v, \flat) \in \mathcal{C}_{\mathcal{H}}$.

Theorem 4. *Let $(\Upsilon, \sigma_{\mathcal{Q}})$ be a 0-$\mathcal{H}$-$\sigma_{\mathcal{Q}}$-complete graphical symmetric space and $\top\colon \Upsilon \to \Upsilon$ be an $\mathcal{H}$-edge-preserving graphical $\mathcal{H}$-(ϑ,φ)-contraction. Suppose that all the conditions of Theorem 3 are satisfied, then $\top$ has a fixed point. In addition, if ${}_k\Upsilon^{\mathcal{H}}_{\top}$ is $\mathcal{H}$-connected and the diagonal $\Delta_{\Upsilon} = \{(\flat,\flat)\colon \flat \in \Upsilon\} \subseteq \mathcal{C}_{\mathcal{H}}$, then the fixed point of $\top$ is unique.*

Proof. The existence of the fixed point u of $\top$ follows from Theorem 3. For the uniqueness of the fixed point, on the contrary, suppose that there is a fixed point $v \in \Upsilon$ of $\top$ and $u \neq v$. Hence, we have $\top^n u = u, \top^n v = v$ for all $n \in \mathbb{N}$. Since, $(u, \top^k u) = (u,u), (v, \top^k v) = (v,v) \in \Delta_{\Upsilon} = \{(\flat,\flat)\colon \flat \in \Upsilon\} \subseteq \mathcal{C}_{\mathcal{H}}$, we have $u, v \in {}_k\Upsilon^{\mathcal{H}}_{\top}$; also, ${}_k\Upsilon^{\mathcal{H}}_{\top}$ is $\mathcal{H}$-connected; hence, suppose that $(u,v) \in \mathcal{C}_{\mathcal{H}}$ (the proof for the case $(v,u) \in \mathcal{C}_{\mathcal{H}}$ is same), then, as $\top$ is a graphical $\mathcal{H}$-(ϑ,φ)-contraction, we have

$$\begin{aligned} \vartheta(\sigma_{\mathcal{Q}}(u,v)) &= \vartheta(\sigma_{\mathcal{Q}}(\top u, \top v)) \\ &\leq \vartheta(\sigma_{\mathcal{Q}}(u,v)) - \varphi(\sigma_{\mathcal{Q}}(u,v)). \end{aligned}$$

The above inequality shows that $\varphi(\sigma_{\mathcal{Q}}(u,v)) = 0$, and since $\varphi \in \Psi$, we must have $\sigma_{\mathcal{Q}}(u,v) = 0$, that is, $u = v$. This contradiction shows that $u \in \Upsilon$ is the unique fixed point of $\top$. $\square$

Remark 11. *In the above theorem, for the uniqueness of the fixed point of $\top$, we assumed the $\mathcal{H}$-connectedness of ${}_k\Upsilon^{\mathcal{H}}_{\top}$ and the inclusion of diagonal Δ_{Υ} in the set $\mathcal{C}_{\mathcal{H}}$. We point out that in Example 7, although the condition of the $\mathcal{H}$-connectedness of ${}_k\Upsilon^{\mathcal{H}}_{\top} = [0,1]$ is satisfied, the inclusion $\Delta_{\Upsilon} \subseteq \mathcal{C}_{\mathcal{H}}$ does not hold, so the fixed point of $\top$ is not unique.*

Remark 12. *Let $(\Upsilon, \sigma_{\mathcal{Q}})$ be a graphical symmetric space. Then, consider the following property:*

(GS4) $\lim_{n\to\infty} \sigma_{\mathcal{Q}}(\flat_n, \flat) = \sigma_{\mathcal{Q}}(\flat,\flat) = 0$ *and* $\lim_{n\to\infty} \sigma_{\mathcal{Q}}(\flat_n, v) = \sigma_{\mathcal{Q}}(v,v) = 0$ *implies* $\flat = v$.

Note that property (GS4) implies property (S); hence, if we replace condition (S) by (GS4), the conclusion of Theorem 3 and Theorem 4 remains true.

Example 8. *Consider the set $\Upsilon = \mathbb{R}$ equipped with the graphs $\mathcal{Q}, \mathcal{H}$, where $E(\mathcal{Q}) = E(\mathcal{H}) = \Delta_{\Upsilon} \cup \{(\flat, v) \in \Upsilon \times \Upsilon\colon 0 \leq v < \flat \leq 1\}$, and consider the function $\sigma_{\mathcal{Q}}\colon \Upsilon \times \Upsilon \to \mathbb{R}$ defined by*

$$\sigma_{\mathcal{Q}}(\flat, v) = \begin{cases} 0, & \text{if } \flat = v; \\ \sin^{-1}(\max\{\flat, v\}), & \text{if } \flat \neq v; \end{cases}$$

