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The Effect of Vertex and Edge Removal on Sombor Index

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Abstract: A vertex degree based topological index called the Sombor index was recently defined in 2021 by Gutman and has been very popular amongst chemists and mathematicians. We determine the amount of change of the Sombor index when some elements are removed from a graph. This is done for several graph elements, including a vertex, an edge, a cut vertex, a pendant edge, a pendant path, and a bridge in a simple graph. Also, pendant and non-pendant cases are studied. Using the obtained formulae successively, one can find the Sombor index of a large graph by means of the Sombor indices of smaller graphs that are just graphs obtained after removal of some vertices or edges. Sometimes, using iteration, one can manage to obtain a property of a really large graph in terms of the same property of many other subgraphs. Here, the calculations are made for a pendant and non-pendant vertex, a pendant and non-pendant edge, a pendant path, a bridge, a bridge path from a simple graph, and, finally, for a loop and a multiple edge from a non-simple graph. Using these results, the Sombor index of cyclic graphs and tadpole graphs are obtained. Finally, some Nordhaus–Gaddum type results are obtained for the Sombor index.

Keywords: Sombor index; vertex removal; edge removal; Nordhaus–Gaddum type result

MSC: 05C07; 05C10; 05C30; 68R10

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1. Introduction

Let $G = (V, E)$ be a graph without loops or multiple edges having the vertex and edge sets as $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $E(G) = \{v_i v_j : v_i, v_j \in V(G)\}$, respectively. We call $|V(G)| = n$ and $|E(G)| = m$ to be the order and the size of G. Sometimes, we use $G(n, m)$ in place of G to emphasize the order and size of G. If v_i and v_j are two end vertices of an edge e of G, this is denoted by $e = v_i v_j$. Then, the vertices v_i and v_j are adjacent, and e is said to be incident with these vertices. Incidency and adjacency are frequently used in spectral graph theory. The degree of $v \in V(G)$ is denoted by $d_G(v)$ or, briefly, dv. The smallest and largest degrees of all vertices will be denoted by δ and Δ, respectively. The neighbourhood of a vertex is defined as $N_G(u) = \{v \in V(G) : v \text{ and u are adjacent in } G\}$. Neighbourhood degree sum of a vertex u is defined as $\delta_G(u) = \sum\limits_{v \in N_G(u)} d_G(v)$. Let $\delta_G(v)$ be the sum of all vertex degrees in the neighbourhood of v.

Graph theory is becoming increasingly popular due to its possible applications in chemistry, pharmacology, physics, neuroscience, network science, and many other areas. Each day, new areas are added to the list. This popularity is due to the fact that a molecule or a social science application can be modeled by a graph. For molecules, one can obtain such a graph by replacing each atom by a vertex and adding an edge between two atoms if there is a chemical bond between those atoms. Graphs obtained in such a way are called chemical (molecular) graphs. Once modeled, we can study this graph by mathematical methods using the existing combinatorial, number theoretical, topological, linear algebraic, etc., methods. At the end of such mathematical calculations, we obtain a number that is characteristic of the graph under consideration. Such numbers are actually invariants and they remain the same under isomorphism. The main step is to establish some exact result, an upper or lower bound, or at least some regression between the obtained mathematical number and the same physico-chemical property of the molecule. This step brings together chemistry and mathematics. The oldest known example is the Wiener index, introduced in 1947 by chemist Harold Wiener to determine the boiling points of some alkane isomers. Today, there are more than 3000 such mathematical formulae to study properties of molecules. Mathematicians call them graph theoretical indices or topological graph indices, and chemists call them molecular descriptors. Today, a serious part of research related to graph theory is published on such descriptors and indices.

2. Materials and Methods

The Sombor index was recently defined in [1] by Gutman as

$$SO(G) = \sum_{e=uv \in E(G)} \sqrt{du^2 + dv^2}.$$

Gutman studied some mathematical and chemical properties of this index in [2,3]. In parallel with these studies, many other researchers also considered various mathematical properties of the Sombor index. In [4], extremal values of the Sombor index were obtained for molecular trees. In [5], the block Sombor index of graphs and their matrix representations were studied. In [6], some mathematical properties of the Sombor index were obtained. In [7], Sombor indices were computed for several networks. In [8], some extremal values were obtained mathematically. In [9], the Sombor index was calculated for c-cyclic graphs. In [10], the mean value of the Sombor index was studied using elementary number theoretical results. In [11], the integer values of the Sombor index were studied by means of extensive use of Pythagorean triples. In [12], some more mathematical properties of the Sombor index were studied. In [13], a spectral study of the Sombor index was done; this new index is related to the graph energy. In [14], Sombor indices of some graph products of some algebraic graphs were considered. Chemical applications of the Sombor index were also considered by some authors. In [15], the Sombor index was calculated for polymers. In [16], Sombor indices of the line graphs of some silicates were studied. In [17], some molecular properties of the Sombor index were obtained. In [18], this index was calculated for some nanotubes. In [19], random hexagonal chains, phenylene chains, and Sombor indices of some chemical graphs were studied. In [20], Sombor indices of chemical graphs were calculated, and their applications to the boiling points of benzenoid hydrocarbons were studied in [21]. Also in [22], the Sombor index was used to predict physicochemical properties of butane derivatives. After defining the Sombor index, its modified version was put forward in [23] and studied in [24–27].

