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Connectedness of Soft-Ideal Topological Spaces

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Abstract: Despite its apparent simplicity, the idea of connectedness has significant effects on topology and its applications. An essential part of the intermediate-value theorem is the idea of connectedness. In many applications, such as population modeling, robotics motion planning, and geographic information systems, connectedness is significant, and it is a critical factor in differentiating between various topological spaces. This study uses soft open sets and the concept of soft ideals as a new class of soft sets to present and explore the ideas of soft connected spaces and strongly soft connected spaces with soft ideals. Also, under certain assumptions regarding the subsequent concepts—soft-ideal connectedness and stronglysoft-ideal connectedness in soft-ideal topological spaces—we characterize this new class of sets by employing soft open sets and soft ideals to examine its fundamental features. Furthermore, we look at a symmetry between our new notions and other existing ones, and this study examines the relationships between these concepts.

Keywords: soft open set; soft idea; soft local function; soft connected; strongly soft connected

MSC: 54A05; 54A10; 54A40

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1. Introduction

Scientific processes, as opposed to tried-and-true methods, must be utilized to answer most empirical issues in technology domains—such as engineering, computer science, and social sciences—that deal with uncertainty and unreliability. In [1], Molodtsov outlined a novel mathematical method known as soft set theory to solve some issues. Molodtsov effectively utilized the soft set theory in [1,2] in several domains, including probability, measurement theory, Riemann integration, game theory, operation research, and smoothness of functions. Considerable advancements in theory and technology have resulted from an exponential growth in the study of and applications of this soft set (see [3–9]). Next, a well-focused investigation of an abstract soft set operator theory with applications to decision-making problems was conducted by Maji et al. [10]. The work of Shabir and Naz focused on the theoretical studies that introduced the idea of soft topological spaces [11]. These spaces are defined over an initial universe that has a predetermined set of soft topological space features. Kandil et al. [12] presented the concept of soft ideals initially. In addition, they established the concept of the soft local function. Several symmetric concepts have been proposed for identifying new soft topologies from old ones, commonly known as soft topological spaces with soft ideals $(X_\varepsilon, \tau, \mathbb{I}_\varepsilon)$. Kandil et al. [13] explored some applications of soft sets in several sectors. Lin [14] developed the concept of connectedness in soft topological spaces. In [13], the authors explored the concept of ($*$-soft separated, $*$-soft connected, and $*_s$-soft connected) sets in soft topological spaces with soft ideals. Al-shami et al. [15] examined notions of connectedness via the class of soft somewhat-open sets. Additionally, Al-shami et al. [16] introduced new types of symmetrical soft connected spaces and infra-soft locally connected spaces and components. Local connection and connectedness are some essential concepts in soft topology. In 1983, Atanassov introduced intuitionistic fuzzy sets [17]. Basic results on intuitionistic fuzzy sets are reported in [18,19]. Yager [20,21] suggested a new form of fuzzy sets called Pythagorean fuzzy sets. Yager [22] introduced the q-rung orthopair fuzzy set for dual universes. Saber Y. et al. [23] discussed the connectedness and stratification of single-valued neutral topological spaces. Li et al. [24] introduced the notion of q-rung picture fuzzy sets. Ajmal et al. [25] investigated the idea of connectedness and local connectedness in fuzzy topological spaces. Refs. [26,27] provide further information about soft connectedness. Research in soft topology and related topics remains active (see, for example, [28,29]), with opportunities for significant contributions. The principal characteristics of these notions are defined and examined by the researchers of [6,15]. Thus, our objectives in this article are to introduce the concept of soft-ideal connected spaces and soft-strongly-ideal connected spaces using ideals, and to extend some significant results on strong connectedness. Also, we aim to ascertain under which conditions soft connectedness, soft-ideal connectedness, and soft -strongly-ideal connectedness are equivalent. Finally, we provide some more examples to illustrate some of the characteristics.

This manuscript is a continuation of the previous works based on symmetric soft connected spaces. The structure of this paper is as follows: Section 2 reviews the main definitions and findings of soft theory. After defining "soft connected" and "soft ideal connected spaces", some of their many extensions are shown in Section 3. With the aid of strong examples, we examine their salient features and make clear the relationships between them. We present the idea of strongly-soft -$\mathbb{I}_\varepsilon$-connected spaces in Section 4 and explore deeply its main characteristics. We show that this concept is comparable to that of "soft connected". Lastly, we provide some insights and pave the way for further research in Section 5.

In this study, topological space and soft topological space, respectively, shall be denoted by the terms used in publications: TS and STS. This essay aims to introduce and explore the notion of soft connected spaces and strongly soft connected spaces with soft ideals.

2. Preliminary

In order to keep this publication self-contained, we will discuss the ideas and conclusions from previous research that are necessary to understand the results that we have found here.

Definition 1 ([1])**.** *Let ε be a set of parameters and X be an initial universe. Let B be a non-null subset of parameters ε and 2^X be the power set of X. A pair (λ, B) indicated by λ_B is a soft set over X_ε, such that λ is a mapping provided by $\lambda : B \to 2^X$. Alternatively stated, a soft set over X_ε is a parameterized collection of subsets of the universe X_ε. For a certain $e \in \varepsilon$, $\lambda(e)$ may be thought of as the set of e-approximate elements of the soft set $(\lambda, \varepsilon) = \lambda_\varepsilon$; also, if $e \notin \varepsilon$, then $\lambda(e) = \phi$, i.e., $\lambda_\varepsilon = \{\lambda(e) : e \in \varepsilon, \lambda : \varepsilon \to 2^X\}$. The collection of all soft sets is represented by $SS(X)_\varepsilon$.*

Definition 2 ([10])**.** *Assume $A_\varepsilon, B_\varepsilon \in SS(X)_\varepsilon$. We call A_ε:*

(a) A soft subset of B_ε, represented by $A_\varepsilon \sqsubseteq B_\varepsilon$, if $A(e) \subseteq B(e)$, for all $e \in \varepsilon$.
(b) An absolute, represented by X_ε, if $A(e) = X$, for all $e \in \varepsilon$.
(c) A null, represented by ϕ_ε, if $A(e) = \phi$, for all $e \in \varepsilon$.

In this instance, A_ε is a soft subset of B_ε and B_ε is a soft superset of A_ε, $A_\varepsilon \sqsubseteq B_\varepsilon$.

Definition 3 ([30])**.**

(a) Asume Δ to be an arbitrary index set and $\Omega = \{(\lambda_\alpha)_\varepsilon : \alpha \in \Delta\}$ to be a subfamily of $SS(X)_\varepsilon$. Then:
  (i) The union of all $(\lambda_\alpha)_\varepsilon$ is the soft set A_ε, where $A(e) = \cup_{\alpha\in\Delta}(\lambda_\alpha)_\varepsilon(e)$ for all $e \in \varepsilon$. We compose $\sqcup_{\alpha\in\Delta}(\lambda_\alpha)_\varepsilon = A_\varepsilon$.
  (ii) The intersection of all $(\lambda_\alpha)_\varepsilon$ is the soft set N_ε, where $N(e) = \cap_{\alpha\in\Delta}(\lambda_\alpha)_\varepsilon(e)$ for all $e \in \varepsilon$. We compose $\sqcap_{\alpha\in\Delta}(\lambda_\alpha)_\varepsilon = N_\varepsilon$.

(b) *A soft set A_ε in (X_ε, τ) is a soft neighborhood of the soft point $x_\varepsilon \in X_\varepsilon$ if there exists a set $B_\varepsilon \in \tau$ such that $x_\varepsilon \in B_\varepsilon \sqsubseteq A_\varepsilon$.*

(c) *A soft set $\lambda_\varepsilon \in SS(X)_\varepsilon$ is a soft point in X_ε if there exist $x \in X$ and $e \in \varepsilon$ such that $\lambda(e) = \{x\}$ and $\lambda(e^c) = \phi$ for all $e^c \in \varepsilon - \{e\}$. This soft point λ_ε is denoted by x_ε.*

Definition 4 ([11]). *Let (X_ε, τ) be a STS and $\lambda_\varepsilon \in SS(X)_\varepsilon$.*

(a) *A complement of a soft set λ_ε, indicated by λ^c_ε, is described as follows: $\lambda^c : \varepsilon \to 2^X$ is a mapping given by $\lambda^c(e) = X_\varepsilon(e) - \lambda(e)$, for all $e \in \varepsilon$, and λ^c is called a soft complement function of λ_ε.*

(b) *A difference of two soft sets λ_ε and A_ε over the common universe X_ε, denoted by $\lambda_\varepsilon - A_\varepsilon$, is the soft set B_ε, for all $e \in \varepsilon$, such that $B(e) = \lambda(e) - A(e)$.*

(c) *The soft interior of λ_ε is $Int(\lambda_\varepsilon) = \sqcup\{A_\varepsilon : A_\varepsilon \in \tau$ and $A_\varepsilon \sqsubseteq \lambda_\varepsilon\}$.*

(d) *The soft closure of λ_ε is $cl(\lambda_\varepsilon) = \sqcap\{A_\varepsilon : A_\varepsilon \in \tau^c$ and $\lambda_\varepsilon \sqsubseteq A_\varepsilon\}$.*

(e) *Let λ_ε be a soft set over X_ε and $x_\varepsilon \in X_\varepsilon$. We say that $x_\varepsilon \in \lambda_\varepsilon$ indicates that x_ε belongs to the soft set λ_ε whenever $x_\varepsilon(e) \in \lambda(e)$, for all $e \in \varepsilon$.*

More details about soft set theory and its applications in various mathematical structures may be found in [8,31,32].

