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Estimates of Some Coefficient Functionals for Close-to-Convex Functions

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Abstract: For a given starlike function $F_\alpha = \frac{z}{1-\alpha z^2}$, $\alpha \in [-1,1]$, the class $\mathcal{C}_0(F_\alpha)$ is defined as follows: an analytic normalized function f belongs to $\mathcal{C}_0(F_\alpha)$ if it satisfies $\mathrm{Re}\frac{zf'(z)}{F_\alpha(z)} > 0$ in the open unit disk Δ. The condition defining this class can be rewritten in the following equivalent form $\mathrm{Re}\{(1-\alpha z^2)f'(z)\} > 0$, $z \in \Delta$. The family $\mathcal{C}_0(F_\alpha)$ is a subclass of the class of close-to-convex functions. The main aim of this paper is to maximize the modulus of a functional which is a linear combination with coefficients symmetric with respect to zero and is defined on the subfamily of $\mathcal{C}_0(F_\alpha)$ of functions with a fixed second coefficient in its Taylor series expansion.

Keywords: close-to-convex function; coefficient problems; successive coefficients; bounds of the coefficient functionals

MSC: 30C45

1. Introduction

Let $\mathcal{A}$ be the class of functions f analytic in the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ normalized by the condition $f(0) = 0 = f'(0) - 1$. Each function f belonging to the class $\mathcal{A}$ has the Taylor series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \Delta. \tag{1}$$

The class of analytic functions of the form

$$q(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad z \in \Delta \tag{2}$$

having a positive real part in Δ (called Carathéodory functions) will be denoted by $\mathcal{P}$. Further, by $\mathcal{S}^*$ we denote the subclass of $\mathcal{A}$ consisting of univalent starlike functions i.e., functions $f \in \mathcal{A}$ such that $\mathrm{Re}\{zf'(z)/f(z)\} > 0$ in the unit disc Δ.

A function $f \in \mathcal{A}$ that satisfies condition $\mathrm{Re}(1 + zf''(z)/f'(z)) > 0$, $z \in \Delta$ is said to be a convex function. By $\mathcal{C}$ we denote the so-called class of close-to-convex functions, that is, the subclass of $\mathcal{A}$ which consists of functions f that satisfy $\mathrm{Re}\frac{f'(z)}{g'(z)} > 0$, $z \in \Delta$ with some convex function g (see [1,2]). The topic of close-to-convex functions is still a topic of interest for researchers (see, for example, [3–6]).

For given $\beta \in (-\pi/2, \pi/2)$ and $g \in \mathcal{S}^*$ the classes $\mathcal{C}_\beta(g)$ are defined as follows ([1]): a function $f \in \mathcal{A}$ belongs to $\mathcal{C}_\beta(g)$ if

$$\mathrm{Re}\left\{\frac{e^{i\beta}zf'(z)}{g(z)}\right\} > 0, \quad z \in \Delta. \tag{3}$$

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Such a function f will be called close-to-convex with argument β with respect to g. Let

$$\mathcal{C}_\beta = \bigcup_{g\in\mathcal{S}^*} \mathcal{C}_\beta(g).$$

It is obvious that

$$\bigcup_{\beta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)} \mathcal{C}_\beta = \mathcal{C}. \tag{4}$$

By varying the function g and the real parameter β in (3), we obtain some interesting subfamilies of the class $\mathcal{C}$. For example,

(i) By selecting $g(z) = \frac{z}{(1-\zeta_1 z)(1-\zeta_2 z)}$, $\zeta_1, \zeta_2 \in \mathbb{C}$, $|\zeta_1| \le 1$, $|\zeta_2| \le 1$, the class $\mathcal{C}_\beta(g)$ leads to the family $\mathcal{C}(\beta, \zeta_1, \zeta_2)$ that was introduced by Lecko in 1994 [7] and has been extensively examined (see [8–10]). This family contains the functions $f \in \mathcal{A}$ for which

$$\mathrm{Re}\left\{e^{i\beta}(1-\zeta_1 z)(1-\zeta_2 z)f'(z)\right\} \ge 0, \quad z \in \Delta.$$

(ii) If we take $g(z) = g_\alpha(z) = \frac{z}{(1-\alpha z)^2}$ with $\alpha \in [0,1]$, then we obtain the family

$$\mathcal{C}(g_\alpha) \equiv \mathcal{C}_0\left(\frac{z}{(1-\alpha z)^2}\right)$$

for which the Fekete–Szegö functional was studied in [11]. Note that for $\alpha = 1$ the class $\mathcal{C}_0\left(z/(1-\alpha z)^2\right)$ reduces to the family $\mathcal{C}_0(k)$ of close-to-convex functions with the argument 0 with respect to the well-known Koebe function $k(z) = z/(1-z)^2$. In [12] bounds of the Fekete–Szegö type functionals of the form

$$\Theta_f(\mu) = a_4 - \mu a_2 a_3$$

and

$$\Phi_f(\mu) = a_2 a_4 - \mu a_3^2$$

were derived for the class $\mathcal{C}_0(k)$. The same family of functions with some additional conditions was considered in [13].

