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# Lagrangian Coupling of Dissipative Electrodynamic Waves with the Thermal Absorption and Emission

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**Abstract:** Electromagnetic wave dissipation is experienced in radiative absorbing-emitting processes and signal transmissions via media. The absorbed wave initiates thermal processes in the conducting medium. Conversely, thermal processes generate electromagnetic waves in the vacuum–material interface region. The two processes do not take place symmetrically, i.e., the incoming and thermalizing electromagnetic spectrum does not occur in the reverse process. The conservation of energy remains in effect, and the loop process "electromagnetic wave–thermal propagation–electromagnetic wave" is dissipative. In the Lagrangian formalism, we provide a unified description of these two interconnected processes. We point out how it involves the origin of the asymmetry.

**Keywords:** Lagrangian formalism; Hamilonian description; generator potentials; electrodynamics; Maxwell's equations; telegrapher's equation; Klein–Gordon equation of the thermal conduction; Lorentz invariant behavior; natural coupling of processes; symmetry breaking effect; asymmetric loop; dissipation; irreversibility

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## 1. Introduction

### 1.1. Motivations and Aims

Electromagnetic radiation and its absorption in an electrically conductive material has been studied from many viewpoints. The reason is that mapping the mechanism of the phenomenon can help in the interpretation of countless physical (involving solid states [1], nanoscale systems [2]), chemical [3], technical (solar cells [4], solar thermal collectors [5], dusty radiative cooling surface [6], and novel materials' absorbing interfaces [7,8]), and even biological processes [9] (biological photovoltaics [10]). The solution is achieved by first looking at the electromagnetic field; it interacts with the material medium, and then the transferred energy creates the process in the medium. This is at least a three-step method. In reality, radiation and material transport energy appear simultaneously in a mutual interaction on the boundary surface. This is why there is another, mathematically interesting side to this radiation absorption problem: how to describe the entire phenomenon in a single summary framework. We will connect the physical fields describing the electromagnetic and thermal process with a coupling member. The expectation is that the mutual interaction will appear in the evolution of the fields, and if we look at the processes separately (at the decoupled case), we can see the well-known behavior of free fields.

The reverse of the absorption process is temperature radiation via the thermal field. On the one hand, this is created immediately during wave absorption; on the other hand, it happens in a non-symmetrical way. This means that the initial electromagnetic field does not recover. Although the total energy is conserved, the frequency spectrum of the thermal radiation is significantly different from the initial spectrum. This description is usually discussed in the context of blackbody radiation. Moreover, we look at processes as if they were separated from each other.

We neglect two processes in our study. One is the perfect reflection of the electromagnetic wave from the surface of the conductor, and the other is the reflection of thermal propagation from the boundary surface within the conducting material. Both processes are lossless from an energetic point of view. Thus, although it occurs simultaneously with the absorption-emission processes, it is irrelevant from the point of view of the coupled processes.

In the current work, the two processes are linked in the Lagrangian–Hamiltonian formalism [11], ensuring the joint operation of the processes. In the implementation of the coupling, there are certain cornerstones, such as the structure of the equations, the shapes of the energy densities and currents, the Joule source term, and the limit cases without coupling, which imply that the new terms that appear have a physical right to exist. In addition, the material parameters are naturally included in the equations. This means that electrical and thermal conductivity, mass density, and specific heat are parameters of the processes. The dependence of electromagnetic emission and absorption on material quality is part of the solution. This is much more than what blackbody radiation can reproduce. In the latter case, the material quality appears in neither absorption nor emission.

In the next section, we summarize the necessary mathematical framework for model creation in several steps.

### *1.2. Lagrangian and Hamiltonian Backgrounds*

The Lagrangian formalism must be formulated for two basic processes: the telegraph type of electrodynamics and the relativistic invariant equation of thermal energy propagation. The first carries the challenge of the appearance of a first derivative term representing dissipation. The latter (instead of the Fourier equation) is needed because both equations must know the condition of finite propagation speed and Lorentz invariance. In the following, we briefly summarize the form of the mathematical equations and the main steps of the solution.

#### 1.2.1. Lagrangian Basics for Electromagnetic Field

In implementing the Lagrangian description of the linear equations of motion, the real challenge is the order of the derivatives. Given that the telegraph equation contains a first-order time derivative, a Lagrangian function cannot be directly specified for measurable physical quantities. The derivation of the telegrapher's equation requires the Lagrange formulation first and then the complete canonical procedure for the Hamiltonian. The telegraph equation contains a second-order space derivative, making it easy to handle in the Lagrangian construction. As we will see later, it can be removed with the half-Fourier transform according to the spatial coordinate

$$\ddot{\varphi} + \alpha\dot{\varphi} - \beta\triangle\varphi = 0 \quad \longrightarrow \quad \ddot{\overline{\varphi}} + \alpha\dot{\overline{\varphi}} - \beta k^2\overline{\varphi} = 0, \tag{1}$$

and we also obtain an oscillator equation. (Here, $\varphi$ represents the physical field; $\alpha$, and $\beta$ are the constant parameters; the $\dot{}$ is the first time derivative, thus the $\ddot{}$ is the second derivative of the function; and $\triangle$ is the Laplacian operator.) Therefore, we only focus on time derivatives.