Definition 5 ([12]). *Let $\mathbb{I}_\varepsilon$ be a non-null collection of soft sets over a universe X_ε. Then, $\mathbb{I}_\varepsilon \subseteq SS(X)_\varepsilon$ is called a soft ideal on X_ε if:*

1. *$A_\varepsilon \in \mathbb{I}_\varepsilon$ and $B_\varepsilon \in \mathbb{I}_\varepsilon$, then $A_\varepsilon \sqcup B_\varepsilon \in \mathbb{I}_\varepsilon$.*
2. *$A_\varepsilon \in \mathbb{I}_\varepsilon$ and $B_\varepsilon \sqsubseteq A_\varepsilon$, then $B_\varepsilon \in \mathbb{I}_\varepsilon$.*

Definition 6 ([12]). *Let (X_ε, τ) be an STS and $\mathbb{I}_\varepsilon$ be a soft ideal over X_ε with the same set of parameters ε. Then, $\overline{D_\varepsilon}^*(\mathbb{I}_\varepsilon, \tau)$ (or $\overline{D}^*_\varepsilon) = \sqcup\{x_\varepsilon \in X_\varepsilon : O_{x_\varepsilon} \sqcap D_\varepsilon \notin \mathbb{I}_\varepsilon$ for all soft open set $O_{x_\varepsilon}\}$ is the soft local function of D_ε with respect to $\mathbb{I}_\varepsilon$ and soft topology τ, where O_{x_ε} is a soft open set containing x_ε.*

Theorem 1 ([33]). *Let (X_ε, τ) be an STS and $\mathbb{I}_\varepsilon$ be a soft ideal over X with the same set of parameters ε. Then, the following statements are equivalent:*

1. *$\tau \sqcap \mathbb{I}_\varepsilon = \phi_\varepsilon$;*
2. *$A_\varepsilon \in \mathbb{I}_\varepsilon$ implies $Int(A_\varepsilon) = \phi_\varepsilon$;*
3. *For all $B_\varepsilon \in \tau$, $B_\varepsilon \sqsubseteq \overline{B}^*_\varepsilon$;*
4. *$X_\varepsilon = \overline{X}^*_\varepsilon$.*

Definition 7 ([1]). *An STS (X_ε, τ) is called soft connected if X cannot be articulated as the soft union of two soft separated sets in (X_ε, τ). If it is not soft connected, (X_ε, τ) is said to be soft disconnected.*

3. Soft-Ideal Connected Spaces

In this section, we introduce and explain the concept of soft-ideal connected spaces. Additionally, we demonstrate that, subject to specific limitations, the product of soft-ideal connected spaces is also a soft-ideal connected space.

Definition 8. *Let (X_ε, τ) be am STS with a soft ideal $\mathbb{I}_\varepsilon$ on X. A soft subset Y_ε of X_ε is soft $\mathbb{I}_\varepsilon$-connected if Y_ε cannot be expressed as a union of two non-soft-ideal sets A_ε and B_ε such that $Y_\varepsilon \sqcap \overline{A}_\varepsilon \sqcap B_\varepsilon = \phi_\varepsilon = A_\varepsilon \sqcap \overline{B}_\varepsilon \sqcap Y_\varepsilon$. A non-soft-ideal set A_ε in X_ε is a subset of X_ε, such that $A_\varepsilon \notin \mathbb{I}_\varepsilon$, and $\overline{A}_\varepsilon$ is a soft closure of A_ε in τ.*

Any soft connected set is soft $\mathbb{I}_\varepsilon$-connected, but the opposite is not always true, as shown below.

Example 1. *Let $(X, \tau_u) = (\mathbb{R}, \tau_u)$ denote the reals with usual topology and $Y = [0,2] \sqcup \{3,4,5\} \subseteq X$. Let $\mathbb{I}_\varepsilon$ denote the ideal of all finite subsets of X_ε and $E = \{e\}$. Let $\tau = \{F \in SS(X, E) : F(e) \in \tau_u, \text{for all } e \in E\}$, be a soft topological space. Then, Y_ε is soft $\mathbb{I}_\varepsilon$-connected but not soft connected.*

Example 2. *Let $X = \{h_1, h_2, h_3\}$, ε={ε_1, ε_2}, and $\tau = \{X_\varepsilon, \phi_\varepsilon, A_\varepsilon, B_\varepsilon\}$, where A_ε and B_ε are soft sets over X defined as follows: $A(\varepsilon_1) = \{h_2\}$, $A(\varepsilon_2) = \{h_1, h_3\}$, and $B(\varepsilon_1) = \{h_1, h_3\}$, $B(\varepsilon_2) = \{h_2\}$. Then, τ is a soft topology on X. Let $\mathbb{I}_\varepsilon = \{\phi_\varepsilon, I_\varepsilon, J_\varepsilon, K_\varepsilon, L_\varepsilon\}$, where $I_\varepsilon, J_\varepsilon, K_\varepsilon, L_\varepsilon$ are soft sets over X defined by $I(\varepsilon_1) = \{h_2\}$, $I(\varepsilon_2) = \{h_1\}$, $J(\varepsilon_1) = \{h_1, h_2\}$, $J(\varepsilon_2) = \{h_3\}$, $K(\varepsilon_1) = \{h_2\}$, $K(\varepsilon_2) = \{h_1, h_3\}$, and $L(\varepsilon_1) = \{h_1\}$, $L(\varepsilon_2) = \{h_3\}$. Then, $\mathbb{I}_\varepsilon$ is a soft ideal on X. Hence, X_ε is not soft connected but it is soft $\mathbb{I}_\varepsilon$-connected.*

Next, we characterize the soft-ideal connected spaces.

Theorem 2. *Let (X_ε, τ) be an STS with a soft ideal $\mathbb{I}_\varepsilon$ on X. Then, the subsequent statements are equivalent:*

1. *X_ε is soft $\mathbb{I}_\varepsilon$-connected.*
2. *X_ε cannot be expressed as a union of two disjoint non-soft-ideal soft open sets.*
3. *X_ε cannot be expressed as a union of two disjoint non-soft-ideal soft closed sets.*

Proof. (1) $\Rightarrow$ (2): Suppose (2) is not true. Then, $X_\varepsilon = A_\varepsilon \sqcup B_\varepsilon$, for some soft subset $A_\varepsilon, B_\varepsilon \notin \mathbb{I}_\varepsilon$ such that $A_\varepsilon, B_\varepsilon$ are disjoint soft open sets. Then, $A_\varepsilon = \overline{A}_\varepsilon$ and $B_\varepsilon = \overline{B}_\varepsilon$, implying $\overline{A}_\varepsilon \sqcap B_\varepsilon = \phi_\varepsilon = A_\varepsilon \sqcap \overline{B}_\varepsilon$. This contradicts (1). Therefore, X_ε is not characterized as a union of two disjoint non-soft-ideal soft open sets.

(2) $\Rightarrow$ (3): Suppose (3) is not true. Then, $X_\varepsilon = A_\varepsilon \sqcup B_\varepsilon$ for some soft subsets $A_\varepsilon, B_\varepsilon \notin \mathbb{I}_\varepsilon$ such that $A_\varepsilon, B_\varepsilon$ are disjoint soft closed sets. Then, $X_\varepsilon = A_\varepsilon \cup B_\varepsilon$, where $A_\varepsilon, B_\varepsilon \notin \mathbb{I}_\varepsilon$, $A_\varepsilon \sqcap B_\varepsilon = \phi_\varepsilon$, and $A_\varepsilon = X_\varepsilon - B_\varepsilon$, $B_\varepsilon = X_\varepsilon - A_\varepsilon$ are soft open sets, which contradicts (2). Therefore, X_ε not characterized as a union of two disjoint non-soft-ideal soft closed sets.

(3) $\Rightarrow$ (1): Suppose X_ε is not soft $\mathbb{I}_\varepsilon$-connected. Then, $X_\varepsilon = A_\varepsilon \cup B_\varepsilon$, for some soft subsets $A_\varepsilon, B_\varepsilon \notin \mathbb{I}_\varepsilon$ such that $\overline{A}_\varepsilon \sqcap B_\varepsilon = \phi_\varepsilon = A_\varepsilon \sqcap \overline{B}_\varepsilon$. This implies that $\overline{A}_\varepsilon \sqsubseteq A_\varepsilon$ and $\overline{B}_\varepsilon \sqsubseteq B_\varepsilon$. Hence, $X_\varepsilon = A_\varepsilon \sqcup B_\varepsilon$, where $A_\varepsilon, B_\varepsilon \notin \mathbb{I}_\varepsilon$, and $A_\varepsilon, B_\varepsilon$ are disjoint soft closed sets; which is a contradiction to (3). So, X_ε is soft $\mathbb{I}_\varepsilon$-connected. □

It is well known that the union of two connected sets is connected if $A_\varepsilon \sqcap B_\varepsilon \neq \phi_\varepsilon$. This result can be generalized as follows.