(iii) The family $\mathcal{C}_0\left(z/(1-z^2)\right)$ was recently examined by Zaprawa et al. in ([14]). The authors pointed out that this class has an interesting symmetry property. Namely, if $f \in \mathcal{C}_0\left(z/(1-z^2)\right)$ and $\tilde{f}(z) = \overline{f(\overline{z})}$ then $\tilde{f} \in \mathcal{C}_0\left(z/(1-z^2)\right)$ and the sets $f(\Delta)$ and $\tilde{f}(\Delta)$ are symmetrical to each other with respect to the real axis.

For $-1 \le \alpha \le 1$ we define the functions

$$F_\alpha(z) = \frac{z}{1-\alpha z^2} = z + \sum_{n=1}^{\infty} \alpha^n z^{2n+1}, \quad z \in \Delta \tag{5}$$

which are starlike for all $\alpha \in [-1,1]$. Consider the class $\mathcal{C}_0(F_\alpha)$. Then (3) takes the form

$$\mathrm{Re}\{(1-\alpha z^2)f'(z)\} > 0, \quad z \in \Delta. \tag{6}$$

It should be noted that $\mathcal{C}_0(F_\alpha)$ becomes the class $\mathcal{C}(0,\alpha,\alpha)$ considered in [7]. Furthermore, if $0 \le \alpha \le 1$ then $\mathcal{C}_0(F_\alpha) \equiv \mathcal{C}_0(\sqrt{\alpha})$, where the symbol $\mathcal{C}_\beta(\alpha)$ ($\alpha \in [0,1]$, $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$) stands for the class of analytic functions that satisfy the condition

$$\mathrm{Re}\left\{e^{i\beta}(1-\alpha^2 z^2)f'(z)\right\} > 0, \quad z \in \Delta.$$

The family $\mathcal{C}_\beta(\alpha)$ was defined in 1989 [15] (see also [16]).

Note that when the α parameter is set, we obtain the already known classes. For example, if $\alpha = 0$ then the condition (6) takes the form

$$\mathrm{Re}\{f'(z)\} > 0, \quad z \in \Delta$$

and, therefore, it defines the class $\mathcal{R}$ of bounded turning (*cf.* [17] Vol. 1, p. 101), while for $\alpha = 1$ the class $C_0(F_\alpha)$ reduces to this, examined in [14].

Assume now that $\alpha \neq 0$ and $\alpha \neq 1$. We will show some examples of the members of the family $\mathcal{C}_0(F_\alpha)$. From the definition (6) of this class, it follows that if $q \in \mathcal{P}$, then there exists $f \in \mathcal{C}_0(F_\alpha)$ that satisfies

$$(1 - \alpha z^2) f'(z) = q(z), \; z \in \Delta. \tag{7}$$

By substituting selected functions from the class $\mathcal{P}$ in place of q we can derive from (7) the functions f that are members of $\mathcal{C}_0(F_\alpha)$. It is obvious that complex homographies of the form $q(z) = \frac{1+az}{1-az}$ with $a \in \mathbb{C}$, $|a| = 1$, are Carathéodory functions. Hence, each of them can be used to generate the function $f \in \mathcal{C}_0(F_\alpha)$. For example, if

$$q(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + 2z^3 + \dots, \tag{8}$$

then f related to this function q by the condition (7) is of the form

$$f(z) = \frac{1}{\alpha - 1}\left[\log\frac{(1-z)^2}{1-\alpha z^2} - \frac{\alpha+1}{2\sqrt{\alpha}}\log\frac{1-\sqrt{\alpha}z}{1+\sqrt{\alpha}z}\right] = z + z^2 + \frac{\alpha+2}{3} + \dots \tag{9}$$

under assumptions that $\alpha \neq 0$ and $\alpha \neq 1$. Putting

$$q(z) = \frac{1-z}{1+z} = 1 - 2z + 2z^2 - 2z^3 + \dots, \tag{10}$$

in (7) we obtain another example of a function f from the class $\mathcal{C}_0(F_\alpha)$ with $\alpha \neq 0$ and $\alpha \neq 1$:

$$f(z) = \frac{1}{\alpha - 1}\left[\log(1+z)^2)(1-\alpha z^2) + \frac{\alpha+1}{2\sqrt{\alpha}}\log\frac{1+\sqrt{\alpha}z}{1-\sqrt{\alpha}z}\right] = z - z^2 + \frac{\alpha+2}{3} + \dots. \tag{11}$$

In this paper, we find estimates of some coefficient functionals in the special subclass of the class $C_0(F_\alpha)$. Note that recently, there have been many articles on univalent functions with a fixed second coefficient in the Taylor series expansion. For example, Li and Sugawa in 2017 [18] considered the class of convex functions with the second coefficient fixed, and Zaprawa et al. (see [12,14]) researched the classes $\mathcal{C}_0\big(z/(1-z^2)\big)$ and $\mathcal{C}_0\big(z/(1-z)^2\big)$ with the same additional assumption (see also [19]). Motivated by this idea, we will focus on the problem of maximizing selected coefficient functionals for $f \in \mathcal{C}_0(F_\alpha)$ with the second coefficient fixed.

