

Article

Generalization Process of the Integrated Mathematical Model Created for the Development of the Production Geometry of Complicated Surfaces

Zsuzsa Balajti 

Institute of Mathematics, University of Miskolc, H-3515 Miskolc, Hungary;
zsuzsanna.ovarine.balajti@uni-miskolc.hu

Abstract: Computer modelling of technical constructions is increasingly carried out using software that includes more detailed knowledge, which requires an increase in the level as well as an expansion of the scope of the geometric knowledge. A significant part of motion transmission mechanisms are worm drive pairs, for which the separation of the parts dealing with the theoretical and practical problems found in the literature can be experienced in numerous instances. Due to the different technical features, in many cases the helical surfaces are not designed and manufactured in a geometrically correct way, or the best solution is not the compulsory chosen. The geometric model describing the production process of the worm surfaces provides the basis for examining the deviation between the surface mathematically determined by the designer and the surface produced. An integrated mathematical kinematic model was developed for the production geometrical analysis of the elements of cylindrical and conical worm gear drive pairs for machining with a traditional thread grinding machine, which causes a serious pitch fluctuating error among several other problems in the case of machining the conical worm. Modelling of the production process of surfaces and the simultaneous study of the manufacturing errors is basically performed with the toolbox of descriptive geometry, including the use of the projective invariants. Knowing the inheritance of the invariants of projective geometry, the aim was the mathematical generalization of the integrated model and the creation of a projective relationship between the reference surfaces of conical and cylindrical spiral surfaces. As a result, the improved constructive geometric model was created, in which the method of analytically creating the projective geometric relationship between the reference surfaces of conical and cylindrical helicoid surfaces has been described for the first time in this article. This procedure is considered the most important result of the present article. Another significance of the further development presented is that during production of the conical helicoid surface, the thread pitch fluctuation has been eliminated. The results obtained, consisting of an improved geometric model, lead to a new geometry of the technological environment regarding the relative position of the cutting tool and the workpiece as well as the relative motion between them.

Keywords: worm gear drives; constructive geometric model; generalization; homogeneous normalized coordinate; central collineation



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1. Introduction

The structure construction of technological constructions [1] and motion transmission mechanisms [2,3], the analytical description of their action [4] and their computer modelling [5], are always carried out with the abstractions of descriptive geometry [6]. A significant group of surfaces with complicated geometry is the worm gear drive pair, whose performance is influenced by the input speed, the viscosity and type of the lubricating oil, the load, the shape of the worm, and the geometry of the worm gear [7]. In the same way that the importance of mathematical proficiency increases with the computerization of calculation procedures [8–10], higher-level descriptive geometry skills are also needed to correctly operate increasingly sophisticated modelling software [11–13].

1.1. The Geometric Environment of Mathematical Generalization

The learned theories and constructs of Monge's approach to drawing 3D presentations using any computer-aided design program cannot be replaced by anything else [14]. Knowledge of Monge's concept allows the user to recognize metrical relationships in space and gives the freedom to choose between different visualization possibilities [15].

Mathematics reveals many geometries for abstractly describing physical phenomena [16,17], solving various tasks, and handling emerging problems [18,19]. There are many examples of how, as long as the plan of buildings [20], the design of mechanical motion transmission structures [21] and their control measurements [22] are made in the Euclidean spatial model, it is possible to reach from one point on Earth to another point using spherical geometry [23]. In the examination of the laws of eye imaging, the laws to which references are created belong to geometry in the traditional sense, but, at the same time, distances and angles do not remain constant during the central projection as a transformation [24]. Accordingly, in the case of central projection, the laws of classical geometry cannot be applied. During the geometric research work of Szabó and Kunkli, in one of their articles, they show how, along the lines of Szabó's earlier theorems, an image obtained by a central projection from a photograph of a photograph, which in most cases is no longer a central projection, can be generated from an image taken from a photograph by applying a well-chosen affinity [25]. In the case of the application of perspective, it is naturally offered to supplement the plane with an infinitely distant straight line, and the space with the infinitely distant plane. The geometry that best models reality should always be used to solve the actual problem.

One of the most salient essential points of the Erlangen program is that every geometry is the theory of the invariants of an algebraic transformation group, and vice versa, every algebraic transformation group has a geometry [26]. The study of the invariants of the transformation group of geometry is itself a geometry [27]. According to the algebraic group of transformations of geometry, two geometric shapes are equivalent if one shape can be transformed into the other using one of the algebraic group of transformations. According to this theory, Euclidean geometry is a particular case of projective geometry [18], and projective geometry is a generalization of Euclidean geometry that includes the Euclidean geometric system.

In addition to being significant in the realistic planar representation of spatial objects, synthetically formed projective geometry also has a strict logical structure. The study of this rigorous logical structure has been presented in an article from the point of view that it could even be a model of Hegelian logic according to certain aspects [28]. The basic tools of analytically further developed projective geometry are homogeneous coordinates and linear transformations, the knowledge of which is essential for computer graphics [29].

For all these reasons, the knowledge and study of projective geometry is a prerequisite for computer modelling and graphics research [30].

A practical solution for the visual representation of the conscious geometric structure of spatial objects based on strict mathematical foundations and logic are computer-enhanced descriptive geometry (CeDG), which is a scientific approach to the computerization of three-dimensional (3D) geometric systems using descriptive geometric procedures, which is compatible with today's technological conditions for modelling. One contribution to the science field is the timeless geometric approach in a timely way, which guarantees mathematical precision with a timely method by inheriting the laws of projective geometric invariants bearing the signs of symmetric duality [31].

1.2. Production Geometry and Technological Background

In addition to displaying the spatial motion-transmitting drive pairs, the basis for creating a model suitable for characterizing relative motions is also descriptive geometry [32]. Geometrical modelling of the technological processes that correspond to reality is essential for the correct examination of motion transmission mechanisms. The mathematical developments and abstractions developed are determined by whether the created abstraction

describes the operation of the investigated technology. The topics of mathematical developments are always determined in parallel with the direction of technological development.

In many cases, the mathematical definition of a complex surface differs from the surface created by machining. The differences are mainly caused by the surface of the machining tools created by the technological environment and the real practicable path of movement that can be created. Such a case occurs, for example, when machining with a grinding wheel, as the reduction in the center distance due to the wear of the working surface of the grinder causes profile distortion. Furthermore the variable center distance between the tool shaft and the worm shaft during the machining of the conical worm would also require a continuous change in the tool profile. The simultaneous examinations of the geometry and the trajectory of the tool are the mathematical basis of several engineering research works [33,34]. This is also the case with the working surfaces of cylindrical or conical worms, namely that the defined helicoid surface differs from the surface machined at the existing technological conditions. The integrated model [35] was developed to analyze these deviations. For the analysis of the production geometry and computing of the elements of the conical and cylindrical worm gear drives, the coordinate systems extended with the geometrical parameters are shown in Figure 1, which is prepared for the traditional thread grinding machine.

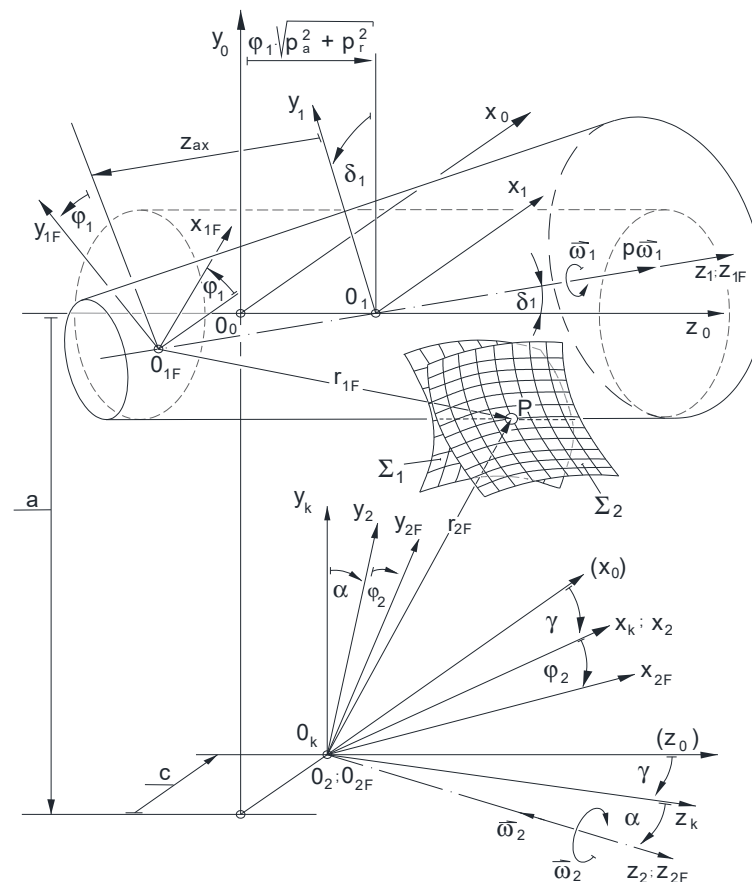


Figure 1. The applied coordinate systems supplemented with reference surfaces for the general study of the production theory of the elements of the drives of conical ($\delta_1 \equiv \delta > 0^0$) and cylindrical ($\delta_1 \equiv \delta = 0^0$) worms produced on a traditional thread grinding machine, which is prepared based on Figure 4.1. of reference [35].

