symmetry

MDPI

Review

Canonical Construction of Invariant Differential Operators: A Review

Vladimir K. Dobrev

Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 72 Tsarigradsko Chaussee, 1784 Sofia, Bulgaria; dobrev@inrne.bas.bg or vkdobrev@yahoo.com

Abstract: In the present paper, we review the progress of the project of the classification and construction of invariant differential operators for non-compact, semisimple Lie groups. Our starting point is the class of algebras which we called earlier 'conformal Lie algebras' (CLA), which have very similar properties to the conformal algebras of Minkowski space-time, though our aim is to go beyond this class in a natural way. For this purpose, we introduced recently the new notion of a *parabolic relation* between two non-compact, semi-simple Lie algebras $\mathcal{G}$ and $\mathcal{G}'$ that have the same complexification and possess maximal parabolic subalgebras with the same complexification.

Keywords: invariant differential operators; non-compact semisimple Lie groups; conformal Lie algebras

check for updates

Citation: Dobrev, V.K. Canonical Construction of Invariant Differential Operators: A Review. *Symmetry* **2024**, *16*, 151. https://doi.org/10.3390/sym16020151

Academic Editor: Michal Hnatič

Received: 3 November 2023
Revised: 3 December 2023
Accepted: 13 December 2023
Published: 27 January 2024

1. Introduction and Preliminaries

Invariant differential operators play a very important role in the description of physical symmetries—starting from the early occurrences in the Maxwell, d'Allembert and Dirac, equations to the latest applications of (super)differential operators in conformal field theory, supergravity and string theory (for reviews, cf., e.g., [1,2]). Thus, it is important for applications in physics to systematically study such operators. For more relevant references cf., e.g., [3–73], and others throughout the text. Especially, we would like to point out the book [74] which contains a section devoted to groups of conformal transformations of curved spacetime.

In a recent paper [75], we started the systematic explicit construction of invariant differential operators. We gave an explicit description of the building blocks, namely, the *parabolic subgroups and subalgebras* from which the necessary representations are induced. Thus, we have set the stage for the study of different non-compact groups. Up to 2016, relevant references may be found in our monograph [76] and also in [77–111].

Our canonical construction is applicable also to *quantum groups*, *super groups*, to *(super-) Virasoro and Kac-Moody algebras*, see our monographs: [112–114].

Preliminaries

Let G be a semi-simple, non-compact Lie group, and K a maximal compact subgroup of G. Then, we have an *Iwasawa decomposition* $G = KA_0N_0$, where A_0 is an Abelian simply connected vector subgroup of G and N_0 is a nilpotent simply connected subgroup of G preserved by the action of A_0. Furthermore, let M_0 be the centralizer of A_0 in K. Then, the subgroup $P_0 = M_0A_0N_0$ is a *minimal parabolic subgroup* of G. A *parabolic subgroup* $P = M'A'N'$ is any subgroup of G which contains a minimal parabolic subgroup.

Furthermore let $\mathcal{G}, \mathcal{K}, \mathcal{P}, \mathcal{M}, \mathcal{A}, \mathcal{N}$ denote the Lie algebras of G, K, P, M, A, N, resp.

For our purposes, we shall be restrict *maximal parabolic subgroups* $P = MAN$, i.e., rank$A = 1$, resp., to *maximal parabolic subalgebras* $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ with dim $\mathcal{A} = 1$.

Let ν be a (non-unitary) character of A, $\nu \in \mathcal{A}^*$, parameterized by a real number d, called the *conformal weight* or energy.

Furthermore, let μ fix a discrete series representation D^μ of M on the Hilbert space V_μ, or the finite-dimensional (non-unitary) representation of M with the same Casimirs.

We call the induced representation $\chi = \mathrm{Ind}_P^G(\mu \otimes \nu \otimes 1)$ an *elementary representation* of G [24]. (These are called *generalized principal series representations* (or *limits thereof*) in [115].) Their spaces of functions are

$$\mathcal{C}_\chi = \{\mathcal{F} \in C^\infty(G, V_\mu) | \mathcal{F}(gman) = e^{-\nu(H)} \cdot D^\mu(m^{-1})\, \mathcal{F}(g)\} \tag{1}$$

where $a = \exp(H) \in A'$, $H \in \mathcal{A}'$, $m \in M'$, $n \in N'$. The representation action is the *left regular action*:

$$(\mathcal{T}^\chi(g)\mathcal{F})(g') = \mathcal{F}(g^{-1}g'), \quad g, g' \in G. \tag{2}$$

- An important ingredient in our considerations are the *highest/lowest-weight representations* of $\mathcal{G}^\mathbb{C}$. These can be realized as (factor-modules of) Verma modules V^Λ over $\mathcal{G}^\mathbb{C}$, where $\Lambda \in (\mathcal{H}^\mathbb{C})^*$, $\mathcal{H}^\mathbb{C}$ is a Cartan subalgebra of $\mathcal{G}^\mathbb{C}$ and weight $\Lambda = \Lambda(\chi)$ is determined uniquely from χ [76].

Actually, since our ERs may be induced from finite-dimensional representations of $\mathcal{M}$ (or their limits) the Verma modules are always reducible. Thus, it is more convenient to use *generalized Verma modules* $\tilde{V}^\Lambda$ such that the role of the highest/lowest-weight vector v_0 is taken by the (finite-dimensional) space $V_\mu\, v_0$. For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight d. Relatedly, for the intertwining differential operators, only the reducibility with regard to non-compact roots is essential.

- Another main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called *multiplets* [76]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the *vertices* of which correspond to the reducible ERs and the *lines (arrows)* between the vertices correspond to intertwining operators. The explicit parameterization of the multiplets and of their ERs is important in understanding of the situation. The notion of multiplets was introduced in [116] and applied to representations of $SO_o(p,q)$ and $SU(2,2)$, resp., induced from their minimal parabolic subalgebras. Then it was applied to the conformal superalgebra [117], to infinite-dimensional (super)algebras [113] and to quantum groups [112]. (For other applications, we refer to [114].)

In fact, the multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consist of the pair (β, m), where β is a (non-compact) positive root of $\mathcal{G}^\mathbb{C}$, $m \in \mathbb{N}$, such that the *BGG Verma module reducibility condition* (for highest-weight modules) is fulfilled:

$$(\Lambda + \rho, \beta^\vee) = m, \quad \beta^\vee \equiv 2\beta/(\beta, \beta) \tag{3}$$

where ρ is half the sum of the positive roots of $\mathcal{G}^\mathbb{C}$. When the above holds, then the Verma module with shifted weight $V^{\Lambda - m\beta}$ (or $\tilde{V}^{\Lambda - m\beta}$ for GVM and β non-compact) is embedded in the Verma module V^Λ (or $\tilde{V}^\Lambda$). This embedding is realized by a singular vector v_s determined by a polynomial $\mathcal{P}_{m,\beta}(\mathcal{G}^-)$ in the universal enveloping algebra $(U(\mathcal{G}_-))\, v_0$, and $\mathcal{G}^-$ is the subalgebra of $\mathcal{G}^\mathbb{C}$ generated by the negative root generators [118]. More explicitly, [76,119], $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, v_0$ (or $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, V_\mu\, v_0$ for GVMs). Then, there exists [76,119] an **intertwining differential operator**

$$\mathcal{D}_{m,\beta} : \mathcal{C}_{\chi(\Lambda)} \longrightarrow \mathcal{C}_{\chi(\Lambda - m\beta)} \tag{4}$$

given explicitly by:

$$\mathcal{D}_{m,\beta} = \mathcal{P}_{m,\beta}(\widehat{\mathcal{G}^-}) \tag{5}$$

where $\widehat{\mathcal{G}^-}$ denotes the *right action* on the functions $\mathcal{F}$.

In most of these situations, the invariant operator $\mathcal{D}_{m,\beta}$ has a non-trivial invariant kernel in which a subrepresentation of $\mathcal{G}$ is realized. Thus, studying the equations with trivial RHS is also very important:

$$\mathcal{D}_{m,\beta}\, f = 0, \qquad f \in \mathcal{C}_{\chi(\Lambda)}, \tag{6}$$

For example, in many physical applications, in the case of first-order differential operators, i.e., for $m = m_\beta = 1$, these equations are called *conservation laws*, and the elements $f \in \ker \mathcal{D}_{m,\beta}$ are called *conserved currents*.

The above construction also works for the *subsingular vectors v_{ssv}* of Verma modules. Such vectors are also expressed by a polynomial $\mathcal{P}_{ssv}(\mathcal{G}^-)$ in the universal enveloping algebra: $v^s_{ssv} = \mathcal{P}_{ssv}(\mathcal{G}^-)\, v_0$, cf. [120]. Thus, there exists a *conditionally invariant differential operator* given explicitly by $\mathcal{D}_{ssv} = \mathcal{P}_{ssv}(\widehat{\mathcal{G}^-})$, and a *conditionally invariant differential equation*; for many more details, see [121]. (Note that these operators (equations) are not of the first order.)

In our exposition below, we shall use the so-called Dynkin labels:

$$m_i \equiv (\Lambda + \rho, \alpha_i^\vee), \quad i = 1, \dots, n, \tag{7}$$

where $\Lambda = \Lambda(\chi)$, ρ is half the sum of the positive roots of $\mathcal{G}^\mathbb{C}$.

We shall use also the so-called Harish–Chandra parameters:

$$m_\beta \equiv (\Lambda + \rho, \beta)\ , \tag{8}$$

where β is any positive root of $\mathcal{G}^\mathbb{C}$. These parameters are redundant, since they are expressed in terms of the Dynkin labels; however, some statements are best formulated in their terms. (Clearly, both the Dynkin labels and Harish–Chandra parameters have their origin in the BGG reducibility condition (3).).

Finally, we shall introduce the notion of 'parabolically related non-compact semisimple Lie algebras' [122]. This notion is not part of our procedure for constructing invariant differential operators, but just a tool to extend the construction from one Lie algebra to another.

Definition 1. *Let $\mathcal{G}, \mathcal{G}'$ be two non-compact semi-simple Lie algebras with the same complexification $\mathcal{G}^\mathbb{C} \cong \mathcal{G}'^\mathbb{C}$. We call them* **parabolically related** *if they have parabolic subalgebras $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$, $\mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$, such that $\mathcal{M}^\mathbb{C} \cong \mathcal{M}'^\mathbb{C}$ ($\Rightarrow \mathcal{P}^\mathbb{C} \cong \mathcal{P}'^\mathbb{C}$).*$\diamondsuit$

Certainly, there are many such parabolic relationships for any given algebra $\mathcal{G}$. Furthermore, two algebras $\mathcal{G}, \mathcal{G}'$ may be parabolically related via different parabolic subalgebras.

The paper is organized as follows. In Section 2, we consider the case of the pseudo-orthogonal algebras $so(p,q)$ which are parabolically related to the conformal algebra $so(n,2)$ for $p + q = n + 2$. In Section 3, we consider the CLA $su(n,n)$ and the parabolically related $sl(2n,\mathbb{R})$, and for $n = 2k : su^*(4k)$. In Section 4, we consider the CLA $sp(n)$ and—for $n = 2r$—the parabolically related $sp(r,r)$. In Section 5, we consider the algebras $so^*(2n)$ (which are CLA when n is even) and the parabolically related algebras. In Section 6, we consider the CLA $E_{7(-25)}$ and the parabolically related $E_{7(7)}$. In Section 7, we consider the hermitian symmetric case $E_{6(-14)}$ and the parabolically related $E_{6(6)}$ and $E_{6(2)}$. In Section 8, we consider the algebra F_4 and its real forms F_4' and F_4''. In Section 9, we consider the algebra $G_{2(2)}$.

We would also like to list some more recent relevant references [123–229].

2. Conformal Algebras $so(n,2)$ and Parabolically Related Algebras

The most widely used algebras are the *conformal algebras* $\boldsymbol{so(n,2)}$ in n-dimensional Minkowski space-time. In that case, there is a maximal *Bruhat decomposition* [230] that has direct physical meaning:

$$so(n,2) = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N} \oplus \tilde{\mathcal{N}}, \qquad (9)$$
$$\mathcal{M} = so(n-1,1), \quad \dim \mathcal{A} = 1, \quad \dim \mathcal{N} = \dim \tilde{\mathcal{N}} = n$$

where $so(n-1,1)$ is the *Lorentz algebra* of n-dimensional Minkowski space-time, the subalgebra $\mathcal{A} = so(1,1)$ represents the *dilatations* and the conjugated subalgebras $\mathcal{N}$, $\tilde{\mathcal{N}}$ are the algebras of *translations* and *special conformal transformations*, both being isomorphic to n-dimensional Minkowski space-time.

Another physically important feature is that the algebras $so(n,2)$ have discrete series representations. We recall that by the Harish–Chandra criterion [231], these are groups where the following holds:

$$\operatorname{rank} G = \operatorname{rank} K,$$

where K is the *maximal compact subgroup* of the non-compact group G.

Furthermore, the algebras $so(n,2)$ belong to the class of *Hermitian symmetric spaces*. The practical criterion is that in these cases, the *maximal compact subalgebra* $\mathcal{K}$ is of the form:

$$\mathcal{K} \;=\; so(2) \oplus \mathcal{K}' \qquad (10)$$

The Lie algebras from this class are as follows:

$$so(n,2), \;\; sp(n,R), \;\; su(m,n), \;\; so^*(2n), \;\; E_{6(-14)}, \;\; E_{7(-25)} \qquad (11)$$

These groups/algebras have *highest/lowest-weight representations*, and relatedly, *holomorphic* discrete series representations.

We label the signature of the ERs of $\mathcal{G}$ as follows:

$$\begin{aligned} &\chi = \{\, n_1, \ldots, n_{\tilde{h}};\, c\,\}, \quad n_j \in \mathbb{Z}/2\,, \quad c = d - \tfrac{n}{2}\,, \quad \tilde{h} \equiv [\tfrac{n}{2}], \\ &|n_1| < n_2 < \cdots < n_{\tilde{h}}\,, \quad n \text{ even}\,, \\ &0 < n_1 < n_2 < \cdots < n_{\tilde{h}}, \quad n \text{ odd}\,, \end{aligned} \qquad (12)$$

where the last entry of χ labels the characters of $\mathcal{A}$, and the first $\tilde{h}$ entries are labels of the finite-dimensional nonunitary irreps of $\mathcal{M} \cong so(n-1,1)$.

The reason to use the parameter c instead of d is that the parametrization of the ERs in the multiplets is given in a simple intuitive way (cf. [232]):

$$\begin{aligned} \chi_1^{\pm} &= \{\epsilon n_1, \ldots, n_{\tilde{h}};\, \pm n_{\tilde{h}+1}\}, \quad n_{\tilde{h}} < n_{\tilde{h}+1}\,, \\ \chi_2^{\pm} &= \{\epsilon n_1, \ldots, n_{\tilde{h}-1},\, n_{\tilde{h}+1};\, \pm n_{\tilde{h}}\} \\ \chi_3^{\pm} &= \{\epsilon n_1, \ldots, n_{\tilde{h}-2}, n_{\tilde{h}}, n_{\tilde{h}+1};\, \pm n_{\tilde{h}-1}\} \\ &\ldots \\ \chi_{\tilde{h}}^{\pm} &= \{\epsilon n_1, n_3, \ldots, n_{\tilde{h}},\, n_{\tilde{h}+1};\, \pm n_2\} \\ \chi_{\tilde{h}+1}^{\pm} &= \{\epsilon n_2, \ldots, n_{\tilde{h}},\, n_{\tilde{h}+1};\, \pm n_1\} \\ \epsilon &= \begin{cases} \pm, & n \textit{ even} \\ 1, & n \textit{ odd} \end{cases} \end{aligned} \qquad (13)$$

Furthermore, we denote by $\tilde{C}_i^{\pm}$ the representation space with signature $\chi_i^{\pm}$.

The number of ERs in the corresponding multiplets is equal to

$$|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| \,/\, |W(\mathcal{M}^{\mathbb{C}}, \mathcal{H}_m^{\mathbb{C}})| \;=\; 2(1+\tilde{h}) \qquad (14)$$

where $\mathcal{H}^{\mathbb{C}}$, $\mathcal{H}_m^{\mathbb{C}}$ are Cartan subalgebras of $\mathcal{G}^{\mathbb{C}}$, $\mathcal{M}^{\mathbb{C}}$, resp. This formula is valid for the main multiplets of all conformal Lie algebras.

Now, in Figure 1, we show the general *even* case $so(p,q)$, $p+q = 2h+2$-even [76,232].

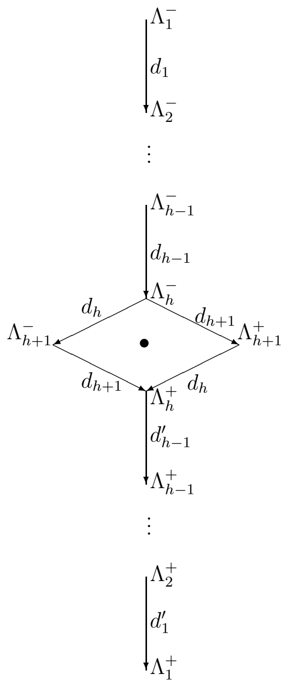


Figure 1. Diagram for the cases $so(p,q)$, $p+q=2h+2$, even, showing only the differential operators, while the integral operators are assumed as symmetry w.r.t. the bullet in the centre.

In Figure 2, we show the general *odd* case $so(p,q)$, $p+q=2h+3$-odd [76,232].

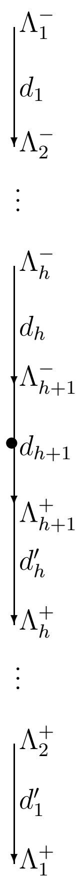


Figure 2. The cases $so(p,q)$, $p+q=2h+3$, showing only the differential operators, while the integral operators are assumed as symmetry w.r.t. the bullet in the centre.

The ERs in the multiplet are related by *intertwining integral and differential operators*. The *integral operators* were introduced by Knapp and Stein [233]. They correspond to elements of the restricted Weyl group of $\mathcal{G}$. These operators intertwine the pairs $\tilde{\mathcal{C}}_i^{\pm}$

$$G_i^{\pm} \ : \ \tilde{\mathcal{C}}_i^{\mp} \longrightarrow \tilde{\mathcal{C}}_i^{\pm}, \quad i \ = \ 1, \ldots, 1 + \tilde{h} \tag{15}$$

The *intertwining differential operators* correspond to non-compact positive roots of the root system of $so(n+2, \mathbb{C})$, cf. [76]. (In the current context, compact roots of $so(n+2, \mathbb{C})$ are those that are roots also of the subalgebra $so(n, \mathbb{C})$, the rest of the roots are non-compact.) The degrees of these intertwining differential operators are given just by the differences of the c entries [76]:

$$\begin{aligned} &\deg d_i = \deg d_i' = n_{\tilde{h}+2-i} - n_{\tilde{h}+1-i}, \qquad i = 1, \ldots, \tilde{h}, \quad \forall n \\ &\deg d_{\tilde{h}+1} = n_2 + n_1, \quad n \text{ even} \end{aligned} \tag{16}$$

where d_h' is omitted from the first line for $(p+q)$ even.

Matters are arranged so that in every multiplet only the ER with signature χ_1^- contains a *finite-dimensional nonunitary subrepresentation* in a subspace $\mathcal{E}$. The latter corresponds to the finite-dimensional unitary irrep of $so(n+2)$ with signature $\{n_1, \ldots, n_{\tilde{h}}, n_{\tilde{h}+1}\}$. The subspace $\mathcal{E}$ is annihilated by the operator G_1^+, and is the image of the operator G_1^-.

- *Interlude:*

We mention one more special feature of $so(n, 2)$, namely that the complexification of the maximal compact subgroup is isomorphic to the complexification of the first two factors of the Bruhat decomposition:

$$\mathcal{K}^{\mathbb{C}} \ = \ so(n, \mathbb{C}) \oplus so(2, \mathbb{C}) \ \cong \ so(n-1, 1)^{\mathbb{C}} \oplus so(1,1)^{\mathbb{C}} = \mathcal{M}^{\mathbb{C}} \oplus \mathcal{A}^{\mathbb{C}} . \tag{17}$$

The coincidence of the complexification of the semi-simple subalgebras

$$\mathcal{K}'^{\mathbb{C}} \ = \ \mathcal{M}^{\mathbb{C}} \tag{18}$$

means that the sets of finite-dimensional (nonunitary) representations of $\mathcal{M}$ are in 1-to-1 correspondence with the finite-dimensional (unitary) representations of $\mathcal{K}'$. The latter leads to the fact that the corresponding induced representations are representations of finite $\mathcal{K}$-type [234].

It turns out that some of the hermitian-symmetric algebras share the above-mentioned special properties of $so(n, 2)$. This subclass consists of

$$so(n, 2), \ \ sp(n, \mathbb{R}), \ \ su(n, n), \ \ so^*(4n), \ \ E_{7(-25)} \tag{19}$$

with the corresponding analogs of Minkowski space-time V being

$$\mathbb{R}^{n-1,1}, \ \ \mathrm{Sym}(n, \mathbb{R}), \ \ \mathrm{Herm}(n, \mathbb{C}), \ \ \mathrm{Herm}(n, \mathbb{Q}), \ \ \mathrm{Herm}(3, \mathbb{O}) \tag{20}$$

In view of applications to physics, we proposed to call these algebras '*conformal Lie algebras*' (or groups).

We summarize the algebras parabolically related to conformal Lie algebras with maximal parabolics fulfilling (18) in Table 1 below. Also, some non-CLAs are included.

There, $sl(n, \mathbb{C})_{\mathbb{R}}$ denotes $sl(n, \mathbb{C})$ as a real Lie algebra (thus, $(sl(n, \mathbb{C})_{\mathbb{R}})^{\mathbb{C}} = sl(n, \mathbb{C}) \oplus sl(n, \mathbb{C})$); e_6 denotes the compact real form of E_6 ; and we have imposed restrictions to avoid coincidences or degeneracies due to well-known isomorphisms: $so(1,2) \cong sp(1, \mathbb{R}) \cong su(1,1)$, $so(2,2) \cong so(1,2) \oplus so(1,2)$, $su(2,2) \cong so(4,2)$, $sp(2, \mathbb{R}) \cong so(3,2)$, $sp(1,1) \cong so(4,1)$, $so^*(4) \cong so(3) \oplus so(2,1)$, $so^*(8) \cong so(6,2)$.

Table 1. Table of conformal Lie algebras (CLA) $\mathcal{G}$ with $\mathcal{M}$-factor fulfilling (18) and the corresponding parabolically related algebras $\mathcal{G}'$; we display also some non-CLA cases.

$\mathcal{G}$	$\mathcal{K}$	$\mathcal{M}$ $\dim V$	$\mathcal{G}'$	$\mathcal{M}'$
$so(n,2)$ $n \geq 3$	$so(n) \oplus so(2)$	$so(n-1,1)$ n	$so(p,q)$, $p+q=n+2$	$so(p-1,q-1)$
$su(n,n)$ $n \geq 3$	$u(n) \oplus su(n)$	$sl(n,\mathbb{C})_{\mathbb{R}}$ n^2	$sl(2n,\mathbb{R})$ $su^*(2n)$, $n=2k$	$sl(n,\mathbb{R}) \oplus$ $sl(n,\mathbb{R})$ $su^*(2k) \oplus$ $su^*(2k)$
$sp(n,\mathbb{R})$ rank $= n \geq 3$	$u(n)$	$sl(n,\mathbb{R})$ $n(n+1)/2$	$sp(r,r)$, $n=2r$	$su^*(2r)$, $n=2r$
$so^*(4n)$ $n \geq 3$	$u(2n)$	$su^*(2n)$ $n(2n-1)$	$so(2n,2n)$	$sl(2n,\mathbb{R})$
$E_{7(-25)}$	$e_6 \oplus so(2)$	$E_{6(-26)}$ 27	$E_{7(7)}$	$E_{6(6)}$
below not CLA				
$so^*(10)$	$u(5)$	$su(3,1) \oplus su(2)$ 13	$so(5,5)$	$sl(4,\mathbb{R}) \oplus$ $sl(2,\mathbb{R})$
$E_{6(-14)}$	$so(10) \oplus so(2)$	$su(5,1)$ 21	$E_{6(6)}$ $E_{6(2)}$	$sl(6,\mathbb{R})$ $su(3,3)$
F'_4	$sp(3) \oplus su(2)$	$sl(3,\mathbb{R}) \oplus$ $sl(2,\mathbb{R})$ 20		
F''_4	$so(9)$	$so(7)$ 15		
$G_{2(2)}$	$su(2) \oplus su(2)$	0 min. $sl(2,\mathbb{R})$ max. 6 min. 5 max.		