To achieve the aim of the Lagrangian construction, from the above reason, we start our examination from the equation of motion for the damped harmonic oscillator as follows:

$$\ddot{x} + 2\lambda\dot{x} + \omega^2 x = 0, \tag{2}$$

where $x$ is the displacement, $m$ is the mass, $\lambda$ is a specific damping factor, and $\omega$ is the angular frequency. Due to the first-order time derivative (the presence of this nonselfadjoint operator), for the measurable quantity $x$, we need to apply a generator potential of $q$ [12] as follows:

$$x = \ddot{q} - 2\lambda\dot{q} + \omega^2 q. \tag{3}$$

The assignment rule is that the measurable quantity must be expressed using the adjoint of the operators in Equation (2). The operators act on the generator potential function. A suitable choice of Lagrangian may be a quadratic expression as follows:

$$L = \frac{1}{2}\left(\ddot{q} - 2\lambda\dot{q} + \omega^2 q\right)^2. \tag{4}$$

The telegrapher equation in Equation (2) can be obtained after the variation performed by variable $q$ and the substitution of the definition of $x$ in Equation (3).

Since the Lagrange function contains first- and second-order time derivatives, both $q$ and $\dot{q}$ are considered general coordinates, and they have separate general canonically conjugated momenta by definition [13,14]:

$$q_1 = q, \tag{5}$$

and

$$q_2 = \dot{q}. \tag{6}$$

In our case, the calculated expression for the momentum of $p_1$ is

$$p_1 = \frac{\partial L}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}} = 4\lambda^2\dot{q} - \dddot{q} - \omega^2\dot{q} - 2\lambda\omega^2 q. \tag{7}$$

Similarly, we obtain the momentum of $p_2$ as

$$p_2 = \frac{\partial L}{\partial \ddot{q}} = \ddot{q} - 2\lambda\dot{q} + \omega^2 q. \tag{8}$$

Then, the Hamiltonian can be also calculated. In the first step, we obtain the Hamiltonian by the derivatives of the potential function of $q$:

$$\begin{aligned} H &= p_1\dot{q}_1 + p_2\dot{q}_2 - L \\ &= 2\lambda^2\dot{q}^2 - \dddot{q}\,\dot{q} - \omega^2\dot{q}^2 + \frac{1}{2}\ddot{q}^2 - \frac{1}{2}\omega^4 q^2. \end{aligned} \tag{9}$$

In the following step, we substitute the potential function with the coordinates of $q_1$, $q_2$ and the momenta $p_1$, $p_2$. We arrive at the canonical formulation of the Hamiltonian as

$$H = \frac{1}{2}p_2^2 - \omega^2 p_2 q_1 + p_1 q_2 + 2\lambda p_2 q_2. \tag{10}$$

To preserve the energy-like unit of the Hamiltonian, we need to transform the coordinates and the momenta applied in a special example. This is the mathematical framework in the Lagrangian formulation of the electromagnetic field.

#### 1.2.2. Lagrangian Basics for the Thermal Field

The problem of thermal energy propagation with a finite propagation speed is a stubborn topic that goes back a long time. Maxwell [15] already drew attention to this deficiency of the Fourier equation. Later, based on different ideas, Cattaneo [16] and Vernotte [17] independently tried to construct telegraph-type equations, saying that they guaranteed a finite propagation speed. This led to the particular problem, in that the second law of thermodynamics is usually not fulfilled. To resolve this, solutions including entropy were created [18–22]. Further investigations pointed out that the equation also defining the propagation mechanism has a ballistic–diffusive construction [23]. This results in a rather complicated solution. Therefore, it is understandable why in the current general formulation, we choose the Lorentz invariant negative mass term Klein–Gordon formulation [24], which is capable of both ballistic and diffusive propagation, and in which the related mathematical structure is not complicated. Thus, the calculus of variations can be easily applied.

Another important aspect is the Lagrangian formulation of thermal phenomena. Since the Fourier equation describes most of the phenomena well, several attempts have been made to do so [21,25,26]. The first-order derivative causes the construction problem. However, we now use the Lorentz invariant solution. It will be seen that, in this case, the mathematical formulation does not carry such difficulties.

To describe the thermal field, we use a mathematical framework that is suitable for coupling Lorentz invariant fields. This condition always ensures the description under the same physical criteria. If it is not necessary to maintain the Lorentz invariant condition, the Fourier heat conduction equation is obtained in the limiting case.

The Lorentz invariant equation of motion of the thermal field is a Klein–Gordon equation with a negative mass term [24]. This equation contains only second-order time and space derivatives as follows:

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} - Dy = 0, \tag{11}$$

where $c$ is the speed of light and $D$ is a constant. We do not need to introduce generator potentials. The Lagrange function is only a first-order derivative expression of the original field variable.

$$L = \frac{1}{2}\frac{1}{c^2}\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 + \frac{1}{2}D\,y^2. \tag{12}$$

Due to the first-order time derivative, the calculation of the canonical momentum is

$$p = \frac{\partial L}{\partial \frac{\partial y}{\partial t}} = \frac{1}{c^2}\frac{\partial y}{\partial t}, \tag{13}$$

by which the obtained Hamiltonian is

$$H = p\frac{\partial y}{\partial t} - L = \frac{1}{2}\frac{1}{c^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 - \frac{1}{2}D\,y^2. \tag{14}$$

This essentially means that the usual procedure can be applied.