The applied coordinate systems of the integrated mathematical kinematical model are defined as follows:

- $K_0 (x_0, y_0, z_0)$: the fixed coordinate system of the machining machine tool.
- $K_1 (x_1, y_1, z_1)$: the coordinate system fixed to a linearly moving table.

- $K_{1F} (x_{1F}, y_{1F}, z_{1F})$: the rotating coordinate system affixed to the helicoidal surface by parameters η and ϑ .
- $K_2 (x_2, y_2, z_2)$: the stationary coordinate system attached to the tool or gear.
- $K_{2F} (x_{2F}, y_{2F}, z_{2F})$: the rotary coordinate system fixed to the tool or gear with axis z_{2F} .
- $K_{20} (x_{20}, y_{20}, z_{20})$: the coordinate system of the generating curve of the tool surface.
- $K_k (x_k, y_k, z_k)$: the auxiliary coordinate system.
- $O_0, O_1, O_2, O_{1F}, O_{2F}, O_k$: the origins of coordinate systems according to their subscripts.

Based on the literature [35], the geometric parameters used in the integrated mathematical kinematic model are as follows:

- a — y -direction coordinate of the origin O_2 of the tool coordinate system in the stationary coordinate system K_0 , namely the center distance [mm].
- c —the x coordinate of the origin O_2 of the tool's coordinate system in the stationary coordinate system K_0 , the value of tool displacement (e.g., the radius of the dedendum cylinder or base cylinder in the case of a convolute or involute worm) [mm].
- α_o —tilting angle of the tool into the groove of the helicoidal surface in the characteristic section [$^\circ$].
- $\gamma_o = \gamma$ —lead angle of worm's reference surface [$^\circ$].
- $p_o = p$ —pitch on the reference surface of the worm [mm].
- p_a —screw parameter in the axial direction [mm].
- p_r —screw parameter in the radial direction [mm].
- φ_1 —angle of rotation of helical surface (enveloping, movement parameter) [$^\circ$].
- φ_2 —angle of rotation of the tool (enveloping, movement parameter) [$^\circ$].
- ω_1 —angular velocity of the worm [s^{-1}].
- ω_2 —angular velocity of the tool [s^{-1}].

The z_{F1} is the axis of cylindrical worms or hobs, the adjusted z_{1F} is the axis of conical worms or hobs, and the z_{F2} is the axis of wheels or tools of worms. The developed integrated model is also suitable for the analysis of the properties of worm gear drives in the case of the axis angle chosen as $\gamma = -90^\circ$.

Conical and cylindrical helicoid surfaces are machined on the traditional thread grinder, where the common generatrix of the reference surfaces are adjusted to the path of the grinder, according to the scheme of the integrated model. For this, it is necessary that the reference cone is tilted with its half-opening angle to the generatrix of the reference cylinder laying on the tool path, i.e., the conical helicoid surface is machined with axis displacement, as shown in Figure 2.

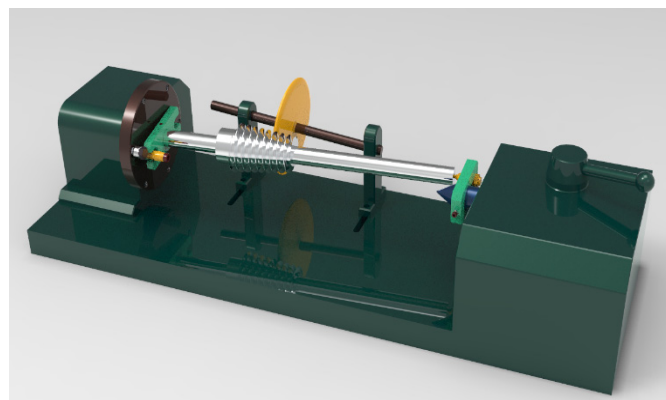


Figure 2. The schematic arrangement of the machining of the conical worm on a traditional thread grinding machine from [36].

In this arrangement, the conical worm with an adjusted axis is driven through a lathe dog driven by a drive pin. The contact of the reference surface of the adjusted conical worm with the grinding wheel has been taken on the ellipse path of a cone; therefore,

the changing radius relative to the axis of the conical worm creates a pitch fluctuation. Eliminating these challenges of thread pitch fluctuation, the determination of the profile of the drive pin has been counted in an explicit form with the use of expediently chosen descriptive geometric projections and the use of inherited projective invariants, if a point contact has been considered between the lathe dog and the drive pin [36].

The elimination of this challenge in a natural mathematical way was one of the reasons for the creation of the improvement presented here, according to which the reference surfaces of the worms are placed on a common axis instead of the common generatrix.

Another main reason for the development presented in this article is the case of the special matrix representation of the coordinate transformation applied in gear theory [37]. In many cases of mechanical engineering applications, the special case of matrix representation of coordinate transformations is used, i.e., with the application of a 4×4 matrix, the new Cartesian coordinates of a point can be expressed with the old Cartesian coordinates as follows:

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & d_1 \\ a_{21} & a_{22} & a_{23} & d_2 \\ a_{31} & a_{32} & a_{33} & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix} \quad (1)$$

where the submatrix a_{nm} ($n, m = 1, 2, 3$) includes the direction cosines, and the d_n column contains the origin of the original frame registered in the new frame. An analytical form of Euclidean geometry interpreted as a special case of projective geometry is used during the geometric modelling of the motion-transition drive pairs [18]. Looking for the possibility of further development of the integrated mathematical kinematic model, the possibility of extending it to the projective space characterized by homogeneous coordinates was examined.

During the further development of the manufacturing geometry of the conical and cylindrical helicoidal surfaces, the need arose for the matrix algebraic formulation of the relationship between the reference cone and the reference cylinder surfaces, which later aimed to unify the testing method of their machining tools.

In this publication, the method of creating the process of the transformation relationship between the reference surfaces interpreted in the projective space model, which has not been published until now, is presented.

2. Constructive Geometric Model Created as a Further Development of the Integrated Model

In the literature created for contact analysis of the cylindrical and conical worm gear drive pairs and the development of the production geometry of the drive pair elements, there was often a separation of the theoretical discussion and practical problems, so the integrated model was developed to solve this problem [35]. The constructive geometric model is a further mathematical generalization of this integrated model, which has been created to examine the cylindrical and conical worm gear drives and develop the production geometry of these drive pair elements.

In the constructive geometric model, as indicated by arrows in Figure 3, the axis of the conical helicoid has been adjusted to the axis of the cylindrical helicoid with the aim of writing the central collineation relationship between their reference surfaces. The relationship between the coordinate systems used in the constructive geometric model and the geometric parameters for the production geometrical examination of the cylindrical or conical worms or hobs on the common z_{F1} axis, as well as the gears or worm machining tools located on the z_{F2} axis, are also shown in Figure 3.