Although the diagram in Figure 1 is valid for arbitrary $so(p,q)$ (even $p+q>5$) due to the parabolic relatedness, the contents are very different. (The same remark holds for the diagram in Figure 2 valid for $so(p,q)$ for odd $p+q \leq 5$.) We comment only on the ER with signature χ_1^+. In all cases, it contains a UIR of $so(p,q)$ realized on an invariant subspace $\mathcal{D}$ of the ER χ_1^+. That subspace is annihilated by the operator G_1^-, and is the image of the operator G_1^+. (Other ERs contain more UIRs.)

If $pq \in 2\mathbb{N}$, the mentioned UIR is a discrete series representation. (Other ERs contain more discrete series UIRs.)

And if $q=2$, the invariant subspace $\mathcal{D}$ is the direct sum of two subspaces $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$, in which a *holomorphic discrete series representation* and its conjugate *anti-holomorphic discrete series representation*, resp., are realized. Note that the corresponding *lowest-weight GVM* is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate *highest-weight GVM* is infinitesimally equivalent to the anti-holomorphic discrete series.

Note that $\deg d_i$, $\deg d'_i$ are Harish–Chandra parameters corresponding to the non-compact positive roots of $so(n+2,\mathbb{C})$. From these, only $\deg d_1$ corresponds to a simple root; i.e., it is a Dynkin label.

Above, we considered $so(n,2)$ for $n > 2$. The case $n = 2$ is reduced to $n = 1$ since $so(2,2) \cong so(1,2) \oplus so(1,2)$. The case $so(1,2)$ is special and must be treated separately. But, in fact, it is contained in what we presented already. In that case, the multiplets contain only *two ERs* which may be depicted by the *top pair* $\chi_1^\pm$ in the pictures that we presented. And they have the properties that we described for $so(n,2)$ with $n > 2$. The case $so(1,2)$ was given already in 1946-7 independently by Gel'fand et al. [235] and Bargmann [236].

3. The Lie Algebra $su(n,n)$ and Parabolically Related Algebras

Let $\mathcal{G} = su(n,n)$, $n \geq 2$. The maximal compact subgroup is $\mathcal{K} \cong u(1) \oplus su(n) \oplus su(n)$, while $\mathcal{M} = sl(n,\mathbb{C})_{\mathbb{R}}$. The number of ERs in the corresponding multiplets is equal to

$$|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| \;/\; |W(\mathcal{M}^{\mathbb{C}}, \mathcal{H}_m^{\mathbb{C}})| = \binom{2n}{n}$$

The signature of the ERs of $\mathcal{G}$ is

$$\chi = \{ n_1, \ldots, n_{n-1}, n_{n+1} \ldots, n_{2n-1}; c \}, \quad n_j \in \mathbb{N}, \quad c = d - n$$

The Knapp–Stein restricted Weyl reflection is given by

$$G_{KS} : \mathcal{C}_\chi \longrightarrow \mathcal{C}_{\chi'}, \quad \chi' = \{(n_1, \ldots, n_{n-1}, n_{n+1}, \ldots, n_{2n-1})^*; -c\}$$
$$(n_1, \ldots, n_{n-1}, n_{n+1}, \ldots, n_{2n-1})^* \doteq (n_{n+1}, \ldots, n_{2n-1}, n_1, \ldots, n_{n-1})$$

Below, in Figures 3 and 4, we give the diagrams for $su(n,n)$ for $n = 3,4$ [237]. (The case $n = 2$ is already considered since $su(2,2) \cong so(4,2)$.) These are also diagrams for the parabolically related $sl(2n,\mathbb{R})$, and for $n = 2k$, these are also diagrams for the parabolically related $su^*(4k)$ [122].

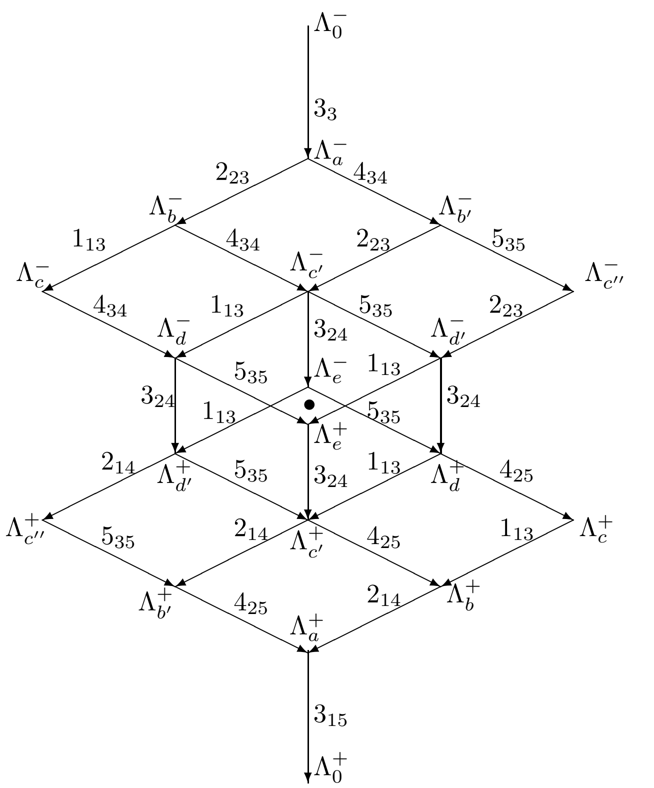


Figure 3. Pseudo-unitary symmetry su(3, 3) The pseudo-unitary symmetry $su(p,p)$ is similar to conformal symmetry in p^2 dimensional space, and for p = 2 coincides with the 4-dimensional conformal case. By parabolic relation the $su(3,3)$diagram above is valid also for $sl(6,R)$.

We use the following conventions. Each intertwining differential operator is represented by an arrow accompanied by a symbol $i_{j...k}$ encoding the root $\beta_{j...k}$ and the number $m_{\beta_{j...k}}$ which is involved in the BGG criterion.

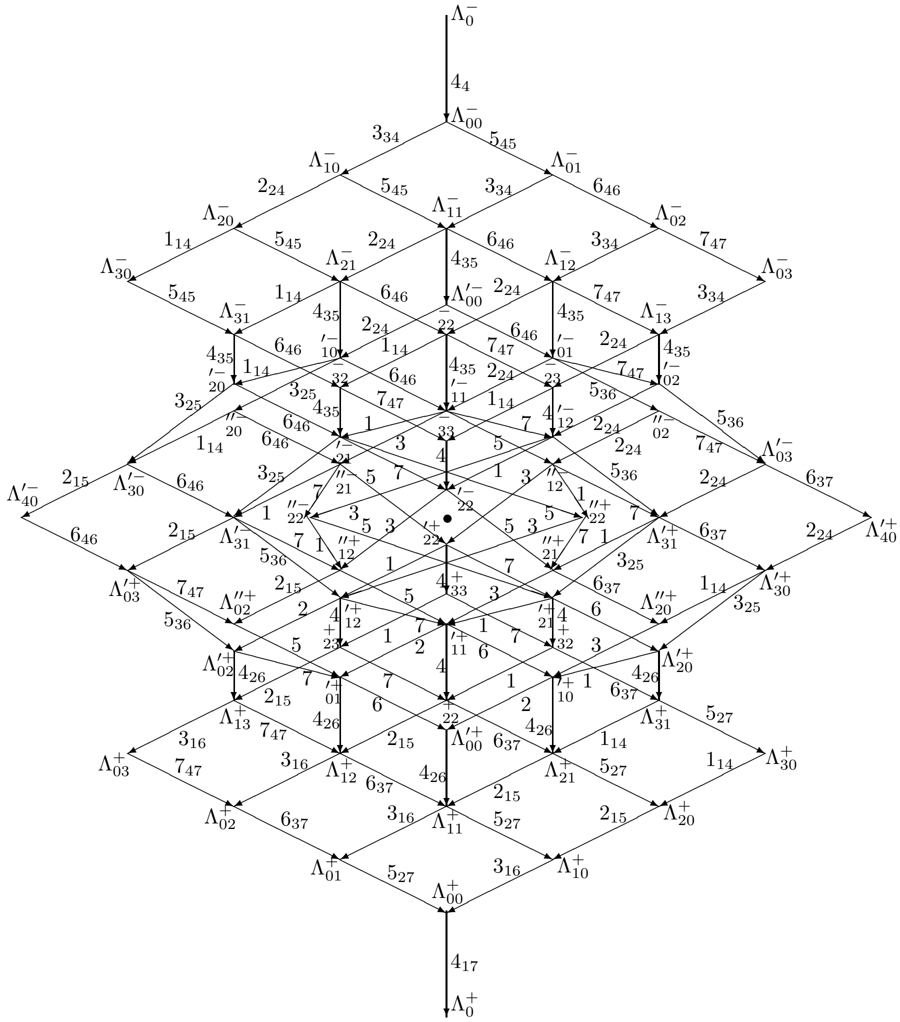


Figure 4. Pseudo-unitary symmetry in 16-dimensional space. By parabolic relation the $su(4,4)$ diagram above is valid also for $sl(8,R)$ and $su^*(8)$.

4. The Lie Algebras $sp(n,\mathbb{R})$ and $sp(\frac{n}{2},\frac{n}{2})$ (n-even)

Let $n \geq 2$. Let $\mathcal{G} = sp(n,\mathbb{R})$ be the split real form of $sp(n,\mathbb{C}) = \mathcal{G}^{\mathbb{C}}$. The maximal compact subgroup is $\mathcal{K} \cong u(1) \oplus su(n)$, while $\mathcal{M} = sl(n,\mathbb{R})$. The number of ERs in the corresponding multiplets is

$$|W(\mathcal{G}^{\mathbb{C}},\mathcal{H}^{\mathbb{C}})| \,/\, |W(\mathcal{M}^{\mathbb{C}},\mathcal{H}_m^{\mathbb{C}})| \;=\; 2^n$$

The signature of the ERs of $\mathcal{G}$ is

$$\chi \;=\; \{\, n_1, \ldots, n_{n-1} \,;\, c \,\}, \qquad n_j \in \mathbb{N}\,,$$

The Knapp–Stein Weyl reflection acts as follows:

$$G_{KS} \,:\, \mathcal{C}_\chi \;\longrightarrow\; \mathcal{C}_{\chi'}, \chi' \;=\; \{\, (n_1, \ldots, n_{n-1})^*; \, -c \,\},$$
$$(n_1, \ldots, n_{n-1})^* \;\doteq\; (n_{n-1}, \ldots, n_1)$$

Below, in Figures 5–8, we give pictorially the multiplets for $sp(n,\mathbb{R})$ for $n = 3,4,5,6$ [238]. (The case $n = 2$ is already considered since $sp(2,\mathbb{R}) \cong so(3,2)$.) For $n = 2r$, these are also multiplets for $sp(r,r)$, $r = 1,2,3$ [122]. (The case $n = 2, r = 1$ is already considered due to $sp(1,1) \cong so(4,1)$ and the parabolic relation between $so(3,2)$ and $so(4,1)$.)

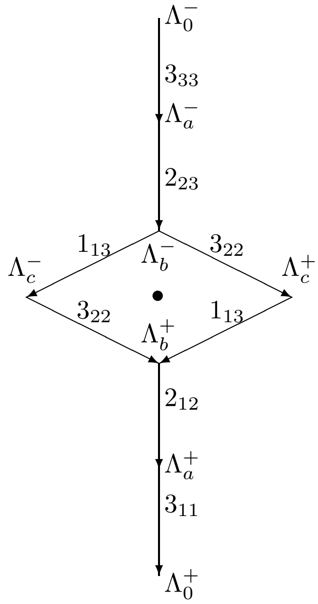


Figure 5. Main multiplets for $Sp(3,\mathbb{R})$.

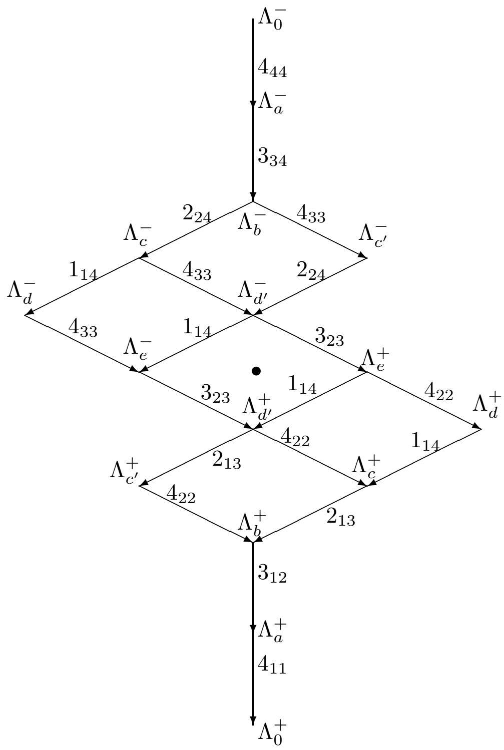


Figure 6. Main multiplets for $sp(4,\mathbb{R})$ and $sp(2,2)$.

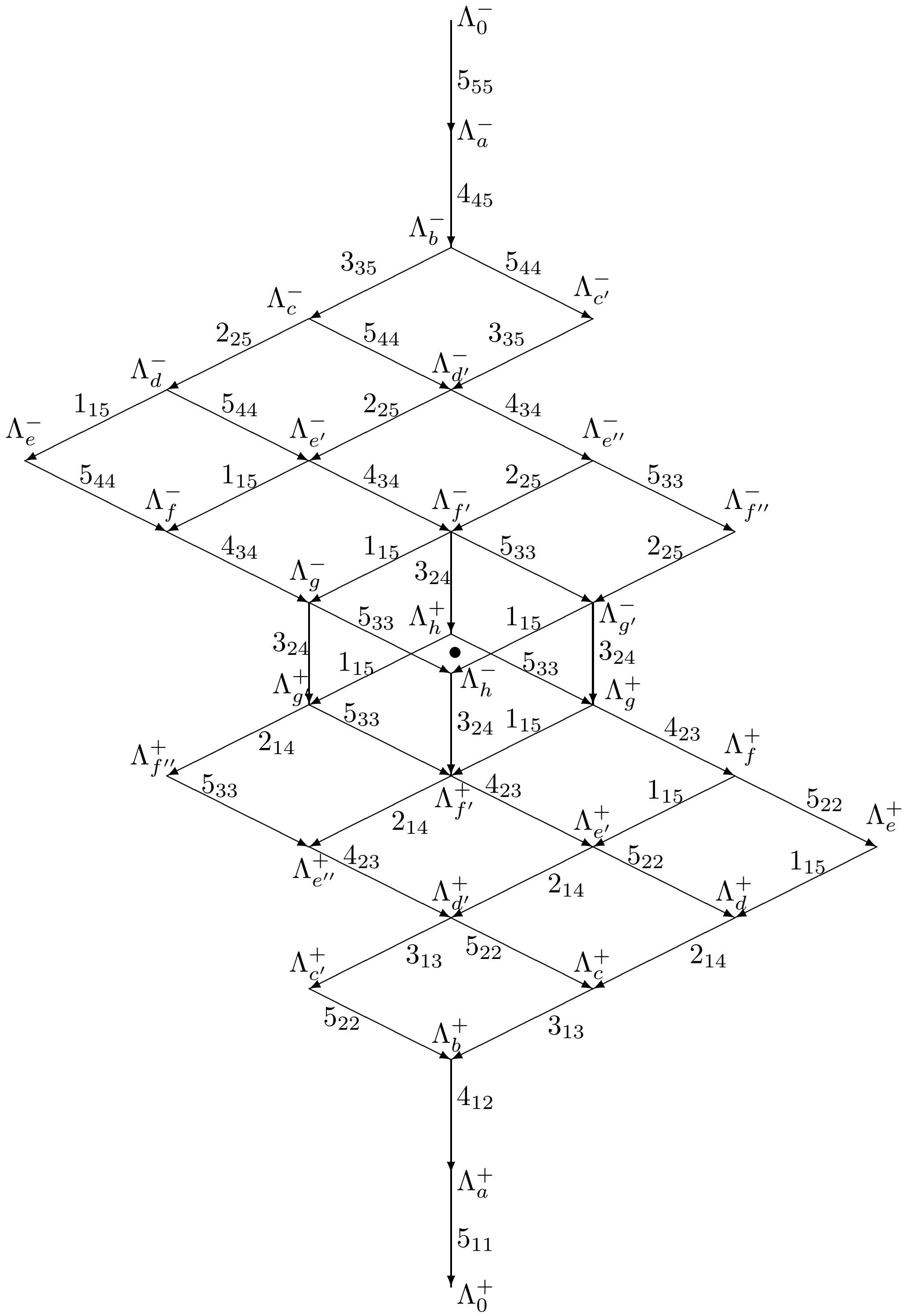


Figure 7. Main multiplets for $Sp(5,\mathbb{R})$.

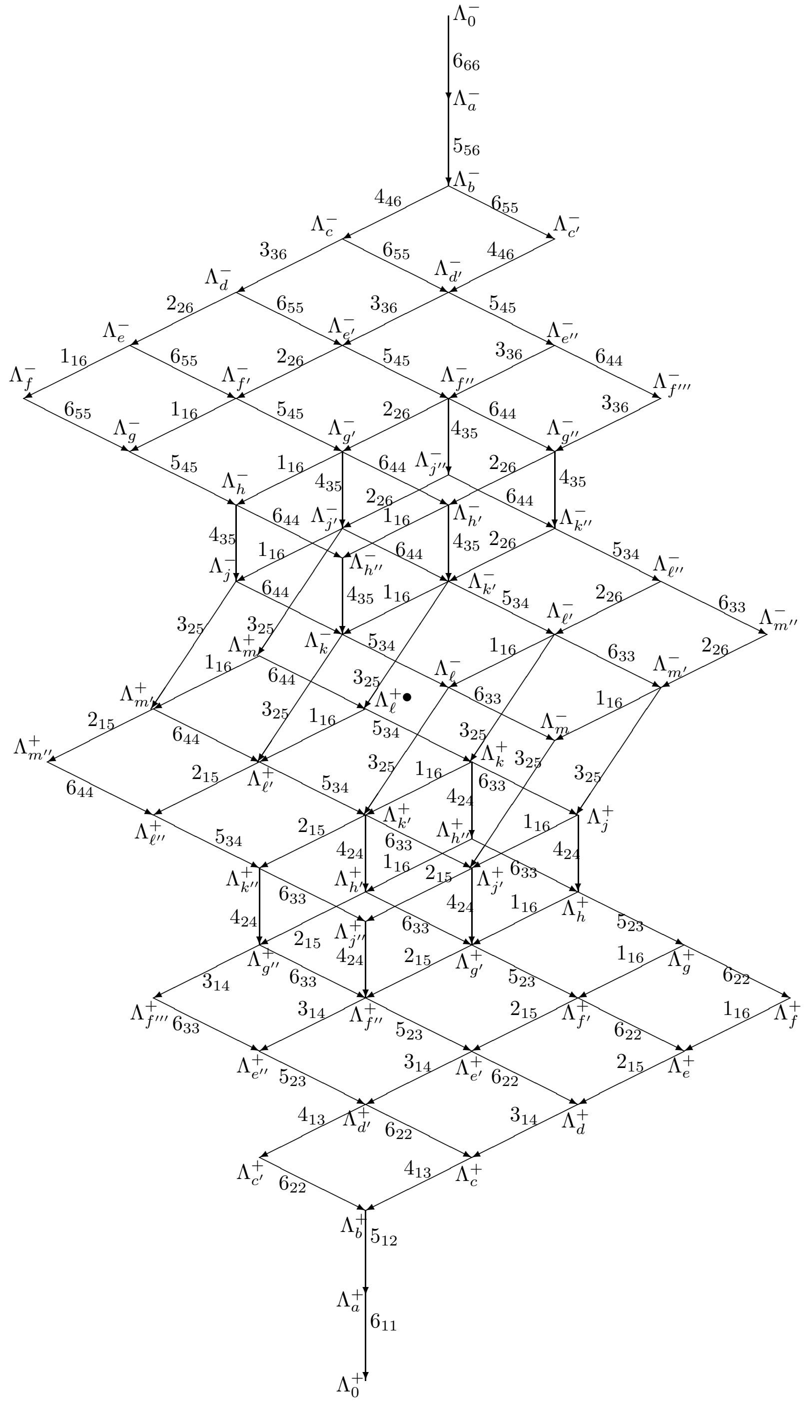


Figure 8. Main multiplets for $Sp(6,\mathbb{R})$ and $sp(3,3)$.

5. The Lie Algebra $so^*(2n)$

The Lie algebra $\mathcal{G} = so^*(2n)$ is given by

$$\begin{aligned} so^*(2n) &\doteq \{X \in so(2n,\mathbb{C}) : J_n C X = X J_n C\} \\ &= \{\ X = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid\ a, b \in gl(n,\mathbb{C}), \\ &\quad\ {}^t a = -a,\ \ b^\dagger = b\ \}\ . \end{aligned}$$

$\dim_R\ \mathcal{G} = n(2n-1)$, $\operatorname{rank}\mathcal{G} = n$.

The maximal compact subalgebra is $\mathcal{K} \cong u(n)$. Thus, $\mathcal{G} = so^*(2n)$ has discrete series representations and highest/lowest-weight representations. The split rank is $r \equiv [n/2]$.

The maximal parabolic subalgebras have $\mathcal{M}$-factors as follows [75]:

$$\begin{aligned} \mathcal{M}_j^{\max} \quad &= \quad so^*(2n-4j) \oplus su^*(2j), \\ & \qquad j = 1, \ldots, r \,. \end{aligned} \tag{21}$$

5.1. Case of $so^*(12)$

For even $n = 2r$, we may choose a *maximal* parabolic $\mathcal{P} = \mathcal{M}\mathcal{A}\mathcal{N}$ such that $\mathcal{A} \cong so(1,1)$, $\mathcal{M} = \mathcal{M}_r^{\max} = su^*(n)$. We note also that

$$\mathcal{K}^{\mathbb{C}} \cong u(1)^{\mathbb{C}} \oplus sl(n,\mathbb{C}) \cong \mathcal{A}^{\mathbb{C}} \oplus \mathcal{M}^{\mathbb{C}}$$

Thus, with this choice we utilize the property which distinguishes the class of 'conformal Lie algebras' to which class the algebras $so^*(4r)$ belong.

Furthermore, we restrict ourselves to $\mathcal{G} = so^*(12)$.

We label the signature of the ERs of $\mathcal{G}$ as follows:

$$\begin{aligned} \chi \quad &= \quad \{\, n_1, n_2, n_3, n_4, n_5;\, c\,\}, \\ & \qquad n_j \in \mathbb{Z}_+\,, \quad c = d - \tfrac{15}{2} \end{aligned}$$

where the last entry of χ labels the characters of $\mathcal{A}$, and the first five entries are labels of the finite-dimensional (nonunitary) irreps of $\mathcal{M} = su^*(6)$ when all $n_j > 0$ or limits of the latter when some $n_j = 0$.

Below, we shall use the following conjugation on the finite-dimensional entries of the signature:

$$(n_1, n_2, n_3, n_4, n_5)^* \;\doteq\; (n_5, n_4, n_3, n_2, n_1)\,.$$

The ERs in the multiplet are related also by intertwining integral operators introduced in [KnSt]. These operators are defined for any ER, the general action being:

$$\begin{aligned} & G_{KS} \,:\, \mathcal{C}_\chi \longrightarrow \mathcal{C}_{\chi'}\,, \\ & \chi \;=\; \{\, n_1, \ldots, n_5;\, c\,\}, \\ & \chi' \;=\; \{\, (n_1, \ldots, n_5)^*;\, -c\,\}. \end{aligned}$$

Furthermore, we give the correspondence between the signatures χ and the highest weight Λ. The connection is through the Dynkin labels:

$$m_i \;\equiv\; (\Lambda + \rho, \alpha_i^\vee) \;=\; (\Lambda + \rho, \alpha_i)\,, \quad i = 1, \ldots, 5,$$

where $\Lambda = \Lambda(\chi)$, ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$. The explicit connection is

$$\begin{aligned} n_i \quad &= \quad m_i, \\ c \quad &= \quad -\tfrac{1}{2}(m_1 + 2m_2 + 3m_3 + 4m_4 + \\ & \qquad +2m_5 + 3m_6) \end{aligned}$$

Finally, we recall that according to [75], the above considerations are applicable also for the algebra $so(6,6)$ with parabolic $\mathcal{M}$-factor $sl(6,\mathbb{R})$.

The main multiplets are in 1-to-1 correspondence with the finite-dimensional irreps of $so^*(12)$; i.e., they are labeled by the six positive Dynkin labels $m_i \in \mathbb{N}$.