Theorem 3. *Let (X_ε, τ) be an STS with a soft ideal $\mathbb{I}_\varepsilon$ on X and A_ε, B_ε be two soft $\mathbb{I}_\varepsilon$-connected sets with $A_\varepsilon \sqcap B_\varepsilon \notin \mathbb{I}_\varepsilon$. Then, $A_\varepsilon \sqcup B_\varepsilon$ is soft $\mathbb{I}_\varepsilon$-connected.*

Proof. Suppose $A_\varepsilon \sqcup B_\varepsilon$ is not soft $\mathbb{I}_\varepsilon$-connected. Then, $A_\varepsilon \sqcup B_\varepsilon = C_\varepsilon \sqcup D_\varepsilon$, where $C_\varepsilon, D_\varepsilon \notin \mathbb{I}_\varepsilon$ and $(A_\varepsilon \sqcup B_\varepsilon) \sqcap \overline{C}_\varepsilon \sqcap D_\varepsilon = \phi_\varepsilon$ and $(A_\varepsilon \sqcup B_\varepsilon) \sqcap C_\varepsilon \sqcap \overline{D}_\varepsilon = \phi_\varepsilon$. We have $(A_\varepsilon \sqcap B_\varepsilon) = (C_\varepsilon \sqcap A_\varepsilon \sqcap B_\varepsilon) \sqcup (D_\varepsilon \sqcap A_\varepsilon \sqcap B_\varepsilon) \notin \mathbb{I}_\varepsilon$. Then, either $C_\varepsilon \sqcap A_\varepsilon \sqcap B_\varepsilon \notin \mathbb{I}_\varepsilon$ or $D_\varepsilon \sqcap A_\varepsilon \sqcap B_\varepsilon \notin \mathbb{I}_\varepsilon$. Suppose $C_\varepsilon \sqcap A_\varepsilon \sqcap B_\varepsilon \notin \mathbb{I}_\varepsilon$, then $C_\varepsilon \sqcap A_\varepsilon \notin \mathbb{I}_\varepsilon$ and $C_\varepsilon \sqcap B_\varepsilon \notin \mathbb{I}_\varepsilon$. Since $A_\varepsilon = (C_\varepsilon \sqcap A_\varepsilon) \sqcup (D_\varepsilon \sqcap A_\varepsilon)$ is a soft $\mathbb{I}_\varepsilon$-connected, either $C_\varepsilon \sqcap A_\varepsilon \in \mathbb{I}_\varepsilon$ or $D_\varepsilon \sqcap A_\varepsilon \in \mathbb{I}_\varepsilon$. As $C_\varepsilon \sqcap A_\varepsilon \notin \mathbb{I}_\varepsilon$, we have $D_\varepsilon \sqcap A_\varepsilon \in \mathbb{I}_\varepsilon$. Similarly, we have $D_\varepsilon \sqcap B_\varepsilon \in \mathbb{I}_\varepsilon$. So, $D_\varepsilon = (D_\varepsilon \sqcap A_\varepsilon) \sqcup (D_\varepsilon \sqcap B_\varepsilon) \in \mathbb{I}_\varepsilon$ and hence $D \in \mathbb{I}_\varepsilon$, which is a contradiction. Hence, $A_\varepsilon \sqcup B_\varepsilon$ is soft $\mathbb{I}_\varepsilon$-connected. □

Theorem 4. *Let (X_ε, τ) be an STS with soft ideal $\mathbb{I}_\varepsilon$ on X_ε with $\tau \sqcap \mathbb{I}_\varepsilon = \phi_\varepsilon$. Then, X_ε is soft $\mathbb{I}_\varepsilon$-connected if X_ε is soft connected.*

Proof. It is enough to prove that if X_ε is soft $\mathbb{I}_\varepsilon$-connected, then it is soft connected. Suppose X_ε is not soft connected. Then, $X_\varepsilon = A_\varepsilon \sqcup B_\varepsilon$, where A_ε, B_ε are non-null soft open sets and $\overline{A}_\varepsilon \sqcap B_\varepsilon = \phi_\varepsilon = A_\varepsilon \sqcap \overline{B}_\varepsilon$. Since $\tau \sqcap \mathbb{I}_\varepsilon = \phi_\varepsilon$, we have $A_\varepsilon, B_\varepsilon \notin \mathbb{I}_\varepsilon$. So, X_ε is not soft $\mathbb{I}_\varepsilon$-connected, which produces a contradiction. Consequently, X_ε is soft connected. □

Theorem 5. *Let (X_ε, τ) be an STS with soft ideal $\mathbb{I}_\varepsilon$ on X_ε. If $A_\varepsilon \sqsubseteq X_\varepsilon$ is soft $\mathbb{I}_\varepsilon$-connected and $A_\varepsilon \sqsubseteq B_\varepsilon \sqsubseteq \overline{A}^*_\varepsilon$) (closure of A_ε in τ^*), then B_ε is soft $\mathbb{I}_\varepsilon$-connected.*

Proof. Suppose that B_ε is not soft $\mathbb{I}_\varepsilon$-connected. Then, $B_\varepsilon = C_\varepsilon \sqcup D_\varepsilon$, where $C_\varepsilon, D_\varepsilon \notin \mathbb{I}_\varepsilon$ and $B_\varepsilon \sqcap \overline{C}_\varepsilon \sqcap D_\varepsilon = \phi_\varepsilon = B_\varepsilon \sqcap C_\varepsilon \sqcap \overline{D}_\varepsilon$. Now, we have $A_\varepsilon = (A_\varepsilon \sqcap C_\varepsilon) \sqcup (A_\varepsilon \sqcap D_\varepsilon)$. Since A_ε is soft $\mathbb{I}_\varepsilon$-connected, either $A_\varepsilon \sqcap C_\varepsilon \in \mathbb{I}_\varepsilon$ or $A_\varepsilon \sqcap D_\varepsilon \in \mathbb{I}_\varepsilon$. Suppose $A_\varepsilon \sqcap D_\varepsilon \in \mathbb{I}_\varepsilon$ and let $x_\varepsilon \in D_\varepsilon - A_\varepsilon$. Then, for every neighborhood V_ε of x_ε, $V_\varepsilon \sqcap A_\varepsilon \notin \mathbb{I}_\varepsilon$. As $V_\varepsilon \sqcap A_\varepsilon = (V_\varepsilon \sqcap A_\varepsilon \sqcap C_\varepsilon) \sqcup (V_\varepsilon \sqcap A_\varepsilon \sqcap D_\varepsilon) \notin \mathbb{I}_\varepsilon$, we have $V_\varepsilon \sqcap A_\varepsilon \sqcap C_\varepsilon \notin \mathbb{I}_\varepsilon$. In particular, $V_\varepsilon \sqcap A_\varepsilon \sqcap C_\varepsilon \neq \phi_\varepsilon$. Then, $V_\varepsilon \sqcap C_\varepsilon \neq \phi_\varepsilon$ and hence $x_\varepsilon \in C_\varepsilon$. Therefore, $x_\varepsilon \in D_\varepsilon - A_\varepsilon$, implying that $x_\varepsilon \in C_\varepsilon$, which is a contradiction to $B_\varepsilon \sqcap \overline{C}_\varepsilon \sqcap D_\varepsilon = \phi_\varepsilon$. Hence, $D_\varepsilon - A_\varepsilon = \phi_\varepsilon$ and $D_\varepsilon \sqsubseteq A_\varepsilon$. Therefore, $D_\varepsilon = D_\varepsilon \sqcap A_\varepsilon \in \mathbb{I}_\varepsilon$, which is a contradiction. Consequently, B_ε is soft $\mathbb{I}_\varepsilon$-connected. □

The above theorem is not true if we replace $*$-closure with closure. We give the following example.

Example 3. *Let $(X, \tau_u) = (\mathbb{R}, \tau_u)$ be the real line with usual topology. Let $A = [0,1] \cup \{x : x$ be rational, $4 < x < 5\}$ and let $\mathbb{I}_\varepsilon$ be the ideal of measure zero sets X_ε and $E = \{\mathbb{Z}\}$. Let $\tau = \{F \in SS(X,E) : F(e) \in \tau_u$ for all $e \in E\}$ be soft topological. Then, A_ε is soft $\mathbb{I}_\varepsilon$-connected, but $\overline{A} = [0,1] \cup [4,5]$ is not soft $\mathbb{I}_\varepsilon$-connected.*

Theorem 6. *Let (X_ε, τ) and (Y_ε, σ) be STSs and let $\mathbb{I}_\varepsilon$ be a soft ideal on X_ε. Let $f_{pu} : (X_\varepsilon, \tau) \to (Y_\varepsilon, \sigma)$ be soft continuous surjective. If (X_ε, τ) is soft $\mathbb{I}_\varepsilon$-connected, then (Y_ε, σ) is soft $f_{pu}(\mathbb{I}_\varepsilon)$-connected.*