#### 1.2.3. The Construction of the Coupling Term

The dissipative electrodynamic energy balance equation [27] can be deduced from Maxwell's equation. Ampère's law involves the conductive current; thus, the Joule dissipation of $-\sigma\mathbf{E}^2$ appears in the description. The dissipated electromagnetic energy is a source for the thermal process, so this term must appear in the thermal equation. We will see in detail that in the previously written formulation of the telegrapher equation, a similar structure of the potential field generates the measurable electrodynamic field $(\ddot{q} - 2\lambda\dot{q} + \omega^2 q) \sim \mathbf{E}$, by which the related Lagrangian should have a cubic structure as follows:

$$L \sim \mathbf{E}^2 y \sim (\ddot{q} - 2\lambda\dot{q} + \omega^2 q)^2 y. \tag{15}$$

By applying the variation by $y$ as follows:

$$\frac{\partial L}{\partial y} \sim (\ddot{q} - 2\lambda\dot{q} + \omega^2 q)^2 \sim \mathbf{E}^2, \tag{16}$$

we obtain the Joule heat in the thermal equation in Equation (11). This justifies the choice of the coupling term.

The result of the variation according to $q$ contributes to the movement of the electromagnetic field. According to the Euler–Lagrange equation, the individual terms are as follows:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L}{\partial \ddot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial q} \sim \frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathbf{E}y) + 2\lambda\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{E}y) + \omega^2(\mathbf{E}y) \tag{17}$$

The calculation shows that the thermal field appears in the equation of electrodynamics. Thus, the coupling is mutual. The transformation into thermal energy appears as a natural process, and conversely, the thermal field can create electromagnetic behavior.

### *1.3. Projecting the Structure of the Physical Model*

To achieve the outlined physics goal, we first describe the individual spaces. First, we formulate the Lagrangian function for the telegraph equation of the electromagnetic field in contact with an electrically conductive material in Sections 2 and 3. The Lorentz invariant negative mass term thermal propagation equation and the related Lagrangian function are presented in Section 4. In Section 5, we construct the coupling of the electromagnetic field with the thermal field using the Joule term, which expresses the energy dissipation of the electromagnetic field. This Lagrangian term gives the Joule heat in the thermal equation; on the other hand, a term in the telegraph equations describes the electric and magnetic field strengths. The Hamiltonian function of the fields containing the joint time evolution and energy relation is given in Sections 6 and 7. The absorption emission process is not symmetric. The spectrum of the incoming electromagnetic wave differs from the emission spectrum. The equations reflect this fact. In this way, we obtain an answer to dissipation and irreversibility appearing in the processes simultaneously. We elaborate on this in the discussion in Section 8. Finally, we give a brief summary of the aims and results of the article in Section 9.

## 2. The Energy Balance of the Electrodynamic Field in Conducting Media

When electrical currents exist in a conducting material, Maxwell's equations have the following forms:

$$\nabla \times \mathbf{B} = \mu_0\sigma\mathbf{E} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}, \tag{18}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{19}$$

where the electric field strength is $\mathbf{E}(x,t)$, the magnetic field strength is $\mathbf{B}(x,t)$, $\sigma$ is the electric conductivity, $\varepsilon_0$ is the free space permittivity, and $\mu_0$ is the magnetic permeability. Two important branches of the equations can be deduced.

One is that a telegraph equation can be derived for both fields $\mathbf{E}$ and $\mathbf{B}$, i.e.,

$$0 = \ddot{\mathbf{E}} + \frac{\sigma}{\varepsilon_0}\dot{\mathbf{E}} - \frac{1}{\varepsilon_0\mu_0}\triangle\mathbf{E}, \tag{20}$$

and

$$0 = \ddot{\mathbf{B}} + \frac{\sigma}{\varepsilon_0}\dot{\mathbf{B}} - \frac{1}{\varepsilon_0\mu_0}\triangle\mathbf{B}, \tag{21}$$

These are the equations of motion of the physical fields and the propagating damping oscillators. The structure of the equations is very important due to the appearance of the first-order time derivative. It will therefore be necessary to introduce potentials in the Lagrange description.

The other one concerns the energy balance:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2\right) + \frac{1}{\mu_0}\mathrm{div}(\mathbf{E}\times\mathbf{B}) = -\sigma\mathbf{E}^2. \tag{22}$$

Here, the term on the right-hand side is the Joule heat, the energy loss from the electromagnetic field. The electromagnetic field theory does not say anything about the energy lost from its original point of view. Our goal is to show how it becomes a source of thermal field energy.