Let $f \in \mathcal{A}$ be of the form (1). From the definition of the class $\mathcal{C}_0(F_\alpha)$ it follows that

$$f \in \mathcal{C}_0(F_\alpha) \iff (1 - \alpha z^2) f'(z) \in \mathcal{P}. \tag{12}$$

In other words

$$(1 - \alpha z^2) f'(z) = q(z),$$

where $q \in \mathcal{P}$. If the function q has the series expansion (2), then comparing the coefficients with the same powers of the variable z leads to the following equality

$$na_n - \alpha(n-2)a_{n-2} = p_{n-1}, \quad n = 1, 2, 3, \dots,$$

where $|p_n| \leq 2$. In particular, we obtain that

$$2a_2 = p_1$$

so $|f''(0)| \leq 2$. For a given number $p \in [-2, 2]$ we define the subclass of $\mathcal{C}_0(F_\alpha)$ with the second coefficient fixed, $a_2 = p/2$, as follows

$$\mathcal{C}_0(F_\alpha, p) = \{f \in \mathcal{C}_0(F_\alpha) : f''(0) = p\}. \tag{13}$$

Therefore, if $f \in \mathcal{C}_0(F_\alpha, p)$, then

$$a_2 = p/2 \quad \text{and} \quad p_1 = p.$$

2. Auxiliary Lemmas

In this section, some results are collected on the coefficients of the functions belonging to the Carathéodory class $\mathcal{P}$. We start with the result known as Carathéodory–Toeplitz theorem (see for example [20]).

Lemma 1 (Carathéodory–Toeplitz theorem)**.** *If q has the power series expansion of the form $q(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, with $p_n \in \mathbb{C}$ then $q \in \mathcal{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & p_1 & p_2 & \dots & p_n \\ p_{-1} & 2 & p_1 & \dots & p_{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ p_{-n} & p_{-n+1} & p_{-n+2} & \dots & 2 \end{vmatrix}, \tag{14}$$

where $p_{-j} = \overline{p_j}$ $(j \geq 1)$, are non-negative for all $n \geq 1$. Moreover, if $D_j > 0$ for each $j = 1, 2, \dots, k-1$ and if $D_k = 0$, then the function q is of the form:

$$q(z) = \sum_{j=1}^{k} \beta_j \frac{1 + \epsilon_j z}{1 - \epsilon_j z}, \quad \beta_j > 0, \ \ |\epsilon_j| = 1 \ \ (j - 1, 2, \dots k), \ \ \epsilon_l \neq \epsilon_m \ (l \neq m). \tag{15}$$

As we can see, the function q in (15) is a linear combination of special complex homographies. Note that from $q(0) = 1$ it follows that the coefficients β_j of this combination satisfy $\beta_1 + \dots + \beta_k = 1$.

The Carathéodory–Toeplitz theorem is a very good tool for creating examples of Carathéodory functions. For example, let us consider a special case of the Toeplitz determinant

$$D_2 = \begin{vmatrix} 2 & p_1 & p_2 \\ \overline{p_1} & 2 & p_1 \\ \overline{p_2} & \overline{p_1} & 2 \end{vmatrix} = 8 - 4|p_1|^2 - 2|p_2|^2 + \overline{p_1}^2 p_2 + p_1^2 \overline{p_2}.$$

If p_1, p_2 are both real, then D_2 takes the form $D_2 = 2(2 - p_2)(2 + p_2 - p_1^2)$ and, consequently, $D_2 = 0$ if and only if $p_2 = 2$ or $p_2 = p_1^2 - 2$. Using Lemma 1, Li and Sugawa [18] put the following assertion related to the latter case.

Lemma 2. *Let $q \in \mathcal{P}$ and $q(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, with $p_1 \in \mathbb{R}$ and $p_2 = p_1^2 - 2$. Then, q has a form*

$$q(z) = \frac{1 - z^2}{1 - p_1 z + z^2} = 1 + p_1 z + (p_1^2 - 2) z^2 + (p_1^3 - 3p_1) z^3 + \dots . \tag{16}$$

Remark 1. *In [18] the authors noticed that if $q \in \mathcal{P}$ with $p_1 \in \mathbb{R}$ and $p_2 = 2$, then*

$$q(z) = (1-\gamma)\frac{1+z}{1-z} + \gamma\frac{1-z}{1+z} \tag{17}$$

where $\gamma \in [0,1]$ is some constant.

Lemma 3. *Let $q \in \mathcal{P}$ and $q(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, with $p_1 \in [-2,2]$ and $p_2 = 2$. Then, q is of the form*

$$q(z) = \frac{1}{2}\Big(1+\frac{p_1}{2}\Big)\frac{1+z}{1-z} + \frac{1}{2}\Big(1-\frac{p_1}{2}\Big)\frac{1-z}{1+z} = 1 + p_1 z + 2z^2 + p_1 z^3 + 2z^4 + \dots . \tag{18}$$

Proof. Under assumptions $p_1 \in [-2,2]$ and $p_2 = 2$ it follows from Remark 1, that there exists $\gamma \in [0,1]$, that satisfy (17). The function q given in (17) has the following power series expansion

$$q(z) = 1 + (2-4\gamma)z + 2z^2 + \dots .$$

From this we immediately have $p_1 = 2 - 4\gamma$ and, consequently, $\gamma = \frac{1}{2}(1 - \frac{p_1}{2})$ with $\gamma \in [0,1]$. Hence, the result follows. □

The next lemma is due to Libera and Złotkiewicz ([21,22]) and has also been proven by using the Carathéodory–Toeplitz theorem.