Then, $\sigma_{\mathcal{Q}}$ is a graphical symmetry on Υ, and $(\Upsilon, \sigma_{\mathcal{Q}})$ is a 0-$\mathcal{H}$-$\sigma_{\mathcal{Q}}$-complete graphical symmetric space. Let $\top\colon \Upsilon \to \Upsilon$ be a mapping defined by $\top\flat = \sin\flat$ for all $\flat \in \Upsilon$ and let $\vartheta, \varphi\colon [0,\infty) \to [0,\infty)$ be two functions defined by $\vartheta(t) = t$ for all $t \in \mathbb{R}$ and

$$\varphi(t) = \begin{cases} t - \sin(t), & \text{if } t \in [0, \pi/2]; \\ \frac{t}{2}, & \text{otherwise.} \end{cases}$$

Since $\top$ is an increasing function on $[0,1]$, $\top$ is $\mathcal{H}$-preserving. For any $\flat_0 \in [0,1]$, the orbit $O(\top, \flat_0)$ is bounded with respect to both ϑ and φ, and $(\flat_0, \top\flat_0) \in E(\mathcal{H})$. Also, if an $\mathcal{H}$-termwise-connected $\top$-Picard sequence $\{\flat_n\}$ is $\sigma_{\mathcal{Q}}$-convergent to $z \in \Upsilon$, then we must have $z = 0$ and $(\flat_n, z) \in \mathcal{C}_{\mathcal{H}}$ for all $n \geq 1$. Also, if an $\mathcal{H}$-termwise-connected $\top$-Picard sequence $\{\flat_n\}$ has two limits u and v where $u \in \Upsilon, v \in \top(\Upsilon)$, then $u = v = 0$. Hence, all the conditions of Theorem 2 are satisfied, and the quadruple $(\Upsilon, \sigma_{\mathcal{Q}}, \mathcal{H}, \top)$ possesses the property (S). Thus, the fixed point of $\top$ exists. Here, ${}_k\Upsilon_{\top}^{\mathcal{H}} = [0,1]$; hence, ${}_k\Upsilon_{\top}^{\mathcal{H}}$ is $\mathcal{H}$-connected, and $\Delta_{\Upsilon} \subseteq \mathcal{C}_{\mathcal{H}}$. Thus, all the conditions of Theorem 4 are satisfied, hence, by Theorem 4, $\top$ has a unique fixed point. Indeed, $\mathcal{F}(\top) = \{0\}$.

A graphical metric space $(\Upsilon, d_{\mathcal{Q}})$ is called $\mathcal{H}$-complete (for details, see Shukla et al. [6]) if every $\mathcal{H}$-termwise-connected sequence in Υ converges in Υ. In view of Remark 1 and the fact that Shukla et al. [6] assumed that Δ_{Υ} was always contained in $E(\mathcal{H})$, we obtain the following corollary of Theorem 4.

Corollary 1. *Let Υ be a nonempty set endowed with a graph $\mathcal{Q}$ such that $(\Upsilon, d_{\mathcal{Q}})$ is an $\mathcal{H}$-complete graphical metric space, and let $\top\colon \Upsilon \to \Upsilon$ be an $\mathcal{H}$-edge-preserving mapping. Suppose that the following conditions hold:*

(A) There exist $\vartheta, \varphi \in \Psi$ such that

$$\vartheta(d_{\mathcal{Q}}(\top\flat, \top v)) \leq \vartheta(d_{\mathcal{Q}}(\flat, v)) - \varphi(d_{\mathcal{Q}}(\flat, v)), \text{ for all } \flat, v \in \Upsilon \text{ with } (\flat, v) \in \mathcal{C}_{\mathcal{H}};$$

(B) There exists $\flat_0 \in \Upsilon$ such that $\max\{\vartheta(d_{\mathcal{Q}}(\flat, v)), \varphi(d_{\mathcal{Q}}(\flat, v))\} \leq K$ for all $\flat, v \in O(\top, \flat_0)$ with $(\flat, v) \in \mathcal{C}_{\mathcal{H}}$, where $K \in [0,\infty)$ is a fixed number and $(\flat_0, \top\flat_0) \in \mathcal{C}_{\mathcal{H}}$;

(C) If an $\mathcal{H}$-termwise-connected $\top$-Picard sequence $\{z_n\}$ converges in Υ, then there exist a limit $z \in \Upsilon$ of $\{z_n\}$ and $n_0 \in \mathbb{N}$ such that $(z_n, z) \in \mathcal{C}_{\mathcal{H}}$ for all $n > n_0$ or $(z, z_n) \in \mathcal{C}_{\mathcal{H}}$ for all $n > n_0$.

Then, there exists $u \in \Upsilon$ such that the $\top$-Picard sequence $\{\flat_n\}$ with initial value $\flat_0 \in \Upsilon$ is $\mathcal{H}$-termwise-connected and convergent to $\top^r u$ for all $r \in \mathbb{N} \cup \{0\}$. In addition, if the quadruple $(\Upsilon, \sigma_{\mathcal{Q}}, \mathcal{H}, \top)$ possesses the property (S) (in the sense of Shukla et al. [6]), then $\top$ has a fixed point in Υ. Also, if ${}_k\Upsilon_{\top}^{\mathcal{H}}$ is $\mathcal{H}$-connected, then the fixed point of $\top$ is unique.