## 3. Lagrangian Background of Telegrapher's Conduction

The goal is to obtain equations similar to the damped oscillator equation. Thus, we calculate the half-Fourier transform of the last terms in Equations (20) and (21):

$$\triangle \longrightarrow -\mathbf{k}^2 = -k_{\text{em}}^2, \tag{23}$$

replacing the Laplacian operator with the electromagnetic wave number of $k_{\text{em}}$. Thus, we obtain a second-order time-dependent ordinary differential equation for the transformed electric field strength $\overline{\mathbf{E}}(k,t)$:

$$\ddot{\overline{\mathbf{E}}} + \frac{\sigma}{\varepsilon_0}\dot{\overline{\mathbf{E}}} + \frac{1}{\varepsilon_0\mu_0}k_{\text{em}}^2\,\overline{\mathbf{E}} = 0, \tag{24}$$

and similarly, for the magnetic field strength $\overline{\mathbf{B}}(k,t)$:

$$\ddot{\overline{\mathbf{B}}} + \frac{\sigma}{\varepsilon_0}\dot{\overline{\mathbf{B}}} + \frac{1}{\varepsilon_0\mu_0}k_{\text{em}}^2\,\overline{\mathbf{B}} = 0. \tag{25}$$

To achieve the damping oscillator form in Equation (2), we introduce the parameters for the damping:

$$2\lambda = \frac{\sigma}{\varepsilon_0} \tag{26}$$

and the angular frequency of the wave:

$$\omega^2 = \frac{1}{\varepsilon_0\mu_0}k_{\text{em}}^2. \tag{27}$$

Applying these parameters, the half-Fourier transformed equation appears as the oscillator equation:

$$\ddot{\overline{\mathbf{E}}} + 2\lambda\dot{\overline{\mathbf{E}}} + \omega^2\,\overline{\mathbf{E}} = 0. \tag{28}$$

A similar equation is valid for the magnetic field strength:

$$\ddot{\overline{\mathbf{B}}} + 2\lambda\dot{\overline{\mathbf{B}}} + \omega^2\,\overline{\mathbf{B}} = 0. \tag{29}$$

Since the mathematical structure is similar to Equation (2), a potential $\overline{\mathbf{q}}(k,t)$ can therefore be defined similarly to Equation (3):

$$\overline{\mathbf{E}} = \ddot{\overline{\mathbf{q}}}_{\text{E}} - 2\lambda\dot{\overline{\mathbf{q}}}_{\text{E}} + \omega^2\overline{\mathbf{q}}_{\text{E}}. \tag{30}$$

The introduced potential, $\overline{\mathbf{q}}_{\text{E}}(k,t)$, denotes the half-Fourier transform of the space-dependent electric potential $\mathbf{q}_{\text{E}}(k,t)$. This can be formulated similarly for the other physical field:

$$\overline{\mathbf{B}} = \ddot{\overline{\mathbf{q}}}_{\text{B}} - 2\lambda\dot{\overline{\mathbf{q}}}_{\text{B}} + \omega^2\overline{\mathbf{q}}_{\text{B}}. \tag{31}$$

The introduction of the potentials $\mathbf{q}_{\text{E}}$ and $\mathbf{q}_{\text{B}}$ follows the algorithm given in Equation (3). We can see that the term expressing the dissipative behavior is involved via the electric and magnetic field strengths and not via the current density. Thus, the description is coherent with the energy balance in Equation (22). We will return to the significance of this in Section 6. Here, we will point out the relationship between these potentials and the classical vector potential of $\mathbf{A}$.

Thus, the Lagrangian can be formulated as follows:

$$L = \frac{1}{2}\left(\ddot{\overline{\mathbf{q}}} - 2\lambda\dot{\overline{\mathbf{q}}} + \omega^2\overline{\mathbf{q}}\right)^2, \tag{32}$$

where $\overline{\mathbf{q}}$ exists in the above sense. This is already a suitable form with which to return the structure of the telegraph equation. However, it is more convincing if the electric and magnetic field-generating potentials are present. In addition, we also want both to appear at the same time. These requirements can be achieved by the Lagrangian expression as follows:

$$\begin{aligned} \overline{L}_{\mathrm{em}} &= \frac{1}{2}\varepsilon_0\left(\ddot{\overline{\mathbf{q}}}_{\mathrm{E}} - 2\lambda\dot{\overline{\mathbf{q}}}_{\mathrm{E}} + \omega^2\overline{\mathbf{q}}_{\mathrm{E}}\right)^2 + \frac{1}{2\mu_0}\left(\ddot{\overline{\mathbf{q}}}_{\mathrm{B}} - 2\lambda\dot{\overline{\mathbf{q}}}_{\mathrm{B}} + \omega^2\overline{\mathbf{q}}_{\mathrm{B}}\right)^2 \\ &= \frac{1}{2}\varepsilon_0\overline{\mathbf{E}}^2 + \frac{1}{2\mu_0}\overline{\mathbf{B}}^2. \end{aligned} \tag{33}$$

The shape of the Lagrangian functions can be different due to the choice of suitable potentials and the applied algebraic expressions. The correct choice is that the derived Hamiltonian functions are identical to a time derivative and a divergence term. We will see in Section 6 that this Lagrangian function is the same as the one obtained in the other frame.