Lemma 4 ([21,22])**.** *If $q \in \mathcal{P}$ is given by (2), then*
(i) $2p_2 = p_1^2 + x(4 - p_1^2)$,
(ii) $4p_3 = p_1^3 + 2p_1(4 - {p_1}^2)x - p_1(4 - {p_1}^2)x^2 + 2(4 - {p_1}^2)(1 - |x|^2)y$,
for some x and y such that $|x| \le 1$, $|y| \le 1$.

Remark 2. *Observe that when (i) in the Lemma 4 occurs with $x = -1$ then the assumption $p_2 = p_1^2 - 2$ with $p_1 \in \mathbb{R}$ in the Lemma 2 is satisfied. Therefore, the form of q can be obtained by Lemma 2. If condition (i) holds with $x = 1$, that is, $p_2 = 2$ with $p_1 \in (-2,2)$, then the assumptions in the Lemma 3 are satisfied. Consequently, we have the function q of the form (18).*

If we put q of the form (16) or (18) in the correspondence (7) then we obtain two functions from the class $\mathcal{C}_0(F_\alpha)$ with the following series expansions, respectively

$$f(z) = z + \frac{1}{2}p_1 z^2 + \frac{1}{3}(\alpha - 2 + p_1^2)z^3 + \frac{1}{4}(p_1\alpha - 3p_1 + p_1^3)z^4 + \dots \tag{19}$$

and

$$f(z) = z + \frac{1}{2}p_1 z^2 + \frac{1}{3}(\alpha+2)z^3 + \frac{1}{4}(\alpha+1)p_1 z^4 + \frac{1}{5}(\alpha^2 + 2\alpha + 2)z^5 + \dots . \tag{20}$$

The next lemma can be obtained as a special case of the result due to Choi, Kim and Sugawa [23]. Let us define

$$Y(a,b,c) = \max_{z\in\overline{\Delta}}\Big(|a + bz + cz^2| + 1 - |z|^2\Big),\ a,b,c \in \mathbb{R}.$$

Lemma 5. *If $ac < 0$, then*

$$Y(a,b,c) = \begin{cases} 1 + |a| + \frac{b^2}{4(1+|c|)}, & |b| < 2(1+|c|) \text{ and } b^2 < -4a(1-c^2)/c, \\ 1 - |a| + \frac{b^2}{4(1-|c|)}, & |b| < 2(1-|c|) \text{ and } b^2 \ge -4a(1-c^2)/c, \\ R(a,b,c), & \text{otherwise,} \end{cases}$$

where

$$R(a,b,c)=\begin{cases}|a|+|b|-|c|, & |ab|\geq|c|(|b|+4|a|),\\ -|a|+|b|+|c|, & |ab|\leq|c|(|b|-4|a|),\\ (|a|+|c|)\sqrt{1-\frac{b^2}{4ac}}, & \text{otherwise.}\end{cases}$$

If $ac\geq 0$, *then*

$$Y(a,b,c)=\begin{cases}|a|+|b|+|c|, & |b|\geq 2(1-|c|),\\ 1+|a|+\frac{b^2}{4(1-|c|)}, & |b|<2(1-|c|)\end{cases}$$

3. Difference of Successive Coefficients

Let us begin this section by recalling the relationship between the coefficients of functions from the classes $\mathcal{C}_0(F_\alpha)$ and $\mathcal{P}$:

$$na_n-\alpha(n-2)a_{n-2}=p_{n-1},\quad n=1,2,3,\dots\ . \tag{21}$$

Combining these dependencies for successive n we obtain

$$2a_2=p_1,\ \ 3a_3=\alpha+p_2,\ \ 4a_4=p_1\alpha+p_3,\ \ 5a_5=\alpha^2+p_2\alpha+p_4 \tag{22}$$

and generally

$$na_n=\sum_{k=1}^{\frac{n}{2}}p_{2k-1}\alpha^{\frac{n}{2}-k}\qquad \text{for } n \text{ even} \tag{23}$$

and

$$na_n=\sum_{k=1}^{\frac{n+1}{2}}p_{2k-2}\alpha^{\frac{n+1}{2}-k}\qquad \text{for } n \text{ odd.} \tag{24}$$

We rewrite the above equalities in a more convenient and clear form. For $m\in\mathbb{N}$ we have

$$a_{2m}=\frac{1}{2m}\sum_{k=1}^{m}p_{2k-1}\alpha^{m-k} \tag{25}$$

and

$$a_{2m-1}=\frac{1}{2m-1}\sum_{k=1}^{m}p_{2k-2}\alpha^{m-k}. \tag{26}$$

For $f\in\mathcal{A}$ of the form (1) we define the coefficient functional that we will deal with in what follows:

$$F_n=a_{n+1}-a_n. \tag{27}$$

Note that F_n is a linear combination of a_{n+1} and a_n with coefficients that are symmetric with respect to 0.