Corollary 2. *Let Υ be a nonempty set endowed with a graph $\mathcal{Q}$ such that $(\Upsilon, d_{\mathcal{Q}})$ is an $\mathcal{H}$-complete graphical metric space, and let $\top\colon \Upsilon \to \Upsilon$ be an $\mathcal{H}$-edge-preserving mapping. Suppose that the following conditions hold:*

(A) There exist $\lambda \in (0,1)$ such that

$$d_{\mathcal{Q}}(\top\flat, \top v) \leq \lambda d_{\mathcal{Q}}(\flat, v), \text{ for all } \flat, v \in \Upsilon \text{ with } (\flat, v) \in C_{\mathcal{H}}.$$

(B) There exists $\flat_0 \in \Upsilon$ such that $(\flat_0, \top\flat_0) \in \mathcal{C}_{\mathcal{H}}$;

(C) If an $\mathcal{H}$-termwise-connected $\top$-Picard sequence $\{z_n\}$ converges in Υ, then there exist a limit $z \in \Upsilon$ of $\{z_n\}$ and $n_0 \in \mathbb{N}$ such that $(z_n, z) \in \mathcal{C}_{\mathcal{H}}$ for all $n > n_0$ or $(z, z_n) \in \mathcal{C}_{\mathcal{H}}$ for all $n > n_0$.

Then, there exists $u \in \Upsilon$ such that the $\top$-Picard sequence $\{\flat_n\}$ with initial value $\flat_0 \in \Upsilon$ is $\mathcal{H}$-termwise-connected and converges to $\top^r u$ for all $r \in \mathbb{N} \cup \{0\}$. In addition, if the quadruple $(\Upsilon, \sigma_{\mathcal{Q}}, \mathcal{H}, \top)$ has the property (S) (in the sense of Shukla et al. [6]), then $\top$ has a fixed point in Υ. Also, if ${}_k\Upsilon_{\top}^{\mathcal{H}}$ is $\mathcal{H}$-connected, then the fixed point of $\top$ is unique.

Proof. Consider the graphical symmetric space $(Y, \sigma_{\mathcal{Q}})$, where $\sigma_{\mathcal{Q}}(\flat, v) = d_{\mathcal{Q}}(\flat, v)$ for all $\flat, v \in Y$. Define $\vartheta, \varphi \colon \mathbb{R} \to [0, \infty)$ by $\vartheta(t) = t$, $\varphi(t) = (1-\lambda)t$ for all $t \in [0, \infty)$. Then, the mapping $\top$ satisfies condition (A) of Theorem 1. Also, by following the process used by Shukla et al. [6], one can easily verify that for $m > n$, $d_{\mathcal{Q}}(\top^n \flat_0, \top^m \flat_0) \leq \dfrac{\lambda^n}{1-\lambda} d_{\mathcal{Q}}(\flat_0, \top \flat_0) < \dfrac{1}{1-\lambda} d_{\mathcal{Q}}(\flat_0, \top \flat_0)$. Hence, there exists $\flat_0 \in Y$ such that $(\flat_0, \top \flat_0) \in \mathcal{C}_{\mathcal{H}}$, and

$$\max\{\vartheta(d_{\mathcal{Q}}(\flat, v)), \varphi(d_{\mathcal{Q}}(\flat, v))\} \leq K\left(= \frac{1}{1-\lambda} d_{\mathcal{Q}}(\flat_0, \top \flat_0)\right)$$

for all $\flat, v \in O(\top, \flat_0)$ with $(\flat, v) \in \mathcal{C}_{\mathcal{H}}$. All other conditions of Corollary 1 are satisfied; hence, $\top$ has a unique fixed point in Y. □

Remark 13. *By following similar arguments as those used by Shukla et al. [6], one can easily derive the corresponding versions of the results of Ran and Reurings [12] and Edelstein [13] in graphical symmetric spaces. Here, we omit the proofs.*

Corollary 3. *Let (Y, b) be a complete b-metric space (see, Czerwik [3]) and $\top \colon Y \to Y$ a mapping such that the following condition is satisfied: for some $\vartheta, \varphi \in \Psi$*

$$\vartheta(b(\top \flat, \top v)) \leq \vartheta(b(\flat, v)) - \varphi(b(\flat, v)), \textit{ for all } \flat, v \in Y.$$

If there exists $\flat_0 \in Y$ such that $\max\{\vartheta(b(\flat, v)), \varphi(b(\flat, v))\} \leq K$ for all $\flat, v \in O(\top, \flat_0)$, where $K \in [0, \infty)$ is a fixed number, then $\top$ has a unique fixed point in Y.

Proof. Consider the graphical symmetric space $(Y, \sigma_{\mathcal{Q}})$, where $\sigma_{\mathcal{Q}}(\flat, v) = b(\flat, v)$ for all $\flat, v \in Y$ and the universal graph $\mathcal{Q}$ where $V(\mathcal{Q}) = Y$, $E(\mathcal{Q}) = Y \times Y$. If we assume $E(\mathcal{H}) = E(\mathcal{Q})$, then it is easy to see that all the conditions of Theorem 4 are satisfied, hence, by Theorem 4, $\top$ has a unique fixed point. □

Corollary 4. *Let (Y, σ) be a 0-σ-complete metric-like space (see, Shukla et al. [11]) and $\top \colon Y \to Y$ a mapping such that the following condition is satisfied: for some $\vartheta, \varphi \in \Psi$*

$$\vartheta(\sigma(\top \flat, \top v)) \leq \vartheta(\sigma(\flat, v)) - \varphi(\sigma(\flat, v)), \textit{ for all } \flat, v \in Y.$$

If there exists $\flat_0 \in Y$ such that $\max\{\vartheta(\sigma(\flat, v)), \varphi(\sigma(\flat, v))\} \leq K$ for all $\flat, v \in O(\top, \flat_0)$, where $K \in [0, \infty)$ is a fixed number, then $\top$ has a unique fixed point in Y.