Proof. Let $f_{pu} : (X_\varepsilon, \tau) \to (Y_\varepsilon, \sigma)$ be a soft continuous surjective map and (X_ε, τ) be soft $\mathbb{I}_\varepsilon$-connected. Assume that (Y_ε, σ) is not soft $f_{pu}(\mathbb{I}_\varepsilon)$-connected; then, $Y_\varepsilon = B_\varepsilon \sqcup C_\varepsilon$, for some $B_\varepsilon, C_\varepsilon \notin f_{pu}(\mathbb{I}_\varepsilon)$, $B_\varepsilon \sqcap C_\varepsilon = \phi_\varepsilon$ and $B_\varepsilon, C_\varepsilon$ are soft open sets. Since f_{pu} is soft continuous, $f^{-1}_{pu}(B_\varepsilon)$, $f^{-1}_{pu}(C_\varepsilon)$ are soft open and $f^{-1}_{pu}(B_\varepsilon) \sqcap f^{-1}_{pu}(C_\varepsilon) = f^{-1}_{pu}(B_\varepsilon \sqcap C_\varepsilon) = f^{-1}_{pu}(\phi_\varepsilon) = \phi_\varepsilon$. Also, $f^{-1}_{pu}(B_\varepsilon), f^{-1}_{pu}(C_\varepsilon) \notin \mathbb{I}_\varepsilon$ (if $f^{-1}_{pu}(B_\varepsilon) \in \mathbb{I}_\varepsilon$, then $B_\varepsilon \in f_{pu}(\mathbb{I}_\varepsilon)$). Now, $X_\varepsilon = f^{-1}_{pu}(B_\varepsilon) \sqcup f^{-1}_{pu}(C_\varepsilon)$, where $f^{-1}_{pu}(B_\varepsilon)$, $f^{-1}_{pu}(C_\varepsilon)$ are soft open sets such that $f^{-1}_{pu}(B_\varepsilon) \sqcap f^{-1}_{pu}(C_\varepsilon) = \phi_\varepsilon$ and $f^{-1}_{pu}(B_\varepsilon), f^{-1}_{pu}(C_\varepsilon) \notin \mathbb{I}_\varepsilon$. Hence, (X_ε, τ) is not soft $\mathbb{I}_\varepsilon$-connected, which is a contradiction to our assumption. Thus, (Y_ε, σ) is soft $f_{pu}(\mathbb{I}_\varepsilon)$-connected. □

Theorem 7. *Let (X_ε, τ) be an STS and $\mathbb{I}_\varepsilon$ be a soft ideal on X_ε and let $A_\varepsilon, B_\varepsilon \sqsubseteq X_\varepsilon$. If A_ε is soft $\mathbb{I}_\varepsilon$-connected and $B_\varepsilon \in \mathbb{I}_\varepsilon$, then $A_\varepsilon \sqcup B_\varepsilon$ is soft $\mathbb{I}_\varepsilon$-connected.*

Proof. If $A_\varepsilon \sqcup B_\varepsilon$ is not soft $\mathbb{I}_\varepsilon$-connected, then there exist soft open sets C_ε and D_ε in X_ε such that:

1. $(A_\varepsilon \sqcup B_\varepsilon) \sqcap \overline{C}_\varepsilon \sqcap D_\varepsilon = \phi_\varepsilon$ and $(A_\varepsilon \sqcup B_\varepsilon) \sqcap C_\varepsilon \sqcap \overline{D}_\varepsilon = \phi_\varepsilon$.
2. $(A_\varepsilon \sqcup B_\varepsilon) \sqcap C_\varepsilon \notin \mathbb{I}_\varepsilon$ and $(A_\varepsilon \sqcup B_\varepsilon) \sqcap D_\varepsilon \notin \mathbb{I}_\varepsilon$.

Since $B_\varepsilon \in \mathbb{I}_\varepsilon$ and $B_\varepsilon \sqcap C_\varepsilon \in \mathbb{I}_\varepsilon$, we have $(A_\varepsilon \sqcap C_\varepsilon) \notin \mathbb{I}_\varepsilon$. Similarly, we have $(A_\varepsilon \sqcap D_\varepsilon) \notin \mathbb{I}_\varepsilon$. Now, $A_\varepsilon = (A_\varepsilon \sqcap C_\varepsilon) \sqcup (A_\varepsilon \sqcap D_\varepsilon)$, which is a contradiction to A_ε being soft $\mathbb{I}_\varepsilon$-connected. Hence, $A_\varepsilon \sqcup B_\varepsilon$ is soft $\mathbb{I}_\varepsilon$-connected. □

Corollary 1. *Let (X_ε, τ) be a STS and let $\mathbb{I}_\varepsilon$ be a soft ideal on X_ε. If A_ε is a soft $\mathbb{I}_\varepsilon$-connected and $X_\varepsilon - A_\varepsilon \in \mathbb{I}_\varepsilon$, then X_ε is a soft $\mathbb{I}_\varepsilon$-connected.*

Now, we show that the product of soft $\mathbb{I}_\varepsilon$-connected spaces is soft $\mathbb{I}_\varepsilon$-connected under some conditions.

Theorem 8. *Let (X_ε, τ) be soft $\mathcal{J}_\varepsilon$-connected and (Y_ε, σ) be soft $\mathcal{H}_\varepsilon$-connected. Assume that $\mathcal{J}_\varepsilon \sqcap \tau$ is closed under arbitrary unions. If $\mathbb{I}_\varepsilon$ is a soft ideal in $X_\varepsilon \times Y_\varepsilon$ containing $p_1^{-1}(\mathcal{J}_\varepsilon)$ and $p_2^{-1}(\mathcal{H}_\varepsilon)$, then $X_\varepsilon \times Y_\varepsilon$ is soft $\mathbb{I}_\varepsilon$-connected.*

Proof. If $X_\varepsilon \in \mathcal{J}_\varepsilon$, then $X_\varepsilon \times Y_\varepsilon$ is in the soft ideal $\mathbb{I}_\varepsilon$ and hence $X_\varepsilon \times Y_\varepsilon$ is soft $\mathbb{I}_\varepsilon$-connected. Now suppose that $X_\varepsilon \notin \mathcal{J}_\varepsilon$ and $X_\varepsilon \times Y_\varepsilon$ is not soft $\mathbb{I}_\varepsilon$-connected. Then, $X_\varepsilon \times Y_\varepsilon = A_\varepsilon \sqcup B_\varepsilon$, where $A_\varepsilon, B_\varepsilon \notin \mathbb{I}_\varepsilon$, $A_\varepsilon \sqcap B_\varepsilon = \phi_\varepsilon$ such that $A_\varepsilon, B_\varepsilon$ are soft open sets in $X_\varepsilon \times Y_\varepsilon$. For each $y_\varepsilon \in Y_\varepsilon$, define $A_\varepsilon(y_\varepsilon) = \{x_\varepsilon \in X_\varepsilon : (x_\varepsilon, y_\varepsilon) \in A_\varepsilon\}$ and $B_\varepsilon(y_\varepsilon) = \{x_\varepsilon \in X_\varepsilon : (x_\varepsilon, y_\varepsilon) \in B_\varepsilon\}$.

Let $C_\varepsilon = \{y_\varepsilon \in Y_\varepsilon : A_\varepsilon(y_\varepsilon) \in \mathcal{J}_\varepsilon\}$ and $D_\varepsilon = \{y_\varepsilon \in Y_\varepsilon : B_\varepsilon(y_\varepsilon) \in \mathcal{J}_\varepsilon\}$.

Then, $X_\varepsilon = A_\varepsilon(y_\varepsilon) \sqcup B_\varepsilon(y_\varepsilon)$. For each y_ε, both $A_\varepsilon(y_\varepsilon)$ and $B_\varepsilon(y_\varepsilon)$ are soft open sets, and $A_\varepsilon(y_\varepsilon) \sqcap B_\varepsilon(y_\varepsilon) = \phi_\varepsilon$. As X_ε is soft $\mathcal{J}_\varepsilon$-connected, either $A_\varepsilon(y_\varepsilon) \in \mathcal{J}_\varepsilon$ or $B_\varepsilon(y_\varepsilon) \in \mathcal{J}_\varepsilon$. In fact, to each $y_\varepsilon \in Y_\varepsilon$, exactly one of $A_\varepsilon(y_\varepsilon)$ and $B_\varepsilon(y_\varepsilon) \in \mathcal{J}_\varepsilon$. $Y_\varepsilon = C_\varepsilon \sqcup D_\varepsilon$ and $C_\varepsilon \sqcap D_\varepsilon = \phi_\varepsilon$. Now, we claim that C_ε is a soft closed set. Fix $y_\varepsilon \in \overline{C}_\varepsilon$. If $A_\varepsilon(y_\varepsilon) \notin \mathcal{J}_\varepsilon$, then $A_\varepsilon(y_\varepsilon) \neq \phi_\varepsilon$. Since A_ε is a soft open set, for each $x_\varepsilon \in A_\varepsilon(y_\varepsilon)$, there exist soft neighborhoods $U_\varepsilon(x_\varepsilon)$ of x_ε and $V_\varepsilon(y_\varepsilon)$ of y_ε such that $(x_\varepsilon, y_\varepsilon) \in U_\varepsilon(x_\varepsilon) \times V_\varepsilon(y_\varepsilon) \sqsubseteq A_\varepsilon$. As $y_\varepsilon \in \overline{C}_\varepsilon$, there is one $y_\varepsilon' \in V_\varepsilon(y_\varepsilon) \sqcap C_\varepsilon$ so $U_\varepsilon(x_\varepsilon) \times \{y_\varepsilon'\} \sqsubseteq A_\varepsilon$ and hence $U_\varepsilon(x_\varepsilon) \sqsubseteq A_\varepsilon(y_\varepsilon')$. As $A_\varepsilon(y_\varepsilon') \in \mathcal{J}_\varepsilon$, we have $U_\varepsilon(x_\varepsilon) \in \mathcal{J}_\varepsilon$. Therefore, $A_\varepsilon(y_\varepsilon) \sqsubseteq \cup_{x_\varepsilon \in A_\varepsilon(y_\varepsilon)} U_\varepsilon(x_\varepsilon) \in \mathcal{J}_\varepsilon$ (by assumption).

Hence, $A_\varepsilon(y_\varepsilon) \in \mathcal{J}_\varepsilon$ and hence $y_\varepsilon = C_\varepsilon$. Thus, C_ε is a soft closed set. Similarly D_ε is a soft closed set. Since Y_ε is soft $\mathcal{H}_\varepsilon$-connected, we have $C_\varepsilon \in \mathcal{H}_\varepsilon$ or $D_\varepsilon \in \mathcal{H}_\varepsilon$.