Using (25) and (26) we derive the formulas for $F_n=a_{n+1}-a_n$ for a function $f\in\mathcal{C}_0(F_\alpha)$ separately for n even and n odd. For $n=2m-1,\ m\in\mathbb{N}$, we obtain

$$\begin{aligned}F_{2m-1}&=a_{2m}-a_{2m-1}\\ &=\frac{1}{2m}\sum_{k=1}^{m}p_{2k-1}\alpha^{m-k}-\frac{1}{2m-1}\sum_{k=1}^{m}p_{2k-2}\alpha^{m-k}\\ &=\alpha^{m-1}\left(\frac{p_1}{2m}-\frac{p_0}{2m-1}\right)+\alpha^{m-2}\left(\frac{p_3}{2m}-\frac{p_2}{2m-1}\right)+\cdots+\left(\frac{p_{2m-1}}{2m}-\frac{p_{2m-2}}{2m-1}\right),\end{aligned} \tag{28}$$

while for $n = 2m,\ m \in \mathbb{N}$, we obtain

$$\begin{aligned} F_{2m} &= a_{2m+1} - a_{2m} \\ &= \frac{1}{2m+1}\sum_{k=1}^{m+1} p_{2k-2}\alpha^{m+1-k} - \frac{1}{2m}\sum_{k=1}^{m} p_{2k-1}\alpha^{m-k} \\ &= \frac{p_0}{2m+1}\alpha^m + \alpha^{m-1}\left(\frac{p_2}{2m+1} - \frac{p_1}{2m}\right) + \alpha^{m-2}\left(\frac{p_4}{2m+1} - \frac{p_3}{2m}\right) + \cdots \\ &+ \left(\frac{p_{2m}}{2m+1} - \frac{p_{2m-1}}{2m}\right). \end{aligned} \tag{29}$$

In this section, we will concentrate on the problem of maximizing the modulus of F_n for $f \in \mathcal{C}_0(F_\alpha, p)$. For the class $\mathcal{C}\left(z/(1-z^2)\right)$, i.e., for $\alpha = 1$, the same problem was studied in ([14], Zaprawa, Trąbka–Więcław) and the following sharp inequalities for $|F_2|$ and $|F_3|$ were given

$$|F_2| \le \begin{cases} \frac{1}{2}(2-p) & \text{when } p \in [-2,1], \\ \frac{1}{6}(2-p)(2p+1) & \text{when } p \in [1,2], \end{cases} \tag{30}$$

and

$$|F_3| \le \begin{cases} \frac{1}{36}(38-17p) & \text{when } p \in [-2,5/3], \\ \frac{1}{12}(2-p)(3p^2+2p-2) & \text{when } p \in [5/3,2]. \end{cases} \tag{31}$$

Let $p \in [-2,2]$ be fixed. In the following theorem, we derive the estimates of $|F_2|$ for the class $\mathcal{C}_0(F_\alpha, p)$.

Theorem 1. *If $f \in \mathcal{C}_0(F_\alpha, p)$, $p \in [-2,2]$, and if f has the form (1), then the following sharp inequality holds*

$$|F_2| = |a_3 - a_2| \le \begin{cases} \frac{1}{6}(4+2\alpha-3p) & \text{when } p \in [-2, p_*], \\ \frac{1}{6}(4-2\alpha+3p-2p^2) & \text{when } p \in [p_*, 2], \end{cases} \tag{32}$$

where $p_* = \frac{1}{2}(3-\sqrt{9-8\alpha})$.

Proof. If $f \in \mathcal{C}_0(z/(1-\alpha z^2), p)$, then

$$(1-\alpha z^2)f'(z) = h(z)$$

for some function $h \in \mathcal{P}$. If h has the series expansion (2), then comparing the coefficients on both sides, we obtain

$$na_n - \alpha(n-2)a_{n-2} = p_{n-1}, \quad f''(0) = 2a_2 = p_1 = p.$$

This gives

$$F_2 = a_3 - a_2 = \frac{1}{3}p_2 - \frac{1}{2}p + \frac{\alpha}{3},$$

where α, p are fixed, $p \in [-2,2]$, $\alpha \in [-1,1]$, and p_2 is a second coefficient of the function $h \in \mathcal{P}$. Therefore, by Lemma 4, we obtain

$$|F_2| = |a_3 - a_2| = \left|\frac{\alpha}{3} + \frac{1}{6}[p^2 + (4-p^2)x] - \frac{p}{2}\right| = \frac{1}{6}|p^2 - 3p + 2\alpha + (4-p^2)x|$$

for some $|x| < 1$. Note that if $p^2 - 3p + 2\alpha \ge 0$, then the maximum of the expression on the right side of the above equality is reached for $x = 1$. If $p^2 - 3p + 2\alpha < 0$, then it is reached for $x = -1$. The square trinomial $w(p) = p^2 - 3p + 2\alpha$ has two roots

$$p_* = \frac{1}{2}(3-\sqrt{9-8\alpha}) \quad \text{and} \quad p^* = \frac{1}{2}(3+\sqrt{9-8\alpha}).$$

Elementary computing shows that for all $\alpha \in [-1,1]$ the first one is an interior point of the interval $[-2,2]$ and the second one is outside of $[-2,2]$. Hence, (32) immediately follows.

The sharpness of the bounds follows from Lemma 4 after taking into account Remark 2. If $p \in [-2, p_*]$, then the equality in (32) occurs for the function f given by (20) with $p_1 = p$, while if $p \in [p_*, 2]$, the equality in (32) is realized by the function f given by (19) with $p_1 = p$. □

We can derive several corollaries from the above theorem. For example, for $\alpha = -1$ we have $p_* = (3-\sqrt{17})/2$ and in this case Theorem 1 yields the following

Corollary 1. *If $f \in \mathcal{C}_0(z/(1+z^2), p)$, $p \in [-2,2]$, has the form (1), then*

$$|F_2| = |a_3 - a_2| \leq \begin{cases} \frac{1}{6}(2-3p) & \text{when } p \in [-2, (3-\sqrt{17})/2], \\ \frac{1}{6}(6+3p-2p^2) & \text{when } p \in [(3-\sqrt{17})/2, 2]. \end{cases}$$

For $\alpha = 0$ we have $p_* = 0$ and Theorem 1 reduces to the following result for the class of functions of bounded turning with the second coefficient fixed. Let us denote this class by $\mathcal{R}_p$.