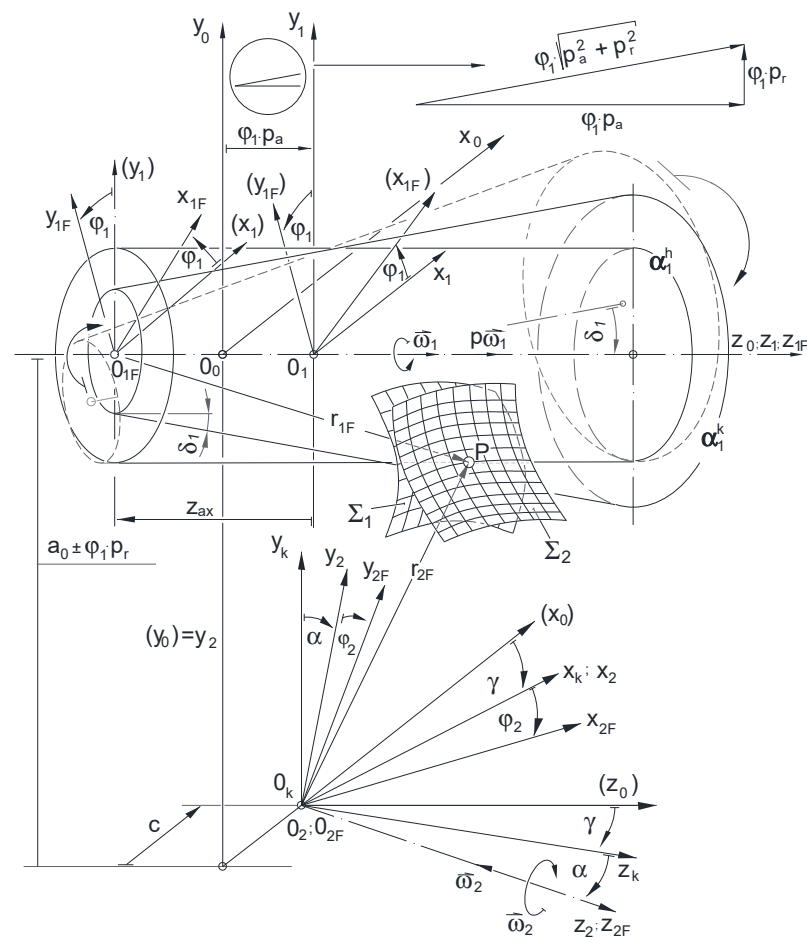


Figure 3. The relationship of the applied coordinate systems to each other for a more general study of the manufacturing theory of the elements of bevel and cylindrical worm gear pairs in the improved constructive geometric model.

In the constructive geometric model, the geometrical analyses of machining are made using an analytical form of Euclidean geometry interpreted as a special case of projective geometry [18].

2.1. Analytical Aspects of the Projective Geometry Applied During the Extension of the Model

This subsection provides a brief overview of the applied parts of projective geometry in elaborating of the scientific development. In the case of the analytical definition of the Euclidean space model, each real coordinate triplet is assigned to a point, which are used to determine the homogeneous coordinates of the points with the following identities:

$$x = \frac{x_1}{x_4}, y = \frac{x_2}{x_4}, z = \frac{x_3}{x_4}, x_4 \neq 0 \quad (2)$$

so then Equation (3) is as follows:

$$P(x, y, z) \rightarrow P(x, y, z, 1) \sim P(x \cdot x_4, y \cdot x_4, z \cdot x_4, x_4) = P(x_1, x_2, x_3, x_4) \quad (3)$$

And, vice versa, in the case of $x_4 \neq 0$ based on Equation (1), Equation (4) is as follows:

$$P(x_1, x_2, x_3, x_4) \sim P(x, y, z, 1) \rightarrow P(x, y, z) \quad (4)$$

Two homogeneous coordinate quadruples define the same point if and only if the corresponding homogeneous coordinates are products of each other by the same scalar.

In the case of the form of a homogeneous coordinate quadruple, Equation (5) is as follows:

$$(x_1, x_2, x_3, x_3 0) \quad (5)$$

Since $x_4 = 0$, the point must be interpreted at an infinite distance in the Cartesian coordinate system in the direction defined by (x_1, x_2, x_3) . In the following, the homogeneous coordinates of point P can also be written in the shorter form (x_K) , where $K = 1, 2, 3, 4$. Each plane $\underline{S}$ of space can be assigned an ordered quadruple consisting of real numbers (u_1, u_2, u_3, u_4) , for which Equation (6) is as follows:

$$\text{rank}(u_1, u_2, u_3, u_4) \neq 0 \quad (6)$$

where a quadruple consisting of all zeros cannot characterize a plane.

Two planes are identical if and only if their respective coordinates are multiples of each other with the same scalar number.

In 3D space, a straight-line l is defined as the intersection of two planes. The analytical form of the intersection line is depicted with a matrix consisting of eight real numbers of the two planes in the following way:

$$\text{rank} \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \end{pmatrix} = 2 \quad (7)$$

where one plane is defined by the quad of numbers $(u_{11} \ u_{12} \ u_{13} \ u_{14})$ and the other plane is defined by the quad of numbers $(u_{21} \ u_{22} \ u_{23} \ u_{24})$, which is $l(u_{\alpha K})$ in short, if $\alpha = 1, 2$.

A point $P(x_1, x_2, x_3, x_4)$ lies on the straight line $(u_{\alpha K})$ where $\alpha = 1, 2$, or in other words the line $(u_{\alpha K})$ passes through on the point $P(x_1, x_2, x_3, x_4)$, if and only if the point P lies on the two planes defining the line, as in Equation (8):

$$\left. \begin{aligned} u_{11} \cdot x_1 + u_{12} \cdot x_2 + u_{13} \cdot x_3 + u_{14} \cdot x_4 &= 0 \\ u_{21} \cdot x_1 + u_{22} \cdot x_2 + u_{23} \cdot x_3 + u_{24} \cdot x_4 &= 0 \end{aligned} \right\} \quad (8)$$

or in short, as in Equation (9):

$$\left. \begin{aligned} u_{11} \cdot x_1 + u_{12} \cdot x_2 + u_{13} \cdot x_3 + u_{14} \cdot x_4 &= 0 \\ u_{21} \cdot x_1 + u_{22} \cdot x_2 + u_{23} \cdot x_3 + u_{24} \cdot x_4 &= 0 \end{aligned} \right\} \quad (9)$$

The projective point transformation is a linear mapping in which the fit of the space elements and the cross ratio of the four points of a line are invariant. That is, a point corresponds to a point, and a straight line corresponds to a straight line. Furthermore, if a space element lies on another space element, then the transformed space element also lies on the transformed space element of other space element. The cross ratio of the four points laying on a straight line is the same as the cross ratio of the four transformed points laying on the transformed straight line.

The projective point transformation that maps the points of the 3D space to the points of the 3D space bijectively maps four non-coplanar points to four non-coplanar points. This special point transformation can be given by a 4×4 regular matrix in the following form:

$$x'_I = c_{IK} \cdot x_K \quad (10)$$

where $|c_{IK}| \neq 0$, and $K = 1, 2, 3, 4$.

Theorem 1. *The set of projective transformations considering the performing of the transformations after each other, which is a transformation, as operations, forms an algebraic group.*

Proof of Theorem 1. The transformation should be given in the matrix form in the following form:

$$(X')^T = C_{IK} \cdot X^T \quad (11)$$

where $|C_{IK}| \neq 0$, and $K = 1, 2, 3, 4$.

By performing two projective transformations one after the other, the theorem can be written in the following forms:

$$(X')^T = C_{IK}^1 \cdot X^T \quad (12)$$

and

$$(X'')^T = C_{IK}^2 \cdot (X')^T \quad (13)$$

where $|C_{IK}^1| \neq 0$, and $|C_{IK}^2| \neq 0$.

Due to the previous equalities, Equation (14) is as follows:

$$(X'')^T = C_{IK}^2 \cdot C_{IK}^1 \cdot X^T \quad (14)$$

in which the $C_{IK}^2 \cdot C_{IK}^1$ is also a regular matrix from the multiplication theorem of the determinants.

Successive execution of projective transformations has associativity due to the similar property of the regular matrices.

There is a unit transformation, which can be given with the unit matrix, and its effect is to provide an identity.

Every transformation has an inverse, and if the original transformation is C described with a matrix, then this is given by the inverse matrix C^{-1} , and its existence as given by C ensures its regularity. $\square$

Definition 1. In the projective space, the set of points whose homogeneous coordinates fulfil the following equation:

$$a_{IK} \cdot x_I \cdot x_K = 0 \quad (I, K = 1, 2, 3, 4) \quad (15)$$

creates a second-order surface.

Definition 2. The (a_{IK}) matrix formed from the coefficients in the equation of the second-order surface is called the basic matrix of the surface.

The matrix of the second-order surfaces is symmetric, and it can be transformed into a symmetric matrix by projective transformations, in which the diagonal has a value of only +1, −1, or 0 by successive application of projective transformations, which is called the normalized form of the matrix.