The number of ERs/GVMs in the corresponding multiplets is [75]

$$\begin{aligned} & |W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| / |W(\mathcal{K}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| = \\ & = |W(so(12,\mathbb{C}))| / |W(sl(6,\mathbb{C}))| = 32 \end{aligned}$$

where $\mathcal{H}$ is a Cartan subalgebra of both $\mathcal{G}$ and $\mathcal{K}$.

They are given explicitly in the Figure 9 below. The pairs $\Lambda^{\pm}$ are symmetric with regard to to the bullet in the middle of the figure—this represents the Weyl symmetry realized by the Knapp–Stein operators

The statements made for the ER with signature χ_0^- in the previous cases also remain valid here. Also, the conjugate ER χ_0^+ contains a unitary discrete series subrepresentation split in highest/lowest-weight representations.

All the above is valid also for the algebra $so(6,6)$, cf. [75]; however, the latter algebra does not have highest/lowest-weight representations.

We present only the main multiplets. The reduced multiplets may be seen in [92].

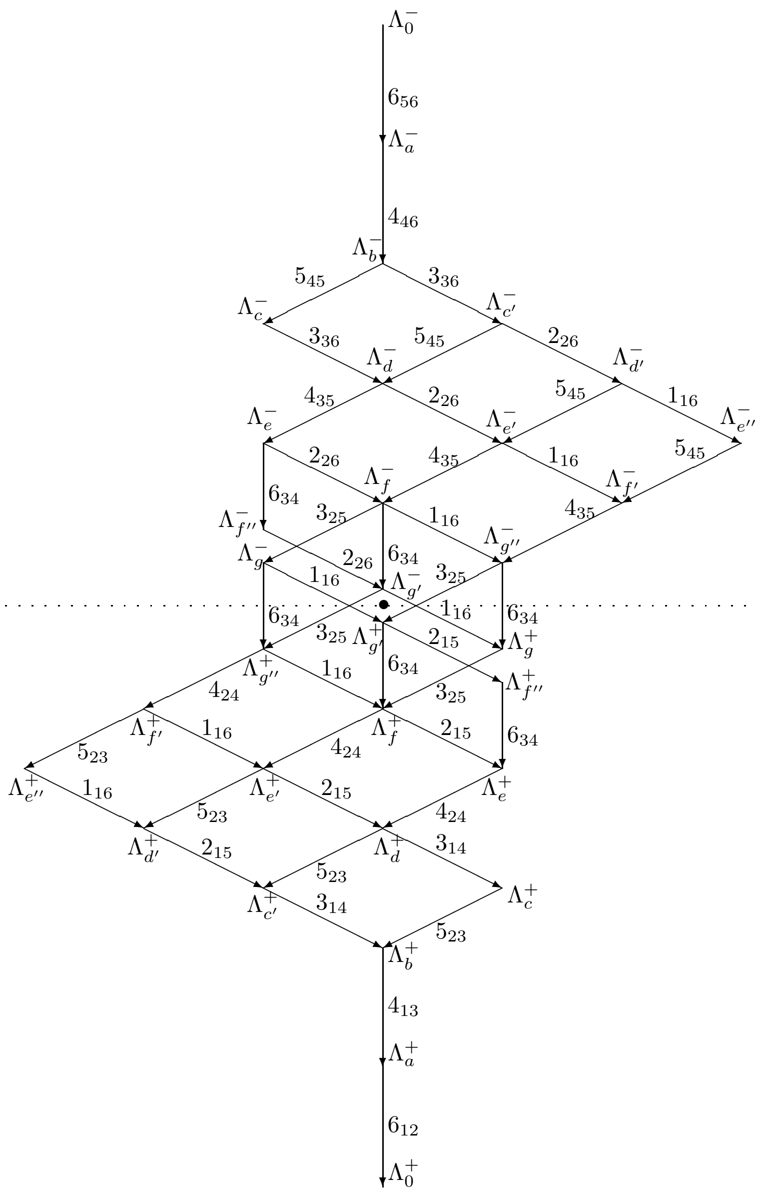


Figure 9. $SO^*(12)$ main multiplets.

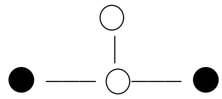
5.2. Case of $so^*(8)$

This case was already considered for the choice of generic maximal parabolic subalgebra of $so(6,2) \cong so^*(8)$, but here, we shall consider a Heisenberg maximal parabolic subalgebra.

We recall that $\mathcal{G} = so^*(8) \cong so(6,2)$ has minimal parabolic:

$$\mathcal{M}_0 = so(3) \oplus so(3) \tag{22}$$

The Satake–Dynkin diagram of $\mathcal{G}$ is



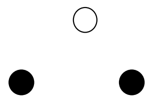
(23)

where, by standard convention, the black dots represent the $so(3)$ subalgebras of $\mathcal{M}_0$.

We shall use the Heisenberg maximal parabolic (21) with $\mathcal{M}$-subalgebra:

$$\mathcal{M} = so^*(4) \oplus so(3) \cong so(2,1) \oplus so(3) \oplus so(3) \tag{24}$$

The Satake–Dynkin diagram of $\mathcal{M}$ is a subdiagram of (23):



(25)

From the above, it follows that the $\mathcal{M}$-compact roots of $\mathcal{G}^{\mathbb{C}}$ are (given in terms of the simple roots):

$$\alpha_{12} = \gamma_1,\ \alpha_{34} = \gamma_3,\ \beta_{34} = \gamma_4 \tag{26}$$

By definition, the above are the positive roots of $\mathcal{M}^{\mathbb{C}}$.

The positive $\mathcal{M}$-noncompact roots of $\mathcal{G}^{\mathbb{C}}$ in terms of the simple roots are:

$$\gamma_{12} = \gamma_1 + \gamma_2,\ \gamma_{13} = \gamma_1 + \gamma_2 + \gamma_3,\ \gamma_2,\ \gamma_{23} = \gamma_2 + \gamma_3, \tag{27a}$$

$$\begin{aligned} &\beta_{12} = \gamma_1 + 2\gamma_2 + \gamma_3 + \gamma_4,\ \beta_{13} = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4, \\ &\beta_{14} = \gamma_1 + \gamma_2 + \gamma_4,\ \beta_{23} = \gamma_2 + \gamma_3 + \gamma_4,\ \beta_{24} = \gamma_2 + \gamma_4 \end{aligned} \tag{27b}$$

where, for convenience, we use the notation $\gamma_{ij} \equiv \alpha_{i,j+1}$

To characterize the Verma modules, we shall first use the Dynkin labels:

$$m_i \equiv (\Lambda + \rho, \gamma_i^\vee), \quad i = 1, \ldots, 4, \tag{28}$$

where ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$. Thus, we shall use:

$$\chi_\Lambda = \{m_1, m_2, m_3, m_4\} \tag{29}$$

Note that when all $m_i \in \mathbb{N}$, then χ_Λ characterizes the finite-dimensional irreps of $\mathcal{G}^{\mathbb{C}}$ and its real forms, in particular, $so^*(8)$. Furthermore, $m_1, m_3, m_4 \in \mathbb{N}$ characterizes the finite-dimensional irreps of the $\mathcal{M}$ subalgebra.

For the $\mathcal{M}$-noncompact roots of $\mathcal{G}^{\mathbb{C}}$, we shall also use the Harish–Chandra parameters:

$$m_{ij} = (\Lambda + \rho, \gamma_{ij}^\vee)\,, \tag{30a}$$

$$\hat{m}_{ij} = (\Lambda + \rho, \beta_{ij}^\vee) \tag{30b}$$

and explicitly in terms of the Dynkin labels (compare (27)):

$$\chi_{HC} = \{ m_{12} = m_1 + m_2,\ m_{13} = m_1 + m_2 + m_3,$$
$$m_2,\ m_{23} = m_2 + m_3, \tag{31a}$$
$$\hat{m}_{12} = m_1 + 2m_2 + m_3 + m_4,$$
$$\hat{m}_{13} = m_1 + m_2 + m_3 + m_4,$$
$$\hat{m}_{14} = m_1 + m_2 + m_4,$$
$$\hat{m}_{23} = m_2 + m_3 + m_4,\ \hat{m}_{24} = m_2 + m_4 \} \tag{31b}$$

Main multiplets of SO*(8)

The main multiplets are in 1-to-1 correspondence with the finite-dimensional irreps of $so^*(8)$; i.e., they are labeled by the four positive Dynkin labels $m_i \in \mathbb{N}$.

We take $\chi_0 = \chi_{HC}$. It has one embedded Verma module with HW $\Lambda_a = \Lambda_0 - m_2\gamma_2$. The number of ERs/GVMs in a main multiplet is 24. We give the whole multiplet as follows:

$$\begin{aligned}
\chi_0 &= \{m_1, m_2, m_3, m_4\} \\
\chi_a &= \{m_{12}, -m_2, m_{23}, m_{2,4}\}, \quad \Lambda_a = \Lambda_0 - m_2\gamma_2 \\
\chi_b &= \{m_2, -m_{12}, m_{13}, m_{12,4}\}, \quad \Lambda_b = \Lambda_a - m_1\gamma_{12} \\
\chi_c &= \{m_{13}, -m_{23}, m_2, m_{24}\}, \quad \Lambda_c = \Lambda_a - m_3\gamma_{23} \\
\chi_d &= \{m_{12,4}, -m_{2,4}, m_{24}, m_2\}, \quad \Lambda_d = \Lambda_a - m_4\beta_{24} \\
\chi_e &= \{m_{23}, -m_{13}, m_{12}, m_{14}\}, \quad \Lambda_e = \Lambda_c - m_1\gamma_{12} \\
\chi_f &= \{m_{2,4}, -m_{12,4}, m_{14}, m_{12}\}, \quad \Lambda_f = \Lambda_b - m_4\beta_{24} \\
\chi_g &= \{m_{14}, -m_{24}, m_{2,4}, m_{23}\}, \quad \Lambda_g = \Lambda_c - m_4\beta_{24} = \Lambda_d - m_3\gamma_{23} \\
\chi_h &= \{m_{24}, -m_{14}, m_{12,4}, m_{13}\}, \quad \Lambda_h = \Lambda_e - m_4\beta_{24} \\
\chi_i &= \{m_3, -m_{13}, m_1, m_{14,2}\}, \quad \Lambda_i = \Lambda_e - m_2\gamma_{13} \\
\chi_j &= \{m_4, -m_{12,4}, m_{14,2}, m_1\}, \quad \Lambda_j = \Lambda_f - m_2\beta_{14} \\
\chi_k &= \{m_{14,2}, -m_{24}, m_4, m_3\}, \quad \Lambda_k = \Lambda_g - m_2\beta_{23}
\end{aligned} \tag{32}$$

$$\begin{aligned}
\chi_k^+ &= \{m_{14,2}, -m_{14}, m_4, m_3\}, \quad \Lambda_k^+ = \Lambda_k - m_1\beta_{12} \\
\chi_j^+ &= \{m_4, -m_{14}, m_{14,2}, m_1\}, \quad \Lambda_j^+ = \Lambda_j - m_3\beta_{12} \\
\chi_i^+ &= \{m_3, -m_{14}, m_1, m_{14,2}\}, \quad \Lambda_i^+ = \Lambda_i - m_4\beta_{12} \\
\chi_h^+ &= \{m_{24}, -m_{14,2}, m_{12,4}, m_{13}\}, \quad \Lambda_h^+ = L_h - m_2\beta_{12} \\
\chi_e^+ &= \{m_{23}, -m_{14,2}, m_{12}, m_{14}\}, \quad \Lambda_e^+ = \Lambda_i^+ - m_2\beta_{24} \\
\chi_f^+ &= \{m_{2,4}, -m_{14,2}, m_{14}, m_{12}\}, \quad \Lambda_f^+ = \Lambda_h^+ - m_3\beta_{14} \\
\chi_g^+ &= \{m_{14}, -m_{14,2}, m_{2,4}, m_{23}\}, \quad \Lambda_g^+ = \Lambda_h^+ - m_1\beta_{23} = \Lambda_k^+ - m_2\gamma_{12} \\
\chi_d^+ &= \{m_{12,4}, -m_{14,2}, m_{24}, m_2\}, \quad \Lambda_d^+ = \Lambda_f^+ - m_1\beta_{23} \\
\chi_c^+ &= \{m_{13}, -m_{14,2}, m_2, m_{24}\}, \quad \Lambda_c^+ = \Lambda_g^+ - m_4\gamma_{13} \\
\chi_b^+ &= \{m_2, -m_{14,2}, m_{13}, m_{12,4}\}, \quad \Lambda_b^+ = \Lambda_e^+ - m_3\beta_{14} = \Lambda_f^+ - m_4\gamma_{13} \\
\chi_a^+ &= \{m_{12}, -m_{14,2}, m_{23}, m_{2,4}\}, \quad \Lambda_a^+ = \Lambda_b^+ - m_1\beta_{23} \\
\chi_0^+ &= \{m_1, -m_{14}, m_3, m_4\}, \quad \Lambda_0^+ = \Lambda_a^+ - m_2\beta_{13}
\end{aligned} \tag{33}$$

We shall label the signature of the ERs of $\mathcal{G}$ also as follows:

$$\chi = [\, n_1, n_2, n_3\, ;\, c\,], \quad c = -\tfrac{1}{2}m_{14,2}, \quad n_1 = m_1,\ n_2 = m_3,\ n_3 = m_4, \tag{34}$$

where the last entry labels the characters of $\mathcal{A}$, and the first three entries of χ are labels of the finite-dimensional irreps of $\mathcal{M}$ when all $n_j > 0$ or limits of the latter when some $n_j = 0$. Note that $m_{14,2} = m_1 + 2m_2 + m_3 + m_4$ is the Harish–Chandra parameter for the highest root β_{12}.

Use of this labeling signatures may be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^\pm &= [m_1, m_3, m_4;\ \pm\tfrac{1}{2} m_{14,2}] \\
\chi_a^\pm &= [m_{12}, m_{23}, m_{2,4};\ \pm\tfrac{1}{2} m_{14}], \\
\chi_b^\pm &= [m_2, m_{13}, m_{12,4};\ \pm\tfrac{1}{2} m_{24}], \\
\chi_c^\pm &= [m_{13}, m_2, m_{24};\ \pm\tfrac{1}{2} m_{12,4}], \\
\chi_d^\pm &= [m_{12,4}, m_{24}, m_2;\ \pm\tfrac{1}{2} m_{13}], \\
\chi_e^\pm &= [m_{23}, m_{12}, m_{14};\ \pm\tfrac{1}{2} m_{2,4}], \\
\chi_f^\pm &= [m_{2,4}, m_{14}, m_{12};\ \pm\tfrac{1}{2} m_{23}], \\
\chi_g^\pm &= [m_{14}, m_{2,4}, m_{23};\ \pm\tfrac{1}{2} m_{12}], \\
\chi_h^\pm &= [m_{24}, m_{12,4}, m_{13};\ \pm\tfrac{1}{2} m_2], \\
\chi_i^\pm &= [m_3, m_1, m_{14,2};\ \pm\tfrac{1}{2} m_4], \\
\chi_j^\pm &= [m_4, m_{14,2}, m_1;\ \pm\tfrac{1}{2} m_3], \\
\chi_k^\pm &= [m_{14,2}, m_4, m_3;\ \pm\tfrac{1}{2} m_1]
\end{aligned} \tag{35}$$

The ERs in the multiplet are alsonrelated by intertwining integral operators introduced in [233]. These operators are defined for any ER, the general action being:

$$\begin{aligned}
&G_{KS} \,:\, \mathcal{C}_\chi \longrightarrow \mathcal{C}_{\chi'}\,, \\
&\chi = [\,n_1, n_2, n_3;\ c\,], \qquad \chi' = [\,n_1, n_2, n_3;\ -c\,].
\end{aligned} \tag{36}$$

The main multiplets are given explicitly in Figure 10. The pairs $\chi^\pm$ are symmetric with regard to to the dashed line in the middle the figure—this represents the Weyl symmetry realized by the Knapp–Stein operators $G_{KS} \,:\, \mathcal{C}_{\chi^\mp} \longrightarrow \mathcal{C}_{\chi^\pm}$.

Some comments are in order.

Matters are arranged so that in every multiplet only the ER with signature χ_0^- contains a finite-dimensional nonunitary subrepresentation in a finite-dimensional subspace $\mathcal{E}$. The latter corresponds to the finite-dimensional irrep of $so^*(8)$ with signature $[m_1, \ldots, m_4]$. The subspace $\mathcal{E}$ is annihilated by the operator G^+ , and is the image of the operator G^- . The subspace $\mathcal{E}$ is annihilated also by the intertwining differential operator acting from χ_0^- to χ_a^- . When all $m_i = 1$, then dim $\mathcal{E} = 1$, and in that case, $\mathcal{E}$ is also the trivial one-dimensional UIR of the whole algebra $\mathcal{G}$. Furthermore, in that case, the conformal weight is zero: $d = \frac{5}{2} + c = \frac{7}{2} - \frac{1}{2}(m_1 + 2m_2 + m_3 + m_4)_{|m_i=1} = 0$.

In the conjugate ER χ_0^+, there is a unitary discrete series subrepresentation in an infinite-dimensional subspace $\mathcal{D}$ with conformal weight $d = \frac{5}{2} + c = \frac{7}{2} + \frac{1}{2}(m_1 + 2m_2 + m_3 + m_4)$. It is annihilated by the operator G^-, and is the image of the operator G^+.

Thus, for $so^*(8)$ the ER with signature χ_0^+ contains both a holomorphic discrete series representation and a conjugate anti-holomorphic discrete series representation. The direct sum of the holomorphic and the antiholomorphic representations spaces form the invariant subspace $\mathcal{D}$ mentioned above. Note that the corresponding lowest-weight GVM is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate highest-weight GVM is infinitesimally equivalent to the anti-holomorphic discrete series.

In Figure 10, we use the notation: $\Lambda^\pm = \Lambda(\chi^\pm)$. Each intertwining differential operator is represented by an arrow accompanied either by a symbol i_{jk} encoding the root γ_{jk} and the number $m_{\gamma_{jk}}$ which is involved in the BGG criterion, or a symbol $i_{\hat{jk}}$ encoding the root β_{jk} and the number $m_{\beta_{jk}}$ from BGG.

Finally, we recall that according to [122], the above considerations are applicable also for the algebra $so(p,q)$ (with $p + q = 8$, $p \geq q \geq 3$) with maximal Heisenberg parabolic subalgebra: $\mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$, $\mathcal{M}' = so(p-2, q-2) \oplus sl(2, \mathbb{R})$.

We present only the main multiplets. The reduced multiplets may be seen in [239].

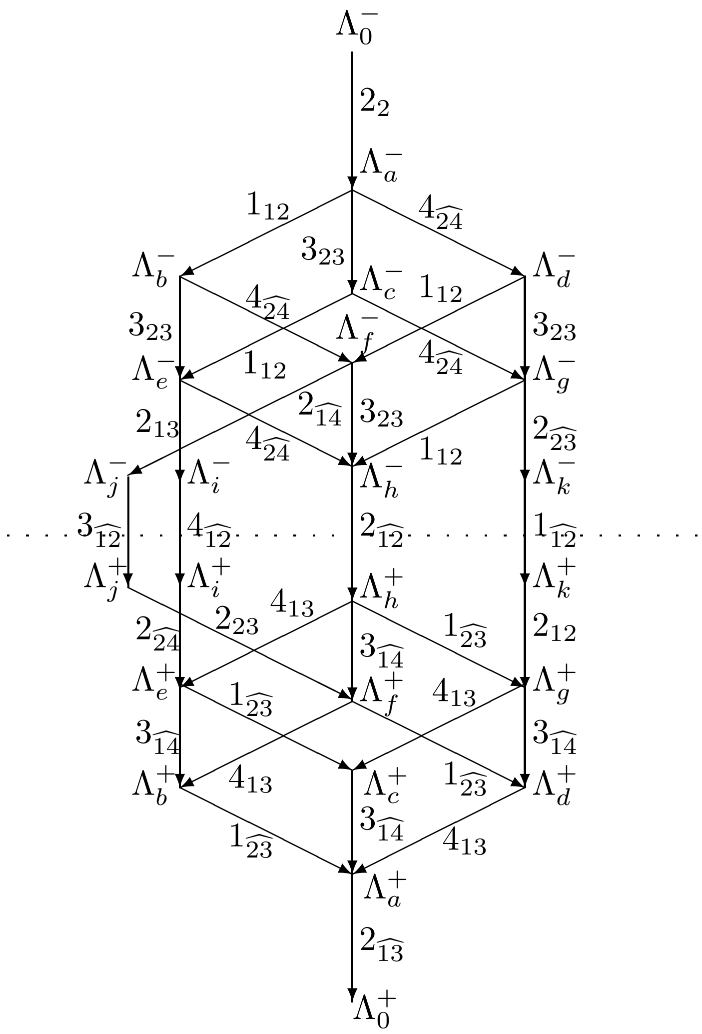


Figure 10. Main multiplets for $SO^*(8)$ using induction from maximal Heisenberg parabolic.

5.3. Case of $so^*(10)$

The case $so^*(10)$ is also special. It is not of the class of 'conformal Lie algebras' but belongs to the wider class of Hermitian symmetric spaces as described in the Introduction.

The maximal parabolic subalgebras of $so^*(2n)$ have $\mathcal{M}$-factors as follows [75]:

$$\mathcal{M}_j^{\max} \ = \ so^*(2n-4j) \oplus su^*(2j), \quad j = 1, \ldots, r\,. \tag{37}$$

The $\mathcal{N}^\pm$ factors in the maximal parabolic subalgebras have dimensions $\dim\,(\mathcal{N}_j^\pm)^{\max} \ = \ j(4n-6j-1)$.

The case $j=1$ is special. In this case, we have a maximal Heisenberg parabolic with $\mathcal{M}$-factor

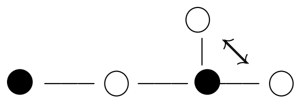
$$\mathcal{M}^{\max}_{\text{Heisenberg}} \ = \ so^*(2n-4) \oplus su(2) \tag{38}$$

which we use in this section.

Furthermore, we restrict ourselves to our case of study $\mathcal{G} \ = \ so^*(10)$ with minimal parabolic:

$$\mathcal{M}_0 \ = \ so(2) \oplus so(3) \oplus so(3) \tag{39}$$

The Satake–Dynkin diagram of $\mathcal{G}$ is:



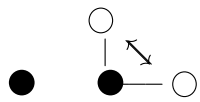
(40)

where, by standard convention, the black dots represent the $so(3)$ subalgebras of $\mathcal{M}_0$ and the left-right arrow represents the $so(2)$ subalgebra of $\mathcal{M}_0$.

We shall use the Heisenberg maximal parabolic (37) with $\mathcal{M}$-subalgebra:

$$\mathcal{M} \;=\; so^*(6) \oplus so(3) \cong su(3,1) \oplus su(2) \tag{41}$$

The Satake–Dynkin diagram of $\mathcal{M}$ is a subdiagram of (40):



(42)

where the single black dot represents the $so(3)$ subalgebra, while the connected part of the diagram represents the $su(3,1)$ subalgebra.

From the above follows that the $\mathcal{M}$-compact roots of $\mathcal{G}^{\mathbb{C}}$ are (given in terms of the simple roots):

$$\alpha_{12} = \gamma_1, \tag{43a}$$

$$\begin{aligned} &\alpha_{34} = \gamma_3,\ \alpha_{45} = \gamma_4,\ \beta_{45} = \gamma_5, \\ &\alpha_{35} = \gamma_3 + \gamma_4,\ \beta_{34} = \gamma_3 + \gamma_4 + \gamma_5,\ \beta_{35} = \gamma_3 + \gamma_5 \end{aligned} \tag{43b}$$

By definition, the above are the positive roots of $\mathcal{M}^{\mathbb{C}}$, namely: $su(2)^{\mathbb{C}}$ (43a) and $su(3,1)^{\mathbb{C}} = sl(4,\mathbb{C})$ (43b).