For an r-regular graph G, the Sombor index is equal to $SO(G) = \sqrt{2}mr$. Also, as the sum of the degrees of vertices is twice the number of edges, we can restate this as $SO(G) = 2\sqrt{2}m^2/n$.

In this paper, we use the ingenious methods that are used in mathematics to calculate large mathematical objects by means of smaller objects that are easier to calculate. These methods are vertex and edge removal. Here, we shall determine how much gthe Sombor index changes when an edge or a vertex is deleted.

3. Results

3.1. Effect of Vertex Removal on the Sombor Index

We now determine how much $SO(G)$ changes when a vertex is deleted from G. According to the enumarations we do with different graphs, there are two different cases where the vertex to be deleted is pendant or not. We shall see those two cases seperately below.

Theorem 1. *Let $v \in V(G)$ be a vertex of degree $dv > 1$. Then*

$$\begin{aligned} SO(G) - SO(G-v) &= \sum_{\substack{vw \in E(G) \\ w \in N_G(v)}} \sqrt{dv^2 + dw^2} + \sum_{\substack{uw \in E(G) \\ u \in N_G(v) \\ d(w,v)=2}} \left[\sqrt{du^2+dw^2} - \sqrt{(du-1)^2+dw^2}\right] \\ &+ \sum_{\substack{uw \in E(G) \\ u,w \in N_G(v)}} \left[\sqrt{du^2+dw^2} - \sqrt{(du-1)^2+(dw-1)^2}\right]. \end{aligned}$$

Proof. From the definition of the Sombor index, we can partition the edges of G into four families: (i) $uw \in E(G)$ such that $v \neq u, w \notin N_G(v)$, (ii) $uw \in E(G)$ such that $u \in N_G(v)$, $d(w,v) = 2$, (iii) $uw \in E(G)$ so that $u, w \in N_G(v)$, and (iv) $vw \in E(G)$ such that $w \in N_G(v)$. By means of this edge partition, we can alternatively rephrase $SO(G)$ as

$$\begin{aligned} SO(G) &= \sum_{\substack{uw \in E(G) \\ v \neq u, w \notin N_G(v)}} \sqrt{du^2+dw^2} + \sum_{\substack{uw \in E(G) \\ u \in N_G(v) \\ d(w,v)=2}} \sqrt{du^2+dw^2} \\ &+ \sum_{\substack{uw \in E(G) \\ u,w \in N_G(v)}} \sqrt{du^2+dw^2} + \sum_{\substack{vw \in E(G) \\ w \in N_G(v)}} \sqrt{dv^2+dw^2}. \end{aligned}$$

If we remove a non-pendant vertex v from the graph G, then the edge partition of $G - v$ would be (i) $uw \in E(G)$ such that $v \neq u, w \notin N_G(v)$, (ii) $uw \in E(G)$ such that $u \in N_G(v)$, $d(w,v) = 2$, and (iii) $uw \in E(G)$ such that $u, w \in N_G(v)$. That is, only the edges of type (iv) will disappear. Hence, the Sombor index of the remaining graph $G - v$ is

$$\begin{aligned} SO(G-v) &= \sum_{\substack{uw \in E(G) \\ v \neq u, w \notin N_G(v)}} \sqrt{du^2+dw^2} + \sum_{\substack{uw \in E(G) \\ u \in N_G(v) \\ d(w,v)=2}} \sqrt{(du-1)^2+dw^2} \\ &+ \sum_{\substack{uw \in E(G) \\ u,w \in N_G(v)}} \sqrt{(du-1)^2+(dw-1)^2}. \end{aligned}$$

The desired result follows. □

Using this theorem, we can directly deduce an upper bound for the change in the Sombor index when a non-pendant vertex is deleted from a graph:

Corollary 1. *Let $v \in V(G)$ be of degree $dv > 1$. Let $\delta_G(v)$ be as above. Let $A = \sqrt{2\delta^2 - 2\delta + 1}$. If there are t pairs of vertices in the neighbourhood of v forming an edge of G, then*

$$SO(G) - SO(G-v) \leq (dv + 2t - \delta_G(v))A + \sqrt{2}[t(1-\delta-\Delta) + \delta_G(v)\Delta]. \tag{1}$$