Corollary 2. *If $f \in \mathcal{R}_p$, $p \in [-2,2]$, has the form (1), then*

$$|F_2| = |a_3 - a_2| \leq \begin{cases} \frac{1}{6}(4-3p) & \text{when } p \in [-2, 0], \\ \frac{1}{6}(4+3p-2p^2) & \text{when } p \in [0, 2]. \end{cases}$$

Finally note that if $\alpha = 1$, then $p_* = 1$ and Theorem 1 yields the result of [14] which was recalled in (30).

Let us now turn to the estimates of the functional $|F_3| = |a_4 - a_3|$. We start with the results in the classes with specific values of the α parameter. If $\alpha = -1$, then we have $F_{-1}(z) = \frac{z}{1+z^2}$. For class $\mathcal{C}_0(\frac{z}{1+z^2}, p)$, we have the following result.

Theorem 2. *If $f \in \mathcal{C}_0(\frac{z}{1+z^2}, p)$, $p \in [-2,2]$, has the form (1), then the following inequalities hold*

$$|F_3| = |a_4 - a_3| \leq \begin{cases} \frac{2}{3}(p^2-2)\sqrt{\frac{-\frac{1}{3}p^2-p+\frac{4}{3}}{3p^4-8p^3-12p^2+16p}} & \text{when } p \in [-2, s_1), \\ \frac{1}{4}p^3 - \frac{1}{3}p^2 - p + 1 & \text{when } p \in [s_1, -\frac{1}{3}], \\ \frac{1}{36}(25p+38) & \text{when } p \in (-\frac{1}{3}, \frac{12}{13}), \\ \frac{1}{36}(p+14) & \text{when } p \in [\frac{12}{13}, \frac{5}{3}), \\ \frac{1}{12}(-3p^3+4p^2+12p-12) & \text{when } p \in [\frac{5}{3}, 2], \end{cases} \tag{33}$$

where s_1 is the unique root of the polynomial $W(p) = 3p^3 - 8p^2 - 12p + 16$ in the interval $(-2, 0)$.

Proof. By using (25) and (26) we obtain for $p \in [-2,2]$

$$12|F_3| = |a_4 - a_3| = |-3p + 3p_3 + 4 - 4p_2|.$$

Applying Lemma 4 results in the following equality

$$12|F_3| = \left| -3p + \frac{3}{4}\left[p^3 + 2pxt - px^2t + 2y(1-|x|^2)t\right] + 4 - 2[p^2 + xt] \right|$$

for some x, y such that $|x| < 1$ and $|y| < 1$ and where $t = 4 - p^2$. From this, we obtain at once that for $p = -2$ as for $p = 2$, an estimate $|F_3| \leq \frac{1}{3}$ holds. In case $p = -2$ the equality holds for the function given by (19) with $\alpha = -1$ and $p = -2$, that is, for $f(z) = \frac{1}{2}\log\frac{(z+1)^2}{z^2+1}$.

In case $p = 2$ the equality is realized by the function given by (20) with $\alpha = -1$ and $p = 2$, that is, by $f(z) = \frac{1}{2}\log\frac{z^2+1}{(z-1)^2}$.

Now, assume that $p \in (-2, 2)$. Further, after computing and simplifying, we obtain from the last equality that

$$12|F_3| \leq \frac{3}{2}(4-p^2)\Big(|a+bx+cx^2|+1-|x|^2\Big), \tag{34}$$

where

$$a = \frac{3p^3-8p^2-12p+16}{6(4-p^2)};\ \ b = p-\frac{4}{3};\ \ c = -\frac{p}{2}. \tag{35}$$

Now, we will apply Lemma 5. For $p \in (-2, s_1)$, we have $ac < 0$. Moreover, some calculations show that each of the following conditions from Lemma 5 is false in this case:

$$b^2 < -4a(1-c^2)/c,\ \ |b| < 2(1-|c|),\ \ |ab| \geq |c|(|b|+4|a|),\ \ |ab| \leq |c|(|b|-4|a|).$$

It follows at once that for $p \in (-2, s_1)$

$$12|F_3| \leq \frac{3}{2}(4-p^2)(|a|+|c|)\sqrt{1-\frac{b^2}{4ac}}$$

with a, b, c, given by (35). After calculating we obtain that

$$|F_3| \leq \frac{2}{3}(p^2-2)\sqrt{\frac{-\frac{1}{3}p^2-p+\frac{4}{3}}{3p^4-8p^3-12p^2+16p}}.$$