Definition 3. The rank of the matrix of the second-order surface is the sum of the number of coefficients +1 and −1 in the matrix's normalized form, which is denoted by R .

Definition 4. The signature of the matrix of the second-order surface is the difference of the number of coefficients +1 and −1 taken in absolute value in the matrix's normalized form, which is denoted by S .

The rank and signature of the matrix of the second-order surface are invariant when the projective transformation is performed, so the classification of second-order surfaces has always been performed in a coordinate system in which the surface can be written in normal form, and the classification is based on the R rank and S signature. Cones and cylinders belong to the same class of second order surfaces when the cases $R = 3$ and $S = 1$ occur.

2.2. The Method for Creating a Relationship Between the Reference Surfaces of the Worms

In the constructive geometric model created for the production geometry development of conical and cylindrical worm gear drive pairs, the reference surfaces of the worms are placed according to Figure 4.

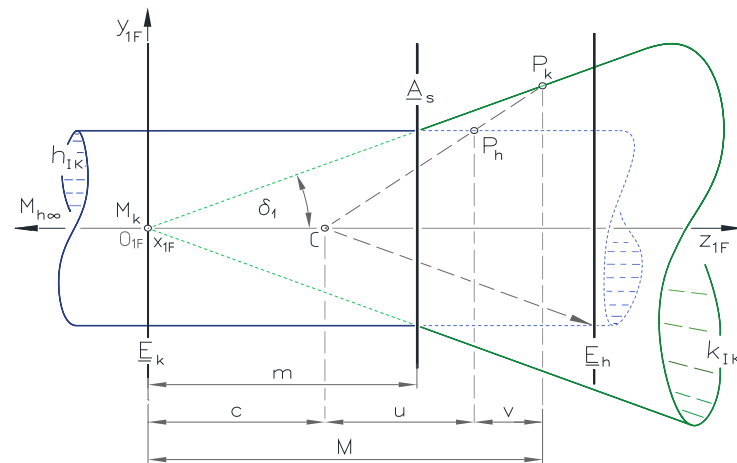


Figure 4. The sections of the reference cylinder and the reference cone in the coordinate plane $[yz]$, in the case of the cylinder vertex $M_{h\infty}$ is infinity distant on the z_{1F} , and vertex M_k is positioned at the origin O_{1F} .

The further developed constructive geometric model of the reference surfaces of the conical and cylindrical worms, with the exact rotation cone and cylinder with their axes laying on the z_{1F} coordinate axis, provide the possibility to create a central collineation relationship.

During the development of the constructive geometry model, the vertex M_k of the reference cone was placed at the origin O_{1F} of the K_{1F} coordinate system to facilitate the procedure of the creation of the relationship between the reference cone and the reference cylinder.

The cone vertex M_k and the cylinder vertex $M_{h\infty}$ with an infinite distance on the axis z_{1F} correspond to each other. The fixed plane A_s is the plane of the common circle of the reference cylinder and the reference cone. Furthermore, the center C of the central collineation is any point on the z_{1F} axis with homogeneous coordinates $C(0, 0, c, 1)$ in such a way that the center C can be any point on the axis z_{1F} except for the cylinder and cone vertices as well as the point laying on the fixed plane A_s . The arranger straight line passing through the point P_h of the cylinder surface and the collineation center C intersects the cone surface at the point P_k . The point P_h and the point P_k could be assigned to each other in the central collinear relationship. Those arranger straight lines laying at the center point C , which are parallel to the cone generator straight lines, intersect the cylinder at the points of the circle, the points of which are located on the plane E_h parallel to the fixed plane A_s .

During the central collineation mapping, the points of the E_h plane are transformed into points in infinity, and disappear into infinity, which is why it is called the vanishing plane. The vanishing plane E_h is as far from the center C as the vanishing plane E_k of the cone is from the fixed plane A_s . As shown in Figure 3, the half-opening angle of the right circular cone is the angle δ_1 . The height of the cone between the fixed plane A_s and the vertex M_k is denoted by m . The radius of the common circle of the right circular cylinder and the right circular cone laying on the fixed plane A_s is denoted by r . The radius of the circle of the point P_k laying on the cone is denoted by R . In the normed homogeneous Cartesian coordinate system, the coordinates of the points of cylinder (marked by x_I and x_K) complete the following relation:

$$h_{IK} \cdot x_I \cdot x_K = 0 \quad (I, K = 1, 2, 3, 4) \quad (16)$$

which can also be written in the matrix format, as follows:

$$\mathbf{X}^T \cdot \mathbf{H} \cdot \mathbf{X} = 0 \quad (17)$$

In the case of calculations with coordinates, it is advisable to write the matrix format with the coordinates in the following way:

$$\begin{pmatrix} x_1^h & x_2^h & x_3^h & x_4^h \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2 \end{pmatrix} \cdot \begin{pmatrix} x_1^h \\ x_2^h \\ x_3^h \\ x_4^h \end{pmatrix} = 0 \quad (18)$$

In the normed homogeneous Cartesian coordinate system, the coordinates of the points of cone (marked by x_I and x_K) complete the following relation:

$$k_{IK} \cdot x_I \cdot x_K = 0 (I, K = 1, 2, 3, 4) \quad (19)$$

which can also be written in the matrix format, as follows:

$$\mathbf{X}^T \cdot \mathbf{K} \cdot \mathbf{X} = 0 \quad (20)$$

In the case of calculations with coordinates, it is advisable to write the matrix format with the coordinates in the following way:

$$\begin{pmatrix} x_1^k & x_2^k & x_3^k & x_4^k \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{r^2}{m^2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1^k \\ x_2^k \\ x_3^k \\ x_4^k \end{pmatrix} = 0 \quad (21)$$

where Equation (22) is as follows:

$$\frac{r^2}{m^2} = tg^2 \delta_1 \quad (22)$$

which means that the following constructive geometrical model:

$$tg^2 \delta_1 = p_r^2 / p_a^2 \quad (23)$$

is clearly fulfilled.

2.2.1. Determination of the Transformation Matrix of the Mapping of the Conical Worm Reference Surface to the Cylindrical Worm Reference Surface

Based on the similar triangles that arise in the case of the placement outlined in Figure 3, the following relationships can be determined:

$$R/(M - c) = r/u = (R - r)/v \quad (24)$$

from which the parameter u can be expressed as follows:

$$u = \frac{r \cdot (M - c)}{R} \quad (25)$$

The z coordinate of the point P_h of the reference cylinder can be determined based on the relationships shown in Figure 3

$$z_{ph} = c + u = c + \frac{r \cdot (M - c)}{R} = \frac{R \cdot c + r \cdot M - r \cdot c}{R} = \frac{c \cdot (R - r) + r \cdot M}{R} \quad (26)$$

Table 1 shows the corresponding points between the reference surfaces of the worms, which have been applied in the procedure to determine the matrix that produces the surface point of the cone.

Table 1. The corresponding points between the reference cylinder and the reference cone.

Cylinder Points		Cone Points
$A_h = (r, 0, m, 1)$	$\Leftrightarrow$	$A_k = (r, 0, m, 1)$
$B_h = (-r, 0, m, 1)$	$\Leftrightarrow$	$B_k = (-r, 0, m, 1)$
$D_h = (0, -r, m, 1)$	$\Leftrightarrow$	$D_k = (0, -r, m, 1)$
$M_h = (0, 0, 1, 0)$	$\Leftrightarrow$	$M_k = (0, 0, 0, 1)$
$P_h = (0, r, z_{ph}, 1)$	$\Leftrightarrow$	$P_k = (0, R, M, 1)$

It is known that every point is also characterized by a non-zero scalar multiple of its homogeneous coordinates. For defining the matrix of the cone, the $\{E_1(1, 0, 0, 0), E_2(0, 1, 0, 0), E_3(0, 0, 1, 0), E_4(0, 0, 0, 1), E_5(1, 1, 1, 1)\}$ points were mapped to the points $\{A_k, B_k, D_k, M_k, P_k\}$ of the cone, whose homogeneous coordinates must be chosen such that the sum of the corresponding coordinates of the first four points are the corresponding coordinates of the fifth point to be. The values of the scalars $\lambda_1, \lambda_2, \lambda_3$, and λ_4 must be determined accordingly, as follows:

$$\left. \begin{aligned} A_k &= \lambda_1 \cdot (r, 0, m, 1) \\ B_k &= \lambda_2 \cdot (-r, 0, m, 1) \\ D_k &= \lambda_3 \cdot (0, -r, m, 1) \\ M_k &= \lambda_4 \cdot (0, 0, 0, 1) \\ P_k &= (0, R, M, 1) \end{aligned} \right\} \quad (27)$$

The coordinates of the point P_k are determined as a result of the solutions of the system of equations, relating to the proportionality factors λ_I as unknowns summarized in Table 2.