Case (1): If $C_\varepsilon \in \mathcal{H}_\varepsilon$, then $X_\varepsilon \times C_\varepsilon \in \mathbb{I}_\varepsilon$. Take $M_\varepsilon = \sqcup\{B_\varepsilon(y_\varepsilon) : y_\varepsilon \in D_\varepsilon\} \in \mathcal{J}_\varepsilon \sqcap \tau$ (assumption). So, $M_\varepsilon \times Y_\varepsilon \in \mathbb{I}_\varepsilon$ and $(X_\varepsilon \times C_\varepsilon) \sqcup (M_\varepsilon \times Y_\varepsilon) \in \mathbb{I}_\varepsilon$. Fix $(x_\varepsilon, y_\varepsilon) \in B_\varepsilon$. If $y_\varepsilon \in C_\varepsilon$, then $(x_\varepsilon, y_\varepsilon) \in X_\varepsilon \times C_\varepsilon$. If $y_\varepsilon \notin C_\varepsilon$, then $y_\varepsilon \in D_\varepsilon$ and $x_\varepsilon \in B_\varepsilon(y_\varepsilon) \sqsubseteq M_\varepsilon$. Therefore, $(x_\varepsilon, y_\varepsilon) \in M_\varepsilon \times Y_\varepsilon$. Hence, $B_\varepsilon \sqsubseteq (X_\varepsilon \times C_\varepsilon) \sqcup (M_\varepsilon \times Y_\varepsilon)$. Hence, $B_\varepsilon \in \mathbb{I}_\varepsilon$, which contradicts the fact that $B_\varepsilon \notin \mathbb{I}_\varepsilon$.

Case (2): If $D_\varepsilon \in \mathcal{H}_\varepsilon$, then $X_\varepsilon \times D_\varepsilon \in \mathbb{I}_\varepsilon$. As in case (1), we obtain a contradiction. Thus, $X_\varepsilon \times Y_\varepsilon$ is soft $\mathbb{I}_\varepsilon$-connected. □

Now, we introduce the definition of soft connected component C_ε and discuss the product of soft $\mathbb{I}_\varepsilon$-connected spaces.

Definition 9. *Let (X_ε, τ) be an STS and $\mathbb{I}_\varepsilon$ be a soft ideal on X_ε. A soft connected component C_ε of X_ε with respect to τ is said to be a soft-ideal component in X_ε if $C_\varepsilon \in \mathbb{I}_\varepsilon$.*

Theorem 9. *Let (X_ε, τ) be soft $\mathcal{J}_\varepsilon$-connected and (Y_ε, σ) be soft $\mathcal{H}_\varepsilon$-connected. Assume that any union of soft-ideal components is a member of $\mathcal{J}_\varepsilon$. If $\mathbb{I}_\varepsilon$ is a soft ideal in $X_\varepsilon \times Y_\varepsilon$ containing $p_1^{-1}(\mathcal{J}_\varepsilon)$ and $p_2^{-1}(\mathcal{H}_\varepsilon)$, then $X_\varepsilon \times Y_\varepsilon$ is soft $\mathbb{I}_\varepsilon$-connected.*

Proof. If $X_\varepsilon \in \mathcal{J}_\varepsilon$, then $X_\varepsilon \times Y_\varepsilon$ is in the soft ideal $\mathbb{I}_\varepsilon$. Hence, $X_\varepsilon \times Y_\varepsilon$ is soft $\mathbb{I}_\varepsilon$-connected. Suppose that $X_\varepsilon \notin \mathcal{J}_\varepsilon$, and $X_\varepsilon \times Y_\varepsilon$ is not soft $\mathbb{I}_\varepsilon$-connected. Then, $X_\varepsilon \times Y_\varepsilon = A_\varepsilon \sqcup B_\varepsilon$, where $A_\varepsilon, B_\varepsilon \notin \mathbb{I}_\varepsilon$, $A_\varepsilon \sqcap B_\varepsilon = \phi_\varepsilon$ and $A_\varepsilon, B_\varepsilon$ are soft open sets. For every soft component C_ε of X_ε and D_ε of Y_ε, $C_\varepsilon \times D_\varepsilon$ is a soft connected subset of $X_\varepsilon \times Y_\varepsilon$ and hence $C_\varepsilon \times D_\varepsilon \sqsubseteq A_\varepsilon$ or $C_\varepsilon \times D_\varepsilon \sqsubseteq B_\varepsilon$. . . . . . (1). For every component D_ε of Y_ε, write

$$\widetilde{A}_\varepsilon = \cup\{C_\varepsilon : C_\varepsilon \text{ is a soft component of } X_\varepsilon \text{ and } C_\varepsilon \times D_\varepsilon \sqsubseteq A_\varepsilon\},$$
$$\widetilde{B}_\varepsilon = \cup\{C_\varepsilon : C_\varepsilon \text{ is a soft component of } X_\varepsilon \text{ and } C_\varepsilon \times D_\varepsilon \sqsubseteq B_\varepsilon\}.$$

Now, we claim that $\widetilde{A}_\varepsilon$ is a soft open set. Let $x_\varepsilon \in \widetilde{A}_\varepsilon$. Then, there exists a soft component C_ε of X_ε such that $x_\varepsilon \in C_\varepsilon$ and $C_\varepsilon \times D_\varepsilon \sqsubseteq A_\varepsilon$. Fix $y_\varepsilon \in D_\varepsilon$. Therefore, $(x_\varepsilon, y_\varepsilon) \in C_\varepsilon \times D_\varepsilon \sqsubseteq A_\varepsilon$. Since A_ε is soft open, there exist soft neighborhoods $U_\varepsilon(x_\varepsilon)$ and $V_\varepsilon(y_\varepsilon)$ of x_ε and y_ε, respectively, such that $U_\varepsilon(x_\varepsilon) \times V_\varepsilon(y_\varepsilon) \sqsubseteq A_\varepsilon$. If $x_\varepsilon \in \overline{\widetilde{B}}_\varepsilon$, then $U_\varepsilon(x_\varepsilon) \sqcap \widetilde{B}_\varepsilon \neq \phi_\varepsilon$. Let $x_\varepsilon' \in U_\varepsilon(x_\varepsilon) \sqcap \widetilde{B}_\varepsilon$, i.e., $x_\varepsilon' \in U_\varepsilon(x_\varepsilon) \sqcap C_\varepsilon'$, for some soft component C_ε', where $C_\varepsilon' \times D_\varepsilon \sqsubseteq B_\varepsilon$. Let $(x_\varepsilon', y_\varepsilon) \in U_\varepsilon(x_\varepsilon') \times V_\varepsilon'(y_\varepsilon) \sqsubseteq B_\varepsilon$, where $U_\varepsilon(x_\varepsilon'), V_\varepsilon'(y_\varepsilon)$ are some soft neighborhoods of x_ε' and y_ε, respectively. Then, $(x_\varepsilon', y_\varepsilon) \in (U_\varepsilon(x_\varepsilon) \sqcap U_\varepsilon(x_\varepsilon')) \times (V_\varepsilon(y_\varepsilon) \sqcap V_\varepsilon'(y_\varepsilon)) \sqsubseteq A_\varepsilon \sqcap B_\varepsilon$, which is a contradiction to $A_\varepsilon \sqcap B_\varepsilon = \phi_\varepsilon$. Therefore, $x_\varepsilon \in \widetilde{A}_\varepsilon$ implies that x_ε is not a limit of the soft point of $\widetilde{B}_\varepsilon$. That is, $\widetilde{A}_\varepsilon$ is a soft open set. Similarly, $\widetilde{B}_\varepsilon$ is a soft open set. Thus, $X_\varepsilon = \widetilde{A}_\varepsilon \sqcup \widetilde{B}_\varepsilon$, $\widetilde{A}_\varepsilon$, and $\widetilde{B}_\varepsilon$ are soft open sets, so exactly one of $\widetilde{A}_\varepsilon$ or $\widetilde{B}_\varepsilon$ is in $\mathcal{J}_\varepsilon$, because $X_\varepsilon \notin \mathcal{J}_\varepsilon$.

$$\text{Let } D'_\varepsilon = \{D_\varepsilon \sqsubseteq Y_\varepsilon : D_\varepsilon \text{ is a soft component of } Y_\varepsilon \text{ and } \widetilde{A}_\varepsilon \in \mathcal{J}_\varepsilon\},$$
$$D''_\varepsilon = \{D_\varepsilon \sqsubseteq Y_\varepsilon : D_\varepsilon \text{ is a soft component of } Y_\varepsilon \text{ and } \widetilde{B}_\varepsilon \in \mathcal{J}_\varepsilon\}.$$

Write $H_\varepsilon = \sqcup_{D_\varepsilon \in D'_\varepsilon} D_\varepsilon$ and $K_\varepsilon = \sqcup_{D_\varepsilon \in D''_\varepsilon} D_\varepsilon$. Then, $Y_\varepsilon = H_\varepsilon \sqcup K_\varepsilon$ and $H_\varepsilon \sqcap K_\varepsilon = \phi_\varepsilon$.