Proof. From Theorem 1

$$\begin{aligned} SO(G) - SO(G-v) &\leq \sum_{\substack{vw \in E(G) \\ w \in N_G(v)}} \sqrt{2}\Delta + \sum_{\substack{uw \in E(G) \\ u \in N_G(v) \\ d(w,v)=2}} \left[\sqrt{2}\Delta - \sqrt{(\delta-1)^2+\delta^2}\right] \\ &+ \sum_{\substack{uw \in E(G) \\ u,w \in N_G(v)}} \left[\sqrt{2}\Delta - \sqrt{2}(\delta-1)\right] \\ &= dv\sqrt{2}\Delta + (\delta_G(v) - dv - 2t)\left(\sqrt{2}\Delta - \sqrt{2\delta^2-2\delta+1}\right) \\ &+ t\left(\sqrt{2}(\Delta-\delta+1)\right) \\ &= dv\sqrt{2}\Delta + \delta_G(v)\sqrt{2}\Delta - \delta_G(v)\sqrt{2\delta^2-2\delta+1} - dv\sqrt{2}\Delta \\ &+ dv\sqrt{2\delta^2-2\delta+1} - 2\sqrt{2}t\Delta + 2t\sqrt{2\delta^2-2\delta+1} + t\sqrt{2}\Delta \\ &- t\delta\sqrt{2} + t\sqrt{2}. \end{aligned}$$

Since $A = \sqrt{2\delta^2 - 2\delta + 1}$, we have

$$SO(G) - SO(G - v) \leq A(dv + 2t - \delta_G(v)) + \sqrt{2}(t(1 - \delta - \Delta) + \delta_G(v)\Delta).$$

□

Corollary 2. *Let G be a tree and let $v \in V(G)$ be of degree $dv > 1$. Let $\delta_G(v)$ be as above. If there are t pairs of vertices in the neighbourhood of v forming an edge of G, then*

$$SO(G) - SO(G - v) \leq dv + 2t - \delta_G(v) + \sqrt{2}\Delta(\delta_G(v) - t).$$

The proof depends on the fact that in a tree, $\delta = 1$ and, hence, $A = 1$. Note that Corollary 2 is also valid when the graph has at least one pendant vertex but is not a tree.

Now, we give results for deleting a pendant vertex from a graph:

Theorem 2. *If $v \in V(G)$ is a pendant vertex, then*

$$SO(G) - SO(G - v) = \sqrt{1 + dw^2} + \sum_{\substack{uw \in E(G) \\ u \in N_G(v) \\ d(w,v)=2}} \left[\sqrt{du^2 + dw^2} - \sqrt{(du - 1)^2 + dw^2}\right].$$

That is, the formula in Theorem 1 simplifies.

Corollary 3. *If $v \in V(G)$ is a pendant vertex and u is its support vertex, then*

$$SO(G) - SO(G - v) \leq \sqrt{2}\Delta du - (du - 1)A$$

where A is given in Corollary 1.

Proof. Using Theorem 2 and the formula $A = \sqrt{2\delta^2 - 2\delta + 1}$, we have

$$\begin{aligned} SO(G) - SO(G - v) &= \sqrt{1 + dw^2} + \sum_{\substack{uw \in E(G) \\ u \in N_G(v) \\ d(w,v)=2}} \left[\sqrt{du^2 + dw^2} - \sqrt{(du - 1)^2 + dw^2}\right] \\ &\leq \sum_{\substack{uw \in E(G) \\ w \in N_G(v)}} \sqrt{2}\Delta + \sum_{\substack{uw \in E(G) \\ u \in N_G(v) \\ d(w,v)=2}} \left[\sqrt{2}\Delta - \sqrt{(\delta - 1)^2 + \delta^2}\right] \\ &\leq \sqrt{2}\Delta dv + (du - 1)\left[\sqrt{2}\Delta - A\right]. \end{aligned}$$

Hence, the result follows. □

3.2. Effect of Edge Removal on the Sombor Index

In this section, we will determine the change in the Sombor index when we remove an edge from graph G. First, we check the effect of deleting a pendant edge.