Table 2. Equations for the λ_I unknowns and solutions.

Equations		Solutions
$r \cdot \lambda_1 - r \cdot \lambda_2 = 0$	$\Rightarrow$	$\lambda_1 = \lambda_2 = \frac{M}{m}$
$-r \cdot \lambda_3 = R$	$\Rightarrow$	$\lambda_3 = -\frac{R}{r} = -\frac{M}{m}$
$m \cdot \lambda_1 + m \cdot \lambda_2 + m \cdot \lambda_3 = M$	$\Rightarrow$	
$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$	$\Rightarrow$	$\lambda_4 = \frac{m-M}{m}$

The columns of the matrix of the transformation $\{E_1, E_2, E_3, E_4, E_5\} \rightarrow \{A_k, B_k, D_k, M_k, P_k\}$ have been formed by the values of the homogeneous coordinates multiplied by λ_I , after the appropriate transformations.

$$U = \begin{pmatrix} \frac{M \cdot r}{m} & -\frac{M \cdot r}{m} & 0 & 0 \\ 0 & 0 & \frac{M \cdot r}{m} & 0 \\ M & M & -M & 0 \\ \frac{M}{m} & \frac{M}{m} & -\frac{M}{m} & \frac{m-M}{m} \end{pmatrix} = \begin{pmatrix} R & -R & 0 & 0 \\ 0 & 0 & R & 0 \\ M & M & -M & 0 \\ \frac{M}{m} & \frac{M}{m} & -\frac{M}{m} & \frac{m-M}{m} \end{pmatrix} \quad (28)$$

The coordinates of the point P_h are determined as a result of the solutions of the system of equations, relating to the proportionality factors λ_I as unknowns summarized in Table 2. The coordinates of the point P_k are determined as a result of the solutions of the system of equations, relating to the proportionality factors ν_I as unknowns summarized in Table 3.

Table 3. Equations for the ν_I unknowns and solutions.

Title 1	Title 2	Title 3
$r \cdot \nu_1 - r \cdot \nu_2 = 0$	$\Rightarrow$	$\nu_1 = 1$
$-r \cdot \nu_3 = r$	$\Rightarrow$	$\nu_2 = 1$
$m \cdot \nu_1 + m \cdot \nu_2 + m \cdot \nu_3 + \nu_4 = z_{ph}$	$\Rightarrow$	$\nu_3 = -1$
$\nu_1 + \nu_2 + \nu_3 = 1$	$\Rightarrow$	$\nu_4 = z_{ph} - m = \frac{c \cdot (R-r)}{R}$

The columns of the matrix of the transformation $\{E_1, E_2, E_3, E_4, E_5\} \rightarrow \{A_h, B_h, D_h, M_h, P_h\}$ have been formed by the values of the homogeneous coordinates multiplied by ν_I , as follows:

$$\mathbf{T} = \begin{pmatrix} r & -r & 0 & 0 \\ 0 & 0 & r & 0 \\ m & m & -m & c \cdot (R-r)/R \\ 1 & 1 & -1 & 0 \end{pmatrix} \quad (29)$$

The matrix of the central collineation can be created from the combination of the $\mathbf{U}$ and $\mathbf{T}$ matrices; thus, the matrix of the transformation from the reference cone surface to the reference cylinder surface has the following relation:

$$\mathbf{C}_{hk} = \mathbf{T} \cdot \mathbf{U}^{-1} \quad (30)$$

The determinant of the $\mathbf{U}$ matrix is required to determine the $\mathbf{C}_{hk}$, which can be given in the following form:

$$\det|\mathbf{U}| = -2 \cdot R^2 \cdot M \cdot (m - M)/m \quad (31)$$

The inverse of matrix $\mathbf{U}$ has been determined using the following relationship:

$$\mathbf{U}^{-1} = \text{adj}(\mathbf{U}) / \det(\mathbf{U}) \quad (32)$$

where $\text{adj}(\mathbf{U})$ refers to the adjoint of a matrix $\mathbf{U}$. The determined inverse of matrix $\mathbf{U}$ is the following matrix:

$$\mathbf{U}^{-1} = \begin{pmatrix} \frac{1}{2R} & \frac{1}{2R} & \frac{1}{2M} & 0 \\ -\frac{1}{2R} & \frac{1}{2R} & \frac{1}{2M} & 0 \\ 0 & \frac{1}{R} & 0 & 0 \\ 0 & 0 & \frac{-1}{m-M} & \frac{m}{m-M} \end{pmatrix} \quad (33)$$

The transformation matrix of the transition from the surface points of the reference cylinder to the surface points of the reference cone is determined by the following relation:

$$\mathbf{C}_{hk} = \mathbf{T} \cdot \mathbf{U}^{-1} \quad (34)$$

To calculate the element a_{33} of the matrix $\mathbf{C}_{kh}$, it is possible to arrive at a simpler form by using the ratio $\text{tg} \alpha = R/M = r/Mm$ and the similar triangles shown in Figure 3; therefore, the steps of this derivation can be performed as follows:

$$a_{33} = \frac{m}{M} - \frac{c \cdot (R-r)}{R} \cdot \frac{1}{m-M} = \frac{r}{R} - \frac{c \cdot (R-r)}{R \cdot (m-M)} = \left. \begin{aligned} \frac{r \cdot (m-M) - c \cdot (R-r)}{R \cdot m - R \cdot M} &= \frac{r \cdot m - r \cdot M - c \cdot (R-r)}{R \cdot m - R \cdot M} = \\ \frac{r \cdot m - R \cdot m - c \cdot (R-r)}{r \cdot M - R \cdot M} &= \frac{m \cdot (r-R) + c \cdot (r-R)}{M \cdot (r-R)} = \frac{m+c}{M} \end{aligned} \right\} \quad (35)$$

Then, the matrix of the central collineation relationship in the case of the reference surface of the conical worm to the reference surface of the cylindrical worm can be written in the following form:

$$\mathbf{C}_{hk} = \begin{pmatrix} \frac{r}{R} & 0 & 0 & 0 \\ 0 & \frac{r}{R} & 0 & 0 \\ 0 & 0 & \frac{m+c}{R} & \frac{c \cdot (R-r) \cdot m}{R \cdot (m-M)} \\ 0 & 0 & \frac{1}{M} & 0 \end{pmatrix} \quad (36)$$

By applying the appropriate relationships in Figure 3, the matrix of the central collineation relationship can be changed to the following simplified form:

$$\mathbf{C}_{hk} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{m+c}{m} & -c \\ 0 & 0 & \frac{1}{m} & 0 \end{pmatrix} \quad (37)$$

and, thus, its application is simplified.

2.2.2. Determination of the Transformation Matrix of the Mapping of the Cylindrical Worm Reference Surface to the Conical Worm Reference Surface

The central collinearity matrix can be created from the combination of the $\mathbf{T}$ and $\mathbf{U}$ matrices according to the following form:

$$\mathbf{C}_{kh} = \mathbf{U} \cdot \mathbf{T}^{-1} \quad (38)$$

The determinant of the $\mathbf{T}$ matrix required to determine the $\mathbf{C}_{kh}$ is as follows:

$$\det|\mathbf{T}| = 2 \cdot r^2 \cdot c \cdot (R - r) / R \quad (39)$$

The inverse of matrix $\mathbf{T}$ has been determined using the following relationship:

$$\mathbf{T}^{-1} = \text{adj}(\mathbf{T}) / \det(\mathbf{T}) \quad (40)$$

where $\text{adj}(\mathbf{T})$ refers to the adjoint of a matrix $\mathbf{T}$. The determined inverse of matrix $\mathbf{T}$ is the following matrix:

$$\mathbf{T}^{-1} = \begin{pmatrix} \frac{1}{2r} & \frac{1}{2r} & 0 & \frac{1}{2} \\ -\frac{1}{2r} & \frac{1}{2r} & 0 & \frac{1}{2} \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & \frac{R}{c \cdot (R-r)} & \frac{-R \cdot m}{c \cdot (R-r)} \end{pmatrix} \quad (41)$$