We claim that H_ε is a soft closed set. Fix $h_\varepsilon \in \overline{H}_\varepsilon$. Let D_ε be the soft component of Y_ε such that $h_\varepsilon \in D_\varepsilon$. Suppose $h_\varepsilon \notin H_\varepsilon$. Then, $D_\varepsilon \notin D'_\varepsilon$ implies that $\widetilde{A}_\varepsilon \notin \mathcal{J}_\varepsilon$, so $\widetilde{B}_\varepsilon \in \mathcal{J}_\varepsilon$. By (1) and our assumption, there is a soft component C_ε of X_ε such that $C_\varepsilon \notin \mathcal{J}_\varepsilon$, $C_\varepsilon \times D_\varepsilon \sqsubseteq A_\varepsilon$. Fix a member $c_\varepsilon \in C_\varepsilon$. Then, $(c_\varepsilon, h_\varepsilon) \in A_\varepsilon$. Since A_ε is a soft open set, there exist soft neighborhoods $U_\varepsilon(c_\varepsilon)$ and $V_\varepsilon(h_\varepsilon)$ of $c_\varepsilon, h_\varepsilon$ in $X_\varepsilon, Y_\varepsilon$, respectively, such that $(c_\varepsilon, h_\varepsilon) \in U_\varepsilon(c_\varepsilon) \times V_\varepsilon(h_\varepsilon) \sqsubseteq A_\varepsilon$. So there is a member $h'_\varepsilon \in V_\varepsilon(h_\varepsilon) \sqcap H_\varepsilon$ and there is a component H'_ε of Y_ε such that $h'_\varepsilon \in H'_\varepsilon$ and $\widetilde{A'}_\varepsilon = \cup\{C_\varepsilon : C_\varepsilon \text{ is a soft component of } X_\varepsilon \text{ and } C_\varepsilon \times H'_\varepsilon \sqsubseteq A_\varepsilon\} \in \mathcal{J}_\varepsilon$. Then, $(c_\varepsilon, h'_\varepsilon) \in U_\varepsilon(c_\varepsilon) \times V_\varepsilon(h_\varepsilon) \sqsubseteq A_\varepsilon$. Therefore, $C_\varepsilon \times H'_\varepsilon \sqsubseteq A_\varepsilon$ and $C_\varepsilon \in \mathcal{J}_\varepsilon$, because $C_\varepsilon \sqsubseteq \widetilde{A'}_\varepsilon \in \mathcal{J}_\varepsilon$. This contradicts $C_\varepsilon \notin \mathcal{J}_\varepsilon$. Therefore, $h_\varepsilon \in H_\varepsilon$, i.e., $H_\varepsilon = \overline{H}_\varepsilon$ and H_ε is a soft closed set. Similarly K_ε is a soft closed set. Thus, $Y_\varepsilon = H_\varepsilon \sqcup K_\varepsilon$, where $H_\varepsilon, K_\varepsilon$ are soft open sets and $H_\varepsilon \sqcap K_\varepsilon = \phi_\varepsilon$. Since Y_ε is soft $\mathcal{H}_\varepsilon$-connected, either $H_\varepsilon \in \mathcal{H}_\varepsilon$ or $K_\varepsilon \in \mathcal{H}_\varepsilon$. Without sacrificing generality, we may assume $H_\varepsilon \in \mathcal{H}_\varepsilon$. Then, $X_\varepsilon \times H_\varepsilon \in \mathbb{I}_\varepsilon$.

Take $M_\varepsilon = \sqcup_{D_\varepsilon \in D''_\varepsilon} \widetilde{B}_\varepsilon \in \mathcal{J}_\varepsilon$ (by assumption). So, $M_\varepsilon \times Y_\varepsilon \in \mathbb{I}_\varepsilon$ and hence $(X_\varepsilon \times H_\varepsilon) \sqcup (M_\varepsilon \times Y_\varepsilon) \in \mathbb{I}_\varepsilon$. It is enough to prove that $B_\varepsilon \sqsubseteq (X_\varepsilon \times H_\varepsilon) \sqcup (M_\varepsilon \times Y_\varepsilon)$. Fix $(x_\varepsilon, y_\varepsilon) \in B_\varepsilon$. Then, there exist soft components C_ε and D_ε such that $(x_\varepsilon, y_\varepsilon) \in C_\varepsilon \times D_\varepsilon \sqsubseteq B_\varepsilon$. If $y_\varepsilon \in H_\varepsilon$, then $(x_\varepsilon, y_\varepsilon) \in X_\varepsilon \times H_\varepsilon$. If $y_\varepsilon \notin H_\varepsilon$, then $y_\varepsilon \in K_\varepsilon = \sqcup_{D_\varepsilon \in D''_\varepsilon} D_\varepsilon$ and hence $x_\varepsilon \in C_\varepsilon \sqsubseteq \widetilde{B}_\varepsilon \sqsubseteq M_\varepsilon$ for some $D_\varepsilon \in D''_\varepsilon$. Therefore, $(x_\varepsilon, y_\varepsilon) \in M_\varepsilon \times Y_\varepsilon$. Hence, $B_\varepsilon \sqsubseteq (X_\varepsilon \times H_\varepsilon) \sqcup (M_\varepsilon \times Y_\varepsilon) \in \mathbb{I}_\varepsilon$. This is a contradiction to $B_\varepsilon \notin \mathbb{I}_\varepsilon$. Hence, $X_\varepsilon \times Y_\varepsilon$ is soft $\mathbb{I}_\varepsilon$-connected. □

4. Strongly Soft $\mathbb{I}_\varepsilon$-Connected

We present and define the idea of strongly soft $\mathbb{I}_\varepsilon$-connected spaces and characterize the concept of strongly soft $\mathbb{I}_\varepsilon$-connected spaces in this part. Additionally, we show that the image of the strongly soft $\mathbb{I}_\varepsilon$-connected set is preserved by the soft continuous surjective maps.

Definition 10. *Let (X_ε, τ) be an STS with a soft ideal $\mathbb{I}_\varepsilon$ on X_ε. A subset A_ε of X_ε is called strongly soft $\mathbb{I}_\varepsilon$-connected if there is a soft connected subset B_ε of X_ε such that $A_\varepsilon = B_\varepsilon \sqcup C_\varepsilon$, where $C_\varepsilon \in \mathbb{I}_\varepsilon$.*

Now, we show that every strongly soft $\mathbb{I}_\varepsilon$-connected set is soft $\mathbb{I}_\varepsilon$-connected.

Theorem 10. *Assume that (X_ε, τ) is an STS with a soft ideal $\mathbb{I}_\varepsilon$ on X_ε. Let $(X_\varepsilon, \tau, \mathbb{I}_\varepsilon)$ be strongly soft $\mathbb{I}_\varepsilon$-connected. Then, $(X_\varepsilon, \tau, \mathbb{I}_\varepsilon)$ is soft $\mathbb{I}_\varepsilon$-connected.*

Proof. Suppose that $(X_\varepsilon, \tau, \mathbb{I}_\varepsilon)$ is strongly soft $\mathbb{I}_\varepsilon$-connected and $X_\varepsilon = B_\varepsilon \sqcup C_\varepsilon$, where B_ε is soft connected and $C_\varepsilon \in \mathbb{I}_\varepsilon$. Let $X_\varepsilon = D'_\varepsilon \sqcup D''_\varepsilon$, where D'_ε and D''_ε are soft open sets and $D'_\varepsilon \sqcap D''_\varepsilon = \phi_\varepsilon$. Then, $B_\varepsilon = (D'_\varepsilon \sqcap B_\varepsilon) \sqcup (D''_\varepsilon \sqcap B_\varepsilon)$ and $D'_\varepsilon \sqcap B_\varepsilon = \phi_\varepsilon$ or $D''_\varepsilon \sqcap B_\varepsilon = \phi_\varepsilon$. Then, $D'_\varepsilon \sqsubseteq X_\varepsilon - B_\varepsilon$ or $D''_\varepsilon \sqsubseteq X_\varepsilon - B_\varepsilon$. Therefore, $D'_\varepsilon \sqsubseteq C_\varepsilon$ or $D''_\varepsilon \sqsubseteq C_\varepsilon$ and $D'_\varepsilon \in \mathbb{I}_\varepsilon$ or $D''_\varepsilon \in \mathbb{I}_\varepsilon$. Hence, X_ε is soft $\mathbb{I}_\varepsilon$-connected. □

The converse of the above theorem is not true.

Example 4. *Let $X = \{h_1, h_2, h_3, h_4, h_5\}$, $\varepsilon = \{\varepsilon_1\}$, and $\tau = \{X_\varepsilon, \phi_\varepsilon, A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon\}$, where A_ε, B_ε, C_ε, and D_ε are soft sets over X defined as follows: $A(\varepsilon_1) = \{h_2\}$, $B(\varepsilon_1) = \{h_3, h_4\}$, $C(\varepsilon_1) = \{h_2, h_3, h_4\}$, and $D(\varepsilon_1) = \{h_1, h_3, h_4, h_5\}$. Then, τ defines a soft topology on X. Let $\mathbb{I}_\varepsilon = \{\phi_\varepsilon, I_\varepsilon\}$, where I_ε is a soft set over X defined by $I(\varepsilon_1) = \{h_2\}$. Then, $\mathbb{I}_\varepsilon$ defines a soft ideal on X. Clearly, X_ε is not soft connected and not soft $\mathbb{I}_\varepsilon$-connected. Now, if $Y = \{h_1, h_3, h_4, h_5\} \sqsubseteq X$, $\varepsilon = \{\varepsilon_1\}$ and $\tau_Y = \{Y_\varepsilon, \phi_\varepsilon, E_\varepsilon\}$, where $E(\varepsilon_1) = \{h_3, h_4\}$, then τ_Y defines a soft topology on Y.*

Then, Y_ε is soft connected and $X_\varepsilon = Y_\varepsilon \sqcup E_\varepsilon$ and $E_\varepsilon \in \mathbb{I}_\varepsilon$. Therefore, X_ε is strongly soft $\mathbb{I}_\varepsilon$-connected and not soft $\mathbb{I}_\varepsilon$-connected.