The positive $\mathcal{M}$-noncompact roots of $\mathcal{G}^{\mathbb{C}}$ in terms of the simple roots are

$$\begin{aligned} &\gamma_{12} = \gamma_1 + \gamma_2,\ \gamma_{13} = \gamma_1 + \gamma_2 + \gamma_3,\ \ \gamma_{14} = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4, \\ &\gamma_2,\ \gamma_{23} = \gamma_2 + \gamma_3,\ \gamma_{24} = \gamma_2 + \gamma_3 + \gamma_4, \end{aligned} \tag{44a}$$

$$\begin{aligned} &\beta_{12} = \gamma_1 + 2\gamma_2 + 2\gamma_3 + \gamma_4 + \gamma_5,\ \beta_{13} = \gamma_1 + \gamma_2 + 2\gamma_3 + \gamma_4 + \gamma_5, \\ &\beta_{14} = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5,\ \beta_{15} = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_5, \\ &\beta_{23} = \gamma_2 + 2\gamma_3 + \gamma_4 + \gamma_5,\ \ \beta_{24} = \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5, \\ &\beta_{25} = \gamma_2 + \gamma_3 + \gamma_5 \end{aligned} \tag{44b}$$

where, for convenience, we use the notation $\gamma_{ij} \equiv \alpha_{i,j+1}$

To characterize the Verma modules, we shall use first the Dynkin labels:

$$m_i \;\equiv\; (\Lambda + \rho, \gamma_i^\vee), \quad i = 1, \ldots, 5, \tag{45}$$

where ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$. Thus, we shall use:

$$\chi_\Lambda \;=\; \{m_1, m_2, m_3, m_4, m_5\} \tag{46}$$

Note that when all $m_i \in \mathbb{N}$, then χ_Λ characterizes the finite-dimensional irreps of $\mathcal{G}^{\mathbb{C}}$ and its real forms, in particular, $so^*(10)$. Furthermore, $m_1 \in \mathbb{N}$ characterizes the finite-dimensional irreps of the $su(2)$ subalgebra, while the set of positive integers $\{m_3, m_4, m_5\}$ characterizes the finite-dimensional irreps of $su(3,1)$.

For the $\mathcal{M}$-noncompact roots of $\mathcal{G}^{\mathbb{C}}$, we shall also use the Harish–Chandra parameters:
.

$$m_{ij} = (\Lambda + \rho, \gamma_{ij}^{\vee}) , \tag{47a}$$
$$\hat{m}_{ij} = (\Lambda + \rho, \beta_{ij}^{\vee}) \tag{47b}$$

and explicitly in terms of the Dynkin labels (compare (44)):

$$\begin{aligned}\chi_{HC} = \{ & m_{12} = m_1 + m_2,\ m_{13} = m_1 + m_2 + m_3, \\ & m_{14} = m_1 + m_2 + m_3 + m_4, \quad m_2, \\ & m_{23} = m_2 + m_3,\ m_{24} = m_2 + m_3 + m_4, \qquad (48a) \\ & \hat{m}_{12} = m_1 + 2m_2 + 2m_3 + m_4 + m_5, \\ & \hat{m}_{13} = m_1 + m_2 + 2m_3 + m_4 + m_5, \\ & \hat{m}_{14} = m_1 + m_2 + m_3 + m_4 + m_5, \\ & \hat{m}_{15} = m_1 + m_2 + m_3 + m_5, \\ & \hat{m}_{23} = m_2 + 2m_3 + m_4 + m_5, \\ & \hat{m}_{24} = m_2 + m_3 + m_4 + m_5, \\ & \hat{m}_{25} = m_2 + m_3 + m_5 \} \qquad (48b)\end{aligned}$$

Main multiplets of SO*(10)

The main multiplets are in 1-to-1 correspondence with the finite-dimensional irreps of $so^*(10)$; i.e., they are labeled by the five positive Dynkin labels $m_i \in \mathbb{N}$.

We take $\chi_0 = \chi_{HC}$. It has one embedded Verma module with HW $\Lambda_a = \Lambda_0 - m_2\gamma_2$. The number of ERs/GVMs in a main multiplet is 40. We give the whole multiplet as follows:

$$\begin{aligned}
\chi_0 &= \{m_1, m_2, m_3, m_4, m_5\} \\
\chi_a &= \{m_{12}, -m_2, m_{23}, m_4, m_5\}, \quad \Lambda_a = \Lambda_0 - m_2\gamma_2 \\
\chi_b &= \{m_2, -m_{12}, m_{13}, m_4, m_5\}, \quad \Lambda_b = \Lambda_a - m_1\gamma_{12} \\
\chi_c &= \{m_{13}, -m_{23}, m_2, m_{34}, m_{3,5}\}, \quad \Lambda_c = \Lambda_a - m_3\gamma_{23} \\
\chi_d &= \{m_{23}, -m_{13}, m_{12}, m_{34}, m_{3,5}\}, \quad \Lambda_d = \Lambda_b - m_3\gamma_{23} = \Lambda_c - m_1\gamma_{12} \\
\chi_e &= \{m_{14}, -m_{24}, m_2, m_3, m_{35}\}, \quad \Lambda_e = \Lambda_c - m_4\gamma_{24} \\
\chi_f &= \{m_{13,5}, -m_{23,5}, m_2, m_{35}, m_3\}, \quad \Lambda_f = \Lambda_c - m_5\beta_{25} \\
\chi_g &= \{m_3, -m_{13}, m_1, m_{24}, m_{23,5}\}, \quad \Lambda_g = \Lambda_d - m_2\gamma_{13} \\
\chi_h &= \{m_{24}, -m_{14}, m_{12}, m_3, m_{35}\}, \quad \Lambda_h = \Lambda_d - m_4\gamma_{24} \\
\chi_i &= \{m_{23,5}, -m_{13,5}, m_{12}, m_{35}, m_3\}, \quad \Lambda_i = \Lambda_d - m_5\beta_{25} = \Lambda_f - m_1\gamma_{13} \\
\chi_j &= \{m_{15}, -m_{25}, m_2, m_{3,5}, m_{34}\}, \quad \Lambda_j = \Lambda_e - m_5\beta_{25} \qquad (49) \\
\chi_k &= \{m_{34}, -m_{14}, m_1, m_{23}, m_{25}\}, \quad \Lambda_k = \Lambda_g - m_4\gamma_{24} = \Lambda_h - m_2\gamma_{13} \\
\chi_l &= \{m_{3,5}, -m_{13,5}, m_1, m_{25}, m_{23}\}, \quad \Lambda_l = \Lambda_g - m_5\beta_{25} \\
\chi_m &= \{m_{25}, -m_{15}, m_{12}, m_{3,5}, m_{34}\}, \quad \Lambda_m = \Lambda_h - m_5\beta_{25} \\
\chi_n &= \{m_{15,3}, -m_{25,3}, m_{23}, m_5, m_4\}, \quad \Lambda_n = \Lambda_j - m_3\beta_{24} \\
\chi_p &= \{m_4, -m_{14}, m_1, m_2, m_{25,3}\}, \quad \Lambda_p = \Lambda_k - m_3\gamma_{14} \\
\chi_q &= \{m_{35}, -m_{15}, m_1, m_{23,5}, m_{24}\}, \quad \Lambda_q = \Lambda_k - m_5\beta_{25} \\
\chi_r &= \{m_5, -m_{13,5}, m_1, m_{25,3}, m_2\}, \quad \Lambda_r = \Lambda_l - m_3\beta_{15} \\
\chi_s &= \{m_{25,3}, -m_{15,3}, m_{13}, m_5, m_4\}, \quad \Lambda_s = \Lambda_m - m_3\beta_{24} \\
\chi_t &= \{m_{15,23}, -m_{25,3}, m_3, m_5, m_4\}, \quad \Lambda_t = \Lambda_n - m_2\beta_{23}
\end{aligned}$$

$$
\begin{aligned}
\chi_p^+ &= \{m_4, -m_{15}, m_1, m_2, m_{25,3}\}, \quad \Lambda_p^+ = \Lambda_p - m_5\beta_{12} \\
\chi_q^+ &= \{m_{35}, -m_{15,3}, m_1, m_{23,5}, m_{24}\}, \quad \Lambda_q^+ = \Lambda_q - m_3\beta_{12} \\
\chi_r^+ &= \{m_5, -m_{15}, m_1, m_{25,3}, m_2\}, \quad \Lambda_r^+ = \Lambda_r - m_4\beta_{12} \\
\chi_s^+ &= \{m_{25,3}, -m_{15,23}, m_{13}, m_5, m_4\}, \quad \Lambda_s^+ = \Lambda_s - m_2\beta_{12} \\
\chi_t^+ &= \{m_{15,23}, -m_{15,3}, m_3, m_5, m_4\}, \quad \Lambda_t^+ = \Lambda_t - m_1\beta_{12} \\
\chi_k^+ &= \{m_{34}, -m_{15,3}, m_1, m_{23}, m_{25}\}, \quad \Lambda_k^+ = \Lambda_p^+ - m_3\beta_{25} \\
\chi_l^+ &= \{m_{3,5}, -m_{15,3}, m_1, m_{25}, m_{23}\}, \quad \Lambda_l^+ = \Lambda_q^+ - m_4\beta_{15} = \Lambda_r^+ - m_3\gamma_{24} \\
\chi_m^+ &= \{m_{25}, -m_{15,23}, m_{12}, m_{3,5}, m_{34}\}, \quad \Lambda_m^+ = \Lambda_q^+ - m_2\beta_{24} = \Lambda_s^+ - m_3\gamma_{13} \\
\chi_n^+ &= \{m_{15,3}, -m_{15,23}, m_{23}, m_5, m_4\}, \quad \Lambda_n^+ = \Lambda_s^+ - m_1\beta_{23} = \Lambda_t^+ - m_2\gamma_{12} \\
\chi_h^+ &= \{m_{24}, -m_{15,23}, m_{12}, m_3, m_{35}\}, \quad \Lambda_h^+ = \Lambda_k^+ - m_2\beta_{24} = \Lambda_m^+ - m_5\gamma_{14} \\
\chi_g^+ &= \{m_3, -m_{15,3}, m_1, m_{24}, m_{23,5}\}, \quad \Lambda_\omega^+ = \Lambda_l^+ - m_5\gamma_{14} \\
\chi_i^+ &= \{m_{23,5}, -m_{15,23}, m_{12}, m_{35}, m_3\}, \quad \Lambda_\tau^+ = \Lambda_l^+ - m_2\beta_{24} = \Lambda_m^+ - m_4\beta_{15} \\
\chi_j^+ &= \{m_{15}, -m_{15,23}, m_2, m_{3,5}, m_{34}\}, \quad \Lambda_\pi^+ = \Lambda_m^+ - m_1\beta_{23} = \Lambda_n^+ - m_3\gamma_{13} \\
\chi_e^+ &= \{m_{14}, -m_{15,23}, m_2, m_3, m_{35}\}, \quad \Lambda_\alpha^+ = \Lambda_h^+ - m_1\beta_{23} \\
\chi_d^+ &= \{m_{23}, -m_{15,23}, m_{12}, m_{34}, m_{3,5}\}, \quad \Lambda_\beta^+ = \Lambda_g^+ - m_2\beta_{24} = \Lambda_i^+ - m_5\gamma_{14} \\
\chi_f^+ &= \{m_{13,5}, -m_{15,23}, m_2, m_{35}, m_3\}, \quad \Lambda_\gamma^+ = \Lambda_i^+ - m_1\beta_{23} = \Lambda_j^+ - m_4\beta_{15} \\
\chi_\delta^+ &= \{m_{14}, -m_{15,23}, m_2, m_3, m_{35}\}, \quad \Lambda_\delta^+ = \Lambda_j^+ - m_5\gamma_{14} = \chi_e^+ \quad !!! \\
\chi_b^+ &= \{m_2, -m_{15,23}, m_{13}, m_4, m_5\}, \quad \Lambda_\epsilon^+ = \Lambda_\beta^+ - m_3\beta_{14} \\
\chi_c^+ &= \{m_{13}, -m_{15,23}, m_2, m_{34}, m_{3,5}\}, \quad \Lambda_\lambda^+ = \Lambda_d^+ - m_1\beta_{23} \\
\chi_\nu^+ &= \{m_{13}, -m_{15,23}, m_2, m_{34}, m_{3,5}\}, \quad \Lambda_\nu^+ = \Lambda_f^+ - m_5\gamma_{14} = \chi_c^+ \quad !!! \\
\chi_a^+ &= \{m_{12}, -m_{15,23}, m_{23}, m_4, m_5\}, \quad \Lambda_\kappa^+ = \Lambda_b^+ - m_1\beta_{23} = \Lambda_c^+ - m_3\beta_{14} \\
\chi_0^+ &= \{m_1, -m_{15,3}, m_3, m_4, m_5\}, \quad \Lambda_0^+ = \Lambda_a^+ - m_2\beta_{13}
\end{aligned} \tag{50}
$$

We shall label the signature of the ERs of $\mathcal{G}$ also as follows:

$$\chi = [\, n\,;\, c;\, n_1, n_2, n_3], \qquad n \in \mathbb{N}, \quad c = -\tfrac{1}{2} m_{15,23}, \quad n_j = m_{j+2} \in \mathbb{Z}_+\,, \tag{51}$$

where the first entry $n = m_1$ labels the finite-dimensional irreps of $su(2)$, the second entry labels the characters of $\mathcal{A}$, the last three entries of χ are labels of the finite-dimensional (nonunitary) irreps of $\mathcal{M} = su(3,1)$ when all $n_j > 0$ or limits of the latter when some $n_j = 0$. Note that $m_{15,23} = m_1 + 2m_2 + 2m_3 + m_4 + m_5$ is the Harish–Chandra parameter for the highest root β_{12}.

Using this labeling, signatures may be given in the following pair-wise manner:

$$
\begin{aligned}
\chi_0^\pm &= [m_1; m_3, m_4, m_5;\ \pm\tfrac{1}{2} m_{15,23}] \\
\chi_a^\pm &= [m_{12}; m_{23}, m_4, m_5;\ \pm\tfrac{1}{2} m_{15,3}] \\
\chi_b^\pm &= [m_2; m_{13}, m_4, m_5;\ \pm\tfrac{1}{2} m_{25,3}] \\
\chi_c^\pm &= [m_{13}; m_2, m_{34}, m_{3,5};\ \pm\tfrac{1}{2} m_{15}] \\
\chi_d^\pm &= [m_{23}; m_{12}, m_{34}, m_{3,5};\ \pm\tfrac{1}{2} m_{25}] \\
\chi_e^\pm &= [m_{14}; m_2, m_3, m_{35};\ \pm\tfrac{1}{2} m_{13,5}] \\
\chi_f^\pm &= [m_{13,5}; m_2, m_{35}, m_3;\ \pm\tfrac{1}{2} m_{14}]
\end{aligned}
$$

$$
\begin{aligned}
\chi_g^{\pm} &= [m_3; m_1, m_{24}, m_{23,5};\ \pm\tfrac{1}{2}m_{35}] \\
\chi_h^{\pm} &= [m_{24}; m_{12}, m_3, m_{35};\ \pm\tfrac{1}{2}m_{23,5}] \\
\chi_i^{\pm} &= [m_{23,5}; m_{12}, m_{35}, m_3;\ \pm\tfrac{1}{2}m_{24}] \\
\chi_j^{\pm} &= [m_{15}; m_2, m_{3,5}, m_{34};\ \pm\tfrac{1}{2}m_{13}] \\
\chi_k^{\pm} &= [m_{34}; m_1, m_{23}, m_{25};\ \pm\tfrac{1}{2}m_{3,5}] \\
\chi_l^{\pm} &= [m_{3,5}; m_1, m_{25}, m_{23};\ \pm\tfrac{1}{2}m_{34}] \\
\chi_m^{\pm} &= [m_{25}; m_{12}, m_{3,5}, m_{34};\ \pm\tfrac{1}{2}m_{23}] \\
\chi_n^{\pm} &= [m_{15,3}; m_{23}, m_5, m_4;\ \pm\tfrac{1}{2}m_{12}] \\
\chi_p^{\pm} &= [m_4; m_1, m_2, m_{25,3};\ \pm\tfrac{1}{2}m_5] \\
\chi_q^{\pm} &= [m_{35}; m_1, m_{23,5}, m_{24};\ \pm\tfrac{1}{2}m_3] \\
\chi_r^{\pm} &= [m_5; m_1, m_{25,3}, m_2;\ \pm\tfrac{1}{2}m_4] \\
\chi_s^{\pm} &= [m_{25,3}; m_{13}, m_5, m_4;\ \pm\tfrac{1}{2}m_2] \\
\chi_t^{\pm} &= [m_{15,23}; m_3, m_5, m_4;\ \pm\tfrac{1}{2}m_1]
\end{aligned}
$$

The ERs in the multiplet are also related by intertwining integral operators introduced in [233]. These operators are defined for any ER, the general action being:

$$
\begin{aligned}
& G_{KS} \ :\ \mathcal{C}_\chi \longrightarrow \mathcal{C}_{\chi'}\ , \\
& \chi = [\,n\,;\,n_1, n_2, n_3;\,c\,], \qquad \chi' = [\,n\,;\,n_1, n_2, n_3;\,-c\,].
\end{aligned} \tag{52}
$$

The main multiplets are given explicitly in Figure 11. The pairs $\chi^{\pm}$ are symmetric with regard to to the bullet in the middle of the figure—this represents the Weyl symmetry realized by the Knapp–Stein operators: $G_{KS}\ :\ \mathcal{C}_{\chi^{\mp}} \longrightarrow \mathcal{C}_{\chi^{\pm}}$.

Some comments are in order.

Matters are arranged so that in every multiplet only the ER with signature χ_0^- contains a finite-dimensional nonunitary subrepresentation in a finite-dimensional subspace $\mathcal{E}$. The latter corresponds to the finite-dimensional irrep of $so^*(10)$ with signature $\{m_1, \ldots, m_5\}$. The subspace $\mathcal{E}$ is annihilated by the operator G^+ , and is the image of the operator G^- . The subspace $\mathcal{E}$ is also annihilated by the intertwining differential operator acting from χ_0^- to χ_a^- . When all $m_i = 1$ then dim $\mathcal{E} = 1$, and in that case, $\mathcal{E}$ is also the trivial one-dimensional UIR of the whole algebra $\mathcal{G}$. Furthermore, in that case, the conformal weight is zero: $d = \frac{7}{2} + c = \frac{7}{2} - \frac{1}{2}(m_1 + 2m_2 + 2m_3 + m_4 + m_5)_{|_{m_i=1}} = 0$.

In the conjugate ER χ_0^+, there is a unitary discrete series subrepresentation in an infinite-dimensional subspace $\mathcal{D}$. It is annihilated by the operator G^-, and is the image of the operator G^+.

Thus, for $so^*(10)$ the ER with signature χ_0^+ contains both a holomorphic discrete series representation and a conjugate anti-holomorphic discrete series representation. The direct sum of the holomorphic and the antiholomorphic representations spaces form the invariant subspace $\mathcal{D}$ mentioned above. Note that the corresponding lowest-weight GVM is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate highest weight GVM is infinitesimally equivalent to the anti-holomorphic discrete series.

Finally, we recall that according to [122], the above considerations are applicable also for the algebra $so(p,q)$ (with $p + q = 10$, $p \geq q \geq 2$) with maximal Heisenberg parabolic subalgebra: $\mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$, $\mathcal{M}' = so(p-2, q-2) \oplus sl(2, \mathbb{R})$.

We present only the main multiplets. The reduced multiplets may be seen in [240].

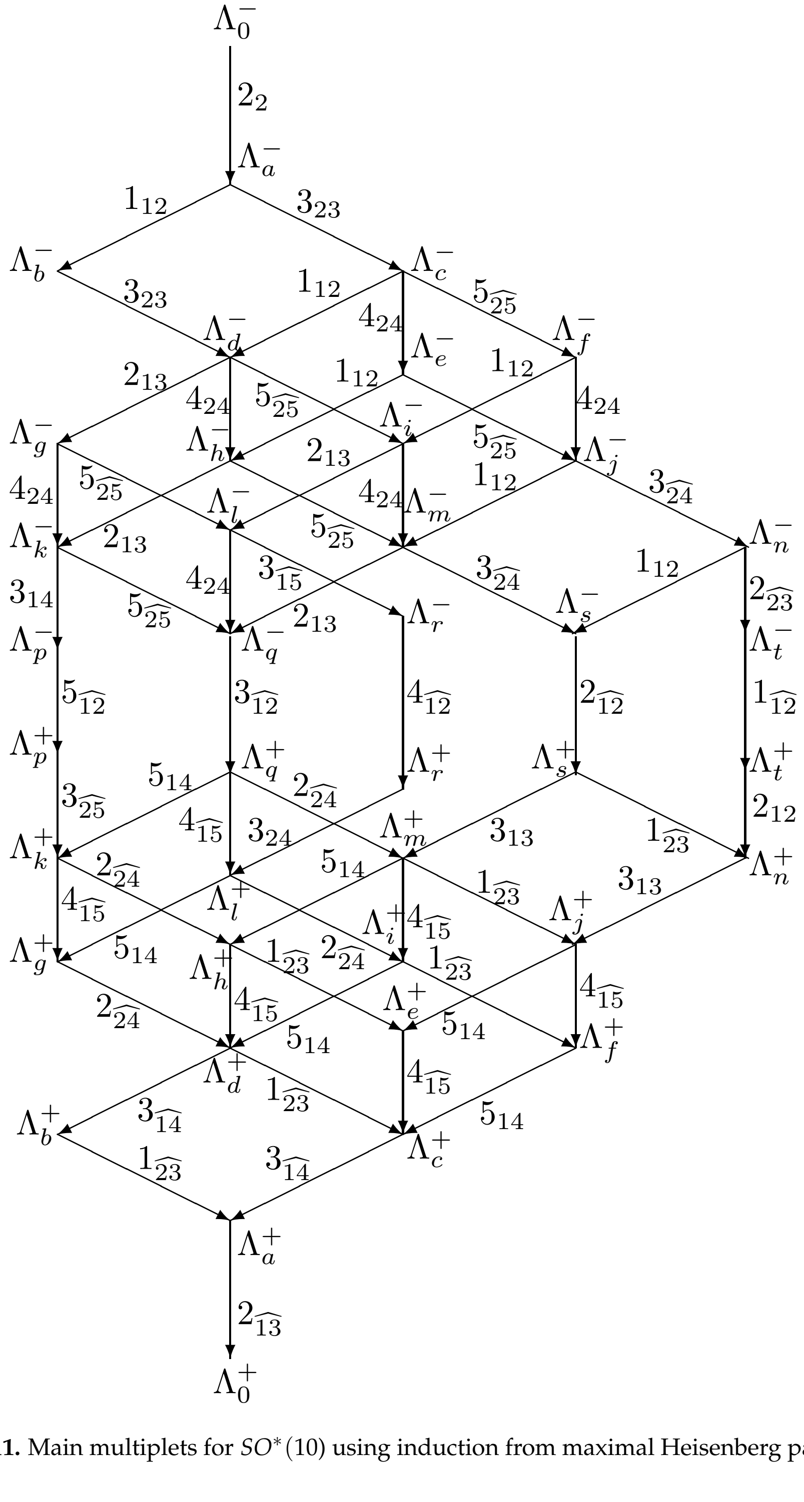
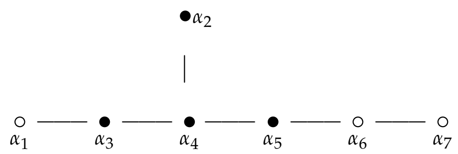


Figure 11. Main multiplets for $SO^*(10)$ using induction from maximal Heisenberg parabolic.

6. The Lie Algebras $E_{7(-25)}$ and $E_{7(7)}$

Let $\mathcal{G} = E_{7(-25)}$. The maximal compact subgroup is $\mathcal{K} \cong e_6 \oplus so(2)$, while $\mathcal{M} \cong E_{6(-6)}$.

The Satake diagram [241] is:



The signatures of the ERs of $\mathcal{G}$ are:

$$\chi = \{ n_1, \ldots, n_6; c \}, \qquad n_j \in \mathbb{N} .$$

expressed through the Dynkin labels:

$$n_i = m_i, \qquad c = -\tfrac{1}{2}(m_{\tilde{\alpha}} + m_7) = -\tfrac{1}{2}(2m_1 + 2m_2 + 3m_3 + 4m_4 + 3m_5 + 2m_6 + 2m_7)$$

The same signatures can be used for the parabolically related exceptional Lie algebra $E_{7(7)}$ (with $\mathcal{M}$-factor $E_{6(6)}$).

The noncompact roots of the complex algebra E_7 are:

$$\begin{aligned}
&\alpha_7, \ \alpha_{17}, \ \ldots, \alpha_{67} , \\
&\alpha_{1,37}, \alpha_{2,47}, \alpha_{17,4}, \alpha_{27,4}, \\
&\alpha_{17,34}, \alpha_{17,35}, \alpha_{17,36}, \alpha_{17,45}, \alpha_{17,46}, \\
&\alpha_{27,45}, \alpha_{27,46}, \\
&\alpha_{17,25,4}, \alpha_{17,26,4}, \alpha_{17,35,4}, \alpha_{17,36,4}, \\
&\alpha_{17,26,45}, \alpha_{17,36,45}, \\
&\alpha_{17,26,35,4}, \alpha_{17,26,45,4}, \\
&\alpha_{17,16,35,4} = \tilde{\alpha} ,
\end{aligned}$$

given through the simple roots α_i:

$$\begin{aligned}
&\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \ i < j, \\
&\alpha_{ij,k} = \alpha_{k,ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j + \alpha_k , \quad i < j , \qquad \text{etc.}
\end{aligned}$$

The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of E_7; i.e., they will be labeled by the seven positive Dynkin labels $m_i \in \mathbb{N}$.