The transformation matrix of the transition from the surface points of the reference cylinder to the surface points of the reference cone is determined by the following relation:

$$\mathbf{C}_{hk} = \mathbf{U} \cdot \mathbf{T}^{-1} \quad (42)$$

which, by substitution, yields the following equality:

$$\mathbf{C}_{kh} = \begin{pmatrix} R & -R & 0 & 0 \\ 0 & 0 & R & 0 \\ M & M & -M & 0 \\ \frac{M}{m} & \frac{M}{m} & -\frac{M}{m} & \frac{m-M}{m} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2r} & \frac{1}{2r} & 0 & \frac{1}{2} \\ -\frac{1}{2r} & \frac{1}{2r} & 0 & \frac{1}{2} \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & \frac{R}{c \cdot (R-r)} & \frac{-R \cdot m}{c \cdot (R-r)} \end{pmatrix} \quad (43)$$

The central collineation matrix that maps the points of the reference surface of the conical worm to the points of the reference surface of the cylindrical worm can be determined using the following form:

$$\mathbf{C}_{kh} = \begin{pmatrix} \frac{R}{r} & 0 & 0 & 0 \\ 0 & \frac{R}{r} & 0 & 0 \\ 0 & 0 & 0 & M \\ 0 & 0 & \frac{R \cdot (m-M)}{m \cdot c \cdot (R-r)} & \frac{M}{m} - \frac{R \cdot (m-M)}{c \cdot (R-r)} \end{pmatrix} \quad (44)$$

By applying the appropriate relationships in Figure 3, the matrix of the central collineation relationship can be changed to the following simplified form:

$$\mathbf{C}_{kh} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & m \\ 0 & 0 & -\frac{1}{c} & \frac{m+c}{c} \end{pmatrix} \quad (45)$$

and, thus, its application is simplified.

3. Results

In the integrated production geometry model for the conical and cylindrical worm drive pairs of the motion transmission mechanisms, the normalized homogeneous coordinates are used. In the improved constructive geometry model, the central collineation relationship between the reference surfaces of the conical and cylindrical worms in the projective geometric space model has been determined in matrix form using the toolbox of matrix algebra.

In the case of the arrangement according to the constructive geometric model, the differences between the test methods for the machining of cylindrical and conical worm surfaces have been reduced.

3.1. The Matrix of the Central Collineation Relationship Between the Reference Surfaces of the Conical and Cylindrical Worms

The central collineation relationship has been written in both directions.

3.1.1. Determination of the Reference Cylinder Surface of the Worm Based on the Specified Reference Cone Surface of the Spiroid Worm, in the Case of the Placement of the Worms on a Common Axis

The reference cylinder surface points of the cylindrical worm can be produced by the projective transformation of the reference cone surface points of the conical worm with the $\mathbf{C}_{hk}$ matrix defined in the constructive geometric model with the following relationship:

$$\begin{pmatrix} x_1^h \\ x_2^h \\ x_3^h \\ x_4^h \end{pmatrix} = \mathbf{C}_{hk} \cdot \begin{pmatrix} x_1^k \\ x_2^k \\ x_3^k \\ x_4^k \end{pmatrix} \quad (46)$$

Equation (47) is as follows:

$$(x_1^h \ x_2^h \ x_3^h \ x_4^h) = (x_1^k \ x_2^k \ x_3^k \ x_4^k) \cdot \mathbf{C}_{hk}^T \quad (47)$$

The transformed points in the equation of the cylinder (18) yields the following relation:

$$(x_1^k \ x_2^k \ x_3^k \ x_4^k) \cdot \mathbf{C}_{hk}^T \cdot \mathbf{H} \cdot \mathbf{C}_{hk} \cdot \begin{pmatrix} x_1^k \\ x_2^k \\ x_3^k \\ x_4^k \end{pmatrix} = 0 \quad (48)$$

This relationship should result in the equation of the reference cone. From the relationship above, the necessary examination involves the following matrix product:

$$\mathbf{C}_{hk}^T \cdot \mathbf{H} \cdot \mathbf{C}_{hk} \quad (49)$$

The matrices written with their elements are in the following form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{m+c}{r} & \frac{1}{m} \\ 0 & 0 & -c & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{m+c}{r} & -c \\ 0 & 0 & \frac{1}{m} & 0 \end{pmatrix} \quad (50)$$

which can be described in the following formulation after multiplying the first two matrices as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{r^2}{m} \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{m+c}{r} & -c \\ 0 & 0 & \frac{1}{m} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (51)$$

The result, according to relation (21), is matrix $\mathbf{K}$ of the reference cone created by its homogeneous coordinates.

3.1.2. Determination of the Reference Cone Surface of the Spiroid Worm Based on the Specified Reference Cylinder Surface of the Worm, in the Case of the Placement of the Worms on a Common Axis

The reference cone surface points of the spiroid worm can be produced by the projective transformation of the reference cylinder surface points of the cylindrical worm with the $\mathbf{C}_{kh}$ matrix defined in the constructive geometric model with the following relationship:

$$\begin{pmatrix} x_1^k \\ x_2^k \\ x_3^k \\ x_4^k \end{pmatrix} = \mathbf{C}_{kh} \cdot \begin{pmatrix} x_1^h \\ x_2^h \\ x_3^h \\ x_4^h \end{pmatrix} \quad (52)$$

Equation (53) is as follows:

$$(x_1^k \ x_2^k \ x_3^k \ x_4^k) = (x_1^h \ x_2^h \ x_3^h \ x_4^h) \cdot \mathbf{C}_{kh}^T \quad (53)$$

The transformed points in the equation of the cone (21) yield the following relation:

$$(x_1^h \ x_2^h \ x_3^h \ x_4^h) \cdot \mathbf{C}_{kh}^T \cdot \mathbf{K} \cdot \mathbf{C}_{kh} \cdot \begin{pmatrix} x_1^h \\ x_2^h \\ x_3^h \\ x_4^h \end{pmatrix} = 0 \quad (54)$$

This relationship should result in the equation of the reference cylinder. From the relationship above, the necessary examination involves the following matrix product:

$$\mathbf{C}_{kh}^T \cdot \mathbf{K} \cdot \mathbf{C}_{kh} \quad (55)$$

The matrices written with their elements are in the following form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{c} \\ 0 & 0 & m & \frac{m+c}{c} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{r^2}{m^2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & m \\ 0 & 0 & -\frac{1}{c} & \frac{m+c}{c} \end{pmatrix} \quad (56)$$

which can be described in the following formulation after multiplying the first two matrices:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{r^2}{m} & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & m \\ 0 & 0 & -\frac{1}{c} & \frac{m+c}{c} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r^2 \end{pmatrix} \quad (57)$$

The result, according to relation (18), is matrix **H** of the reference cylinder of the cylindrical worm, created by its homogeneous coordinates.

3.2. Elimination of Thread Pitch Fluctuation During the Machining of the Conical Worm Surface

In the constructive geometric model created for the production geometry development of conical and cylindrical worm gear drive pairs, the reference surfaces of the worms are placed according to Figure 4.

In accordance with the arrangement outlined in Figure 5, it is necessary to rearrange the technological environment.

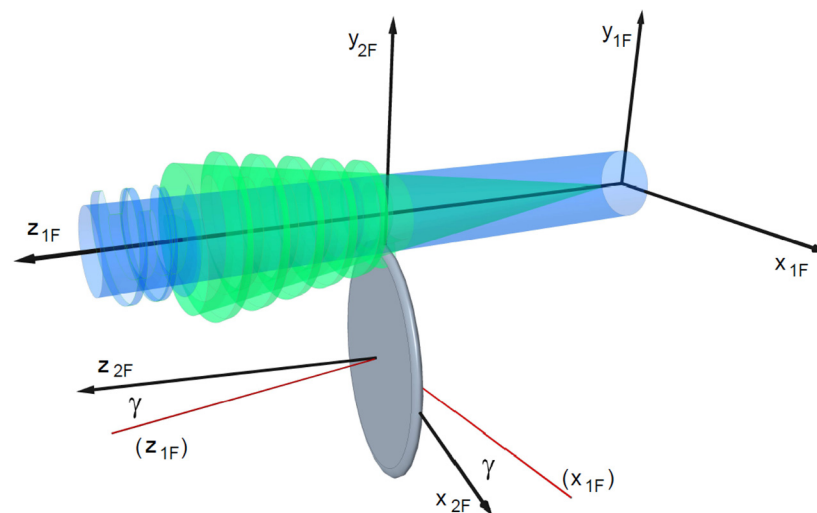


Figure 5. The sections of the reference cylinder and the reference cone in the coordinate plane [yz], in the case where the cylinder vertex $M_{h\infty}$ is infinity distance on the z_{1F} and vertex M_k is positioned at the origin O_{1F} .