Now, let $p \in [s_1, 0)$. We have to consider two cases here because, in each of them, Lemma 5 will be used in a different form. Namely, if $p \in [s_1, -\frac{1}{3})$ then we have

$$ac \geq 0 \quad \text{and} \quad |b| \geq 2(1-|c|)$$

while for $p \in [-\frac{1}{3}, 0)$:

$$ac \geq 0 \quad \text{and} \quad |b| < 2(1-|c|).$$

By Lemma 5 and after some computation we obtain for $p \in [s_1, -\frac{1}{3})$:

$$12|F_3| \leq \frac{3}{2}(4-p^2)(|a|+|b|+|c|) = 3p^3-4p^2-12p+12$$

and hence the appropriate result of Theorem 2 yields. If $p \in [-\frac{1}{3}, 0)$ then we compute that

$$12|F_3| \leq \frac{3}{2}(4-p^2)\left(1+|a|+\frac{b^2}{4(1-|c|)}\right) = \frac{1}{3}(25p+38)$$

and the result follows. Now, consider the case $p \in [0, \frac{12}{13})$. Then, we have

$$ac < 0 \quad \text{and} \quad |b| < 2(1+|c|) \quad \text{and} \quad b^2 < -4a(1-c^2)/c.$$

Thus, we obtain

$$12|F_3| \leq \frac{3}{2}(4-p^2)\left(1+|a|+\frac{b^2}{4(1+|c|)}\right) = \frac{1}{3}(25p+38).$$

The interval $[\frac{12}{13}, \frac{5}{3})$ also has to be broken into two parts. If $p \in [\frac{12}{13}, s_2)$, where s_2 is the only root of the polynomial $W(p) = 3p^3-8p^2-12p+16$ in the interval $(0, 1)$, then

$$ac < 0 \quad \text{and} \quad |b| < 2(1-|c|) \quad \text{and} \quad b^2 \geq -4a(1-c^2)/c$$

and consequently,

$$|F_3| \leq \frac{3}{24}(4-p^2)\left(1-|a|+\frac{b^2}{4(1-|c|)}\right) = \frac{1}{36}(p+14).$$

However, for $p \in [s_2, \frac{5}{3})$ we have

$$ac \geq 0 \quad \text{and} \quad |b| < 2(1-|c|).$$

So then

$$12|F_3| \leq \frac{3}{2}(4-p^2)\left(1+|a|+\frac{b^2}{4(1-|c|)}\right)$$

which gives the same result as we had in the previously considered interval. Finally, we have to deal with the case, if $p \in [\frac{5}{3}, 2)$. Then,

$$ac \geq 0 \quad \text{and} \quad |b| \geq 2(1-|c|).$$

This results in the following inequality:

$$12|F_3| \leq \frac{3}{2}(4-p^2)(|a|+|b|+|c|) = -3p^3+4p^2+12p-12.$$

□

In the next theorem, we derive the bounds of $|F_3|$ for the class $\mathcal{R}_p$.

Theorem 3. *If $f \in \mathcal{R}_p$, $p \in [-2,2]$ then the following inequalities hold*

$$|F_3| \leq \begin{cases} \frac{1}{12}(2p^2-3p)\sqrt{\frac{-8p^3+12p^2-24p+16}{3p^4-8p^3}} & \text{when } p \in [-2,t_1), \\ \frac{1}{12}(3p^3-4p^2-9p+8) & \text{when } p \in [t_1,-\frac{1}{3}], \\ \frac{1}{18}(13-8p) & \text{when } p \in (-\frac{1}{3},0), \\ \frac{1}{18}(13-2p) & \text{when } p \in [0,\frac{5}{3}), \\ \frac{1}{24}(-6p^3+8p^2+18p-16) & \text{when } p \in [\frac{5}{3},2], \end{cases} \tag{36}$$

where t_1 is the only root of the polynomial $T(p) = 12p^3-32p^2-4/3p+16$ in the interval $(-1,0)$.

Proof. From Lemma 4:

$$12|F_3| = \left|\frac{3}{4}\left[p^3+2px(4-p^2)-px^2(4-p^2)+2y(1-|x|^2)(4-p^2)\right]-2[p^2+x(4-p^2)]\right|$$

for some x, y such that $|x|<1$ and $|y|<1$. For $p=-2$ and $p=2$, we immediately obtain$|F_3| \leq \frac{7}{6}$ and $|F_3| \leq \frac{1}{6}$, respectively. The equalities in these bounds are realized by the functions $f(z) = \log(z+1)^2 - z$ and $f(z) = \log\frac{1}{(1-z)^2} - z$, respectively.

Let $p \in (-2,2)$. Then, after some calculations we obtain

$$12|F_3| \leq \frac{3}{2}(4-p^2)\left(|a+bx+cx^2|+1-|x|^2\right), \tag{37}$$

where

$$a = \frac{3p^3-8p^2}{6(4-p^2)}; \;\; b = p-\frac{4}{3}; \;\; c = -\frac{p}{2}. \tag{38}$$