Theorem 3. *If $e = uv \in E(G)$ is a pendant edge with pendant vertex v, then*

$$SO(G) - SO(G - e) = \sqrt{du^2 + 1} - \sum_{\substack{uw \in E(G) \\ v \neq w \in N_G(u)}} \left[\sqrt{(du - 1)^2 + dw^2} - \sqrt{du^2 + dw^2}\right].$$

Proof. Using the definition of $SO(G)$, we reorganize it as

$$\begin{aligned} SO(G) &= \sum_{\substack{rs\in E(G) \\ u\neq r,s\notin N_G(u)}} \sqrt{dr^2+ds^2} + \sum_{\substack{uw\in E(G) \\ w\in N_G(u)}} \sqrt{du^2+dw^2} \\ &= \sum_{\substack{rs\in E(G) \\ u\neq r,s\notin N_G(u)}} \sqrt{dr^2+ds^2} + \sum_{\substack{uw\in E(G) \\ v\neq w\in N_G(u)}} \sqrt{du^2+dw^2} + \sqrt{du^2+1}. \end{aligned}$$

If we remove a pendant edge $e = uv$ with pendant vertex v, then we get

$$SO(G-e) = \sum_{\substack{rs\in E(G) \\ u\neq r,s\notin N_G(u)}} \sqrt{dr^2+ds^2} + \sum_{\substack{uw\in E(G) \\ v\neq w\in N_G(u)}} \sqrt{(du-1)^2+dw^2}.$$

Hence, the result follows. □

Theorem 3 implies that it is possible to obtain the maximum value of the decrease in the Sombor index when a pendant edge is deleted from the graph:

Corollary 4. *For a graph G and a pendant edge $e = uv$ with pendant vertex v, we have*

$$SO(G) - SO(G-e) \leq (du-1)(\sqrt{2}\Delta - A) + \sqrt{\Delta^2+1}.$$

Proof. By Theorem 3, we have

$$\begin{aligned} SO(G) - SO(G-e) &= \sum_{\substack{uw\in E(G) \\ v\neq w\in N_G(u)}} \left[\sqrt{du^2+dw^2} - \sqrt{(du-1)^2+dw^2}\right] + \sqrt{du^2+1} \\ &\leq \sum_{\substack{uw\in E(G) \\ v\neq w\in N_G(u)}} (\sqrt{2}\Delta - A) + \sqrt{\Delta^2+1}. \end{aligned}$$

Since there are $du - 1$ edges in the neighbourhood of the vertex u, the result follows. □

The next result gives a similar formula for the amount of change in the Sombor index of a graph when a non-pendant edge is deleted:

Theorem 4. *Let $e = uv \in E(G)$ be a non-pendant edge. Then*

$$\begin{aligned} SO(G) - SO(G-e) &= \sqrt{du^2+dv^2} - \sum_{\substack{uw\in E(G) \\ v\neq w\in N_G(u)}} \left[\sqrt{(du-1)^2+dw^2} - \sqrt{du^2+dw^2}\right] \\ &\quad - \sum_{\substack{vw\in E(G) \\ u\neq w\in N_G(v)}} \left[\sqrt{(dv-1)^2+dw^2} - \sqrt{dv^2+dw^2}\right]. \end{aligned}$$

Proof. By the definition of the Sombor index, we can group the edges in G as follows:

$$\begin{aligned} SO(G) &= \sum_{\substack{rs\in E(G) \\ u\neq r\notin N_G(u) \\ v\neq s\notin N_G(v)}} \sqrt{dr^2+ds^2} + \sum_{\substack{uw\in E(G) \\ v\neq w\in N_G(u)}} \sqrt{du^2+dw^2} \\ &\quad + \sum_{\substack{vw\in E(G) \\ u\neq w\in N_G(v)}} \sqrt{dv^2+dw^2} + \sqrt{du^2+dv^2}. \end{aligned}$$

If we remove a non-pendant edge e from the graph G, the Sombor index of graph $G-e$ becomes

$$
\begin{aligned}
SO(G-e) &= \sum_{\substack{rs\in E(G)\\ u\neq r\notin N_G(u)\\ v\neq s\notin N_G(v)}} \sqrt{dr^2+ds^2} + \sum_{\substack{uw\in E(G)\\ v\neq w\in N_G(u)}} \sqrt{(du-1)^2+dw^2} \\
&+ \sum_{\substack{vw\in E(G)\\ u\neq w\in N_G(v)}} \sqrt{(dv-1)^2+dw^2}.
\end{aligned}
$$

Hence, the result is obtained. □

The following result giving the maximum amount of change in the Sombor index of a graph in terms of the size of the graph when a non-pendant edge is deleted from the graph can be deduced from the above results:

Corollary 5. *Let e be a non-pendant edge in G. Then*

$$SO(G) - SO(G-e) \leq \sqrt{2}m\Delta - (m-1)A.$$

Proof. We have

$$
\begin{aligned}
SO(G)-SO(G-e) &= \sum_{\substack{uw\in E(G)\\ v\neq w\in N_G(u)}} \left[\sqrt{du^2+dw^2} - \sqrt{(du-1)^2+dw^2}\right] \\
&+ \sum_{\substack{vw\in E(G)\\ u\neq w\in N_G(v)}} \left[\sqrt{dv^2+dw^2} - \sqrt{(dv-1)^2+dw^2}\right] + \sqrt{du^2+dv^2} \\
&\leq \sum_{\substack{uw\in E(G)\\ v\neq w\in N_G(u)}} \left(\sqrt{2}\Delta - A\right) + \sum_{\substack{vw\in E(G)\\ u\neq w\in N_G(v)}} \left(\sqrt{2}\Delta - A\right) + \sqrt{2}\Delta \\
&= (m-1)\left(\sqrt{2}\Delta - A\right) + \sqrt{2}\Delta
\end{aligned}
$$