In summary, it can be stated that the presented procedure is an important result in the theoretical and practical fields simultaneously. When extending the integrated production geometry model, further developed in the direction of mathematical generalization, to the projective space model, the matrix algebraic determination of the central collineation relationship created between the reference surfaces of the conical and cylindrical worms results in the general desirability of the production geometry studies. The placing of the conical and cylindrical reference surfaces in a central collineation relationship results in the elimination of thread pitch fluctuations when machining a conical worm surface.

4. Discussion

The formation of the result was influenced by some constructive articles, several of which have an effect on the further research goal.

The increasingly diverse technical constructions are constantly created with new technological solutions [38]. The variety of renewable technological solutions that create continuously developing components is also increasing [39], as a result of which the machining procedures must also be continuously adapted and developed based on mathematical analysis [40].

Among the many directions of machine modelling, significant results were also obtained with regard to the thickness of the material and how the machining is affected by the cutting vibration [41], which also shows a new research perspective.

The main purpose of model making was achieved, for example, in the industrial application, when the surface of the tooth side of the wheel in contact with the cylindrical worm was created on a theoretical basis with a three-dimensional CAD software (Creo Parametric 2.0 Primer) model, in such a way that it is suitable for comparing the contact patterns of the analytical and experimental tooth surfaces by manufacturing the worm wheel using the CAM process. The production method of the approximation required in industry has been validated [42], which is a recognition of all theoretical ideas, and an achievable goal.

The development of tool geometry involves descriptive geometry-based mathematical modelling, on the basis of which computer visualization helps control production accuracy, and this is one of the research projects that can be continued [43]. Technological properties determine the geometry of the surfaces produced during the production of the components. Reducing the deviation from the mathematically defined surface, resulting in an increase in the production accuracy, is the constant goal of engineering research, so its further development is also an open question [44]. Knowledge of the precise geometric relationships of the machining process of the elements of the motion transmission mechanisms is necessary for computer modelling, the systematization of which is exemplary in one research [45]. The use of more and more advanced modelling software requires an increasingly higher level of geometrical knowledge, which should be linked to info-communication [46]. The study of projective invariants can also be an open question in the preparation and motion study of the 3D model of the astronomical clock using stereographic projection [47].

A study has already been prepared to examine the bearing pattern of the spiroid worm gear drive [48], as well as to examine the curvature interference boundary line [49].

In the available literature on the manufacturing geometry of the elements of conical and helical worm drive pairs, the theoretical parts are often separated from the parts discussing practical problems; intending to connect these parts, the integrated manufacturing geometry model for the technological properties of the traditional thread grinder was created [35].

In order to establish the systematic programmability of the elements of the conical and cylindrical worm gear drive pairs, further mathematical generalization of the integrated mathematical model was set as the goal of the research work.

Then, as a second step forward, the basis for the development of the production geometry theory necessary for the geometrically correct machining of cylindrical and conical worm surfaces with a constant thread pitch involved a constructive geometric model designed for the kinematics of a changing technological center distance. Along with this, the idea of organizing the elements of conical and cylindrical worm gear drives was created. To create an analytical relationship between the reference surfaces of conical and cylindrical screws, an extension to the projective space model was necessary.

As a third step, in the constructive geometric model, the analytical form of the central collineation relationship between the reference surfaces of the conical and cylindrical worms defined by homogeneous coordinates was created, which provided a clear point transformation, the purpose of which is uniform treatment, which is a serious step forward in achieving manageability in the system.

A study was also made on the geometry of motion in the extended Euclidean space, which supports the correct choice of the research direction [50].

An open question for further research is the definition of the machining of helicoid surfaces and their uniform description in the constructive geometric model.

5. Conclusions

The creation of the central collineation relationship of the reference surfaces of conical and cylindrical snails in the extended Euclidean space is a significant step in the direction of mathematical generalization. The constructive production geometry model expands the production of worm surfaces with a constant thread pitch with the possibility of a variable technological wheelbase, which ensures geometrically correct production. The relationship defined in the extended Euclidean space is the basis for uniform programmability. The subject of further research work is to synchronize the processing of worms in the same form.

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References

- Wang, Y.; Xiong, L.; Feng, D.; Liu, X.; Zhao, S. Research Progress on the Manufacturing of Screw-Shaped Parts in Screw Compressors. *Appl. Sci.* **2024**, *14*, 1945. [\[CrossRef\]](#)
- Bendefy, A.; Horák, P. Cylindrical Gears with Changing Ratio. *Period. Polytech. Mech. Eng.* **2017**, *61*, 130–134. [\[CrossRef\]](#)
- Stanojević, M.; Tomović, R.; Ivanović, L.; Stojanović, B. Critical Analysis of Design of Ravigneaux Planetary Gear Trains. *Appl. Eng. Lett.* **2022**, *7*, 32–44. [\[CrossRef\]](#)
- Sztankovics, I. Analytical Determination of High-Feed Turning Procedures by the Application of Constructive Geometric Modeling. *FME Trans.* **2024**, *52*, 173–185. [\[CrossRef\]](#)
- Tolvaly-Roşca, F.; Forgó, Z.; Máté, M. Evaluation of a Mixed CAD Gear Modeling from Time and Precision Point of View. *Procedia Technol.* **2015**, *19*, 28–33. [\[CrossRef\]](#)
- Ladrón-de-Gueva-Munoz, M.C.; Alonso-García, M.; de-Cózar-Macías, Ó.D.; Blázquez-Parra, E.B. The Place of Descriptive Geometry in the Face of Industry 4.0 Challenges. *Symmetry* **2023**, *15*, 2190. [\[CrossRef\]](#)
- Skulić, A.; Stojanović, B.; Radosavljević, S.; Veličković, S. Experimental Determination of Worm Gearing Efficiency. *Appl. Eng. Lett.* **2019**, *4*, 115–119. [\[CrossRef\]](#)
- Bodzás, S.; Szanyi, G. Geometrical- and Tooth Contact Analysis of the Areas and Perimeters on the Archimedean Worm Wheels Depending on the Modification of the Axial Module. *Acta Polytech. Hung.* **2024**, *21*, 105–125. [\[CrossRef\]](#)
- Máté, M.; Tolvaly-Roşca, F.; Hodgyai, N.; Drăgoi, M.V. A new approach of defining the grinding wheel profile of the Gear Hob's Rake Face. In Proceedings of the IEEE Joint 22nd International Symposium on Computational Intelligence and Informatics and 8th International Conference on Recent Achievements in Mechatronics, Automation, Computer Science and Robotics (CINTI-MACRO 2022), Budapest, Hungary, 21–22 November 2022.
- Zhao, K.; Yang, J.; Li, Y.; Teng, B.; Ding, N. Design method and parameter optimization of small angle rake face of hourglass worm gear hob. *J. Adv. Mech. Des. Syst. Manuf.* **2024**, *18*, 1–18. [\[CrossRef\]](#)
- Simon, V. Multiobjective optimization of hypoid gears to improve operating characteristics. *Mech. Mach. Theory* **2019**, *146*, 905–914. [\[CrossRef\]](#)
- Dóka, T.; Horák, P. An Approach to Creating a Simple Digital Twin for Optimizing a Small Electric Concept Vehicle Drivetrain. In Proceedings of the 34th International ECMS Conference on Modelling and Simulation, European Council for Modelling and Simulation (ECMS), Wildau, Germany, 9–12 June 2020; pp. 328–333. [\[CrossRef\]](#)
- Hegedűs, G. Manufacturing Parameters Determination on Ball Nut Grinding. *Des. Mach. Struct.* **2015**, *5*, 33–38.
- Torcoratoru, C. Exploring Descriptive Geometry Through Simple Applications with Tikz Euclide: A Powerful Tool in Latex. *J. Ind. Des. Eng. Graph.* **2024**, *19*, 31–38.
- Mladinic, P.; Radovic, N. *Descriptive Geometry, Perspective, Monge's Procedure, Axonometry*, 1st ed.; PROVEN Grupa d.o.o.: Zagreb, Serbia, 2019; p. 170.
- Jenkovszky, L.; Lake, M.J.; Soloviev, V. János Bolyai, Carl Friedrich Gauss, Nikolai Lobachevsky and the New Geometry: Foreword. *Symmetry* **2023**, *15*, 707. [\[CrossRef\]](#)
- Wildberger, N.J. Universal Hyperbolic Geometry, Sydpoints and Finite Fields: A Projective and Algebraic Alternative. *Universe* **2018**, *4*, 3. [\[CrossRef\]](#)
- Ledneczki, P.; Monár, E. Projective geometry in engineering. *Period. Polytech. Mech. Eng.* **1995**, *39*, 43–60.
- Molnár, E.; Szirmai, J. Non-Euclidean Crystal Geometry. *S. Bohemia Math. Lett.* **2022**, *30*, 28–40.