The number of ERs in the corresponding multiplets is equal to

$$|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| \,/\, |W(\mathcal{K}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})| = 56$$

The multiplets are given in Figure 12 [122,242].

The Knapp–Stein operators $G^{\pm}_{\chi}$ act pictorially as reflections with regard to the bullet intertwining each $\mathcal{T}^{-}_{\chi}$ member with the corresponding $\mathcal{T}^{+}_{\chi}$ member.

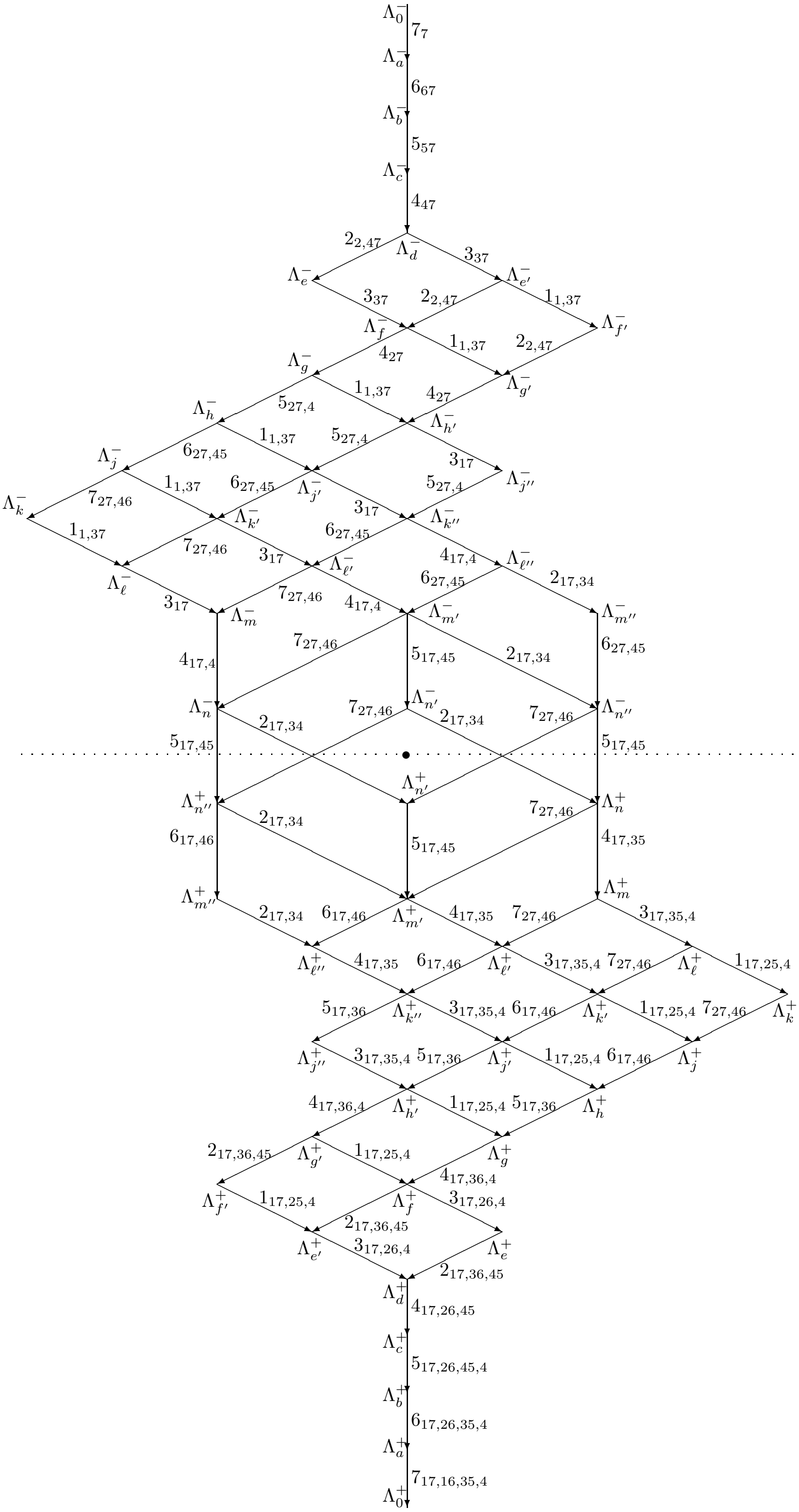


Figure 12. Main type for $E_{7(-25)}$.

7. The Lie Algebras $E_{6(-14)}$, $E_{6(6)}$ and $E_{6(2)}$

Let $\mathcal{G} = E_{6(-14)}$. The maximal compact subalgebra is $\mathcal{K} \cong so(10) \oplus so(2)$, while $\mathcal{M} \cong su(5,1)$.

The Satake diagram [241] is:

$$\underbrace{\begin{array}{ccccccccc} & & & & \circ\,\alpha_2 & & & & \\ & & & & | & & & & \\ \circ & \text{——} & \bullet & \text{——} & \bullet & \text{——} & \bullet & \text{——} & \circ \\ \alpha_1 & & \alpha_3 & & \alpha_4 & & \alpha_5 & & \alpha_6 \end{array}} \tag{53}$$

The signature of the ERs of $\mathcal{G}$ is:

$$\chi = \{\, n_1, n_3, n_4, n_5, n_6;\, c\,\}\,, \quad c = d - \tfrac{11}{2}\,. \tag{54}$$

expressed through the Dynkin labels as:

$$n_i = m_i, \quad -c = \tfrac{1}{2} m_{\tilde{\alpha}} = \tfrac{1}{2}(m_1 + 2m_2 + 2m_3 + 3m_4 + 2m_5 + m_6) \tag{55}$$

The same signatures can be used for the parabolically related exceptional Lie algebras $E_{6(6)}$ and $E_{6(2)}$ with $\mathcal{M}$–factors $sl(6,\mathbb{R})$ and $su(3,3)$, resp.

Furthermore, we need the noncompact roots of the complex algebra E_6:

$$\begin{gathered} \alpha_2,\ \alpha_{14},\ \alpha_{15},\ \alpha_{16},\ \alpha_{24},\ \alpha_{25},\ \alpha_{26} \\ \alpha_{2,4},\ \alpha_{2,45},\ \alpha_{2,46},\ \alpha_{25,4},\ \alpha_{15,4},\ \alpha_{26,4} \\ \alpha_{16,4},\ \alpha_{15,34},\ \alpha_{26,45},\ \alpha_{16,34},\ \alpha_{16,45} \\ \alpha_{16,35},\ \alpha_{16,35,4},\ \alpha_{16,25,4} = \tilde{\alpha} \end{gathered} \tag{56}$$

The multiplets of the main type are in 1-to-1 correspondence with the finite-dimensional irreps of $\mathcal{G}$; i.e., they will be labeled by the six positive Dynkin labels $m_i \in \mathbb{N}$.

Since these algebras do not belong to the class of conformal Lie algebras (CLA), the number of ERs/GVMs in the multiplet is not given by formula (14). It turns out that each such multiplet contains 70 ERs/GVMs—see Figure 13 [122,243]. Another difference with the CLA class is that, pictorially, the the Knapp–Stein operators $G^{\pm}_{\chi}$ act as reflections with regard to the dotted line separating the $\mathcal{T}^{-}_{\chi}$ members from the $\mathcal{T}^{+}_{\chi}$ members (and not as reflections with regard to a central dot (bullet) as in the CLA cases).

Note that there are five cases when the embeddings correspond to the highest root $\tilde{\alpha}: V^{\Lambda^-} \longrightarrow V^{\Lambda^+}$, $\Lambda^+ = \Lambda^- - m_{\tilde{\alpha}}\,\tilde{\alpha}$. In these five cases, the weights are denoted as: $\Lambda^{\pm}_{k''}$, $\Lambda^{\pm}_{k'}$, $\Lambda^{\pm}_{\tilde{k}}$, $\Lambda^{\pm}_{k}$, $\Lambda^{\pm}_{k^o}$; then, $m_{\tilde{\alpha}} = m_1, m_3, m_4, m_5, m_6$, resp. Thus, their action coincides with the action of the Knapp–Stein operators G^{+}_{χ} which, in the above five cases, degenerate to differential operators as we discussed for $so(3,2)$.

Note that the figure has the standard E_6 symmetry, namely, conjugation exchanging indices $1 \longleftrightarrow 6$, $3 \longleftrightarrow 5$.

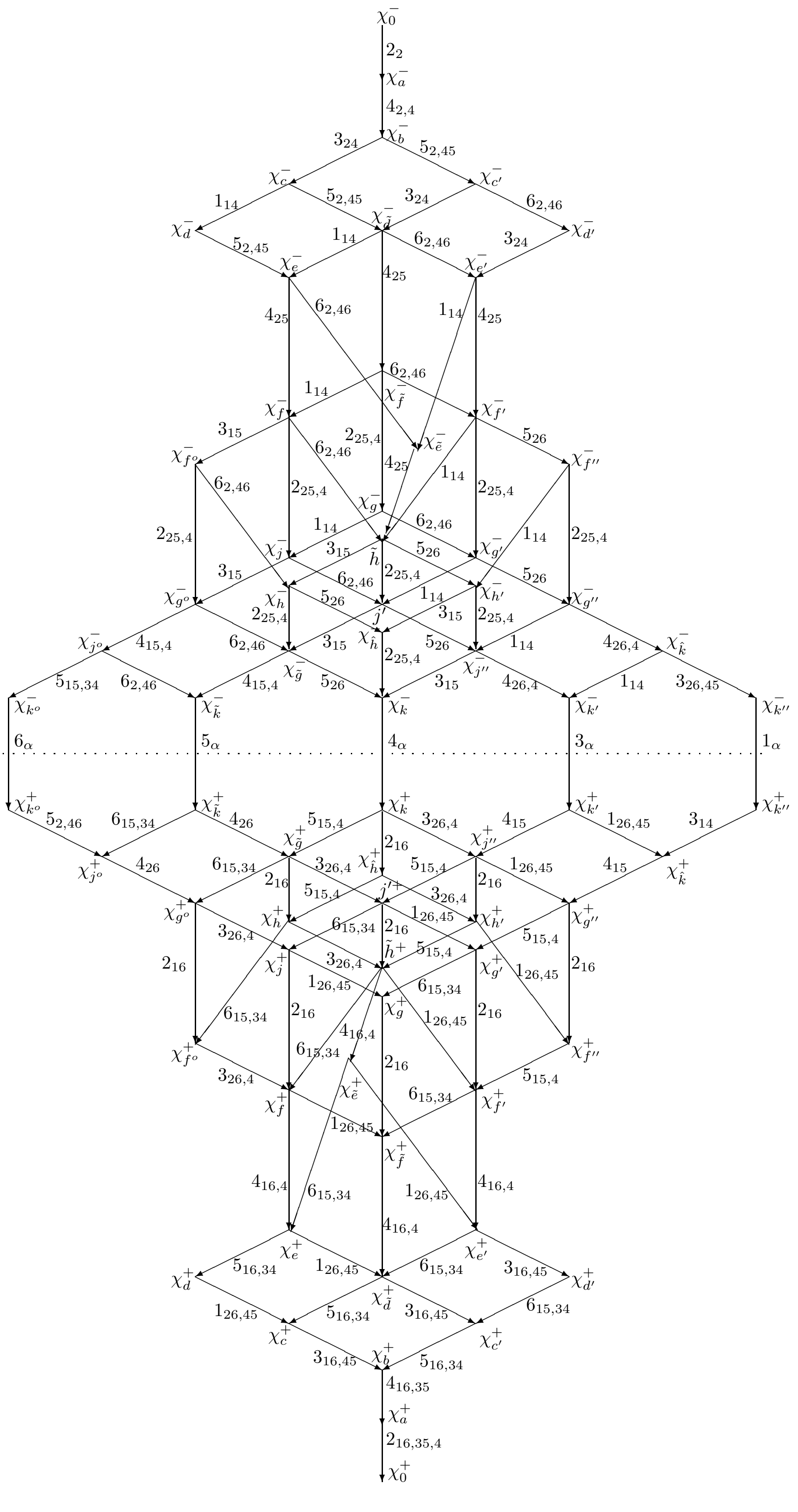


Figure 13. Main Type for $E_{6(-24)}$.

8. The Lie Algebra F_4

The complex Lie algebra F_4 has two real forms denoted by F_4' and F_4''.

The first F_4' is the split real form (denoted also as $F_{4(4)}$). It has discrete series representations since $\mathrm{rank}\, F_4' = \mathrm{rank}\, \mathcal{K} = 4$, where $\mathcal{K} = sp(3) \oplus su(2)$ is the maximal compact subalgebra.

The real form F_4' has several parabolic subalgebras. We shall consider a maximal (also called Heisenberg) parabolic subalgebra:

$$\begin{aligned} \mathcal{P} &= \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}, \\ \mathcal{M} &= sl(3,\mathbb{R}) \oplus sl(2,\mathbb{R}), \\ \dim \mathcal{A} &= 1, \qquad \dim \mathcal{N} = 20 \end{aligned}$$

Note that in what follows we shall use the case when the $sl(3,\mathbb{R})$ subalgebra is formed by the two short roots of F_4, and the $sl(2,\mathbb{R})$ subalgebra is formed by a long root of F_4. The other (equivalent in our considerations) possibility is to flip the short and the long roots.

The embedding diagram is given in Figure 14.

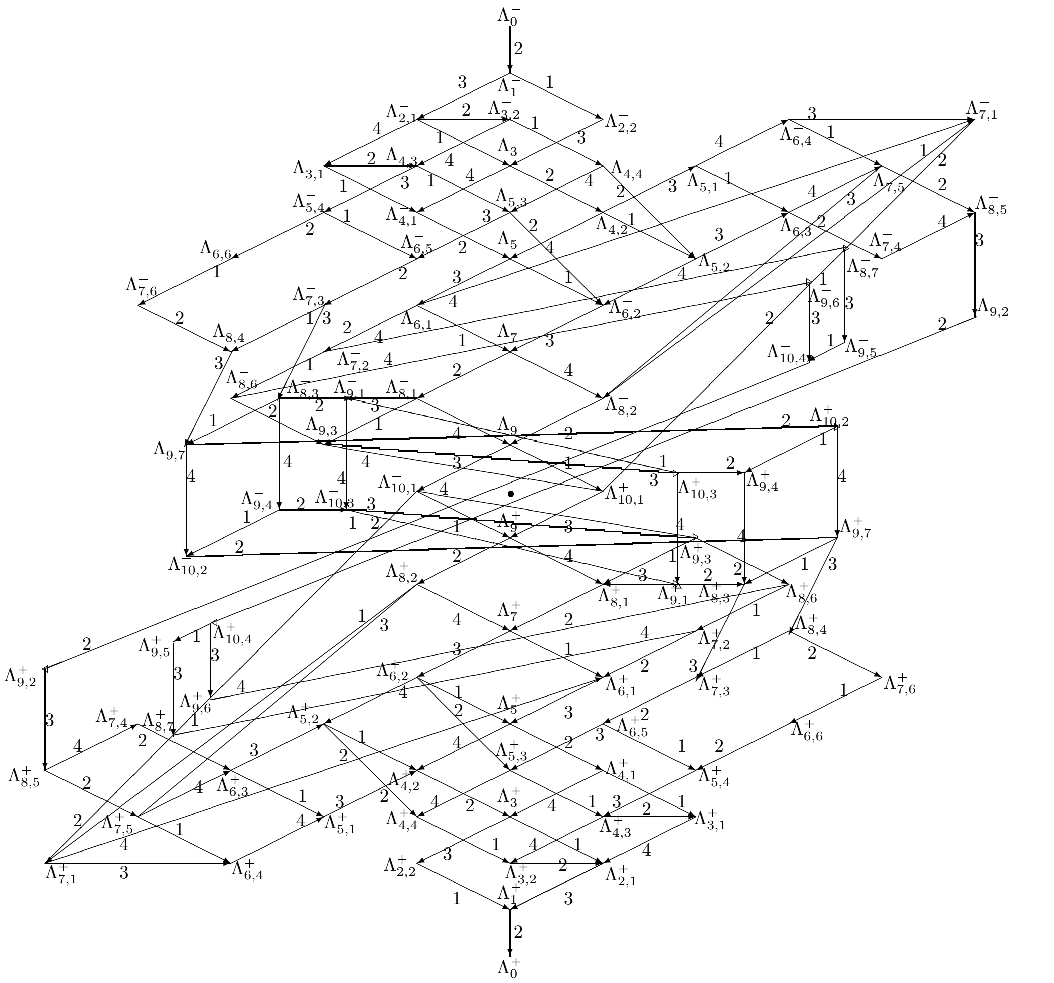


Figure 14. Multiplets for the real split form F_4' using maximal parabolic with $\mathcal{M} = sl(3,\mathbb{R})_{shortroots} \oplus sl(2,\mathbb{R})_{longroots}$.

The other (split rank one) real form of F_4 is denoted as F_4'', sometimes as $F_{4(-20)}$. This real form also has discrete series representations since $\mathcal{K} = so(9)$. The minimal (also maximal) parabolic $\mathcal{P}$ and the corresponding Bruhat decomposition are:

$$\begin{aligned} \mathcal{P} &= \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}, \qquad \mathcal{M} = so(7) \\ \mathcal{G} &= \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}^+ \oplus \mathcal{N}^- \end{aligned}$$

Each main multiplet contains 24 GVMs (ERs) an777d is given in Figure 15.

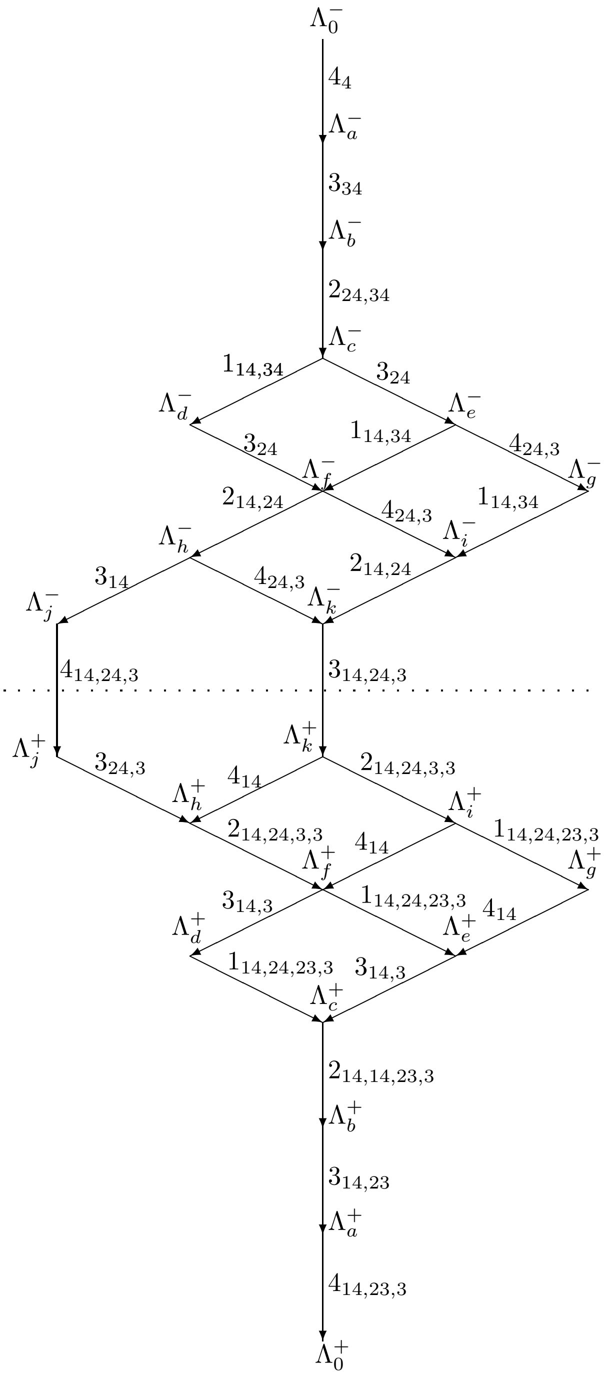


Figure 15. Main multiplets for F_4''.

9. The Case of Lie Algebra $G_{2(2)}$

Let $\mathcal{G}^{\mathbb{C}} = G_2$, with Cartan matric: $(a_{ij}) = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$, simple roots α_1, α_2 with products: $(\alpha_1, \alpha_1) = 3(\alpha_2, \alpha_2) = -2(\alpha_1, \alpha_2)$. We choose $(\alpha_2, \alpha_2) = 2$; then, $(\alpha_1, \alpha_1) = 6, (\alpha_1, \alpha_2) = -3$. As we know, G_2 is 14-dimensional. The positive roots are:

$$\Delta^+ = \{\alpha_1,\ \alpha_2,\ \alpha_1 + \alpha_2,\ \alpha_1 + 2\alpha_2,\ \alpha_1 + 3\alpha_2,\ 2\alpha_1 + 3\alpha_2\} \tag{57}$$

We shall use the orthonormal basis $\varepsilon_1, \varepsilon_2, \varepsilon_3$. In its terms for the simple roots, we may choose:

$$\alpha_1 = \varepsilon_1 - 2\varepsilon_2 + \varepsilon_3, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3 \tag{58}$$

With the chosen normalization, the roots α_1, $\alpha_1 + 3\alpha_2 = \varepsilon_1 + \varepsilon_2 - 2\varepsilon_3$, $2\alpha_1 + 3\alpha_2 = 2\varepsilon_1 - \varepsilon_2 - \varepsilon_3$ have a length of 6, while α_2, $\alpha_1 + \alpha_2 = \varepsilon_1 - \varepsilon_2$, $\alpha_1 + 2\alpha_2 = \varepsilon_1 - \varepsilon_3$ have a length of 2. Another way to write the roots in general is $\beta = (a_1, a_2, a_3)$ under the condition $a_1 + a_2 + a_3 = 0$. Then,

$$\begin{aligned} &\alpha_1 = (1,-2,1), \;\; \alpha_1 + 3\alpha_2 = (1,1,-2), \;\; 2\alpha_1 + 3\alpha_2 = (2,-1,-1), \\ &\alpha_2 = (0,1,-1), \;\; \alpha_1 + \alpha_2 = (1,-1,0), \;\; \alpha_1 + 2\alpha_2 = (1,0,-1) \end{aligned} \tag{59}$$

The dual roots are: $\alpha_1^\vee = \alpha_1/3$, $\alpha_2^\vee = \alpha_2$, $(\alpha_1 + \alpha_2)^\vee = \alpha_1 + \alpha_2 = 3\alpha_1^\vee + \alpha_2^\vee$, $(\alpha_1 + 2\alpha_2)^\vee = \alpha_1 + 2\alpha_2 = 3\alpha_1^\vee + 2\alpha_2^\vee$, $(\alpha_1 + 3\alpha_2)^\vee = (\alpha_1 + 3\alpha_2)/3 = \alpha_1^\vee + \alpha_2^\vee$, $(2\alpha_1 + 3\alpha_2)^\vee = (2\alpha_1 + 3\alpha_2)/3 = 2\alpha_1^\vee + \alpha_2^\vee$.

The Weyl group of G_2 is the dihedral group of order 12. This follows from the fact that $(s_1 s_2)^6 = 1$, where s_1, s_2 are the two simple reflections.

The algebra G_2 has one non-compact real form: $\mathcal{G} = G_{2(2)}$, which is naturally split. Its maximal compact subalgebra is $\mathcal{K} = su(2) \oplus su(2)$. Thus, $\mathcal{G} = G_{2(2)}$ has discrete series representations. The complimentary space $\mathcal{Q}$ is eight-dimensional. The positive root system of $\mathcal{K}^{\mathbb{C}}$ consists of $\alpha_2 = (0,1,-1)$, $2\alpha_1 + 3\alpha_2 = (2,-1,-1)$ (chosen to be orthogonal to each other).

The minimal parabolic of $\mathcal{G}$ is:

$$\begin{aligned} &\mathcal{P}_0 = \mathcal{M}_0 \oplus \mathcal{A}_0 \oplus \mathcal{N}_0 \\ &\mathcal{M}_0 = 0, \quad \dim \mathcal{A}_0 = 2, \quad \dim \mathcal{N}_0 = 6 \end{aligned} \tag{60}$$

There are two isomorphic maximal parabolic subalgebras of $\mathcal{G}$ which are of Heisenberg type:

$$\begin{aligned} &\mathcal{P}_k = \mathcal{M}_k \oplus \mathcal{A}_k \oplus \mathcal{N}_k, \quad k = 1,2, \\ &\mathcal{M}_k = sl(2,\mathbb{R})_k, \quad \dim \mathcal{A}_k = 1, \quad \dim \mathcal{N}_k = 5 \end{aligned} \tag{61}$$

where $sl(2,\mathbb{R})_k$ inherits from $\mathcal{G}^{\mathbb{C}}$ the simple root α_k ($k = 1,2$). Equivalently, the$\mathcal{M}_k$-compact root of $\mathcal{G}^{\mathbb{C}}$ is α_k ($k = 1,2$). In each case, the remaining five positive roots of $\mathcal{G}^{\mathbb{C}}$ are $\mathcal{M}_k$-noncompact.