## 4. Construction of the Thermal Field

The Lorentz invariant Lagrangian of the thermal propagation is

$$L_{\mathrm{T}} = \frac{1}{2}\frac{1}{c^2}\left(\frac{\partial T}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial T}{\partial x}\right)^2 + \frac{1}{2}\frac{\omega_{0\mathrm{T}}^2}{c^2}T^2, \tag{34}$$

where the introduced parameter of $\omega_{0\mathrm{T}}$ involves the thermodynamic quantities, i.e., the specific heat $c_{\mathrm{v}}$ (exactly $\varrho C_{\mathrm{v}}$), and the thermal conductivity $\kappa$:

$$\omega_{0\mathrm{T}} = \frac{c^2 c_{\mathrm{v}}}{2\kappa}. \tag{35}$$

It can be seen that in this construction, we can directly specify the Lagrange function for $T$ describing the thermal space. The derived equation of motion is a Lorentz invariant equation:

$$\frac{1}{c^2}\frac{\partial^2 T}{\partial t^2} - \frac{\partial^2 T}{\partial x^2} - \frac{\omega_{0\mathrm{T}}^2}{c^2}T = 0. \tag{36}$$

The examination of the dispersion relation and the group velocity of this equation

$$\omega(k) = \sqrt{c^2k^2 - \frac{c^4c_{\mathrm{v}}^2}{4\kappa^2}} \quad \longrightarrow \quad \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{c}{\sqrt{1 - \frac{c^2c_{\mathrm{v}}^2}{4\kappa^2k^2}}} \tag{37}$$

shows that in the limit $c \longrightarrow \infty$, it turns into the dispersion relation and the group velocity of the classical Fourier heat conduction:

$$\omega(k) = -\mathrm{i}\frac{\kappa}{c_{\mathrm{v}}}k^2 \quad \longrightarrow \quad \frac{\mathrm{d}\omega}{\mathrm{d}k} = -2\mathrm{i}\frac{\kappa}{c_{\mathrm{v}}}k. \tag{38}$$

For further use, it is worth taking the half-Fourier transform of Equation (36):

$$\frac{1}{c^2}\frac{\partial^2\overline{T}}{\partial t^2} + \frac{1}{2}k^2\overline{T} - \frac{\omega_{0\mathrm{T}}^2}{c^2}\overline{T} = 0. \tag{39}$$

Furthermore, if the unit of the terms in the equation is the same as the Fourier line of heat conduction, we can treat it as a relation describing thermal propagation. The unit of measure of the thermal Lagrange function must be $\mathrm{J/m^3}$; then, we can connect it with the field of electrodynamics. For this purpose, we must apply a multiplier. The selection of this is unclear and so we formally introduce a factor as follows:

$$\kappa C_{\mathrm{emT}}, \tag{40}$$

where $C_{\mathrm{emT}}$ is the parameter to be determined later. The unit of it is s/K. With this, the unit is a correct Lagrange function:

$$\overline{L}_{th} = \kappa C_{\mathrm{emT}} \left[ \frac{1}{2} \frac{1}{c^2} \left( \frac{d\overline{T}}{dt} \right)^2 - \left( \frac{1}{2} k^2 - \frac{1}{2} \frac{\omega_{0\mathrm{T}}^2}{c^2} \right) \overline{T}^2 \right]. \tag{41}$$

The obtained Euler–Lagrange equation is

$$\frac{1}{c^2} \frac{d^2 \overline{T}}{dt^2} + \left( k^2 - \frac{\omega_{0\mathrm{T}}^2}{c^2} \right) \overline{T} = 0. \tag{42}$$

The above multiplication factor does not affect the form of the equation.

We can define the canonical momentum as usual:

$$\overline{T} \quad \longrightarrow \quad \overline{p}_{\mathrm{th}} = \frac{d\overline{L}_{\mathrm{th}}}{d\frac{d\overline{T}}{dt}} = \frac{\kappa C_{\mathrm{emT}}}{c^2} \frac{d\overline{T}}{dt}. \tag{43}$$

Applying the canonical variables $\overline{T}$ and $\overline{p}_{\mathrm{th}}$, the Hamiltonian of the thermal part is

$$\overline{H}_{\mathrm{th}} = \overline{p}_{\mathrm{th}} \frac{d\overline{T}}{dt} - \overline{L}_{\mathrm{th}} = \kappa C_{\mathrm{emT}} \left[ \frac{1}{2} \frac{1}{c^2} \left( \frac{d\overline{T}}{dt} \right)^2 + \left( \frac{1}{2} k^2 - \frac{1}{2} \frac{\omega_{0\mathrm{T}}^2}{c^2} \right) \overline{T}^2 \right]. \tag{44}$$

## 5. Coupling of the Electrodynamic and the Thermal Fields

The loss of the electrodynamic energy (volumetric energy density per unit time) $\sigma \mathbf{E}^2$ appears in the thermal field as a source. This makes it possible for the electromagnetic effect to appear in the equation of the thermal field. The electric field strength can be replaced using Equation (30). This energy dissipation term (source for the thermal field) can be deduced from an interacting term as follows:

$$\sigma \mathbf{E}^2 \quad \longrightarrow \quad \sigma \left( \ddot{\overline{\mathbf{q}}}_{\mathrm{E}} - 2\lambda \dot{\overline{\mathbf{q}}}_{\mathrm{E}} + \omega^2 \overline{\mathbf{q}}_{\mathrm{E}} \right)^2 C_{\mathrm{emT}} \overline{T}, \tag{45}$$

This formulation agrees with the presumed mathematical formula of the cubic-order interaction in Equation (15). We modify this expression somewhat for future use.