Proof. Consider the graphical symmetric space $(Y, \sigma_{\mathcal{Q}})$, where $\sigma_{\mathcal{Q}}(\flat, v) = \sigma(\flat, v)$ for all $\flat, v \in Y$ and the universal graph $\mathcal{Q}$ where $V(\mathcal{Q}) = Y$, $E(\mathcal{Q}) = Y \times Y$. If we assume $E(\mathcal{H}) = E(\mathcal{Q})$, then it is easy to see that all the conditions of Theorem 2 are satisfied, hence, by Theorem 2, the $\top$ Picard sequence $\{\flat_n\}$ with initial value $\flat_0 \in Y$ is convergent to $\top^r u$ and $\sigma(\top^r u, \top^r u) = 0$ for all $r \in \mathbb{N} \cup \{0\}$. Hence, by Remark 1 of Shukla et al. [11], we obtain $\top^r u = u$ for all $r \in \mathbb{N}$; thus, $\top$ has a fixed point in Y. The uniqueness of the fixed point follows from the fact that the graph $\mathcal{H}$ is universal. □

Let (Y, d) be a symmetric space (see Wilson [7], Alshehri et al. [5] and references therein). Since in symmetric spaces, the triangular inequality does not hold, as a replacement, the following property called (W3) is used at several places:

$$\lim_{n\to\infty} d(\flat_n, \flat) = 0 \text{ and } \lim_{n\to\infty} d(\flat_n, v) = 0 \text{ imply } \flat = v \tag{4}$$

We derive the following corollary in symmetric spaces.

Corollary 5. *Let (Υ, d) be a symmetric space (see Alshehri et al. [5] and references therein) such that for every d-Cauchy sequence $\{\flat_n\}$ in Υ, there exists $v \in \Upsilon$ such that $\lim_{n\to\infty} d(\flat_n, v) = 0$. Suppose $\top\colon \Upsilon \to \Upsilon$ is a mapping such that the following condition is satisfied: for some $\vartheta, \varphi \in \Psi$*

$$\vartheta(d(\top\flat, \top v)) \leq \vartheta(d(\flat, v)) - \varphi(d(\flat, v)), \text{ for all } \flat, v \in \Upsilon.$$

If there exists $\flat_0 \in \Upsilon$ such that $\max\{\vartheta(d(\flat, v)), \varphi(d(\flat, v))\} \leq K$ for all $\flat, v \in O(\top, \flat_0)$, where $K \in [0, \infty)$ is a fixed number, and suppose property (W3) is satisfied, then $\top$ has a unique fixed point in Υ.

Proof. Consider the graphical symmetric space $(\Upsilon, \sigma_{\mathcal{Q}})$, where $\sigma_{\mathcal{Q}}(\flat, v) = d(\flat, v)$ for all $\flat, v \in \Upsilon$ and the universal graph $\mathcal{Q}$ where $V(\mathcal{Q}) = \Upsilon$, $E(\mathcal{Q}) = \Upsilon \times \Upsilon$. Define the subgraph $\mathcal{H}$ of $\mathcal{Q}$ by: $E(\mathcal{H}) = \Upsilon \times \Upsilon$. Then, it is easy to see that conditions (A) and (B) of Theorem 2 are satisfied. Note that condition (W3) implies that condition (GS4) holds; hence, by Remark 12, the quadruple $(\Upsilon, \sigma_{\mathcal{Q}}, \mathcal{H}, \top)$ possesses the property (S). Hence, by Theorem 3, the mapping $\top$ has a fixed point in Υ. The uniqueness of the fixed point follows from the fact that the graph $\mathcal{H}$ is universal. □

A point $u \in \Upsilon$ is called a periodic point of order $k \in \mathbb{N}$ of a mapping $\top\colon \Upsilon \to \Upsilon$ if $\top^k u = u$. If $(\Upsilon, \sigma_{\mathcal{Q}})$ is a graphical symmetric space, then each edge of the graph can be considered weighted, where the weight of edge $(\flat, v) \in E(\mathcal{Q})$ is given by the quantity $\sigma_{\mathcal{Q}}(\flat, v)$. We prove the following periodic-point theorem:

Theorem 5. *Let $(\Upsilon, \sigma_{\mathcal{Q}})$ be a graphical symmetric space, $\top\colon \Upsilon \to \Upsilon$ be an $\mathcal{H}$-edge-preserving graphical $\mathcal{H}$-(ϑ, φ)-contraction and ϑ be nondecreasing. If there exists a point in ${}_k\Upsilon^{\mathcal{H}}_{\top}$ such that its corresponding edge is with minimum weight in ${}_k\Upsilon^{\mathcal{H}}_{\top}$, then $\top$ has a periodic point of order k. In addition, if ${}_k\Upsilon^{\mathcal{H}}_{\top}$ is $\mathcal{H}$-connected and the diagonal $\Delta_\Upsilon = \{(\flat, \flat)\colon \flat \in \Upsilon\} \subseteq \mathcal{C}_{\mathcal{H}}$, then the periodic point of $\top$ is unique.*