The positive roots of $\mathcal{G}^{\mathbb{C}}$ in terms of the simple roots will be denoted as :

$$\gamma_1 = \alpha_1, \quad \gamma_{13} = \alpha_1 + 3\alpha_2, \quad \gamma_{23} = 2\alpha_1 + 3\alpha_2 \tag{62a}$$

$$\gamma_2 = \alpha_2, \quad \gamma_{11} = \alpha_1 + \alpha_2, \quad \gamma_{12} = \alpha_1 + 2\alpha_2 \tag{62b}$$

(where, as above, in (62a) are the long roots, in (62b) are the short roots).

To characterize the Verma modules, we shall use first the Dynkin labels:

$$m_i \equiv (\Lambda + \rho, \gamma_i^\vee), \quad i = 1,2, \tag{63}$$

where ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$. Thus, we shall use:

$$\chi_\Lambda = \{m_1, m_2\} \tag{64}$$

Note that when all $m_i \in \mathbb{N}$, then χ_Λ characterizes the finite-dimensional irreps of $\mathcal{G}^{\mathbb{C}}$ and its real forms, in particular, $\mathcal{G}$. Furthermore, $m_k \in \mathbb{N}$ characterizes the finite-dimensional irreps of the $\mathcal{M}_k$ subalgebra.

We shall use also the Harish–Chandra parameters:

$$m_{ij} = (\Lambda + \rho, \gamma_{ij}^\vee)\,, \tag{65}$$

and explicitly in terms of the Dynkin labels:

$$\chi_{HC} \;=\; \{\, m_1,\; m_{13} = m_1 + m_2, \quad m_{23} = 2m_1 + m_2, \tag{66a}$$

$$m_2,\; m_{11} = 3m_1 + m_2, \quad m_{12} = 3m_1 + 2m_2 \,\} \tag{66b}$$

9.1. Induction from Minimal Parabolic

Main Multiplets

The main multiplets are in 1-to-1 correspondence with the finite-dimensional irreps of G_2; i.e., they are labeled by the two positive Dynkin labels $m_i \in \mathbb{N}$. When we induce from the minimal parabolic, the main multiplets of $\mathcal{G}$ are the same as for the complexified Lie algebra $\mathcal{G}^{\mathbb{C}}$.

We take $\chi_0 = \chi_{HC}$. It has two embedded Verma modules with HW $\Lambda_1 = \Lambda_0 - m_1\gamma_1$, and $\Lambda_2 = \Lambda_0 - m_2\gamma_2$. The number of ERs/GVMs in a main multiplet is $12 = |W(\mathcal{G}^{\mathbb{C}})|$. We give the whole multiplet as follows:

$$\begin{aligned}
\chi_0 &= \{m_1, m_2; -\tfrac{1}{2}(2m_1+m_2)\} \\
\chi_1 &= \{-m_1, 3m_1+m_2; -\tfrac{1}{2}(m_1+m_2)\}, \quad \Lambda_1 = \Lambda_0 - m_1\gamma_1 \\
\chi_2 &= \{m_1+m_2, -m_2; -\tfrac{1}{2}(2m_1+m_2)\}, \quad \Lambda_2 = \Lambda_0 - m_2\gamma_2 \\
\chi_{12} &= \{-m_1-m_2, 3m_1+2m_2; -\tfrac{1}{2}m_1\}, \quad \Lambda_{12} = \Lambda_1 - m_2\gamma_{11} \\
\chi_{21} &= \{2m_1+m_2, -3m_1-m_2; -\tfrac{1}{2}(m_1+m_2)\}, \quad \Lambda_{21} = \Lambda_2 - m_1\gamma_{13} \\
\chi_{121} &= \{-2m_1-m_2, 3m_1+2m_2; \tfrac{1}{2}m_1\}, \quad \Lambda_{121} = \Lambda_{12} - m_1\gamma_{23} \\
\chi_{212} &= \{2m_1+m_2, -3m_1-2m_2; -\tfrac{1}{2}m_1\}, \quad \Lambda_{212} = \Lambda_{21} - m_2\gamma_{12} \\
\chi_{1212} &= \{-2m_1-m_2, 3m_1+m_2; \tfrac{1}{2}(m_1+m_2)\}, \quad \Lambda_{1212} = \Lambda_{121} - m_2\gamma_{12} \\
\chi_{2121} &= \{m_1+m_2, -3m_1-2m_2; \tfrac{1}{2}m_1\}, \quad \Lambda_{2121} = \Lambda_{212} - m_1\gamma_{23} \\
\chi_{12121} &= \{-m_1-m_2, m_2; \tfrac{1}{2}(2m_1+m_2)\}, \quad \Lambda_{12121} = \Lambda_{1212} - m_1\gamma_{13} \\
\chi_{21212} &= \{m_1, -3m_1-m_2; \tfrac{1}{2}(m_1+m_2)\}, \quad \Lambda_{21212} = \Lambda_{2121} - m_2\gamma_{11} \\
\chi_{121212} &= \{-m_1, -m_2; \tfrac{1}{2}(2m_1+m_2)\} = \chi_{212121}, \\
&\quad \Lambda_{121212} = \Lambda_{12121} - m_2\gamma_2 = \Lambda_{21212} - m_1\gamma_1 \\
&\quad \Lambda_{21} = \Lambda_1 - (3m_1+m_2)\gamma_2 \\
&\quad \Lambda_{212} = \Lambda_{12} - (3m_1+2m_2)\gamma_2 \\
&\quad \Lambda_{2121} = \Lambda_{121} - (3m_1+2m_2)\gamma_2 \\
&\quad \Lambda_{21212} = \Lambda_{1212} - (3m_1+m_2)\gamma_2 \\
&\quad \Lambda_{12} = \Lambda_2 - (m_1+m_2)\gamma_1 \\
&\quad \Lambda_{121} = \Lambda_{21} - (2m_1+m_2)\gamma_1 \\
&\quad \Lambda_{1212} = \Lambda_{212} - (2m_1+m_2)\gamma_1 \\
&\quad \Lambda_{12121} = \Lambda_{2121} - (m_1+m_2)\gamma_1
\end{aligned} \tag{67}$$

where we have included as third entry also the parameter $c = -\frac{1}{2}(2m_1+m_2)$, related to the Harish–Chandra parameter of the highest root (recalling that $m_{\gamma_{23}} = 2m_1+m_2$). It is also related to the conformal weight $d = \frac{3}{2} + c$.

The ER χ_{121212} contains discrete series representation according to the Harish–Chandra criterion [3] (all HC parameters are negative).

These labeling signatures may be given in the following pair-wise manner:

$$\begin{aligned}
\chi_0^{\pm} &= \{\mp m_1, \mp m_2; \pm\tfrac{1}{2}(2m_1+m_2)\} \\
\chi_1^{\pm} &= \{\pm m_1, \mp(3m_1+m_2); \pm\tfrac{1}{2}(m_1+m_2)\}, \\
\chi_2^{\pm} &= \{\mp(m_1+m_2), \pm m_2; \pm\tfrac{1}{2}(2m_1+m_2)\}, \\
\chi_{12}^{\pm} &= \{\pm(m_1+m_2), \mp(3m_1+2m_2); \pm\tfrac{1}{2}m_1\} \\
\chi_{21}^{\pm} &= \{\mp(2m_1+m_2), \pm(3m_1+m_2); \pm\tfrac{1}{2}(m_1+m_2)\} \\
\chi_{121}^{\pm} &= \{\pm(2m_1+m_2), \mp(3m_1+2m_2); \mp\tfrac{1}{2}m_1\},
\end{aligned} \tag{68}$$

where $\chi^-_{\dots} = \chi_{\dots}$ from (67), $\chi_0^+ = \chi_{121212}$, $\chi_1^+ = \chi_{21212}$, $\chi_2^+ = \chi_{12121}$, $\chi_{12}^+ = \chi_{2121}$, $\chi_{21}^+ = \chi_{1212}$, $\chi_{121}^+ = \chi_{212}$.

The ERs in the multiplet are also related by intertwining integral operators introduced in [233]. These operators are defined for any ER, the general action in our situation being:

$$\begin{aligned} &G_{KS} : \mathcal{C}_\chi \longrightarrow \mathcal{C}_{\chi'} \ , \\ &\chi = [\, n_1, n_2;\, c\,], \qquad \chi' = [\, -n_1, -n_2;\, -c\,]. \end{aligned} \tag{69}$$

This action is consistent with the parameterization in (68).

The main multiplets are given explicitly in Figure 16. The pairs $\chi^\pm$ are symmetric with regard to to the bullet in the middle of the picture—this symbolizes the Weyl symmetry realized by the Knapp–Stein operators:

$$G_{KS} \ : \ \mathcal{C}_{\chi^\mp} \longrightarrow \mathcal{C}_{\chi^\pm}.$$

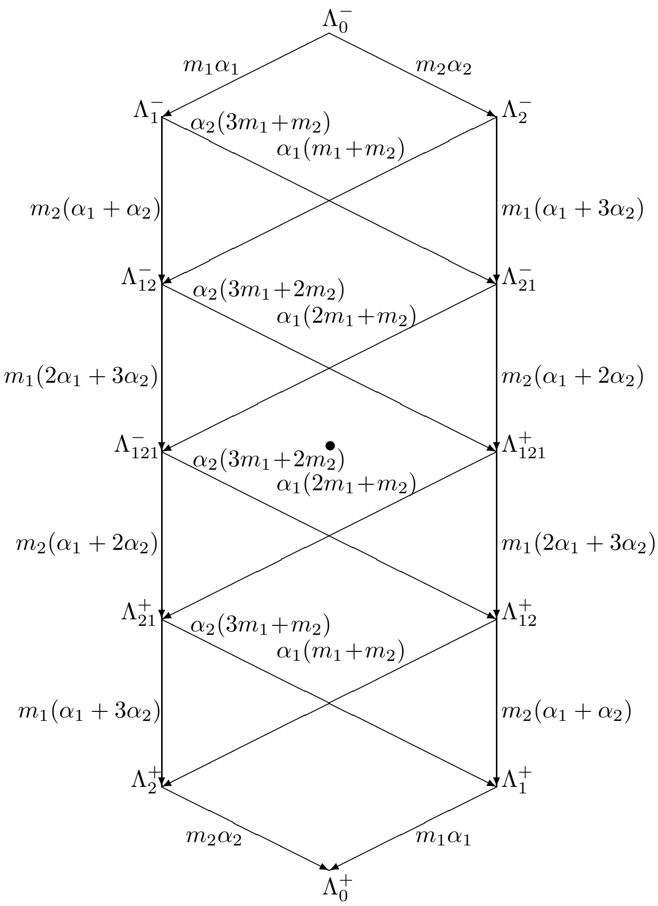


Figure 16. Main multiplets for $G_{2(2)}$ using induction from the minimal parabolic.

Some comments are in order.

Matters are arranged so that in every multiplet only the ER with signature χ_0^- contains a finite-dimensional nonunitary subrepresentation in a finite-dimensional subspace $\mathcal{E}$. The latter corresponds to the finite-dimensional irrep of $G_{2(2)}$ with signature $[m_1, m_2]$. The subspace $\mathcal{E}$ is annihilated by the operator G^+ , and is the image of the operator G^- . When all $m_i = 1$, then dim $\mathcal{E} = 1$, and in that case, $\mathcal{E}$ is also the trivial one-dimensional UIR of the whole algebra $\mathcal{G}$. Furthermore, in that case, the conformal weight is zero: $d = \frac{3}{2} + c = \frac{3}{2} - \frac{1}{2}(2m_1 + m_2)_{|_{m_i=1}} = 0$.

In the conjugate ER χ_0^+ there is a unitary discrete series subrepresentation in an infinite-dimensional subspace $\mathcal{D}$ with conformal weight $d = \frac{3}{2} + c = \frac{7}{2} + \frac{1}{2}(2m_1 + m_2)$. It is annihilated by the operator G^-, and is the image of the operator G^+.

9.2. Induction from Maximal Parabolics

9.2.1. Main Multiplets When Inducing from $\mathcal{P}_1$

When inducing from the maximal parabolic $\mathcal{P}_1 = \mathcal{M}_1 \oplus \mathcal{A}_1 \oplus \mathcal{N}_1$ there is one $\mathcal{M}_1$-compact root, namely, α_1. We take again the Verma module with $\Lambda_{HC} = \Lambda_0^{1-}$. We take $\chi_0^{1-} = \chi_{HC}$. The GVM Λ_0^{1-} has one embedded GVM with HW $\Lambda_2^{1-} = \Lambda_0^{1-} - m_2\gamma_2$, $m_2 \in \mathbb{N}$. Altogether, the main multiplet in this case includes the same number of ERs/GVMs as in (32), so we use the same notation only adding super index 1, namely

$$\begin{aligned}
\chi_0^{1\pm} &= \{\mp m_1, \mp m_2; \pm\tfrac{1}{2}(2m_1+m_2)\} \\
\chi_1^{1\pm} &= \{\pm m_1, \mp(3m_1+m_2); \pm\tfrac{1}{2}(m_1+m_2)\}, \\
\chi_2^{1\pm} &= \{\mp(m_1+m_2), \pm m_2; \pm\tfrac{1}{2}(2m_1+m_2)\}, \\
\chi_{12}^{1\pm} &= \{\pm(m_1+m_2), \mp(3m_1+2m_2); \pm\tfrac{1}{2}m_1\} \\
\chi_{21}^{1\pm} &= \{\mp(2m_1+m_2), \pm(3m_1+m_2); \pm\tfrac{1}{2}(m_1+m_2)\} \\
\chi_{121}^{1\pm} &= \{\pm(2m_1+m_2), \mp(3m_1+2m_2); \mp\tfrac{1}{2}m_1\},
\end{aligned} \tag{70}$$

In addition, in order to avoid coincidence with (35) we must impose in (70) the conditions: $m_1 \notin \mathbb{N}$, $m_1 \notin \mathbb{N}/2$, $m_1 \notin \mathbb{N}/3$.

What is peculiar is that the ERs/GVMs of the main miltiplet (70) actually consist of three submultiplets with intertwining diagrams as follows:

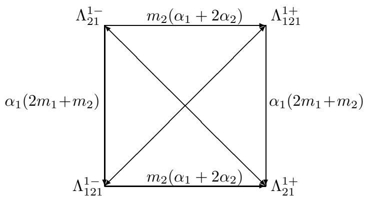
$$\begin{array}{ccc}
\Lambda_0^{1-} & \overset{\mathcal{D}_{\gamma_2}^{m_2}}{\longrightarrow} & \Lambda_2^{1-} \\
\updownarrow & & \updownarrow \quad \text{subtype } (\mathrm{A}_1) \\
\Lambda_0^{1+} & \overset{\mathcal{D}_{\gamma_2}^{m_2}}{\longleftarrow} & \Lambda_2^{1+}
\end{array} \tag{71a}$$

$$\begin{array}{ccc}
\Lambda_1^{1-} & \overset{\mathcal{D}_{\gamma_{11}}^{m_2}}{\longrightarrow} & \Lambda_{12}^{1-} \\
\updownarrow & & \updownarrow \quad \text{subtype } (\mathrm{B}_1) \\
\Lambda_1^{1+} & \overset{\mathcal{D}_{\gamma_{11}}^{m_2}}{\longleftarrow} & \Lambda_{12}^{1+}
\end{array} \tag{71b}$$

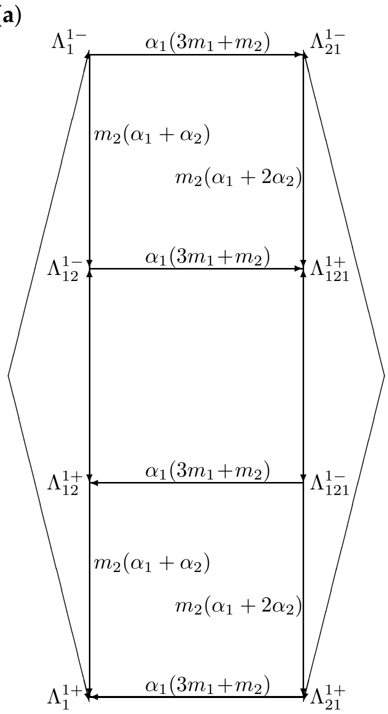
$$\begin{array}{ccc}
\Lambda_{21}^{1-} & \overset{\mathcal{D}_{\gamma_{12}}^{m_2}}{\longrightarrow} & \Lambda_{121}^{1+} \\
\updownarrow & & \updownarrow \quad \text{subtype } (\mathrm{C}_1) \\
\Lambda_{21}^{1+} & \overset{\mathcal{D}_{\gamma_{12}}^{m_2}}{\longleftarrow} & \Lambda_{121}^{1-}
\end{array} \tag{71c}$$

Next, we relax in (70) one of the conditions, namely, we allow $m_1 \in \mathbb{N}/2$, still keeping $m_1 \notin \mathbb{N}$, $m_1 \notin \mathbb{N}/3$. This changes the diagram of subtype (C_1), (71c), as given in Figure 17a below.

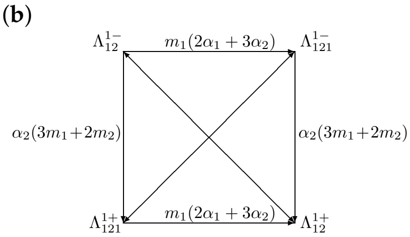
Next, we relax in (70) another condition, namely, we allow $m_1 \in \mathbb{N}/3$, still keeping $m_1 \notin \mathbb{N}$, $m_1 \notin \mathbb{N}/2$. This changes the diagrams of subtypes (B_1) and (C_1) combining them as given in Figure 17b below.



(a)



(b)



(c)

Figure 17. (**a**) Submultiplets type (C_1) for $G_{2(2)}$ using induction from the maximal parabolic $\mathcal{P}1$ for $m_2 \in \mathbb{N}$, $m_1 \notin \mathbb{N}$, $m_1 \in \mathbb{N}/2$, $m_1 \notin \mathbb{N}/3$; the (anti)diagonal arrows represent the KS operators. (**b**) Submultiplets type (B_1)+ (C_1) for $G_{2(2)}$ using induction from the maximal parabolic $\mathcal{P}1$ for $m_2 \in \mathbb{N}$, $m_1 \notin \mathbb{N}$, $m_1 \notin \mathbb{N}/2$, $m_1 \in \mathbb{N}/3$; the up-down arrows represent four pairs of KS operators. (**c**) Submultiplets type (C_2) for $G_{2(2)}$ using induction from the maximal parabolic $\mathcal{P}2$ for $m_1 \in \mathbb{N}$, $m_2 \notin \mathbb{N}$, $m_2 \in \mathbb{N}/2$; the (anti)diagonal arrows represent the KS operators.

9.2.2. Main Multiplets When Induction from $\mathcal{P}_2$

This case is partly dual to the previous one. When inducing from the maximal parabolic $\mathcal{P}_2 = \mathcal{M}_2 \oplus \mathcal{A}_2 \oplus \mathcal{N}_2$, there is one $\mathcal{M}_2$-compact root, namely, α_2. We take again the Verma module with $\Lambda_{HC} = \Lambda_0^{2-}$. We take $\chi_0^{2-} = \chi_{HC}$. The GVM Λ_0^{2-} has one embedded GVM with HW $\Lambda_1^{2-} = \Lambda_0^{2-} - m_1\gamma_1$, $m_1 \in \mathbb{N}$. Altogether, the main multiplet in this case includes the same number of ERs/GVMs as in (32), so we use the same notation only adding super index 2, namely

$$\begin{aligned}
\chi_0^{2\pm} &= \{\mp m_1, \mp m_2; \pm\tfrac{1}{2}(2m_1+m_2)\} \\
\chi_1^{2\pm} &= \{\pm m_1, \mp(3m_1+m_2); \pm\tfrac{1}{2}(m_1+m_2)\}, \\
\chi_2^{2\pm} &= \{\mp(m_1+m_2), \pm m_2; \pm\tfrac{1}{2}(2m_1+m_2)\}, \\
\chi_{12}^{2\pm} &= \{\pm(m_1+m_2), \mp(3m_1+2m_2); \pm\tfrac{1}{2}m_1\} \\
\chi_{21}^{2\pm} &= \{\mp(2m_1+m_2), \pm(3m_1+m_2); \pm\tfrac{1}{2}(m_1+m_2)\} \\
\chi_{121}^{2\pm} &= \{\pm(2m_1+m_2), \mp(3m_1+2m_2); \mp\tfrac{1}{2}m_1\},
\end{aligned} \tag{72}$$

In addition, in order to avoid coincidence with (35) we must impose in (70) the conditions $m_2 \notin \mathbb{N}$, $m_2 \notin \mathbb{N}/2$.

Similarly to the $\mathcal{P}_1$ case, the ERs/GVMs of the main miltiplet (72) actually consist of three submultiplets with intertwining diagrams as follows:

$$\begin{array}{ccc}
\Lambda_0^{2-} & \xrightarrow{\mathcal{D}_{\gamma_1}^{m_1}} & \Lambda_1^{2-} \\
\updownarrow & & \updownarrow \\
\Lambda_0^{2+} & \xleftarrow{\mathcal{D}_{\gamma_1}^{m_1}} & \Lambda_1^{2+}
\end{array} \quad \text{subtype } (\mathrm{A}_2) \tag{73a}$$

$$\begin{array}{ccc}
\Lambda_2^{2-} & \xrightarrow{\mathcal{D}_{\gamma_{13}}^{m_1}} & \Lambda_{21}^{2-} \\
\updownarrow & & \updownarrow \\
\Lambda_2^{2+} & \xleftarrow{\mathcal{D}_{\gamma_{13}}^{m_1}} & \Lambda_{21}^{2+}
\end{array} \quad \text{subtype } (\mathrm{B}_2) \tag{73b}$$

$$\begin{array}{ccc}
\Lambda_{12}^{2-} & \xrightarrow{\mathcal{D}_{\gamma_{23}}^{m_1}} & \Lambda_{121}^{2-} \\
\updownarrow & & \updownarrow \\
\Lambda_{12}^{2+} & \xleftarrow{\mathcal{D}_{\gamma_{23}}^{m_1}} & \Lambda_{121}^{2+}
\end{array} \quad \text{subtype } (\mathrm{C}_2) \tag{73c}$$

Next, we relax in (70) one of the conditions, namely, we allow $m_2 \in \mathbb{N}/2$, still keeping $m_2 \notin \mathbb{N}$. This changes the diagram of subtype (C_2), (73c), as given in Figure 17c.

Funding: This research received no external funding.