We will apply Lemma 5. For $p \in (-2,-\frac{1}{3})$ we have $ac<0$ and none of the conditions

$$b^2 < -4a(1-c^2)/c, \;\; |b| < 2(1-|c|), \;\; |ab| \geq |c|(|b|+4|a|), \;\; |ab| \leq |c|(|b|-4|a|)$$

of Lemma 5 are satisfied. Let us denote by t_1 the only root of the polynomial

$$T(p) = 12p^3 - 32p^2 - \frac{4}{3}p + 16$$

in the interval $(-1, 0)$. After some computation we can check that for $p \in (-2, t_1)$ the inequalities

$$|ab| \geq |c|(|b| + 4|a|) \quad \text{and} \quad |ab| \leq |c|(|b| - 4|a|)$$

are false. Then, from Lemma 5 we obtain

$$12|F_3| \leq \frac{3}{2}(4 - p^2)(|a| + |c|)\sqrt{1 - \frac{b^2}{4ac}}$$

with a, b, c, given by (38). This gives the first result of Theorem 3:

$$|F_3| \leq \frac{1}{12}(2p^2 - 3p)\sqrt{\frac{-8p^3 + 12p^2 - 24p + 16}{3p^4 - 8p^3}}.$$

For $p \in [t_1, -\frac{1}{3})$ we have $|ab| \leq |c|(|b| - 4|a|)$. Therefore, from Lemma 5 we obtain in this case that

$$12|F_3| \leq \frac{3}{2}(4 - p^2)(-|a| + |b| + |c|) = 3p^3 - 4p^2 - 9p + 8$$

which implies the second estimate of our theorem.

Assume that $p \in [-\frac{1}{3}, 0)$. Then, we have

$$ac < 0 \quad \text{and} \quad |b| < 2(1 - |c|) \quad \text{and} \quad b^2 \geq -4a(1 - c^2)/c$$

and consequently,

$$12|F_3| \leq \frac{3}{2}(4 - p^2)\left(1 - |a| + \frac{b^2}{4(1 - |c|)}\right) = \frac{1}{18}(13 - 8p)$$

which gives the next bound.

If $p \in [0, \frac{5}{3})$ then the conditions

$$ac \geq 0 \quad \text{and} \quad |b| < 2(1 - |c|)$$

are satisfied. By Lemma 5 and after some computation we obtain for $p \in [0, \frac{5}{3})$

$$|F_3| \leq \frac{1}{8}(4 - p^2)\left(1 + |a| + \frac{b^2}{4(1 - |c|)}\right) = \frac{1}{9}\left(\frac{13}{2} - p\right).$$

In the last case, i.e., for $p \in [\frac{5}{3}, 2]$ we have

$$ac \geq 0 \quad \text{and} \quad |b| \geq 2(1 - |c|).$$

Therefore,

$$|F_3| \leq \frac{1}{8}(4 - p^2)(|a| + |b| + |c|) = \frac{1}{24}(-6p^3 + 8p^2 + 18p - 16).$$

□

Now let us move on to the consideration of the investigated problem of $|F_3|$ estimation in the general case. Unfortunately, the result was not achieved in the full range of parameters, i.e., for all (p, α) from the rectangle $[-2, 2] \times [-1, 1]$.

Theorem 4. *If* $f \in \mathcal{C}_0(F_\alpha, p)$ $\alpha \in [0, \frac{4}{45}]$ *and* $p \in (-\frac{1}{3}, \frac{12\alpha}{9\alpha-4}]$, *then*

$$|F_3| \leq \frac{1}{36}[(9\alpha - 16)p + 26 - 12\alpha]. \tag{39}$$

Proof. Applying (25) and (26) we obtain that

$$12|F_3| = |3p\alpha + 3p_3 - 4\alpha - 4p_2|, \quad p \in [-2, 2].$$

Therefore, by Lemma 4 we have

$$12|F_3| = \left|3p\alpha + \frac{3}{4}\left[p^3 + 2pxt - px^2t + 2y(1 - |x|^2)t\right] - 4\alpha - 2(p^2 + xt)\right|$$

for some $|x| < 1$ and $|y| < 1$ and where $t = p^2$. After a simple computation we obtain

$$12|F_3| \leq \frac{3}{2}(4 - p^2)\left(|a + bx + cx^2| + 1 - |x|^2\right),$$

where

$$a = \frac{3p^3 - 8p^2 + 12p\alpha - 16\alpha}{6(4 - p^2)}; \quad b = p - \frac{4}{3}; \quad c = -\frac{p}{2}. \tag{40}$$

We will apply Lemma 5. For $\alpha \in (0, 4/45]$ and $p \in \left(-\frac{1}{3}, \frac{12\alpha}{9\alpha-4}\right]$ the following inequalities hold

$$ac < 0 \quad \text{and} \quad |b| < 2(1 - |c|) \quad \text{and} \quad b^2 \geq -4a(1 - c^2)/c.$$

Therefore, with the above assumptions we have

$$|F_3| \leq \frac{3}{2}(4 - p^2)\left(1 - |a| + \frac{b^2}{4(1 - |c|)}\right) = \frac{1}{36}[(9\alpha - 16)p + 26 - 12\alpha] \tag{41}$$

and the desired result (39) follows. □

Note that if α goes to zero, then the bound obtained from Theorem 4, coincides with what we got for $p \in (-\frac{1}{3}, 0)$ in Theorem 3.

4. Conclusions

A key procedure leading to the results presented in this article was to relate the investigated class to the class of Carathéodory functions $\mathcal{P}$. The resulting relationship between the coefficients of both classes, combined with known estimates of the coefficients in the class $\mathcal{P}$ and the use of Lemmas 4 and 5, allowed the studied coefficient functionals to be maximized. An interesting direction for future research seems to be the attempt to estimate other coefficient functionals, for example, some combinations of two successive coefficients or the sum of n initial coefficients of the Taylor series for functions in the studied class.

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