Proof. Suppose $u \in {}_k\Upsilon^{\mathcal{H}}_{\top}$ is such that its corresponding edge is with minimum weight in ${}_k\Upsilon^{\mathcal{H}}_{\top}$, that is,

$$\sigma_{\mathcal{Q}}(u, \top^k u) \leq \sigma_{\mathcal{Q}}(v, \top^k v) \text{ for all } v \in {}_k\Upsilon^{\mathcal{H}}_{\top} \tag{5}$$

where $k \in \mathbb{N}$ is fixed. We show that u is a periodic point of $\top$ of order k. Then, as $u \in {}_k\Upsilon^{\mathcal{H}}_{\top}$, we have $(u, \top^k u) \in \mathcal{C}_{\mathcal{H}}$ and since $\top$ is $\mathcal{H}$-edge-preserving, we have $(\top u, \top^k \top u) \in \mathcal{C}_{\mathcal{H}}$, hence $\top u \in {}_k\Upsilon^{\mathcal{H}}_{\top}$. Let $W_k(v) = \sigma_{\mathcal{Q}}(v, \top^k v)$ for all $v \in \Upsilon$ and suppose that $W_k(u) > 0$ (otherwise, $\sigma_{\mathcal{Q}}(u, \top^k u) = 0$ implies $\top^k u = u$ and we are done) then the inequality (5) shows that $0 < W_k(u) \leq W_k(v)$ for all $v \in {}_k\Upsilon^{\mathcal{H}}_{\top}$.

Since $(u, \top^k u) \in \mathcal{C}_{\mathcal{H}}$, $\top$ is a graphical $\mathcal{H}$-(ϑ, φ)-contraction and $W_k(u) = \sigma_{\mathcal{Q}}(u, \top^k u) > 0$, we have

$$\begin{aligned} \vartheta(\sigma_{\mathcal{Q}}(\top u, \top\top^k u)) &\leq \vartheta(\sigma_{\mathcal{Q}}(u, \top^k u)) - \varphi(\sigma_{\mathcal{Q}}(u, \top^k u)) \\ &< \vartheta(\sigma_{\mathcal{Q}}(u, \top^k u)). \end{aligned}$$

As ϑ is nondecreasing, the above inequality shows that $\sigma_{\mathcal{Q}}(\top u, \top\top^k u) < \sigma_{\mathcal{Q}}(u, \top^k u)$, that is, $W_k(\top u) < W_k(u)$. This is a contradiction because $\top u \in {}_k\Upsilon^{\mathcal{H}}_{\top}$. Hence, we must have $W_k(u) = 0$, that is, $\sigma_{\mathcal{Q}}(u, \top^k u) = 0$. Thus, $\top^k u = u$.

For the uniqueness of the periodic point, on the contrary, suppose that there is a periodic point $v \in \Upsilon$ of $\top$ of order k, and $u \neq v$. Hence, we have $\top^k u = u, \top^k v = v$. Since $(u, \top^k u) = (u, u)$,$(v, \top^k v) = (v, v) \in \Delta_\Upsilon = \{(\flat, \flat)\colon \flat \in \Upsilon\} \subseteq \mathcal{C}_{\mathcal{H}}$ and $\top$ is $\mathcal{H}$-edge-preserving, we have $(\top^n u, \top^n v) \in \mathcal{C}_{\mathcal{H}}$ for all $n \in \mathbb{N}$ and $u, v \in {}_k\Upsilon^{\mathcal{H}}_{\top}$; also, ${}_k\Upsilon^{\mathcal{H}}_{\top}$ is

$\mathcal{H}$-connected; hence, suppose that $(u, v) \in \mathcal{C}_{\mathcal{H}}$ (proof for the case $(v, u) \in \mathcal{C}_{\mathcal{H}}$ is same). As $\top$ is a graphical $\mathcal{H}$-(ϑ, φ)-contraction, we have

$$\begin{aligned} \vartheta(\sigma_{\mathcal{Q}}(u, v)) &= \vartheta(\sigma_{\mathcal{Q}}(\top^k u, \top^k v)) \\ &\leq \vartheta(\sigma_{\mathcal{Q}}(\top^{k-1} u, \top^{k-1} v)) - \varphi(\sigma_{\mathcal{Q}}(\top^{k-1} u, \top^{k-1} v)) \\ &\leq \vartheta(\sigma_{\mathcal{Q}}(\top^{k-1} u, \top^{k-1} v)). \end{aligned}$$

A repetition of this process gives

$$\vartheta(\sigma_{\mathcal{Q}}(u, v)) \leq \vartheta(\sigma_{\mathcal{Q}}(\top u, \top v)) \leq \vartheta(\sigma_{\mathcal{Q}}(u, v)) - \varphi(\sigma_{\mathcal{Q}}(u, v)) < \vartheta(\sigma_{\mathcal{Q}}(u, v)).$$