Data Availability Statement: All used research data is in references and acknowledged where proper.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Maldacena, J.M. Large N Field Theories, String Theory and Gravity. In *Lectures on Quantum Gravity*; Series of the Centro De Estudios Scientificos; Gomberoff, A., Marolf, D., Eds.; Springer: New York, NY, USA, 2005; pp. 91–150.
2. Terning, J. *Modern Supersymmetry: Dynamics and Duality*; International Series of Monographs on Physics # 132; Oxford University Press: Oxford, UK, 2005.
3. Chandra, H. Discrete series for semisimple Lie groups, II. *Acta Math.* **1966**, *116*, 1–111. [CrossRef]
4. Bernstein, I.N.; Gelfand, I.M.; Gelfand, S.I. Structure of representations possessing a highest weight. *Funct. Anal. Appl.* **1971**, *5*, 1–8. (in Russian) [CrossRef]
5. Bernstein, I.N.; Gelfand, I.M.; Gelfand, S.I. Differential operators on the base affine space and a study of g-modules. In *Lie Groups and Their Representations*; Gelfand, I.M., Ed.; Halsted Press: New York, NY, USA, 1975; pp. 21–64.
6. Warner, G. *Harmonic Analysis on Semi-Simple Lie Groups I*; Springer: Berlin/Heidelberg, Germany, 1972.
7. Langlands, R.P. On the classification of irreducible representations of real algebraic groups. *Math. Surv. Monogr.* **1988**, *31*, 101–170.
8. Ferrara, S.; Wess, J.; Zumino, B. Supergauge multiplets and superfields. *Phys. Lett. B* **1974**, *51*, 239. [CrossRef]
9. Ferrara, S.; Zumino, B. Supergauge invariant Yang-Mills theories. *Nucl. Phys. B* **1974**, *79*, 413. [CrossRef]

10. Ferrara, S.; Zumino, B. Transformation properties of the supercurrent. *Nucl. Phys. B* **1975**, *87*, 207. [CrossRef]
11. Zhelobenko, D.P. *Harmonic Analysis on Semisimple Complex Lie Groups*; Nauka: Moscow, Russian, 1974.
12. Kostant, B. *Verma Modules and the Existence of Quasi-Invariant Differential Operators*; Lecture Notes in Mathematics; Dold, A., Eckmann, B., Eds.; Springer: Berlin/Heidelberg, Germany, 1975; Volume 466, pp. 101–128.
13. Sokatchev, E. Projection Operators and Supplementary Conditions for Superfields with an Arbitrary Spin. *Nucl. Phys. B* **1975**, *99*, 96. [CrossRef]
14. Sokatchev, E. Noncommutative Geometry and String Field Theory. *Phys. Lett. B* **1986**, *169*, 209. [CrossRef]
15. Sokatchev, E. Harmonic superparticle. *Class. Quant. Gravity* **1987**, *4*, 237. [CrossRef]
16. Freedman, D.Z.; van Nieuwenhuizen, P.; Ferrara, S. Progress Toward a Theory of Supergravity. *Phys. Rev. D* **1976**, *13*, 3214–3218. [CrossRef]
17. Ferrara, S.; van Nieuwenhuizen, P. The Auxiliary Fields of Supergravity. *Phys. Lett. B* **1978**, *74*, 333. [CrossRef]
18. Wolf, J. *Unitary Representations of Maximal Parabolic Subgroups of the Classical Groups*; Memoirs American Mathematical Society 180; AMS: Providence, RI, USA, 1976.
19. Ademollo, M.; Brink, L.; D'Adda, A.; D'Auria, R.; Napolitano, E.; Sciuto, S.; Giudice, E.D.; Vecchia, P.D.; Ferrara, S.; Gliozzi, F.; et al. Supersymmetric Strings and Color Confinement. *Phys. Lett. B* **1976**, *62*, 105–110. [CrossRef]
20. Ademollo, M.; Brink, L.; D'Adda, A.; D'Auria, R.; Napolitano, E.; Sciuto, S.; Giudice, E.D.; Vecchia, P.D.; Ferrara, S.; Gliozzi, F.; et al. Dual String with U(1) Color Symmetry. *Nucl. Phys. B* **1976**, *111*, 77–110. [CrossRef]
21. Fayet, P.; Ferrara, S. Supersymmetry. *Phys. Rep.* **1977**, *32*, 249. [CrossRef]
22. Wolf, J. *Classification and Fourier Inversion for Parabolic Subgroups with Square Integrable Nilradical*; Memoirs of the American Mathematical Society 225; AMS: Providence, RI, USA, 1979.
23. Knapp, A.W.; Zuckerman, G.J. *Lecture Notes in Math*; Springer: Berlin/Heidelberg, Germany, 1977; Volume 587, pp. 138–159.
24. Dobrev, V.K.; Mack, G.; Petkova, V.B.; Petrova, S.G.; Todorov, I.T. Harmonic Analysis on the n-Dimensional Lorentz Group and Its Application to Conformal Quantum Field Theory. *Lect. Notes Phys.* **1977**, *63*, 1–280.
25. Dobrev, V.K.; Mack, G.; Petkova, V.B.; Petrova, S.G.; Todorov, I.T. On the Clebsch-Gordan Expansion for the Lorentz Group in n Dimensions. *Rep. Math. Phys.* **1976**, *9*, 219–246. [CrossRef]
26. Dobrev, V.K.; Petkova, V.B.; Petrova, S.G.; Todorov, I.T. Dynamical Derivation of Vacuum Operator Product Expansion in Euclidean Conformal Quantum Field Theory. *Phys. Rev. D* **1976**, *13*, 887. [CrossRef]
27. Ogievetsky, V.; Sokatchev, E. On Vector Superfield Generated by Supercurrent. *Nucl. Phys. B* **1977**, *124*, 309–316. [CrossRef]
28. Ogievetsky, V.; Sokatchev, E. Structure of Supergravity Group. *Phys. Lett. B* **1978**, *79*, 222. [CrossRef]
29. Cremmer, E.; Sherk, J.; Ferrara, S. SU(4) Invariant Supergravity Theory. *Phys. Lett. B* **1978**, *74*, 61–64. [CrossRef]
30. Cremmer, E.; Ferrara, S.; Girardello, L.; Proeyen, A.V. Coupling Supersymmetric Yang-Mills Theories to Supergravity. *Phys. Lett. B* **1982**, *116*, 231–237. [CrossRef]
31. Speh, B.; Vogan, D. Reducibility of generalized principal series representations. *Acta Math.* **1980**, *145*, 227–299. [CrossRef]
32. Vogan, D. *Representations of Real Reductive Lie Groups*; Progress in Mathematics; Birkhäuser: Boston, MA, USA; Basel, Switzerland; Stuttgart, Germany, 1981; Volume 15.
33. Enright, T.; Howe, R.; Wallach, N. *Representations of Reductive Groups*; Trombi, P., Ed.; Birkhäuser: Boston, MA, USA, 1983; pp. 97–143.
34. Galperin, A.; Ivanov, E.; Kalitsyn, S.; Ogievetsky, V.; Sokatchev, E. Unconstrained N = 2 Matter, Yang-Mills and Supergravity Theories in Harmonic Superspace. *Class. Quant. Gravity* **1984**, *1*, 469–498; Erratum in *Class. Quant. Gravity* **1985**, *2*, 127. [CrossRef]
35. Dobrev, V.K.; Petkova, V.B. All positive energy unitary irreducible representations of extended conformal supersymmetry. *Phys. Lett. B* **1985**, *162*, 127–132. [CrossRef]
36. Dobrev, V.K.; Petkova, V.B. Group-Theoretical Approach to Extended Conformal Supersymmetry: Function Space Realizations and Invariant Differential Operators. *Fortsch. Phys.* **1987**, *35*, 537–572. [CrossRef]
37. Dobrev, V.K. Characters of the unitarizable highest weight modules over the N = 2 superconformal algebras. *Phys. Lett. B* **1987**, *186*, 43–51. [CrossRef]
38. Delduc, F.; Galperin, A.; Howe, P.S.; Sokatchev, E. A Twistor formulation of the heterotic D = 10 superstring with manifest (8,0) world sheet supersymmetry. *Phys. Rev. D* **1993**, *47*, 578–593. [CrossRef]
39. Truini, P.; Varadarajan, V.S. Quantization of Reductive Lie Algebras: Construction and Universality. *Rev. Math. Phys.* **1993**, *5*, 363. [CrossRef]
40. Jakobsen, H.P. *Lecture Notes in Physics*; Springer, Berlin/Heidelberg, Germany, 1986; Volume 261, pp. 253–265.
41. Kac, V.G.; Wakimoto, M. *Lie Theory and Geometry*; Progress in Mathematics; Birkhäuser: Boston, MA, USA, 1994; Volume 123, pp. 415–456.
42. Kobayashi, T.S. Discrete decomposability of the restriction of $A_q(\lambda)$ with respect to reductive subgroups and its applications. *Inv. Math.* **1994**, *117*, 181–205. [CrossRef]
43. Witten, E. On the Landau-Ginzburg description of N = 2 minimal models. *Int. J. Mod. Phys. A* **1994**, *9*, 4783–4800. [CrossRef]
44. Witten, E. SL(2, Z) action on three-dimensional conformal field theories with abelian symmetry. In *From Fields to Stings: Circumnavigating Theoretical Physics*; Shifman, M., Vainshtein, A., Wheater, J., Eds.; World Scientific: Singapore, 2004; Volume 2, pp. 1173–1200.

45. Argyres, P.C.; Presser, M.; Seiberg, N.; Witten, E. New N = 2 superconformal field theories in four dimensions. *Nucl. Phys. B* **1996**, *461*, 71–84. [CrossRef]
46. Ferrara, S.; Harvey, J.A.; Strominger, A.; Vafa, C. Second-quantized mirror symmetry. *Phys. Lett. B* **1995**, *361*, 59. [CrossRef]
47. Ceresole, A.; Dall'Agata, G.; D'Auria, R.; Ferrara, S. Spectrum of type IIB supergravity on AdS 5× T 11: Predictions on N = 1 SCFT's. *Phys. Rev. D* **2000**, *61*, 066001. [CrossRef]
48. Antoniadis, I.; Ferrara, S.; Minasian, R.; Narain, K.S. R4 couplings in M-and type II theories on Calabi-Yau spaces. *Nucl. Phys. B* **1997**, *507*, 571. [CrossRef]
49. Branson, T.P.; Olafsson, G.; Orsted, B. Spectrum generating operators and intertwining operators for representations induced from a maximal parabolic subgroup. *J. Funct. Anal.* **1996**, *135*, 163–205. [CrossRef]
50. Andrianopoli, L.; Ferrara, S.; Sokatchev, E.; Zupnik, B. Shortening of primary operators in N-extended SCFT_4 and harmonic-superspace analyticity. *Adv. Theor. Math. Phys.* **2000**, *4*, 1149.
51. Ferrara, S.; Maldacena, J.M. Branes, central charges and U duality invariant BPS conditions. *Class. Quant. Gravity* **1998**, *15*, 749–758. [CrossRef]
52. Ferrara, S.; Fronsdal, C. Conformal Maxwell theory as a singleton field theory on AdS, IIB 3-branes and duality. *Class. Quant. Gravity* **1998**, *15*, 2153. [CrossRef]
53. Howe, P.S.; Sokatchev, E.; West, P.C. 3-point functions in N = 4 Yang-Mills. *Phys. Lett. B* **1998**, *444*, 341. [CrossRef]
54. Aharony, O.; Gubser, S.S.; Maldacena, J.M.; Ooguri, H.; Oz, Y. Large N field theories, string theory and gravity. *Phys. Rep.* **2000**, *323*, 183–386. [CrossRef]
55. Eden, B.; Petkou, A.C.; Schubert, C.; Sokatchev, E. Partial non-renormalisation of the stress-tensor four-point function in N = 4 SYM and AdS/CFT. *Nucl. Phys. B* **2001**, *607*, 191. [CrossRef]
56. Dolan, L.; Nappi, C.R.; Witten, E. Conformal operators for partially massless states. *J. High Energy Phys.* **2001**, *110*, 16. [CrossRef]
57. Arutyunov, G.; Eden, B.; Petkou, A.C.; Sokatchev, E. Exceptional non-renormalization properties and OPE analysis of chiral four-point functions in N = 4 SYM4. *Nucl. Phys. B* **2002**, *620*, 380. [CrossRef]
58. Knapp, A.W. *Lie Groups Beyond an Introduction*, 2nd ed.; Progress in Mathematics; Birkhäuser: Boston, MA, USA; Basel, Switzerland; Stuttgart, Germany, 2002; Volume 140.
59. Kac, V.; Roan, S.S.; Wakimoto, M. Quantum Reduction for AFfine Superalgebras. *Comm. Math. Phys.* **2003**, *241*, 307–342. [CrossRef]
60. Ferrara, S.; Sokatchev, E. Universal properties of superconformal OPEs for 1/2 BPS operators in $3 \leqslant D \leqslant 6$. *New J. Phys.* **2002** , *4*, 2. [CrossRef]
61. Kostant, B. Powers of the Euler product and commutative subalgebras of a complex simple Lie algebra. *Inv. Math.* **2004**, *158*, 181–226. [CrossRef]
62. Baur, K.; Wallach, N. Nice parabolic subalgebras of reductive Lie algebras. *Represent. Theory* **2005**, *9*, 1–29. [CrossRef]
63. Gannon, T.; Vasudevan, M. Charges of exceptionally twisted branes. *J. High Energy Phys.* **2005**, *507*, 35. [CrossRef]
64. Carmeli, C.; Cassinelli, G.; Toigo, A.; Varadarajan, V.S. Unitary representations of super Lie groups and applications to the classification and multiplet structure of super particles. *Comm. Math. Phys.* **2006**, *263*, 217. [CrossRef]
65. Duff, M.J.; Ferrara, S. E_6 and the bipartite entanglement of three qutrits. *Phys. Rev. D* **2007**, *76*, 124023. [CrossRef]
66. Faraggi, A.E.; Kounnas, C.; Rizos, J. Spinor-Vector Duality in fermionic Z2× Z2 heterotic orbifold models. *Nucl. Phys. B* **2007**, *774*, 208–231. [CrossRef]
67. Kinney, J.; Maldacena, J.M.; Minwalla, S.; Raju, S. An Index for 4 dimensional super conformal theories. *Commun. Math. Phys.* **2007**, *275*, 209–254. [CrossRef]
68. Gurrieri, G.; Lukas, L.; Micu, A. Heterotic string compactifications on half-flat manifolds II. *J. High Energy Phys.* **2007**, *712*, 81. [CrossRef]
69. Hofman, D.M.; Maldacena, J. Conformal collider physics: Energy and charge correlations. *J. High Energy Phys.* **2008**, *5*, 12. [CrossRef]
70. Bernardoni, F.; Cacciatori, S.L.; Cerchiai, B.L.; Scotti, A. Mapping the geometry of the E6 group. *J. Math. Phys.* **2008**, *49*, 012107. [CrossRef]
71. Kallosh, R.; Kugo, T. The footprint of $E_{7(7)}$ in amplitudes of $N = 8$ supergravity. *J. High Energy Phys.* **2009**, *901*, 072. [CrossRef]
72. Mizoguchi, S. Localized modes in type II and heterotic singular Calabi-Yau conformal field theories. *J. High Energy Phys.* **2008**, *811*, 022. [CrossRef]
73. Ferrara, S.; Kallosh, R.; Marrani, A. Degeneration of groups of type E_7 and minimal coupling in supergravity. *J. High Energy Phys.* **2012**, *1206*, 074. [CrossRef]
74. Petrov, A. *New Methods in the General Theory of Relativity*; Nauka: Moscow, Russia, 1966; 496p.
75. Dobrev, V.K. Invariant differential operators for non-compact Lie groups: Parabolic subalgebras. *Rev. Math. Phys.* **2008**, *20*, 407–449. [CrossRef]
76. Dobrev, V.K. *Invariant Differential Operators, Volume 1: Noncompact Semisimple Lie Algebras and Groups*; De Gruyter Studies in Mathematical Physics; De Gruyter: Berlin, Germany; Boston, MA, USA, 2016; Volume 35, pp. 408 + xii, ISBN 978-3-11-042764-6.
77. Catto, S.; Choun, Y.S.; Kurt, L. Invariance properties of the exceptional quantum mechanics (F_4) and its generalization to complex Jordan algebras (E_6). In *Lie Theory and Its Applications in Physics* ; Springer Proceedings in Mathematics & Statistics; Springer: Tokyo, Japan, 2013; Volume 36, p. 469.

78. Alkalaev, K.B. Mixed-symmetry tensor conserved currents and AdS/CFT correspondence. *J. Phys. A Math. Theor.* **2013**, *46*, 214007. [CrossRef]
79. Borsten, L.; Duff, M.J.; Ferrara, S.; Marrani, A. Freudenthal Dual Lagrangians. *Class. Quant. Gravity* **2013**, *30*, 235003. [CrossRef]
80. Cacciatori, S.L.; Cerchiai, B.L.; Marrani, A. Magic coset decompositions. *Adv. Theor. Math. Phys.* **2013**, *17*, 1077–1128. [CrossRef]
81. Chicherin, D.; Derkachov, S.; Isaev, A.P. Conformal group: R-matrix and star-triangle relation. *J. High Energy Phys.* **2013**, *1304*, 020. [CrossRef]
82. Cotaescu, I.I. Covariant representations of the de Sitter isometry group. *Mod. Phys. Lett. A* **2013**, *28*, 1350033. [CrossRef]
83. Ferrara, S.; Marrani, A.; Zumino, B. Jordan pairs, E_6 and U-duality in five dimensions. *J. Phys. A* **2013**, *46*, 065402. [CrossRef]
84. Kleinschmidt, A.; Nicolai, H. On higher spin realizations of $K(E_{10})$. *J. High Energy Phys.* **2013**, *1308*, 41. [CrossRef]
85. Kubo, T. On the homomorphisms between the generalized Verma modules arising from conformally invariant system. *J. Lie Theory* **2013**, *23*, 847–883.
86. Neumann, C.; Rehren, K.-H.; Wallenhorst, L. New methods in conformal partial wave analysis. In *Springer Proceedings in Mathematics and Statistics*; Springer: Tokyo, Japan; Berlin/Heidelberg, Germany, 2013; Volume 36, pp. 109–125.
87. Todorov, I.T. Studying Quantum Field Theory. *Bulg. J. Phys.* **2013**, *40*, 93–114.
88. Belitsky, A.V.; Hohenegger, S.; Korchemsky, G.P.; Sokatchev, E.; Zhiboedov, A. From correlation functions to event shapes. *Nucl. Phys. B* **2014**, *884*, 305–343. [CrossRef]
89. Costa, M.S.; Goncalves, V.; Penedones, J. Spinning AdS propagators. *J. High Energy Phys.* **2014**, *1409*, 64. [CrossRef]
90. Dobrev, V.K. Invariant differential operators for non-compact Lie groups: The reduced SU(3,3) multiplets. *Phys. Part. Nucl. Lett.* **2014**, *11*, 864–871. [CrossRef]
91. Dobrev, V.K. Multiplet classification for SU(n,n). *J. Phys. Conf. Ser.* **2014**, *563*, 012008. [CrossRef]
92. Dobrev, V.K. Invariant differential operators for non-compact Lie groups: The SO*(12) case. *J. Phys. Conf. Ser.* **2015**, *597*, 012032. [CrossRef]
93. Godazgar, H.; Godazgar, M.; Hohm, O.; Nicolai, H.; Samtleben, H. Supersymmetric $E_{7(7)}$ exceptional field theory. *J. High Energy Phys.* **2014**, *1409*, 44. [CrossRef]
94. Marrani, A.; Truini, P. Exceptional Lie algebras, SU(3) and Jordan pairs Part 2: Zorn-type representations. *J. Phys. A* **2014**, *47*, 265202. [CrossRef]
95. Matumoto, H. On the homomorphisms between scalar generalized Verma modules. *Compos. Math.* **2014**, *150*, 877–892. [CrossRef]
96. Metsaev, R.R. BRST invariant effective action of shadow fields, conformal fields, and AdS/CFT. *Theor. Math. Phys.* **2014**, *181*, 1548–1565. [CrossRef]
97. Metsaev, R.R. Arbitrary spin conformal fields in (A)dS. *Nucl. Phys. B* **2014**, *885*, 734–771. [CrossRef]
98. Metsaev, R.R. Mixed-symmetry fields in AdS(5), conformal fields, and AdS/CFT. *J. High Energy Phys.* **2015**, *1501*, 077. [CrossRef]
99. Metsaev, R.R. Light-cone AdS/CFT-adapted approach to AdS fields/currents, shadows, and conformal fields. *J. High Energy Phys.* **2015**, *1510*, 110. [CrossRef]
100. Nikolov, M.N.; Stora, R.; Todorov, I.T. Renormalization of massless Feynman amplitudes in configuration space. *Rev. Math. Phys.* **2014**, *26*, 1430002. [CrossRef]
101. Anand, N.; Cantrell, S. The Goldstone equivalence theorem and AdS/CFT. *J. High Energy Phys.* **2015**, *1508*, 50. [CrossRef]
102. Barnich, G.; Bekaert, X.; Grigoriev, M. Notes on conformal invariance of gauge fields. *J. Phys. A* **2015**, *48*, 505402. [CrossRef]
103. Bekaert, X.; Erdmenger, J.; Ponomarev, D.; Sleight, C. Towards holographic higher-spin interactions: Four-point functions and higher-spin exchange. *J. High Energy Phys.* **2015**, *1511*, 149. [CrossRef]
104. Costa, M.S.; Hansen, T. Conformal correlators of mixed-symmetry tensors. *J. High Energy Phys.* **2015**, *1502*, 151. [CrossRef]
105. Dobrev, V.K. Classification of conformal representations induced from the maximal cuspidal parabolic. *Phys. At. Nucl.* **2017**, *80*, 347–352. [CrossRef]
106. Elkhidir, E.; Karateev, D.; Serone, M. General Three-Point Functions in 4D CFT. *J. High Energy Phys.* **2015**, *1501*, 133. [CrossRef]
107. Kleinschmidt, A.; Nicolai, H. Standard model fermions and $K(E_{10})$. *Phys. Lett. B* **2015**, *747*, 251–254. [CrossRef]
108. Vos, G. Generalized additivity in unitary conformal field theories. *Nucl. Phys. B* **2015**, *899*, 91–111. [CrossRef]
109. Xiao, W. Differential equations and singular vectors in Verma modules over sl(n,C). *Acta Math. Sin. Engl. Ser.* **2015**, *31*, 1057–1066. [CrossRef]
110. Zhang, G. Discrete components in restriction of unitary representations of rank one semisimple Lie groups. *J. Funct. Anal.* **2015**, *269*, 3689–3713. [CrossRef]
111. Hijano, E.; Kraus, P.; Perlmutter, E.; Snively, R. Witten Diagrams Revisited: The AdS Geometry of Conformal Blocks. *J. High Energy Phys.* **2016**, *1601*, 146. [CrossRef]
112. Dobrev, V.K. *Invariant Differential Operators, Volume 2: Quantum Groups*; De Gruyter Studies in Mathematical Physics; De Gruyter: Berlin, Germany; Boston, MA, USA, 2017; Volume 39, pp. 394 + xii, ISBN 978-3-11-043543-6/978-3-11-042770-7.
113. Dobrev, V.K. *Invariant Differential Operators, Volume 3: Supersymmetry*; De Gruyter Studies in Mathematical Physics; De Gruyter: Berlin, Germany; Boston, MA, USA, 2018; Volume 49, pp. 218 + viii, ISBN 978-3-11-052-7490/978-3-11-052-6639, ISSN 2194-3532.
114. Dobrev, V.K. *Invariant Differential Operators, Volume 4: AdS/CFT, (Super-)Virasoro and Affine (Super-)Algebras*; De Gruyter Studies in Mathematical Physics; De Gruyter, Berlin, Boston, 2019; Volume 53, pp. 234 + x, ISBN: 3110609681/978-3110609684.