giving the required result. □

3.3. Effect of Bridge Removal on the Sombor Index

In many calculations with graphs, cut vertices and bridges help us to do the calculations much more easily, as they partition the graph into blocks that are much smaller than the given graph. In the following result, we use this method to calculate the Sombor index of some large graphs in terms of Sombor indices of the blocks of the given graphs.

Theorem 5. *Let G be a graph and let $e = uv$ be a bridge in G. Let $d_G u = k+1$ and $d_G v = t+1$. Then*

$$
\begin{aligned}
SO(G)-SO(G-e) &= \sqrt{(k+1)^2+(t+1)^2} + \textstyle\sum_{i=1}^{k}\left(\sqrt{(k+1)^2+du_i^2} - \sqrt{k^2+du_i^2}\right) \\
&+ \textstyle\sum_{j=1}^{t}\left(\sqrt{(t+1)^2+dv_j^2} - \sqrt{t^2+dv_j^2}\right).
\end{aligned}
$$

Proof. Let the two blocks of G connected with the bridge e be G_1 and G_2. Let the neighbours of u apart from v be $u_1,\ u_2,\ \cdots, u_k$, and let the neighbours of v apart from u be $v_1,\ v_2,\ \cdots, v_t$. Let $A = \{xy|\ x,y \in V(G_1),\ x,y \neq u\}$ and $B = \{xy|\ x,y \in V(G_2),\ x,y \neq v\}$. We can organize $SO(G)$ as follows:

$$SO(G) = \sqrt{du^2+dv^2} + \sum_{i=1}^{k}\sqrt{du^2+du_i^2} + \sum_{j=1}^{t}\sqrt{dv^2+dv_j^2} + \sum_{xy\in A\cup B}\sqrt{dx^2+dy^2}.$$

Similarly,

$$SO(G-e)=\sum_{i=1}^{k}\sqrt{(du-1)^2+du_i^2}+\sum_{j=1}^{t}\sqrt{(dv-1)^2+dv_j^2}+\sum_{xy\in A\cup B}\sqrt{dx^2+dy^2}.$$

Hence, the required result is obtained easily after some calculations. □

Theorem 5 can be generalized to some number of bridges seperating some number of blocks.

The difference in Theorem 5 can also be stated in terms of the Sombor indices of the two blocks G_1 and G_2 as follows. The proof is omitted as it is similar to the previous ones:

Corollary 6. *Let G be a graph and let $e=uv$ be a bridge in G, as in Theorem 5. Let $d_Gu=k+1$ and $d_Gv=t+1$. Then*

$$\begin{aligned} SO(G)-SO(G-e) &= \sqrt{(k+1)^2+(t+1)^2}+\textstyle\sum_{i=1}^{k}\sqrt{(k+1)^2+du_i^2} \\ &+\textstyle\sum_{j=1}^{t}\sqrt{(t+1)^2+dv_j^2}+\sum_{xy\in A\cup B}\sqrt{dx^2+dy^2}-SO(G_1)-SO(G_2). \end{aligned}$$

3.4. Effect of Path-Bridge Removal on the Sombor Index

In the following result, we delete a path bridge between two blocks of a graph instead of deleting a bridge:

Theorem 6. *Let G be a graph and let $e=uv$ be a path bridge of length r. That is, between u and v, there are r vertices $w_1,\ w_2,\ \cdots,w_r$ all having degree 2 in G. Let $C=\{w_1,\ w_2,\ \cdots,w_r\}$, $d_Gu=k+1$, and $d_Gv=t+1$. Then, the change in the Sombor index of G when the set C is deleted from G is*

$$\begin{aligned} SO(G)-SO(G-C) &= \sqrt{(k+1)^2+4}+\sqrt{(t+1)^2+4}+2(r-1)\sqrt{2} \\ &+\textstyle\sum_{i=1}^{k}\left[\sqrt{(k+1)^2+du_i^2}-\sqrt{k^2+du_i^2}\right] \\ &+\textstyle\sum_{j=1}^{t}\left[\sqrt{(t+1)^2+dv_j^2}-\sqrt{t^2+dv_j^2}\right]. \end{aligned}$$