Remark 1. *Let (X_ε, τ) be an STS and let $\mathbb{I}_\varepsilon$ be a soft ideal on X_ε. If A_ε is a strongly soft $\mathbb{I}_\varepsilon$-connected set and $B_\varepsilon \in \mathbb{I}_\varepsilon$, then $A_\varepsilon \sqcup B_\varepsilon$ is strongly soft $\mathbb{I}_\varepsilon$-connected.*

We obtain equivalent conditions for connectedness in ideal soft spaces in the following corollary.

Corollary 2. *Let (X_ε, τ) be an STS and let $\mathbb{I}_\varepsilon$ be a soft ideal on X_ε with $\tau \sqcap \mathbb{I}_\varepsilon = \phi_\varepsilon$. Then, the subsequent statements are equivalent. X_ε is:*

1. *Soft connected.*
2. *Soft $\mathbb{I}_\varepsilon$-connected.*
3. *Strongly soft $\mathbb{I}_\varepsilon$-connected.*

Example 5. *Let $X = \{h_1, h_2, h_3\}$, $\varepsilon = \{\varepsilon_1, \varepsilon_2\}$, and $\tau = \{X_\varepsilon, \phi_\varepsilon, A_\varepsilon, B_\varepsilon\}$, where A_ε and B_ε are soft sets over X defined as follows: $A(\varepsilon_1) = \{h_2\}$, $A(\varepsilon_2) = \{h_1, h_3\}$, and $B(\varepsilon_1) = \{h_1, h_3\}$, $B(\varepsilon_2) = \{h_2\}$. Then, τ defines a soft topology on X. Let $\mathbb{I}_\varepsilon = \{\phi_\varepsilon, I_\varepsilon\}$ be a soft set over X defined by $I(\varepsilon_1) = \{h_2\}$, $I(\varepsilon_2) = \{h_1\}$. Then, $\mathbb{I}_\varepsilon$ defines a soft ideal on X. Clearly, X_ε is not soft connected since $\tau \sqcap \mathbb{I}_\varepsilon = \phi_\varepsilon$. Therefore, X_ε is not strongly soft $\mathbb{I}_\varepsilon$-connected.*

It is well known that the union of two connected sets is connected if $A_\varepsilon \sqcap B_\varepsilon \neq \phi_\varepsilon$. This result can be generalized as follows.

Theorem 11. *Let (X_ε, τ) be an STS and let $\mathbb{I}_\varepsilon$ be a soft ideal on X_ε. If B_ε and A_ε are strongly soft $\mathbb{I}_\varepsilon$-connected sets such that $A_\varepsilon \sqcap B_\varepsilon \notin \mathbb{I}_\varepsilon$, then $A_\varepsilon \sqcup B_\varepsilon$ is strongly soft $\mathbb{I}_\varepsilon$-connected.*

Proof. Let B_ε and A_ε be strongly soft $\mathbb{I}_\varepsilon$-connected sets. Then, $A_\varepsilon = H_\varepsilon \sqcup K_\varepsilon$ and $B_\varepsilon = M_\varepsilon \sqcup N_\varepsilon$, where H_ε and M_ε are soft connected sets and $K_\varepsilon, N_\varepsilon \in \mathbb{I}_\varepsilon$. Since $A_\varepsilon \sqcap B_\varepsilon \notin \mathbb{I}_\varepsilon$, the $A_\varepsilon, B_\varepsilon, H_\varepsilon, M_\varepsilon \notin \mathbb{I}_\varepsilon$. Let $D_\varepsilon^{'} = (A_\varepsilon \sqcap B_\varepsilon) \sqcap K_\varepsilon$ and $D_\varepsilon^{''} = (A_\varepsilon \sqcap B_\varepsilon) \sqcap N_\varepsilon$. Then, $D_\varepsilon^{'}, D_\varepsilon^{''} \in \mathbb{I}_\varepsilon$. Therefore, $D_\varepsilon^{'} \sqcup D_\varepsilon^{''} \in \mathbb{I}_\varepsilon$. Now, put $D_\varepsilon = (A_\varepsilon \sqcap B_\varepsilon) \setminus (D_\varepsilon^{'} \sqcup D_\varepsilon^{''})$. Then, $D_\varepsilon \notin \mathbb{I}_\varepsilon$ because $A_\varepsilon \sqcap B_\varepsilon \notin \mathbb{I}_\varepsilon$ and $D_\varepsilon^{'}, D_\varepsilon^{''} \in \mathbb{I}_\varepsilon$. Since $D_\varepsilon \sqsubseteq H_\varepsilon$ and $D_\varepsilon \sqsubseteq M_\varepsilon$, then $D_\varepsilon \sqsubseteq M_\varepsilon \sqcap H_\varepsilon \notin \mathbb{I}_\varepsilon$, which implies that $M_\varepsilon \sqcap H_\varepsilon \neq \phi_\varepsilon$. Hence, $M_\varepsilon \sqcup H_\varepsilon$ is soft connected and $A_\varepsilon \sqcup B_\varepsilon = (M_\varepsilon \sqcup H_\varepsilon) \sqcup C_\varepsilon$, where $C_\varepsilon \sqsubseteq (A_\varepsilon \setminus H_\varepsilon) \sqcup (B_\varepsilon \setminus M_\varepsilon) \sqsubseteq (K_\varepsilon \sqcup N_\varepsilon) \in \mathbb{I}_\varepsilon$. Thus, $C_\varepsilon \in \mathbb{I}_\varepsilon$. Hence, $A_\varepsilon \sqcup B_\varepsilon$ is strongly soft $\mathbb{I}_\varepsilon$-connected. □

Theorem 12. *Let (X_ε, τ) be an STS and let $\mathbb{I}_\varepsilon$ be a soft ideal on X_ε. Let $\overline{I} \in \mathbb{I}_\varepsilon$ for all $I \in \mathbb{I}_\varepsilon$. If $A_\varepsilon \sqsubseteq B_\varepsilon \sqsubseteq \overline{A}_\varepsilon$ such that A_ε is strongly soft $\mathbb{I}_\varepsilon$-connected, then B_ε is also strongly soft $\mathbb{I}_\varepsilon$-connected for all B_ε. In particular, $\overline{A}_\varepsilon$ is strongly soft $\mathbb{I}_\varepsilon$-connected.*

Proof. Suppose A_ε is strongly soft $\mathbb{I}_\varepsilon$-connected. Then, $A_\varepsilon = C_\varepsilon \sqcup D_\varepsilon$, where C_ε is soft connected and $D_\varepsilon \in \mathbb{I}_\varepsilon$. Since $A_\varepsilon \sqsubseteq B_\varepsilon \sqsubseteq \overline{A}_\varepsilon$ and $A_\varepsilon = C_\varepsilon \sqcup D_\varepsilon \sqsubseteq B_\varepsilon$, we have $B_\varepsilon = (C_\varepsilon \sqcap B_\varepsilon) \sqcap (D_\varepsilon \sqcap B_\varepsilon)$, where $C_\varepsilon \sqcap B_\varepsilon$ is soft connected as $C_\varepsilon \sqsubseteq (\overline{C}_\varepsilon \sqcap B_\varepsilon) \sqsubseteq \overline{C}_\varepsilon$ and $\overline{D}_\varepsilon \sqcap B_\varepsilon \in \mathbb{I}_\varepsilon$. Hence, B_ε is also strongly soft $\mathbb{I}_\varepsilon$-connected. As a particular case when A_ε is strongly soft $\mathbb{I}_\varepsilon$-connected, A_ε is strongly soft $\mathbb{I}_\varepsilon$-connected for all $\overline{I} \in \mathbb{I}_\varepsilon$. □

It is well known that the continuous image of a connected set is connected. This result can be generalized as follows.