It would be worthwhile to make this relationship symmetric in terms of the electric and magnetic field strengths. That is why we consider Faraday's law of induction $\nabla \times \mathbf{E} = -\dot{\mathbf{B}}$ in Equation (19). After its Fourier transform, we obtain

$$\mathbf{k} \times \mathbf{E} = \omega \mathbf{B}. \tag{46}$$

From this, we can deduce the well-known relation between the magnitude of electric and magnetic field strengths:

$$c\mathbf{B} = \mathbf{e}_{\mathrm{k}} \times \mathbf{E} \qquad \mathbf{e}_{\mathrm{k}} = \mathbf{k}/|\mathbf{k}| \qquad E = cB. \tag{47}$$

Applying this relation, we can divide the expression of Joule's heat into two equal parts:

$$\sigma\mathbf{E}^2 = \frac{1}{2}\sigma\mathbf{E}^2 + \frac{1}{2}\sigma c^2\mathbf{B}^2 \quad \longrightarrow$$
$$\underbrace{\left(\frac{1}{2}\sigma\left(\ddot{\overline{\mathbf{q}}}_{\mathrm{E}} - 2\lambda\dot{\overline{\mathbf{q}}}_{\mathrm{E}} + \omega^2\overline{\mathbf{q}}_{\mathrm{E}}\right)^2 + \frac{1}{2}\sigma c^2\left(\ddot{\overline{\mathbf{q}}}_{\mathrm{B}} - 2\lambda\dot{\overline{\mathbf{q}}}_{\mathrm{B}} + \omega^2\overline{\mathbf{q}}_{\mathrm{B}}\right)^2\right)C_{\mathrm{emT}}\overline{T}}_{\overline{L}_{\mathrm{Joule}}}, \tag{48}$$

One part pertains to the electric and the other to the magnetic field strengths. This division into two parts allows the Lagrangian function to be symmetric in the electrodynamics variables. This is important from the point of view of mathematical structure. Now, we can formulate the Lagrangian fucntion from the pure thermal and Joule parts:

$$\begin{aligned}
\overline{L}_{\mathrm{th+Joule}} &= C_{\mathrm{emT}}\left[\frac{1}{2}\frac{1}{c^2}\left(\frac{\partial\overline{T}}{\partial t}\right)^2 - \left(\frac{1}{2}k^2 - \frac{1}{2}\frac{\omega_{0\mathrm{T}}^2}{c^2}\right)\overline{T}^2\right] \\
&+ \frac{1}{2}C_{\mathrm{emT}}\,\sigma\left(\ddot{\overline{\mathbf{q}}}_{\mathrm{E}} - 2\lambda\dot{\overline{\mathbf{q}}}_{\mathrm{E}} + \omega^2\overline{\mathbf{q}}_{\mathrm{E}}\right)^2\overline{T} \\
&+ \frac{1}{2}C_{\mathrm{emT}}\,\sigma c^2\left(\ddot{\overline{\mathbf{q}}}_{\mathrm{B}} - 2\lambda\dot{\overline{\mathbf{q}}}_{\mathrm{B}} + \omega^2\overline{\mathbf{q}}_{\mathrm{B}}\right)^2\overline{T}.
\end{aligned} \tag{49}$$

Later, we will use this mathematical form in the calculations, because the steps of the variation calculation will have to be applied according to the $\mathbf{q}_{\mathrm{E}}$ and $\mathbf{q}_{\mathrm{B}}$ spaces. However, for easier interpretation, it is worth displaying it in a more compact form as follows:

$$\overline{L}_{\mathrm{th+Joule}} = \kappa C_{\mathrm{emT}}\left[\frac{1}{2}\frac{1}{c^2}\left(\frac{\partial\overline{T}}{\partial t}\right)^2 - \left(\frac{1}{2}k^2 - \frac{1}{2}\frac{\omega_{0\mathrm{T}}^2}{c^2}\right)\overline{T}^2\right] + C_{\mathrm{emT}}\,\sigma\mathbf{E}^2\overline{T}. \tag{50}$$

The Joule part combines the interaction of the thermal and electrodynamic fields. We obtain the related Euler–Lagrange equation for the thermal transport after the variation in temperature $\overline{T}$:

$$\begin{aligned}
\kappa\left[\frac{1}{c^2}\frac{\partial^2\overline{T}}{\partial t^2} + \left(k^2 - \frac{\omega_{0\mathrm{T}}^2}{c^2}\right)\overline{T}\right] &= \frac{1}{2}\sigma\left(\ddot{\overline{\mathbf{q}}}_{\mathrm{E}} - 2\lambda\dot{\overline{\mathbf{q}}}_{\mathrm{E}} + \omega^2\overline{\mathbf{q}}_{\mathrm{E}}\right)^2 + \frac{1}{2}\sigma c^2\left(\ddot{\overline{\mathbf{q}}}_{\mathrm{B}} - 2\lambda\dot{\overline{\mathbf{q}}}_{\mathrm{B}} + \omega^2\overline{\mathbf{q}}_{\mathrm{B}}\right)^2 \\
&= \sigma\overline{\mathbf{E}}^2.
\end{aligned} \tag{51}$$

We see that the coupling in Equation (45) gives the physically correct thermal equation. The Joule heat appears as a source in the thermal transport equation.