Proof. The edges in G can be partitioned as A and B as in the proof of Theorem 5: $\{uw_1,w_1w_2,w_2w_3,\cdots,w_{r-1}w_r,w_rv\}$, $\{uu_1,uu_2,uu_3,\cdots,uu_k\}$, $\{vv_1,vv_2,vv_3,\cdots,vv_t\}$. Then, the partitioning of $G-C$ would be A and B, $\{uu_1,uu_2,uu_3,\cdots,uu_k\}$, $\{vv_1,vv_2,vv_3,\cdots,vv_t\}$. Considering the fact that the degrees of the end vertices u and v will decrease by one in $G-C$, the proof follows. □

Our next result is about deleting a pendant path from a graph. Let $uv_1v_2v_3\cdots v_r$ be a pendant path in a graph G such that $du=k+1, dv_1=dv_2=dv_{r-1}=dv_r+1=2$. Let us denote the set $\{v_1,\ v_2,\ \cdots,v_{r-1},\ v_r\}$ by T. Then, we have the following result:

Theorem 7. *Let G be a graph and let $T=\{v_1,\ v_2,\ \cdots,v_{r-1},\ v_r\}$ be a pendant path of length r as above. Then, the change in the Sombor index of G when the set T is deleted from G is*

$$\begin{aligned} SO(G)-SO(G-T) &= \textstyle\sum_{i=1}^{k}\left[\sqrt{(k+1)^2+dv_i^2}-\sqrt{k^2+dv_i^2}\right]+\sqrt{(k+1)^2+4} \\ &+2(r-2)\sqrt{2}+\sqrt{5}. \end{aligned}$$

Proof. The edges in G can be partitioned as $\{uv_1,v_1v_2,v_2v_3,\cdots,v_{r-1}v_r\}$, $\{uu_1,uu_2,uu_3,\cdots,uu_k\}$ and $A=\{xy\in E(G-T)|\ x,y\neq u\}$. Similarly, the edges in $G-T$ can be partitioned as $\{uu_1,uu_2,uu_3,\cdots,uu_k\}$ and A. Considering the vertex degrees in G and $G-T$, the result follows. □

As an application of this result, we calculate the Sombor index of a tadpole graph:

Example 1. *Let $G = T_{r,s}$ in Theorem 7. Let T be the pendant path P_s of G so that $G - T = C_r$. Here $k = 2$. Hence, by Theorem 7, we get*

$$\begin{aligned} SO(G) = SO(T_{r,s}) &= SO(C_r) + \sqrt{9+4} + \sqrt{9+4} - \sqrt{4+4} - \sqrt{4+4} + \sqrt{9+4} \\ &\quad +2(s-2)\sqrt{2} + \sqrt{5} \\ &= 2\sqrt{2}(r+s) + 3\sqrt{13} + \sqrt{5 - 8\sqrt{2}} \end{aligned}.$$

3.5. Nordhaus–Gaddum Type Result for the Sombor Index

Let G be a graph and let $\bar{G}$ be its complement. For a vertex v in $V(G)$, $d_G v + d_{\bar{G}} v = n - 1$. Also, for any tree T and for a vertex v in $V(T)$, $d_T v + d_{\bar{T}} v = m$. It is an obvious fact that if G is r-regular, then $\bar{G}$ is $r' = (n-1-r)$-regular. Also for an r-regular graph G, we have $r = 2m/n$. If the end vertices of an edge e are x and y, then this edge is said to be of type $\{dx, dy\}$. Hence, an r-regular graph has $nr/2$ edges of type $\{r, r\}$. Therefore, we have the following result:

Theorem 8. *If G is an r-regular graph, then its Sombor index is*

$$SO(G) = \frac{nr^2\sqrt{2}}{2}.$$

Theorem 8 is enough to show the following Nordhaus–Gaddum type result on the Sombor index:

Theorem 9. *If G is an r-regular graph, then the following relation is satisfied:*

$$SO(G) + SO(\bar{G}) = \frac{n\sqrt{2}}{2}\left[r^2 + (n-1-r)^2\right].$$

Proof. Note that the size of the complement graph $\bar{G}$ is

$$\begin{aligned} m(\bar{G}) &= \tfrac{n(n-1)}{2} - \tfrac{nr}{2} \\ &= \tfrac{n}{2}(n-1-r) \end{aligned}$$

and, hence, we obtain the required relation using the regularity of the complement graph $\bar{G}$:

$$\begin{aligned} SO(\bar{G}) &= \tfrac{n(n-1-r)}{2}(n-1-r)\sqrt{2} \\ &= \tfrac{n\sqrt{2}}{2}(n-1-r)^2. \end{aligned}$$

This proves our required relation. □

The following example gives a nice application of Theorem 9 to calculate the Sombor index of the complement of a cycle graph.