115. Knapp, A.W. *Representation Theory of Semisimple Groups (An Overview Based on Examples)*; Princeton University Press: Princeton, NJ, USA, 1986.
116. Dobrev, V.K. Multiplet classification of the reducible elementary representations of real semisimple Lie groups: The $SO_e(p,q)$ example. *Lett. Math. Phys.* **1985**, *9*, 205–211. [CrossRef]
117. Dobrev, V.K.; Petkova, V.B. On the group-theoretical approach to extended conformal supersymmetry: Classification of multiplets *Lett. Math. Phys.* **1985**, *9*, 287–298. [CrossRef]
118. Dixmier, J. *Enveloping Algebras*; North Holland: New York, NY, USA, 1977.
119. Dobrev, V.K. Canonical construction of differential operators intertwining representations of real semisimple Lie groups. *Rept. Math. Phys.* **1988**, *25*, 159–181.
120. Dobrev, V.K. Subsingular vectors and conditionally invariant (q-deformed) equations *J. Phys. A* **1995**, *28*, 7135–7155. [CrossRef]
121. Dobrev, V.K. Kazhdan-Lusztig polynomials, subsingular vectors, and conditionally invariant (q-deformed) equations. In Proceedings of the Symmetries in Science IX, Bregenz, Austria, 6–10 August 1996; Gruber, B., Ramek, M., Eds.; Plenum Press: New York, NY, USA, 1997; pp. 47–80.
122. Dobrev, V.K. Invariant Differential Operators for Non-Compact Lie Algebras Parabolically Related to Conformal Lie Algebras. *J. High Energy Phys.* **2013**, *2*, 15. [CrossRef]
123. Fioresi, R.; Varadarajan, V.S. Harish-Chandra Highest Weight Representations of Semisimple Lie Algebras and Lie Groups. *J. Lie Theory* **2023**, *33*, 217–252.
124. Fioresi, R.; Zanchetta, F. Deep Learning and Geometric Deep Learning: An introduction for mathematicians and physicists. *Int. J. Geom. Methods Mod. Phys.* **2023**, *20*, 2330006. [CrossRef]
125. Juhl, A. Extrinsic Paneitz operators and Q-curvatures for hypersurfaces. *Differ. Geom. Appl.* **2023**, *89*, 102027. [CrossRef]
126. Disch, J. Generic Gelfand-Tsetlin modules of quantized and classical orthogonal algebras. *J. Algebra* **2023**, *620*, 157–193. [CrossRef]
127. Acosta-Humánez, P.; Barkatou, M.; Sánchez-Cauce, R.; Weil, J.A. Darboux Transformations for Orthogonal Differential Systems and Differential Galois Theory. *SIGMA* **2023**, *19*, 016. [CrossRef]
128. Raza, N.; Salman, F.; Butt, A.R.; Gandarias, M.L. Lie symmetry analysis, soliton solutions and qualitative analysis concerning to the generalized q-deformed Sinh-Gordon equation. *Commun. Nonlinear Sci. Numer. Simul.* **2023**, *116*, 106824. [CrossRef]
129. Isaev, A.P.; Krivonos, S.O.; Provorov, A.A. Split Casimir operator for simple Lie algebras in the cube of ad-representation and Vogel parameters. *Int. J. Mod. Phys. A* **2023**, *38*, 2350037. [CrossRef]
130. Isaev, R.A.P.; Provorov, A.A. Split Casimir operator and solutions of the Yang–Baxter equation for the $osp(M|N)$ and $s\ell(M|N)$ Lie superalgebras, higher Casimir operators, and the Vogel parameters. *Teor. Mat. Fiz.* **2022**, *210*, 259–301. [CrossRef]
131. Isaev, A.P.; Provorov, A.A. Projectors on invariant subspaces of representations $\mathrm{ad}^{\otimes 2}$ of Lie algebras $so(N)$ and $sp(2r)$ and Vogel parameterization. *Teor. Mat. Fiz.* **2021**, *206*, 3–22. [CrossRef]
132. Sasso, E.; Umanità, V. On the relationships between covariance and the decoherence-free subalgebra of a quantum Markov semigroup. *Inf. Dim. Anal. Quant. Probab. Rel. Top.* **2023**, *26*, 2250022. [CrossRef]
133. Aschieri, P.; Fioresi, R.; Latini, E.; Weber, T. Quantum principal bundles and noncommutative differential calculus. *Proc. Sci.* **2022**, *406*, 280.
134. Chuah, M.; Fioresi, R. Levi Factors and Admissible Automorphisms. *Algebr. Represent. Theory* **2022**, *25*, 341–358. [CrossRef]
135. Xie, W.; Li, W. Entanglement properties of random invariant quantum states. *Quant. Inf. Comput.* **2022**, *22*, 901–923. [CrossRef]
136. Eremko, A.; Brizhik, L.; Loktev, V. Algebra of the spinor invariants and the relativistic hydrogen atom. *Ann. Phys.* **2023**, *451*, 169266. [CrossRef]
137. Zhao, Q.; Wang, H.; Li, X.; Li, C. Lie Symmetry Analysis and Conservation Laws for the (2 + 1)-Dimensional Dispersionless B-Type Kadomtsev–Petviashvili Equation. *J. Nonlin. Math. Phys.* **2023**, *30*, 92–113. [CrossRef]
138. Artawan, I.N.; Purwanto, A.; Yuwana, L., Invariants for determining entanglements pattern. *Phys. Scr.* **2022**, *97*, 075106. [CrossRef]
139. Hu, S.; Yeats, K. Completing the c_2 completion conjecture for $p = 2$. *Commun. Num. Theor. Phys.* **2023**, *17*, 343–384. [CrossRef]
140. Bautista, A.; Ibort, A.; Lafuente, J. The sky invariant: A new conformal invariant for Schwarzschild spacetime. *Int. J. Geom. Meth. Mod. Phys.* **2022**, *19*, 2250168. [CrossRef]
141. Khantoul, B.; Bounames, A. Exact solutions for time-dependent complex symmetric potential well. *Acta Polytech.* **2023**, *63*, 132–139. [CrossRef]
142. Weng, Z.H. Two incompatible types of invariants in the octonion spaces. *Int. J. Geom. Meth. Mod. Phys.* **2022**, *19*, 2250161. [CrossRef]
143. Jalalzadeh, S.; Rasouli, S.M.M.; Moniz, P. Shape Invariant Potentials in Supersymmetric Quantum Cosmology. *Universe* **2022**, *8*, 316. [CrossRef]
144. Val'kov, V.V.; Shustin, M.S.; Aksenov, S.V.; Zlotnikov, A.O.; Fedoseev, A.D.; Mitskan, V.A.; Kagan, M.Y. Topological superconductivity and Majorana states in low-dimensional systems. *Phys. Usp.* **2022**, *65*, 2–39. [CrossRef]
145. Jafari, M.; Zaeim, A.; Tanhaeivash, A. Symmetry group analysis and conservation laws of the potential modified KdV equation using the scaling method. *Int. J. Geom. Meth. Mod. Phys.* **2022**, *19*, 2250098. [CrossRef]
146. Marquette, I.; Zelaya, K. On the general family of third-order shape-invariant Hamiltonians related to generalized Hermite polynomials. *Phys. D Nonlinear Phenom.* **2022**, *442*, 133529. [CrossRef]

147. Wang, Y.K.; Ge, L.Z.; Fei, S.M.; Wang, Z.X. A Note on Holevo Quantity of SU(2)-invariant States. *Int. J. Theor. Phys.* **2022**, *61*, 7. [CrossRef]
148. Lavrov, P.M. On gauge-invariant deformation of reducible gauge theories. *Eur. Phys. J. C* **2022**, *82*, 429. [CrossRef]
149. Adeyemo, O.D.; Khalique, C.M. Lie Group Classification of Generalized Variable Coefficient Korteweg-de Vries Equation with Dual Power-Law Nonlinearities with Linear Damping and Dispersion in Quantum Field Theory. *Symmetry* **2022**, *14*, 83. [CrossRef]
150. Liu, W.; Fang, X.; Jing, J.; Wang, A. Gauge invariant perturbations of general spherically symmetric spacetimes. *Sci. China Phys. Mech. Astron.* **2023**, *66*, 210411. [CrossRef]
151. Latorre, A.; Ugarte, L. Abelian J-Invariant Ideals on Nilpotent Lie Algebras. In Proceedings of the International Workshop on Lie Theory and Its Applications in Physics, Sofia, Bulgaria, 20–26 June 2021; Springer Proceedings in Mathematics & Statistics; Springer: Singapore, 2022; Volume 396, pp. 509–514.
152. Vaneeva, O.; Magda, O.; Zhalij, A. Lie Reductions and Exact Solutions of Generalized Kawahara Equations. In Proceedings of the International Workshop on Lie Theory and Its Applications in Physics, Sofia, Bulgaria, 20–26 June 2021; Springer Proceedings in Mathematics & Statistics; Springer: Singapore, 2022; Volume 396, pp. 333–338.
153. Blitz, S. A sharp characterization of the Willmore invariant. *Int. J. Math.* **2023**, *34*, 2350054. [CrossRef]
154. Singh, S.; Nechita, I. Diagonal unitary and orthogonal symmetries in quantum theory: II. Evolution operators. *J. Phys. A* **2022**, *55*, 255302. [CrossRef]
155. Aizawa, N.; Dobrev, V.K. Invariant differential operators for the Jacobi algebra $\mathcal{G}2$. *Mod. Phys. Lett. A* **2022**, *37*, 2250067. [CrossRef]
156. Schaposnik, L.P.; Schulz, S. Triality for Homogeneous Polynomials. *SIGMA* **2021**, *17*, 79. [CrossRef]
157. Bonora, L.; Malik, R.P. BRST and Superfield Formalism—A Review. *Universe* **2021**, *7*, 280. [CrossRef]
158. Watson, C.K.; Julius, W.; Gorban, M.; McNutt, D.D.; Davis, E.W.; Cleaver, G.B. An Invariant Characterization of the Levi-Civita Spacetimes. *Symmetry* **2021**, *13*, 1469. [CrossRef]
159. Sen, I. Analysis of the superdeterministic Invariant-set theory in a hidden-variable setting. *Proc. Roy. Soc. Lond. A* **2022**, *478*, 20210667. [CrossRef]
160. Geloun, J.B.; Ramgoolam, S. All-orders asymptotics of tensor model observables from symmetries of restricted partitions. *J. Phys. A* **2022**, *55*, 435203. [CrossRef]
161. Anjali, S.; Gupta, S. Symplectic gauge-invariant reformulation of a free-particle system on toric geometry. *EPL* **2021**, *135*, 11002.
162. Johansson, M. Low degree Lorentz invariant polynomials as potential entanglement invariants for multiple Dirac spinors. *Ann. Phys.* **2023**, *457*, 169410. [CrossRef]
163. Haddadin, W.I.J. Invariant polynomials and machine learning. *arXiv* **2021**, arXiv:2104.12733.
164. Barnes, G.; Padellaro, A.; Ramgoolam, S. Permutation invariant Gaussian two-matrix models. *J. Phys. A* **2022**, *55*, 145202. [CrossRef]
165. Schnetz, O. Geometries in perturbative quantum field theory. *Commun. Num. Theor. Phys.* **2021**, *15*, 743–791. [CrossRef]
166. Ichikawa, T. Chern-Simons invariant and Deligne-Riemann-Roch isomorphism. *Trans. Am. Math. Soc.* **2021**, *374*, 2987–3005. [CrossRef]
167. Varshovi, A.A. $\star$-cohomology, third type Chern character and anomalies in general translation-invariant noncommutative Yang–Mills. *Int. J. Geom. Meth. Mod. Phys.* **2021**, *18*, 2150089. [CrossRef]
168. Chae, J. A Cable Knot and BPS-Series. *SIGMA* **2023**, *19*, 002. [CrossRef]
169. Brandt, F. Properties of an alternative off-shell formulation of 4D supergravity. *Symmetry* **2021**, *13*, 620. [CrossRef]
170. Abramovich, D.; Chen, Q.; Gross, M.; Siebert, B. Decomposition of degenerate Gromov–Witten invariants. *Compos. Math.* **2020**, *156*, 2020–2075. [CrossRef]
171. Mattingly, B.; Kar, A.; Gorban, M.; Julius, W.; Watson, C.K.; Ali, M.; Baas, A.; Elmore, C.; Lee, J.S.; Shakerin, B.; et al. Curvature Invariants for the Alcubierre and Natário Warp Drives. *Universe* **2021**, *7*, 21. [CrossRef]
172. Mashford, J. A Spectral Calculus for Lorentz Invariant Measures on Minkowski Space. *Symmetry* **2020**, *12*, 1696. [CrossRef]
173. Kac, V.G.; Frajria, P.M.; Papi, P. Invariant Hermitian forms on vertex algebras. *Commun. Contemp. Math.* **2022**, *24*, 2150059. [CrossRef]
174. Thibes, R. BRST analysis and BFV quantization of the generalized quantum rigid rotor. *Mod. Phys. Lett. A* **2021**, *36*, 2150116. [CrossRef]
175. Wang H., Weyl invariant Jacobi forms: A new approach. *Adv. Math.* **2021**, *384* , 107752 [CrossRef]
176. Špenko, Š.; Bergh, M.V. Perverse schobers and GKZ systems. *Adv. Math.* **2022**, *402*, 108307. [CrossRef]
177. Yamani, H.A.; Mouayn, Z. Properties of Shape-Invariant Tridiagonal Hamiltonians. *Theor. Math. Phys.* **2020**, *203*, 380–400, 761–779. [CrossRef]
178. Bahmandoust, P.; Latifi, D. Naturally reductive homogeneous (α,β) spaces. *Int. J. Geom. Meth. Mod. Phys.* **2020**, *17*, 2050117. [CrossRef]
179. Geer, N.; Ha, N.P.; Patureau-Mirand, B. Modified graded Hennings invariants from unrolled quantum groups and modified integral. *J. Pure Appl. Algebra* **2022**, *226*, 106815. [CrossRef]
180. Berceanu, S. Invariant metric on the extended Siegel–Jacobi upper half space. *J. Geom. Phys.* **2021**, *162*, 104049. [CrossRef]
181. Pappas, G. Volume and symplectic structure for ℓ-adic local systems. *Adv. Math.* **2021**, *387*, 107836. [CrossRef]
182. Wang, C.; Wang, X.R.; Guo, C.X.; Kou, S.P. Defective edge states and anomalous bulk-boundary correspondence for topological insulators under non-Hermitian similarity transformation. *Int. J. Mod. Phys. B* **2020**, *34*, 2050146. [CrossRef]

183. Ai, C.; Dong, C.; Lin, X. Some exceptional extensions of Virasoro vertex operator algebras. *J. Algebra* **2020**, *546*, 370–389. [CrossRef]
184. Kumar, S.; Chauhan, B.; Tripathi, A.; Malik, R.P. Massive 4D Abelian 2-form theory: Nilpotent symmetries from the (anti-)chiral superfield approach. *Int. J. Mod. Phys. A* **2022**, *37*, 2250003. [CrossRef]
185. Zenad, M.; Ighezou, F.Z.; Cherbal, O.; Maamache, M. Ladder Invariants and Coherent States for Time-Dependent Non-Hermitian Hamiltonians. *Int. J. Theor. Phys.* **2020**, *59*, 1214–1226. [CrossRef]
186. Ha, N.P. Anomaly-free TQFTs from the super Lie algebra $\mathfrak{sl}(2|1)$. *J. Knot Theor. Ramifications* **2022**, *31*, 2250029. [CrossRef]
187. Allegretti, D.G.L. Stability conditions, cluster varieties, and Riemann-Hilbert problems from surfaces. *Adv. Math.* **2021**, *380*, 107610. [CrossRef]
188. Baseilhac, S.; Roche, P. Unrestricted Quantum Moduli Algebras. I. The Case of Punctured Spheres. *SIGMA* **2022**, *18*, 25. [CrossRef]
189. Fioresi, R.; Latini, E.; Marrani, A. The q-linked complex Minkowski space, its real forms and deformed isometry groups. *Int. J. Geom. Methods Mod. Phys.* **2019**, *16*, 1950009. [CrossRef]
190. Liu, J.G.; Yang, X.J.; Feng, Y.Y. Characteristic of the algebraic traveling wave solutions for two extended (2 + 1)-dimensional Kadomtsev–Petviashvili equations. *Mod. Phys. Lett. A* **2019**, *35*, 2050028. [CrossRef]
191. Biswas, I.; Rayan, S. Homogeneous Higgs and co-Higgs bundles on Hermitian symmetric spaces. *Int. J. Math.* **2020**, *31*, 2050118. [CrossRef]
192. Haouam, I. Analytical Solution of (2+1) Dimensional Dirac Equation in Time-Dependent Noncommutative Phase-Space. *Acta Polytech.* **2020**, *60*, 111–121. [CrossRef]
193. Adler, D.; Gritsenko, V. The D_8-tower of weak Jacobi forms and applications. *J. Geom. Phys.* **2020**, *150*, 103616. [CrossRef]
194. Nigsch, E.A.; Vickers, J.A. A nonlinear theory of distributional geometry. *Proc. Roy. Soc. Lond. A* **2020**, *476*, 20200642. [CrossRef] [PubMed]
195. Abe, S. Weak invariants in dissipative systems: Action principle and Noether charge for kinetictheory. *Phil. Trans. Roy. Soc. Lond. A* **2020**, *378*, 20190196. [CrossRef] [PubMed]
196. Zhang, Y.; Wang, X.P. Mei Symmetry and Invariants of Quasi-Fractional Dynamical Systems with Non-Standard Lagrangians. *Symmetry* **2019**, *11*, 1061. [CrossRef]
197. Gueorguiev, V.G.; Maeder, A. Geometric Justification of the Fundamental Interaction Fields for the Classical Long-Range Forces. *Symmetry* **2021**, *13*, 379. [CrossRef]
198. Bambozzi, F.; Murro, S. On the uniqueness of invariant states. *Adv. Math.* **2021**, *376*, 107445. [CrossRef]
199. González-Prieto, Á. Virtual classes of parabolic SL2(C) -character varieties. *Adv. Math.* **2020**, *368*, 107148. [CrossRef]
200. Krishnaswami, G.S.; Vishnu, T.R. Invariant tori, action-angle variables and phase space structure of the Rajeev-Ranken model. *J. Math. Phys.* **2019**, *60*, 082902. [CrossRef]
201. Dabholkar, A.; Jain, D.; Rudra, A. APS η-invariant, path integrals, and mock modularity. *J. High Energy Phys.* **2019**, *11*, 80. [CrossRef]
202. Lin, D. Seiberg-Witten equation on a manifold with rank-2 foliation. *Proc. Am. Math. Soc.* **2021**, *149*, 4411–4417. [CrossRef]
203. Nozawa, M.; Tomoda, K. Counting the number of Killing vectors in a 3D spacetime. *Class. Quant. Gravity* **2019**, *36*, 155005. [CrossRef]
204. Chen, H. Cohomological invariants of representations of 3-manifold groups. *J. Knot Theor. Ramifications* **2020**, *29*, 2043003. [CrossRef]
205. Xiao, Z.; Yang, Y.; Zhang, Y. The diagram category of framed tangles and invariants of quantized symplectic group. *Sci. China Math.* **2019**, *63*, 689–700. [CrossRef]
206. Zubkov, M.A.; Wu, X. Topological invariant in terms of the Green functions for the Quantum Hall Effect in the presence of varying magnetic field. *Ann. Phys.* **2020**, *418*, 168179; Erratum in *Ann. Phys.* **2021**, *430*, 168510. [CrossRef]
207. Garoufalidis, S.; Zagier, D. Asymptotics of Nahm sums at roots of unity. *Ramanujan J.* **2021**, *55*, 219–238. [CrossRef]
208. Slavnov, A.A. Renormalizability and Unitarity of the Englert–Broute–Higgs–Kibble Model. *Theor. Math. Phys.* **2018**, *197*, 1611–1614. [CrossRef]
209. Halder, A.K.; Paliathanasis, A.; Leach, P.G.L. Noether's Theorem and Symmetry. *Symmetry* **2018**, *10*, 744. [CrossRef]
210. Yeats, K. A Special Case of Completion Invariance for the c2 Invariant of a Graph. *Can. J. Math.* **2018**, *70*, 1416–1435. [CrossRef]
211. Suzuki, S. The universal quantum invariant and colored ideal triangulations. *Algebr. Geom. Topol.* **2018**, *18*, 3363–3402. [CrossRef]
212. Habibullin, I.T.; Khakimova, A.R. A Direct Algorithm for Constructing Recursion Operators and Lax Pairs for Integrable Models. *Theor. Math. Phys.* **2018**, *196*, 1200–1216. [CrossRef]
213. Wheeler, J.T. General relativity as a biconformal gauge theory. *Nucl. Phys. B* **2019**, *943*, 114624. [CrossRef]
214. Helleland, C.; Hervik, S. Real GIT with applications to compatible representations and Wick-rotations. *J. Geom. Phys.* **2019**, *142*, 92–110. [CrossRef]
215. Talamini, V. Canonical bases of invariant polynomials for the irreducible reflection groups of types E_6, E_7, and E_8. *J. Algebra* **2018**, *503*, 590–603. [CrossRef]
216. Benkart, G.; Elduque, A. Cross products, invariants, and centralizers. *J. Algebra* **2018**, *500*, 69–102. [CrossRef]
217. Wang, H. Weyl invariant E_8 Jacobi forms. *Commun. Num. Theor. Phys.* **2021**, *15*, 517–573. [CrossRef]
218. Bunk, S.; Szabo, R.J. Topological insulators and the Kane–Mele invariant: Obstruction and localization theory. *Rev. Math. Phys.* **2019**, *32*, 2050017. [CrossRef]
219. Kauffman, L.H.; Lambropoulou, S. Skein Invariants of Links and Their State Sum Models. *Symmetry* **2017**, *9*, 226. [CrossRef]

220. Fernández-Culma, E.A.; Godoy, Y. Anti-Kählerian Geometry on Lie Groups. *Math. Phys. Anal. Geom.* **2018**, *21*, 8. [CrossRef]
221. Wakamatsu, M.; Kitadono, Y.; Zhang, P.M. The issue of gauge choice in the Landau problem and the physics of canonical and mechanical orbital angular momenta. *Ann. Phys.* **2018**, *392*, 287–322. [CrossRef]
222. Khalfoun, H. $\mathfrak{aff}(1|1)$-Relative cohomology on $\mathbb{R}1|1$. *Int. J. Geom. Meth. Mod. Phys.* **2017**, *14*, 1750174. [CrossRef]
223. Takeuchi, Y. Ambient constructions for Sasakian η-Einstein manifolds. *Adv. Math.* **2018**, *328*, 82–111. [CrossRef]
224. Levchenko, E.A.; Trifonov, A.Y.; Shapovalov, A.V. Symmetries of the One-Dimensional Fokker–Planck–Kolmogorov Equation with a Nonlocal Quadratic Nonlinearity. *Russ. Phys. J.* **2017**, *60*, 284–291. [CrossRef]
225. Chen, J.; Han, M.; Li, Y.; Zeng, B.; Zhou, J. Local density matrices of many-body states in the constant weight subspaces. *Rep. Math. Phys.* **2019**, *83*, 273–292. [CrossRef]
226. Belgun, F.; Cortés, V.; Haupt, A.S.; Lindemann, D. Left-invariant Einstein metrics on $S^3 \times S^3$. *J. Geom. Phys.* **2018**, *128*, 128–139. [CrossRef]
227. Kuessner, T. Fundamental classes of 3-manifold groups representations in SL(4,R). *J. Knot Theor. Ramifications* **2017**, *26*, 1750036. [CrossRef]
228. Weng, Z.H. Spin Angular Momentum of Proton Spin Puzzle in Complex Octonion Spaces. *Int. J. Geom. Meth. Mod. Phys.* **2017**, *14*, 1750102. [CrossRef]
229. Jamal, S.; Paliathanasis, A. Group invariant transformations for the Klein–Gordon equation in three dimensional flat spaces. *J. Geom. Phys.* **2017**, *117*, 50–59. [CrossRef]
230. Bruhat, F. Sur les représentations induites des groupes de Lie. *Bull. Soc. Math. France* **1956**, *84*, 97–205. [CrossRef]
231. Chandra, H. 2.! semi-simple groups IV, V, VI Amer. *Am. J. Math.* **1955**, *77*, 743–777.
232. Dobrev, V.K. Positive energy representations, holomorphic discrete series and finite-dimensional irreps. *J. Phys. A* **2008**, *41*, 425206. [CrossRef]
233. Knapp, A.W.; Stein, E.M. Interwining operators for semisimple groups. *Ann. Math.* **1971**, *93*, 489–578. [CrossRef]
234. Harish-Chandra, Representations of Semisimple Lie Groups VI: Integrable and Square-Integrable Representations. *Am. J. Math.* **1956**,*78*, 1–41.
235. Gelfand, I.M.; Naimark, M.A. Unitary Representations of the Lorentz Group. *Acad. Sci. USSR J. Phys.* **1946**, *10*, 93–94.
236. Bargmann, V. Irreducible unitary representations of the Lorentz group. *Ann. Math.* **1947**, *48*, 568–640. [CrossRef]
237. Dobrev, V.K. Invariant differential operators for non-compact Lie groups: The main su(n, n) cases. *Phys. At. Nucl.* **2013**, *76*, 983–990. [CrossRef]
238. Dobrev, V.K. Invariant Differential Operators for Non-Compact Lie Groups: The Sp(n,R) Case Lie Theory and Its Applications in Physics. In Proceedings of the 9th International Workshop, Varna, Bulgaria, 26 June 2011; Springer Proceedings in Mathematics and Statistics; Springer: Tokyo, Japan; Berlin/Heidelberg, Germany, 2013; Volume 36, pp. 311–335.
239. Dobrev, V.K. Heisenberg parabolic subgroup of SO*(8) and invariant differential operators. In Proceedings of the Workshop on Quantum Geometry, Field Theory and Gravity, Corfu, Greece, 20–27 September 2021; Volume 406. Available online: https://pos.sissa.it/406/303 (accessed on 23 November 2022).
240. Dobrev, V.K. Heisenberg Parabolic Subgroup of SO*(10) and Corresponding Invariant Differential Operators. *Symmetry* **2022**, *14*, 1592. [CrossRef]
241. Satake, I. On Representations and Compactifications of Symmetric Riemannian Spaces. *Ann. Math.* **1960**, *71*, 77–110. [CrossRef]
242. Dobrev, V.K. The exceptional Lie algebra $E_{7(-25)}$: Multiplets and invariant differential operators. *J. Phys. A* **2009**, *42*, 285203. [CrossRef]
243. Dobrev, V.K. Invariant Differential Operators for Non-Compact Lie Groups: The $E_{6(14)}$ case. In Proceedings of the 5th Mathematical Physics Meeting: Summer School and Conference on Modern Mathematical Physics, Belgrade, Serbia, 6–17 July 2008; Dragovich, B., Rakic, Z., Eds.; Institute of Physics: Belgrade, Serbia, 2009; pp. 95–124.