### *5.1. Canonical Variables of the Joule Part*

As in the thermal case, we must calculate the canonical variables to obtain the Hamiltonian of the interacting Joule part. Due to the second derivative structure, not only does the canonical momenta involve further terms than usual, but canonical pairs appear for the first derivative [14]. So, we have four canonical pairs, two for the electric field strength variable

$$\overline{\mathbf{q}}_{\mathrm{E}} \quad \longrightarrow \quad \overline{\mathbf{p}}_{\mathrm{E},1} = \frac{\partial\overline{L}_{\mathrm{Joule}}}{\partial\dot{\overline{\mathbf{q}}}_{\mathrm{E}}} - \frac{d}{dt}\frac{\partial\overline{L}_{\mathrm{Joule}}}{\partial\ddot{\overline{\mathbf{q}}}_{\mathrm{E}}} = -2C_{\mathrm{emT}}\,\sigma\lambda\overline{\mathbf{E}}\,\overline{T} - C_{\mathrm{emT}}\,\sigma\frac{d}{dt}\left(\overline{\mathbf{E}}\,\overline{T}\right), \tag{52}$$

and

$$\dot{\overline{\mathbf{q}}}_{\mathrm{E}} \quad \longrightarrow \quad \overline{\mathbf{p}}_{\mathrm{E},2} = \frac{\partial\overline{L}_{\mathrm{Joule}}}{\partial\ddot{\overline{\mathbf{q}}}_{\mathrm{E}}} = C_{\mathrm{emT}}\,\sigma\overline{\mathbf{E}}\,\overline{T}. \tag{53}$$

Moreover, two relations are for the magnetic field strength variables:

$$\overline{\mathbf{q}}_{\mathrm{B}} \quad \longrightarrow \quad \overline{\mathbf{p}}_{\mathrm{B},1} = \frac{\partial \overline{L}_{\mathrm{Joule}}}{\partial \dot{\overline{\mathbf{q}}}_{\mathrm{B}}} - \frac{d}{dt}\frac{\partial \overline{L}_{\mathrm{Joule}}}{\partial \ddot{\overline{\mathbf{q}}}_{\mathrm{B}}} = -2C_{\mathrm{emT}}\,\sigma c^2 \lambda \overline{\mathbf{B}}\,\overline{T} - C_{\mathrm{emT}}\,\sigma c^2 \frac{d}{dt}\left(\overline{\mathbf{B}}\,\overline{T}\right), \tag{54}$$

and

$$\dot{\overline{\mathbf{q}}}_{\mathrm{B}} \quad \longrightarrow \quad \overline{\mathbf{p}}_{\mathrm{B},2} = \frac{\partial \overline{L}_{\mathrm{Joule}}}{\partial \ddot{\overline{\mathbf{q}}}_{\mathrm{B}}} = C_{\mathrm{emT}}\,\sigma c^2 \overline{\mathbf{B}}\,\overline{T}. \tag{55}$$

*5.2. Hamiltonian of the Joule Part*

The obtained canonical variables enables us to calculate the Joule part of the Hamiltonian:

$$\begin{aligned}
\overline{H}_{\mathrm{Joule}} &= \overline{\mathbf{p}}_{\mathrm{E},1}\dot{\overline{\mathbf{q}}}_{\mathrm{E}} + \overline{\mathbf{p}}_{\mathrm{E},2}\ddot{\overline{\mathbf{q}}}_{\mathrm{E}} + \overline{\mathbf{p}}_{\mathrm{B},1}\dot{\overline{\mathbf{q}}}_{\mathrm{B}} + \overline{\mathbf{p}}_{\mathrm{B},2}\ddot{\overline{\mathbf{q}}}_{\mathrm{B}} - \overline{L}_{\mathrm{Joule}} = \\
&- 2C_{\mathrm{emT}}\,\sigma\lambda\overline{\mathbf{E}}\,\overline{T}\,\dot{\overline{\mathbf{q}}}_{\mathrm{E}} - C_{\mathrm{emT}}\,\sigma\frac{d}{dt}\left(\overline{\mathbf{E}}\,\overline{T}\right)\dot{\overline{\mathbf{q}}}_{\mathrm{E}} + C_{\mathrm{emT}}\,\sigma\overline{\mathbf{E}}\,\overline{T}\,\ddot{\overline{\mathbf{q}}}_{\mathrm{E}} \\
&- 2C_{\mathrm{emT}}\,\sigma\lambda c^2\overline{\mathbf{B}}\,\overline{T}\,\dot{\overline{\mathbf{q}}}_{\mathrm{B}} - C_{\mathrm{emT}}\,\sigma c^2\frac{d}{dt}\left(\overline{\mathbf{B}}\,\overline{T}\right)\dot{\overline{\mathbf{q}}}_{\mathrm{B}} + C_{\mathrm{emT}}\,\sigma c^2\overline{\mathbf{B}}\,\overline{T}\,\ddot{\overline{\mathbf{q}}}_{\mathrm{B}} \\
&- \frac{1}{2}C_{\mathrm{emT}}\,\sigma\overline{\mathbf{E}}^2\overline{T} - \frac{1}{2}C_{\mathrm{emT}}\,\sigma c^2\overline{\mathbf{B}}^2\overline{T}.
\end{aligned} \tag{56}$$

The expression involves the generator potentials $\overline{\mathbf{q}}_{\mathrm{E}}$ and $\overline{\mathbf{q}}_{\mathrm{B}}$. It is correct, but it is useful to apply only one potential field. So, later (see Equation (71)), we turn back for a further discussion of this part of the Hamiltonian.