Example 2. *By Theorem 9, we can write*

$$SO(C_n) + SO(\bar{C_n}) = \tfrac{n\sqrt{2}}{2}\left(n^2 - 4n + 7\right).$$

As C_n has n edges of type $\{2,2\}$, $SO(C_n) = 2n\sqrt{2}$. Therefore, by subtracting this $SO(C_n)$ from Equation (2), we can deduce the Sombor index of the complement of the cycle graph. As $\bar{C_n}$ has $n(n-1)/2$ edges of type $\{n-3, n-3\}$, $SO(\bar{C_n}) = \left(n^3 - 4n^2 + 3n\right)\sqrt{2}/2$, which gives us the same result.

4. Non-Simple Graphs

In this paper, up to now, we have been concerned with simple graphs, which are without loops or multiple edges. As the existence of these rather crude types of vertices causes problems with the combinatorial calculations, most papers restrict themselves to simple graphs. In this section, we shall consider non-simple graphs and deal with the effect of removing a loop or multiple edge from a non-simple graph on the Sombor index. First, we study the effect of deleting a loop:

Theorem 10. *Let G be a non-simple graph having at least one loop and let e be a loop starting and ending at a vertex u of G. Let $d_G u = k$. Then*

$$SO(G) - SO(G-e) = \sum_{uv_i \in E(G)} \left[\sqrt{k^2 + d_G v_i{}^2} - \sqrt{(k-2)^2 + d_G v_i{}^2} \right] + k\sqrt{2}.$$

Proof. As $d_G u = k$ and the edge e contributes 2 to this degree as a loop, there are $k-2$ neighbours of u in G. Let these neighbours be $v_1,\ v_2,\ \cdots,\ v_{k-2}$. The edge partition of G is as follows: (i) the edges $uv_i \in E(G)$ joining the vertex u to its neighbours, (ii) the edges $vw \in E(G)$ such that $v,\ w \neq u$, which has both end vertices different than u, and (iii) $e \in E(G)$. In the graph $G-e$, we have the following edge types: (i) The edges $uv_i \in E(G-e)$, (ii) $vw \in E(G-e)$ such that $v,\ w \neq u$. Now, concerning the vertex degrees in G and $G-e$, we know that $d_G u = k$, $d_{G-e} u = k-2$, and all the remaining vertices have the same degree in both graphs, as the removal of e does not effect their end vertices. Therefore, we have

$$\begin{aligned}
SO(G) &= \sum_{uv_i \in E(G)} \sqrt{d_G u^2 + d_G v_i{}^2} + \sum_{\substack{vw \in E(G) \\ v,\ w \neq u}} \sqrt{d_G v^2 + d_G w^2} + \sqrt{d_G u^2 + d_G u^2} \\
&= \sum_{uv_i \in E(G)} \sqrt{k^2 + d_G v_i{}^2} + \sum_{\substack{vw \in E(G) \\ v,\ w \neq u}} \sqrt{d_G v^2 + d_G w^2} + \sqrt{k^2 + k^2} \\
&= \sum_{uv_i \in E(G)} \sqrt{k^2 + d_G v_i{}^2} + \sum_{\substack{vw \in E(G) \\ v,\ w \neq u}} \sqrt{d_G v^2 + d_G w^2} + k\sqrt{2}
\end{aligned}$$

and

$$\begin{aligned}
SO(G-e) &= \sum_{uv_i \in E(G-e)} \sqrt{d_{G-e} u^2 + d_{G-e} v_i{}^2} + \sum_{\substack{vw \in E(G-e) \\ v,\ w \neq u}} \sqrt{d_{G-e} v^2 + d_{G-e} w^2} \\
&= \sum_{uv_i \in E(G)} \sqrt{(k-2)^2 + d_G v_i{}^2} + \sum_{\substack{vw \in E(G) \\ v,\ w \neq u}} \sqrt{d_G v^2 + d_G w^2} \\
&= \sum_{uv_i \in E(G)} \sqrt{(k-2)^2 + d_G v_i{}^2} + \sum_{\substack{vw \in E(G) \\ v,\ w \neq u}} \sqrt{d_G v^2 + d_G w^2}
\end{aligned}$$

implying the required result. □

Second, we study the effect of deleting one of the multiple edges between two vertices of a graph.

Theorem 11. *Let G be a non-simple graph having l multiple edges between two vertices u and v of G and let these multiple edges be labeled by $e_1,\ e_2,\ \cdots,\ e_l$. Let $d_G u = k$ and $d_G v = t$.*

$$\begin{aligned}
SO(G) - SO(G-e) &= l \times \sqrt{k^2 + t^2} - (l-1) \times \sqrt{(k-1)^2 + (t-1)^2} \\
&\quad + \sum_{uu_i \in E(G)} \left[\sqrt{k^2 + d_G u_i{}^2} - \sqrt{(k-1)^2 + d_G u_i{}^2} \right] \\
&\quad + \sum_{vv_j \in E(G)} \left[\sqrt{t^2 + d_G v_j{}^2} - \sqrt{(t-1)^2 + d_G v_j{}^2} \right].
\end{aligned}$$