This shows that $\varphi(\sigma_{\mathcal{Q}}(u, v)) = 0$, and since $\varphi \in \Psi$, we must have $\sigma_{\mathcal{Q}}(u, v) = 0$, that is, $u = v$. This contradiction shows that $u \in \Upsilon$ is the unique periodic point of $\top$ of order k. □

Corollary 6. *Let $(\Upsilon, \sigma_{\mathcal{Q}})$ be a graphical symmetric space, $\top\colon \Upsilon \to \Upsilon$ be an $\mathcal{H}$-edge-preserving graphical $\mathcal{H}$-(ϑ, φ)-contraction and ϑ be nondecreasing. If there exists a point in ${}_1\Upsilon_{\top}^{\mathcal{H}}$ such that its corresponding edge is with minimum weight in ${}_1\Upsilon_{\top}^{\mathcal{H}}$, then $\top$ has a fixed point in Υ. In addition, if ${}_k\Upsilon_{\top}^{\mathcal{H}}$ is $\mathcal{H}$-connected and the diagonal $\Delta_\Upsilon = \{(\flat, \flat)\colon \flat \in \Upsilon\} \subseteq \mathcal{C}_{\mathcal{H}}$, then the fixed point of $\top$ is unique.*

Proof. Since a fixed point of mapping $\top$ is a periodic point of $\top$ of order one, hence the existence and uniqueness of the fixed point of $\top$ follows from Theorem 5. □

4. Positive Solution of Fractional Periodic Boundary Value Problems

In this section, we apply the results of the previous section to the problem of the existence of positive solutions of a fractional periodic boundary value problem involving Caputo's derivatives of fractional order. We consider the following fractional periodic boundary value problem for positive solutions:

$${}_0^C D_t^\alpha \flat(t) = h(t, \flat(t)), t \in [0, 1] \tag{6}$$

$$\flat(0) = \flat(1) = a, a \geq 0 \tag{7}$$

where ${}_0^C D_t^\alpha$ represents the Caputo derivatives of fractional order α, and $\flat\colon [0, 1] \to \mathbb{R}$ and $h\colon [0, 1] \times \mathbb{R} \to \mathbb{R}$ are some functions. We first state some definitions and facts about the Riemann–Liouville fractional integral and Caputo's derivatives of fractional order.

Definition 8 (Samko et al. [14], Podlubny [15])**.** *The Riemann–Liouville fractional integral of a function $\flat \in C_{\mathbb{R}}[b, c]$ of order $\alpha \in (0, \infty)$ is given by*

$${}_b I_t^\alpha [\flat(t)] = \frac{1}{\Gamma(\alpha)} \int_b^t (t - s)^{\alpha - 1} \flat(s) ds.$$

Definition 9 (Podlubny [15])**.** *Caputo's derivative of a function $\flat\colon [0, \infty) \to \mathbb{R}$ of order $\alpha \in (0, \infty)$ is given by*

$${}_b^C D_t^\alpha \flat(t) = {}_b I_t^{n-\alpha} \left[\frac{d^n}{dt^n} \flat(t) \right] = \frac{1}{\Gamma(n - \alpha)} \int_b^t (t - s)^{n - \alpha - 1} \flat^{(n)}(s) ds$$

where $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of α.

Remark 14. *It is obvious that if $\flat(t) = t^{n-1}$, $n \in \mathbb{N}$, then ${}_b^C D_t^\alpha \flat(t) = 0$.*

Definition 10. *Let $\mathcal{P}$ be the cone in $C_{\mathbb{R}}[0, 1]$ defined by*

$$\mathcal{P} = \{\flat(t) \in C_{\mathbb{R}}[0, 1]\colon \flat(t) \geq 0 \textit{ for all } t \in [0, 1]\}.$$

Then, a function $\flat \in C_{\mathbb{R}}[0,1]$ is called a positive solution of (6) *and* (7) *if* $\flat \in \mathcal{P}$ *and* ${}_0^C D_t^\alpha \flat(t) = h(t, \flat(t)), t \in [0,1]$ *and* $\flat(0) = \flat(1) = a$.

The following lemmas play an important role in establishing the existence results for the periodic boundary value problem (6) and (7).

Lemma 1 (Zhang [16])**.** *Let $\alpha > 0$; then, the fractional differential equation ${}_b^C D_t^\alpha \flat(t) = 0$ has a solution $u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, where $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$ and $n = [\alpha] + 1$.*

Lemma 2 (Zhang [16])**.** *Let $\alpha > 0$; then,*

$$ {}_b I_t^\alpha \left[{}_b^C D_t^\alpha \flat(t) \right] = \flat(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1} $$

where $n = [\alpha] + 1$, $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$.