Theorem 13. *Let (X_ε, τ) and (Y_ε, σ) be a STSs and let $\mathbb{I}_\varepsilon$ be a soft ideal on X_ε. Let $f_{pu} : (X_\varepsilon, \tau) \to (Y_\varepsilon, \sigma)$ be a soft continuous surjection. If (X_ε, τ) is a strongly soft $\mathbb{I}_\varepsilon$-connected, then (Y_ε, σ) is a strongly soft $f_{pu}(\mathbb{I}_\varepsilon)$-connected.*

Proof. Let $f_{pu} : (X_\varepsilon, \tau) \to (Y_\varepsilon, \sigma)$ be a soft continuous surjective map and (X_ε, τ) be strongly soft $\mathbb{I}_\varepsilon$-connected. $X_\varepsilon = B_\varepsilon \sqcup C_\varepsilon$, where B_ε is soft connected and $C_\varepsilon \in \mathbb{I}_\varepsilon$. Therefore,

$Y_\varepsilon = f_{pu}(X_\varepsilon) = f_{pu}(B_\varepsilon \sqcup C_\varepsilon) = f_{pu}(B_\varepsilon) \sqcup f_{pu}(C_\varepsilon)$, where $f_{pu}(B_\varepsilon)$ is soft connected and $f_{pu}(C_\varepsilon) \in f_{pu}(\mathbb{I}_\varepsilon)$. Thus, (Y_ε, σ) is strongly soft $f_{pu}(\mathbb{I}_\varepsilon)$-connected. □

Theorem 14. *Let (X_ε, τ) be strongly soft $\mathcal{J}_\varepsilon$-connected and (Y_ε, σ) be strongly soft $\mathcal{H}_\varepsilon$-connected. If $\mathbb{I}_\varepsilon$ is a soft ideal in $X_\varepsilon \times Y_\varepsilon$ containing $p_1^{-1}(\mathcal{J}_\varepsilon)$ and $p_2^{-1}(\mathcal{H}_\varepsilon)$, then $X_\varepsilon \times Y_\varepsilon$ is strongly soft $\mathbb{I}_\varepsilon$-connected.*

Proof. Suppose that X_ε is strongly soft $\mathcal{J}_\varepsilon$-connected and Y_ε is strongly soft $\mathcal{H}_\varepsilon$-connected. Then, $X_\varepsilon = A_\varepsilon \sqcup H_\varepsilon$ and $X_\varepsilon = B_\varepsilon \sqcup K_\varepsilon$, such that A_ε and B_ε are soft connected subsets of X_ε and Y_ε, respectively, and $H_\varepsilon, K_\varepsilon \in \mathbb{I}_\varepsilon$. Therefore, $X_\varepsilon \times Y_\varepsilon = (A_\varepsilon \times B_\varepsilon) \sqcup [(H_\varepsilon \times Y_\varepsilon) \sqcup (X_\varepsilon \times K_\varepsilon)]$. Since $A_\varepsilon \times B_\varepsilon$ is soft connected with $\tau \times \sigma$ and $H_\varepsilon \times Y_\varepsilon \in \mathbb{I}_\varepsilon$ and $X_\varepsilon \times K_\varepsilon \in \mathbb{I}_\varepsilon$, then $(H_\varepsilon \times Y_\varepsilon) \sqcup (X_\varepsilon \times K_\varepsilon) \in \mathbb{I}_\varepsilon$. Thus, $X_\varepsilon \times Y_\varepsilon$ is strongly soft $\mathbb{I}_\varepsilon$-connected. □

5. Conclusions and Future Work

Shabir and Naz [11] and Çağman et al. [34] have individually shown how a soft topology on a universal set expands the classical (crisp) topology. Investigating this topological generalization is starting to seem intriguing. There are a lot of approaches for generating soft topologies available in the literature that can be used. We broadened our understanding of soft topology by exploring the concepts of soft connected, soft-ideal connected, and strongly soft-ideal connected spaces. The connection of soft-ideal spaces is the foundation of this investigation. We discussed some basic activities in pliable ideal places. A description of a soft-ideal connected space is given, along with an outline of its attributes. These are preliminary results, and additional investigations will examine more aspects of soft-ideal connected spaces. Prospects for further contributions to this trend are created by our study in terms of soft primal topologies and fuzzy soft topologies in soft and classical environments, as well as soft connectedness along with idea structures using generalized rough approximation spaces. This is accomplished by combining these two strategies.

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References

1. Molodtsov, D. Soft set theory-first results. *Comput. Math. Appl.* **1999**, *37*, 19–31. [CrossRef]
2. Molodtsov, D.; Leonov, V.Y.; Kovkov, D.V. Soft sets technique and its application. *Nechetkie Sist. Myagkie Vychisleniya* **2006**, *1*, 8–39.
3. Al-Saadi, H.S.; Aygün, H.; Al-Omari, A. Some notes on soft hyperconnected spaces. *J. Anal.* **2020**, *28*, 351–362. [CrossRef]
4. Al-Omari, A. Soft topology in ideal topological spaces. *Hacet. J. Math. Stat.* **2019**, *48*, 1277–1285. [CrossRef]
5. Al Ghour, S.; Al-Mufarrij, J. Between Soft Complete Continuity and Soft Somewhat-Continuity. *Symmetry* **2023**, *15*, 2056. [CrossRef]
6. Hussain, S. A note on soft connectedness. *J. Egypt. Math. Soc.* **2015**, *23*, 6–11. [CrossRef]
7. Hussain, S. Binary soft connected spaces and an application of binary soft sets in decision making problem. *Fuzzy Inf. Eng.* **2019,** *11*, 506–521. [CrossRef]
8. Al Ghour, S.; Ameen, Z.A. On soft submaximal spaces. *Heliyon* **2022**, *8*, e10574. [CrossRef]
9. Jankovic, D.; Hamlett, T.R. New topologies from old via ideals. *Am. Math. Mon.* **1990**, *97*, 295–310. [CrossRef]
10. Maji, P.K.; Biswas, R.; Roy, A.R. Soft set theory. *Comput. Math. Appl.* **2003**, *45*, 555–562. [CrossRef]
11. Shabir, M.; Naz, M. On soft topolgical spaces. *Comput. Math. Appl.* **2011**, *61*, 1786–1799. [CrossRef]
12. Kandil, A.; Tantawy, O.A.E.; El-Sheikh, S.A.; Abd El-latif, A.M. Soft ideal theory, Soft local function and generated soft topological spaces. *Appl. Math. Inf. Sci.* **2014**, *8*, 1595–1603. [CrossRef]
13. Kandil, A.; Tantawy, O.A.E.; El-Sheikh, S.A.; Abd El-latif, A.M. Soft Connectedness Via Soft Ideals. *J. New Results Sci.* **2014**, *4*, 90–108.
14. Lin, F. Soft connected spaces and soft paracompact spaces. *Int. J. Math. Sci. Eng.* **2013**, *7*, 667–681.

15. Al-shami, T.M.; Mhemdi, A.; Abu-Gdairi, R.; El-Shafei, M.E. Compactness and connectedness via the class of soft somewhat open sets. *AIMS Math.* **2023**, *8*, 815–840. [CrossRef]
16. Al-shami, T.M.; Abo-Tabl, E.-S.A. Connectedness and Local Connectedness on Infra Soft Topological Spaces. *Mathematics* **2021**, *9*, 1759. [CrossRef]
17. Atanassov, K.T. Intuitionistic fuzzy sets. In Proceedings of the VII ITKR's Session, Sofia, Bulgaria, 20–23 June 1983.
18. Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
19. Atanassov, K.T. Review and New Results on Intuitionistic Fuzzy Sets, Mathematical Foundations of Artificial Intelligence Seminar, Sofia, 1988. preprint IM-MFAIS1-88; reprinted in *Int. J. Bioautomation* **2016**, *20*, S7–S16.
20. Yager, R.R. Pythagorean fuzzy subsets. In Proceedings of the 2013 Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, AB, Canada, 24–28 June 2013; pp. 57–61.
21. Yager, R.R. Pythagorean membership grades in multi-criteria decision making. *IEEE Trans. Fuzzy Syst.* **2014**, *22*, 958–965. [CrossRef]
22. Yager, R.R. Generalized orthopair fuzzy sets. *IEEE Trans. Fuzzy Syst.* **2016**, *25*, 1222–1230. [CrossRef]
23. Saber, Y.; Alsharari, F.; Smarandache, F.; Abdel-Sattar, M. Connectedness and Stratification of Single-Valued Neutrosophic Topological Spaces. *Symmetry* **2020**, *12*, 1464. [CrossRef]
24. Li, L.; Zhang, R.T.; Wang, J.; Shang, X.P.; Bai, K.Y. A novel approach to multi-Attribute group decision-making with q-rung picture linguistic information. *Symmetry* **2018**, *10*, 172. [CrossRef]
25. Ajmal, N.; Kohli, J.K. Connectedness and local connectedness in fuzzy topological spaces. *Fuzzy Sets Syst.* **1994**, *61*, 231–237. [CrossRef]
26. Thakur, S.S.; Rajput, A.S. *P*-connectedness between soft sets. *Facta Univ. Ser. Math. Inform.* **2016**, *31*, 335–347. [CrossRef]
27. Thakur, S.S.; Rajput, A.S. Connectedness between soft sets. *New Math. Nat. Comput.* **2018**, *14*, 53–71. [CrossRef]
28. Shahzadi, G.; Akram, M.; Al-Kenani, A.N. Decision-making approach under Pythagorean fuzzy Yager weighted operators. *Mathematics* **2020**, *8*, 70. [CrossRef]
29. Ayyildiz, E. A novel Pythagorean fuzzy multi-criteria decision-making methodology for e-scooter charging station locationselection. *Transp. Res. Part D Transp. Environ.* **2022**, *111*, 103459. [CrossRef]
30. Zorlutuna, I.; Akdağ, M.; Min, W.K.; Atmaca, S. Remarks on soft topological spaces. *Ann. Fuzzy Math. Inform.* **2012**, *3*, 171–185.
31. Al-Omari, A.; Acharjee, S.; Özkoç, M. A new operator of primal topological spaces. *Mathematica* **2023**, *65*, 175–183. [CrossRef]
32. Ali, M.I.; Feng, F.; Liu, X.; Min, W.K.; Shabir, M. On some new operations in soft set theory. *Comput. Math. Appl.* **2009**, *57*, 1547–1553. [CrossRef]
33. Al-Omari, A.; Alqurashi, W. Hyperconnectedness and Resolvability of Soft Ideal Topological Spaces. *Mathematics* **2023**, *11*, 4697. [CrossRef]
34. Çagman, N.; Karataş, S.; Enginoglu, S. Soft topology. *Comput. Math. Appl.* **2011**, *62*, 351–358. [CrossRef]