20. Ding, Z.; Liu, S.; Liao, L.; Zhang, L. A digital construction framework integrating building information modeling and reverse engineering technologies for renovation projects. *Autom. Constr.* **2019**, *102*, 45–58. [\[CrossRef\]](#)
21. Kőrei, A.; Szilágyi, S. Displaying Parametric Curves with Virtual and Physical Tools. *Teach. Math.* **2022**, *25*, 61–73. [\[CrossRef\]](#)
22. Gharib, H.; Kovacs, G. Reliability analysis of marine diesel engines vs. industrial diesel engines: A comparative approach. *Acta Logist. —Int. Sci. J. About Logist.* **2024**, *11*, 325–337. [\[CrossRef\]](#)
23. Mohammed, M.A.A.; Szabó, N.P.; Szűcs, P. Characterization of groundwater aquifers using hydrogeophysical and hydrogeochemical methods in the eastern Nile River area, Khartoum State, Sudan. *Environ. Earth Sci.* **2023**, *82*, 21. [\[CrossRef\]](#)
24. Ábrahám, G.; Wenzelné Gerőfy, K.; Antal, Á.; Kovács, G. *Technical Optics*, 1st ed.; Institute of Mechatronics, Optics, and Mechanical Engineering Informatics, Budapest University of Technology and Economics: Budapest, Hungary, 2015; p. 308, ISBN 978-963-313-202-9.
25. Szabó, J.; Kunkli, R. The Generalization of Szabó's Theorem for Rectangular Cuboids and an Application. *J. Geom. Graph.* **2013**, *17*, 213–222.
26. Molnár, E. The Projective Interpretation of the Eight 3-dimensional Homogeneous Geometries, Contributions to Algebra and Geometry. *Beiträge Zur Algebra Und Geom.* **1997**, *38*, 261–288.
27. Molnár, E.; Prok, I.; Szirmai, J. On Maximal Homogeneous 3-Geometries and Their Visualization. *Universe* **2017**, *3*, 83. [\[CrossRef\]](#)
28. Redding, P. Projective Geometry as a Model for Hegel's Logic. *Logics* **2024**, *2*, 11–30. [\[CrossRef\]](#)
29. Stachel, H. Descriptive Geometry Meets Computer Vision—The Geometry of Two Images. *J. Geom. Graph.* **2006**, *10*, 137–153.
30. Gutiérrez de Ravé, E.; Jiménez-Hornero, F.J. A 3D Descriptive Geometry Problem-Solving Methodology Using CAD and Orthographic Projection. *Symmetry* **2024**, *16*, 476. [\[CrossRef\]](#)
31. Prado-Velasco, M.; García-Ruesgas, L. Intersection and Flattening of Surfaces in 3D Models through Computer-Extended Descriptive Geometry (CeDG). *Symmetry* **2023**, *15*, 984. [\[CrossRef\]](#)
32. Balajti, Z. Development of Production Geometry of Kinematical Drive Pairs. Ph.D. Thesis, University of Miskolc, Miskolc, Hungary, 2007; p. 171. (In Hungarian).
33. Karpuschewski, B.; Kundrák, J.; Felhő, C.; Varga, G.; Sztankovics, I.; Makkai, T.; Borysenko, D. Preliminary Investigation for the Effect of Cutting Tool Edge Geometry in High-Feed Face Milling. In Proceedings of the Vehicle and Automotive Engineering 2: Lecture Notes in Mechanical Engineering, Miskolc, Hungary, 23–25 May 2018; pp. 241–245.
34. Hoggai, N.; Drágoi, M.V.; Tolvaly-Rosca, F.; Máté, M. About the Grinding of Gear Hob's Rake Face. *Pap. Tech. Sci. Int. Sci. Ser. Transylv. Mus. Soc.* **2022**, *16*, 31–35. [\[CrossRef\]](#)
35. Dudás, I. *The Theory and Practice of Worm Gear Drives*; Penton Press: London, UK, 2000.
36. Balajti, Z.; Mátyás, Z. Proposed solution to eliminate pitch fluctuation in case of conical screw surface machining by apex adjustment. *Procedia Manuf.* **2021**, *55*, 266–273. [\[CrossRef\]](#)
37. Litvin, F. *Gear Geometry and Applied Theory*, 2nd ed.; Gear Load Sharing; Cambridge University Press: Cambridge, UK, 2004.
38. Gál, V.; Lukács, Z. Effect of Cooling Channels to the Press Hardening Tools Temperature. In Proceedings of the Vehicle and Automotive Engineering 3: Lecture Notes in Mechanical Engineering, Miskolc, Hungary, 25–26 November 2020; pp. 312–320. [\[CrossRef\]](#)
39. Cabezas, S.; Hegedűs, G.; Bencs, P. Transient heat convection analysis of a single rod in air cross-flow. *Pollack Period.* **2023**, *18*, 23–28. [\[CrossRef\]](#)
40. Sztankovics, I. The Analytical and Experimental Analysis of the Machined Surface Roughness in High-Feed Tangential Turning. *Eng* **2024**, *5*, 1768–1784. [\[CrossRef\]](#)
41. Wang, P.; Bai, Q.; Cheng, K.; Zhao, L.; Ding, H. The Modelling and Analysis of Micro-Milling Forces for Fabricating Thin-Walled Micro-Parts Considering Machining Dynamics. *Machines* **2022**, *10*, 217. [\[CrossRef\]](#)
42. Kawasaki, K.; Tsuji, I. Manufacturing Method for Large Cylindrical Worm Gear Set of ISO Type I on Universal CNC Machine Tools. *J. Manuf. Mater. Process.* **2023**, *7*, 53. [\[CrossRef\]](#)
43. Ábel, J. Computer Based Constructive Geometric and Analytical Development of the Manufacturing Geometry of Worm Gear Drive Pairs. Ph.D. Thesis, University of Miskolc, Miskolc, Hungary, 2023; p. 99. (In Hungarian).
44. Mátyás, Z. Intelligent Manufacturing System and Geometrically Exact Manufacture of the Helicoid Surfaces. Ph.D. Thesis, University of Miskolc, Miskolc, Hungary, 2022; p. 105. (In Hungarian).
45. Dudás, L. Modelling and simulation of a new worm gear drive having point-like contact. *Eng. Comput.* **2013**, *29*, 251–272. [\[CrossRef\]](#)
46. Papp, I.; Zichar, M. 3D Modelling and Printing Interpreted in Terms of Cognitive Infocommunication. In *Cognitive Infocommunications, Theory and Applications Topics in Intelligent Engineering and Informatics*; Springer: Berlin/Heidelberg, Germany, 2019; Volume 13. [\[CrossRef\]](#)
47. Popkonstantinović, B.; Obradović, R.; Stojićević, M.; Jeli, Z.; Cvetković, I.; Vasiljević, I.; Milojević, Z. The Design and Simulation of an Astronomical Clock. *Appl. Sci.* **2021**, *11*, 3989. [\[CrossRef\]](#)
48. Dudás, I.; Balajti, Z. Modelling and development for describing the bearing pattern of spiroid drives. In Proceedings of the Sixth IASTED International Conference on Robotics and Applications, Cambridge, MA, USA, 31 October–2 November 2005; pp. 203–208, ISBN 0-88986-521-3.

-
49. Meng, Q.; Zhao, Y.; Yang, Z. Curvature interference characteristic of conical worm gear. *Forsch Ingenieurwes* **2019**, *83*, 759–773. [[CrossRef](#)]
 50. Glaeser, G.; Stachel, H. Kinematics—Geometry of Motion. In *Open Geometry: OpenGL® + Advanced Geometry*; Springer: Berlin/Heidelberg, Germany, 1999; Volume 451, pp. 229–267. [[CrossRef](#)]

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