Lemma 3. *Let $1 \leq \alpha < 2$, and let $h \colon [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then, $\flat \in C_{\mathbb{R}}[0,1]$ is a solution of* (6) *and* (7) *if and only if it is a solution of the following fractional integral equation*

$$ \flat(t) = a - \frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s, \flat(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s, \flat(s)) ds. $$

Proof. Suppose $\flat(t)$ is a solution of (6) and (7); then, by Lemma 2, we have for some $a_0, a_1 \in \mathbb{R}$:

$$ \begin{aligned} \flat(t) &= a_0 + a_1 t + {}_0 I_t^\alpha \left[{}_0^C D_t^\alpha \flat(t) \right] \\ &= a_0 + a_1 t + {}_0 I_t^\alpha \left[h(t, \flat(t)) \right] \\ &= a_0 + a_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s, \flat(s)) ds. \end{aligned} $$

Therefore,

$$ \flat(t) = a_0 + a_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s, \flat(s)) ds. \tag{8} $$

Applying boundary condition (7) in the above equation, we obtain

$$ a = \flat(0) = a_0; \; a = \flat(1) = a_0 + a_1 + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s, \flat(s)) ds $$

that is, $a_0 = a$ and $a_1 = -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s, \flat(s)) ds$. These values with (8) proves the desired result. □

We next prove that under some particular conditions, the fractional boundary value problem (6) and (7) has a positive solution.

In the rest of the discussion, we assume that $p > 0$ is a fixed number, $r = 2a$ (where a is the constant considered in (7)), $\Upsilon = C_{\mathbb{R}}[0,1]$, and the set $B_{\mathcal{P}}(r)$ is given by

$$ B_{\mathcal{P}}(r) = \{ \flat \in \mathcal{P} \colon \flat(t) \leq r \text{ for all } t \in [0,1] \}. $$

Theorem 6. *Assume that the following conditions are satisfied:*

(i) *There exist a nondecreasing continuous function $\chi \colon [0, \infty) \to [0, \infty)$ such that $\chi(t) < t$ for all $t > 0$ and*

$$ |h(t, \flat(t)) - h(t, v(t))| \leq \frac{\Gamma(1+\alpha)}{4} [\chi(|\flat(t) - v(t)|^p)]^{1/p} \tag{9} $$

for all $\flat, v \in B_{\mathcal{P}}(r)$*;*

(ii) $0 \leq h(t, \flat(t)) \leq a\Gamma(1+\alpha)$ *for all* $t \in [0,1]$ *and* $\flat \in B_{\mathcal{P}}(r)$.

Then, the fractional boundary value problem (6) *and* (7) *has a positive solution.*

Proof. We convert the fractional boundary value problem (6) and (7) into the equivalent fixed-point problem as follows: define $\top\colon \Upsilon \to \Upsilon$ by

$$\top\flat = a - \frac{t}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} h(s,\flat(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s,\flat(s))ds.$$

Then, it is clear that $\flat(t)$ is a solution of (6) and (7) if and only if it is a fixed point of $\top$.

Consider the graphs $\mathcal{Q}$ and $\mathcal{H}$ defined by $V(\mathcal{Q}) = V(\mathcal{H}) = \Upsilon$, $E(\mathcal{Q}) = E(\mathcal{H}) = B_{\mathcal{P}}(r) \times B_{\mathcal{P}}(r)$ and the function $\sigma_{\mathcal{Q}}\colon \Upsilon \times \Upsilon \to \mathbb{R}$ defined by

$$\sigma_{\mathcal{Q}}(\flat, v) = \sup_{t\in[0,1]} |\flat(t) - v(t)|^p \text{ for all } \flat, v \in \Upsilon.$$

Then, $\sigma_{\mathcal{Q}}$ is a graphical symmetry on Υ, and the pair $(\Upsilon, \sigma_{\mathcal{Q}})$ is a 0-$\mathcal{H}$-$\sigma_{\mathcal{Q}}$-complete graphical symmetric space. Let $\vartheta, \varphi\colon [0,\infty) \to [0,\infty)$ be defined by

$$\vartheta(t) = t \text{ and } \varphi(t) = \left(1 - \frac{1}{2^p}\right)t \text{ for all } t \in [0,\infty).$$

Since $p > 0$, we have $\vartheta, \varphi \in \Psi$.

We show that $\top$ is an $\mathcal{H}$-edge-preserving graphical $\mathcal{H}$-(ϑ,φ)-contraction.

It is easy to see that $B_{\mathcal{P}}(r) \neq \emptyset$. Consider $\flat \in B_{\mathcal{P}}(r)$. Since (ii) holds, we have

$$\begin{aligned}
\top\flat &= a - \frac{t}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} h(s,\flat(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s,\flat(s))ds \\
&\geq a - \frac{t}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} \cdot a\Gamma(1+\alpha)ds \\
&= a - \frac{ta\Gamma(1+\alpha)}{\alpha\Gamma(\alpha)} \\
&\geq 0
\end{aligned}$$

for all $t \in [0,1]$. Also,

$$\begin{aligned}
\top\flat &= a - \frac{t}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} h(s,\flat(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s,\flat(s))ds \\
&\leq a + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \cdot a\Gamma(1+\alpha)ds \\
&= a + \frac{t^\alpha a\Gamma(1+\alpha)}{\alpha\Gamma(\alpha)} \\
&\leq 2a = r.
\end{aligned}$$

for all $t \in [0,1]$. Hence, $\top(B_{\mathcal{P}}(r)) \subseteq B_{\mathcal{P}}(r)$. This shows that $\top$ is $\mathcal{H}$-edge-preserving.