Proof. Let G be as stated. Then, there are $k-l$ incident edges to u in addition to the l multiple edges $e_i's$, and there are $t-l$ incident edges to v in addition to the l multiple edges $e_i's$. The edge partition of G is as follows: (i) the multiple edges $e_j \in E(G)$ joining the vertex u to v for $j = 1, 2, \cdots, l$, (ii) the edges $uu_i \in E(G)$ for $i = 1, 2, \cdots, u_{k-l}$ such that $u_i \neq v$, which are different than the multiple edges between u and v, (iii) the edges $vv_j \in E(G)$ for $j = 1, 2, \cdots, v_{t-l}$ such that $v_j \neq u$, which are different than the multiple edges between u and v, and (iv) the remaining edges $rs \in E(G)$ such that r, s are adjacent to neither u nor v. Similarly, the edge partition of $G - e_i$ is as follows: (i) the multiple edges $e_j \in E(G - e_i)$ joining the vertex u to v for $j = 1, 2, \cdots, i-1, i+1, \cdots, l$, that is, all the multiple edges except e_i, which is removed, (ii) the edges $uu_i \in E(G - e_i)$ for $i = 1, 2, \cdots, u_{k-l}$ such that $u_i \neq v$, which are different than the multiple edges between u and v, (iii) the edges $vv_j \in E(G - e_i)$ for $j = 1, 2, \cdots, v_{t-l}$ such that $v_j \neq u$, which are different than the multiple edges between u and v, and (iv) the remaining edges $rs \in E(G - e_i)$ such that r, s are adjacent to neither u nor v.

Now, concerning the vertex degrees in G and $G - e_i$, we know that $d_G u = k$, $d_{G-e_i} u = k - 1$, $d_G v = t$, $d_{G-e_i} v = t - 1$, and all the remaining vertices have the same degree in both graphs, as the removal of e_i does not effect their end vertices. Therefore, we have

$$\begin{aligned}
SO(G) &= \sum_{e_j = uv \in E(G)} \sqrt{d_G u^2 + d_G v^2} + \sum_{uu_i \in E(G)} \sqrt{d_G u^2 + d_G {u_i}^2} \\
&+ \sum_{vv_j \in E(G)} \sqrt{d_G v^2 + d_G {v_j}^2} + \sum_{\substack{rs \in E(G) \\ r,\, s \notin \{u,\, v\}}} \sqrt{d_G r^2 + d_G s^2} \\
&= l \times \sqrt{k^2 + t^2} + \sum_{uu_i \in E(G)} \sqrt{k^2 + d_G {u_i}^2} \\
&+ \sum_{vv_j \in E(G)} \sqrt{t^2 + d_G {v_j}^2} + \sum_{\substack{rs \in E(G) \\ r,\, s \notin \{u,\, v\}}} \sqrt{d_G r^2 + d_G s^2},
\end{aligned}$$

as the first sum in the first line has l summands, and

$$\begin{aligned}
SO(G - e_i) &= \sum_{e_j = uv \in E(G - e_i)} \sqrt{d_{G-e_i} u^2 + d_{G-e_i} v^2} + \sum_{uu_i \in E(G - e_i)} \sqrt{d_{G-e_i} u^2 + d_{G-e_i} {u_i}^2} \\
&+ \sum_{vv_j \in E(G - e_i)} \sqrt{d_{G-e_i} v^2 + d_{G-e_i} {v_j}^2} + \sum_{\substack{rs \in E(G - e_i) \\ r,\, s \notin \{u,\, v\}}} \sqrt{d_{G-e_i} r^2 + d_{G-e_i} s^2} \\
&= (l-1) \times \sqrt{(k-1)^2 + (t-1)^2} + \sum_{uu_i \in E(G)} \sqrt{(k-1)^2 + d_G {u_i}^2} \\
&+ \sum_{vv_j \in E(G)} \sqrt{(t-1)^2 + d_G {v_j}^2} + \sum_{\substack{rs \in E(G) \\ r,\, s \notin \{u,\, v\}}} \sqrt{d_G r^2 + d_G s^2},
\end{aligned}$$

as the first sum in the first line has $l - 1$ summands, implying the required result. □

5. Conclusions

The effects of vertex and edge removal from a graph are useful in calculating some property of large graphs in terms of the same property of a smaller graph. Sometimes, using iteration, one can manage to obtain a property of a really large graph in terms of the same property of many other smaller graphs. Here, the calculations are made for a pendant and non-pendant vertex, a pendant and non-pendant edge, a pendant path, a bridge, and a bridge path in a simple graph. Using these results, Sombor indices of cyclic graphs and tadpole graphs are obtained as an application. Finally, some Nordhaus–Gaddum type results are given for the Sombor index.

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