Suppose $(\flat, v) \in \mathcal{C}_{\mathcal{H}}$; then, by definition $\flat, v \in B_{\mathcal{P}}(r)$, and by (ii), we have

$$\begin{aligned}
|\top\flat - \top v| &= \bigg| -\frac{t}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} h(s,\flat(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s,\flat(s))ds \\
&\quad + \frac{t}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} h(s,v(s))ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(s,v(s))ds \bigg| \\
&\leq \frac{t}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} |h(s,\flat(s)) - h(s,v(s))|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} |h(s,\flat(s)) - h(s,v(s))|ds.
\end{aligned}$$

Using (i) in the above inequality, we obtain

$$\begin{aligned} |\top\flat - \top v| &\leq \frac{\Gamma(1+\alpha)}{4\Gamma(\alpha)}[\chi(\sigma_{\mathcal{Q}}(\flat, v))]^{1/p}\left[t\int_0^1 (1-s)^{\alpha-1}ds + \int_0^t (t-s)^{\alpha-1}ds\right] \\ &= \frac{\Gamma(1+\alpha)}{4\Gamma(\alpha)}[\chi(\sigma_{\mathcal{Q}}(\flat, v))]^{1/p}\left[\frac{t}{\alpha} + \frac{t^\alpha}{\alpha}\right] \\ &\leq \frac{1}{2}[\chi(\sigma_{\mathcal{Q}}(\flat, v))]^{1/p}. \end{aligned}$$

for all $t \in [0,1]$. Suppose that $\sigma_{\mathcal{Q}}(\flat, v) > 0$. Then, the above inequality shows that $\sigma_{\mathcal{Q}}(\top\flat, \top v) \leq \frac{1}{2^p}\chi(\sigma_{\mathcal{Q}}(\flat, v)) < \frac{1}{2^p}\sigma_{\mathcal{Q}}(\flat, v)$. As $\vartheta(t) = t$, $\varphi(t) = \left(1 - \frac{1}{2^p}\right)t$, we have

$$\vartheta(\sigma_{\mathcal{Q}}(\top\flat, \top v)) \leq \vartheta(\sigma_{\mathcal{Q}}(\flat, v)) - \varphi(\sigma_{\mathcal{Q}}(\flat, v)).$$

If $\sigma_{\mathcal{Q}}(\flat, v) = 0$, then the above conditions follows trivially.

Note that for any $\flat_0 \in B_{\mathcal{P}}(r)$, we have $\top\flat_0 \in B_{\mathcal{P}}(r)$; hence, $(\flat_0, \top\flat_0) \in \mathcal{C}_{\mathcal{H}}$. Also, if $\flat_0 \in B_{\mathcal{P}}(r)$ and $\flat, v \in O(\top, \flat_0)$, then $\flat, v \in B_{\mathcal{P}}(r)$, that is, $0 \leq \flat(t), v(t) \leq r$ for all $t \in [0,1]$. Hence,

$$\sigma_{\mathcal{Q}}(\flat, v) = \sup_{t\in[0,1]} |\flat(t) - v(t)|^p \leq (2r)^p.$$

Therefore, $O(\top, \flat_0)$ is $\mathcal{H}$-bounded with respect to both ϑ and φ. It is easy to see that if an $\mathcal{H}$-termwise-connected $\top$-Picard sequence $\{z_n\}$ is $\sigma_{\mathcal{Q}}$-convergent to some $z \in \Upsilon$, then we must have $z_n, z \in B_{\mathcal{P}}(r)$, so by the definition, we have $(z_n, z) \in \mathcal{C}_{\mathcal{H}}$ for all $n \in \mathbb{N}$. Finally, one can see that the quadruple $(\Upsilon, \sigma_{\mathcal{Q}}, \mathcal{H}, \top)$ possesses property (S).

Thus, all the conditions of Theorem 3 are satisfied; hence, $\top$ has a fixed point u which is also a solution of (6) and (7). From the proof of Theorem 3, it is clear that this fixed point u is a limit of an $\mathcal{H}$-termwise-connected $\top$-Picard sequence, so $u \in B_{\mathcal{P}}(r)$; therefore, u is a positive solution of (6) and (7). □

5. Conclusions

In this work, we presented a new idea of graphical symmetric spaces and proved some fixed-point results for $\mathcal{H}$-edge-preserving mappings. The class of these spaces is broad and includes various known classes of distance spaces. The fixed-point results in graphical symmetric spaces are more general than the existing results for such mappings in the terms of both the broader domain of graphical symmetric spaces and the contractive condition applied on the mappings under consideration. In particular, we generalized the concepts of spaces given in [2,3,5–7] and extended the fixed-point results proved by Shukla et al. [6], Ran and Reurings [12] and Edelstein [13]. We showed that the $\mathcal{H}$-edge-preserving mappings and fixed-point results for such mappings in graphical symmetric spaces could be applied to find the positive solutions of fractional periodic boundary value problems. A further investigation of applications of these results (or generalized versions of these results) can be conducted to ensure the solutions of fractional integral inclusion systems, fractional difference equations, etc. Another possible extension of the results of this paper is to establish the fixed figure or fixed circle problem (see, [17]) and to apply it to the related geometric problems.

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