*5.3. Equations of Motion with the Joule Term*

We arrive at the complete description of the electrodynamic and thermal field coupling by the sum of the Lagrangians in Equations (33) and (49):

$$\overline{L}(\overline{\mathbf{q}}, \overline{T}) = \overline{L}_{\mathrm{em}} + \overline{L}_{\mathrm{th+Joule}}. \tag{57}$$

The variation with respect to $\overline{\mathbf{q}}_{\mathrm{E}}$ results in

$$\ddot{\overline{\mathbf{E}}} + \frac{\sigma}{\varepsilon_0}\dot{\overline{\mathbf{E}}} + \omega^2\,\overline{\mathbf{E}} = -\frac{\sigma}{\varepsilon_0}C_{\mathrm{emT}}\left(2\dot{\overline{\mathbf{E}}}\,\dot{\overline{T}} + \overline{\mathbf{E}}\,\ddot{\overline{T}} + \frac{\sigma}{\varepsilon_0}\overline{\mathbf{E}}\,\dot{\overline{T}}\right). \tag{58}$$

This calculation corresponds to the coupling in Equation (17). Similarly, via the variation with respect to $\overline{\mathbf{q}}_{\mathrm{B}}$, we obtain

$$\ddot{\overline{\mathbf{B}}} + \frac{\sigma}{\varepsilon_0}\dot{\overline{\mathbf{B}}} + \omega^2\,\overline{\mathbf{B}} = -\frac{\sigma}{\varepsilon_0}C_{\mathrm{emT}}\left(2\dot{\overline{\mathbf{B}}}\,\dot{\overline{T}} + \overline{\mathbf{B}}\,\ddot{\overline{T}} + \frac{\sigma}{\varepsilon_0}\overline{\mathbf{B}}\,\dot{\overline{T}}\right). \tag{59}$$

The identical structure of the two equations is important because the change in time of the electric and magnetic field strengths is the same. The term on the right side of the equation is the result of the thermal feedback.

*5.4. The Thermally Emitted Outgoing Electromagnetic Wave*

On the other hand, we must remember that here we want to describe the electromagnetic field generated by the thermal field. Therefore, the electromagnetic field does not go through a damping process; however, on the contrary, it goes through an exciting process. We can express this by reversing time in a certain sense. The easiest way to express this in a mathematical context is to reverse the sign of the first-order time changes in the electromagnetic field: **E** and **B**. That is, the above equations describe the emitted outgoing radiation when we write

$$\ddot{\overline{\mathbf{E}}} - \frac{\sigma}{\varepsilon_0}\dot{\overline{\mathbf{E}}} + \omega^2\,\overline{\mathbf{E}} = -\frac{\sigma}{\varepsilon_0}C_{\mathrm{emT}}\left(-2\dot{\overline{\mathbf{E}}}\,\dot{\overline{T}} + \overline{\mathbf{E}}\,\ddot{\overline{T}} + \frac{\sigma}{\varepsilon_0}\overline{\mathbf{E}}\,\dot{\overline{T}}\right), \tag{60}$$

and

$$\ddot{\overline{\mathbf{B}}} - \frac{\sigma}{\varepsilon_0}\dot{\overline{\mathbf{B}}} + \omega^2\,\overline{\mathbf{B}} = -\frac{\sigma}{\varepsilon_0}C_{\mathrm{emT}}\left(-2\dot{\overline{\mathbf{B}}}\,\dot{\overline{T}} + \overline{\mathbf{B}}\,\ddot{\overline{T}} + \frac{\sigma}{\varepsilon_0}\overline{\mathbf{B}}\,\dot{\overline{T}}\right). \tag{61}$$

Previously we could see in Equation (51) that the variation concerning $\overline{T}$ provides the thermal transport, including the Joule heat effect. Finally, we can conclude that with the above relations, the electromagnetic and thermal fields are coupled and that the obtained relations are physically relevant.

## 6. The Dissipative Electrodynamic Hamiltonian Build Up

We show that the potentials introduced are equivalent to the vector potential formulation and are even more suitable in dissipative cases. The mathematical relationship between the fields is not complicated, the Hamiltonians are equivalent expressions, and, in dissipative cases, the expression of the current density does not have to be a known function.

So, we would like to handle the fields together. The electromagnetic field involves both the $\overline{E}_{\mathrm{y}}$ and $\overline{B}_{\mathrm{z}}$ fields; thus, we can express a sum of the field vectors in the form

$$\overline{\mathbf{E}} + c\overline{\mathbf{B}}; \quad \overline{\mathbf{E}} \perp \overline{\mathbf{B}}. \tag{62}$$

The canonical coordinate $\mathbf{q}_2$ may be defined as

$$\overline{\mathbf{q}}_2 = \dot{\overline{\mathbf{q}}} = \dot{\overline{\mathbf{q}}}_{\mathrm{E}} + \frac{1}{c}\dot{\overline{\mathbf{q}}}_{\mathrm{B}} = -\overline{\mathbf{A}}, \tag{63}$$

because it can generate $\overline{\mathbf{E}}$ and $\overline{\mathbf{B}}$ simultaneously, as shown in Equations (30) and (31). Taking the relation in Faraday's law as shown in Equation (46), due to the linear connection between $\overline{\mathbf{E}}$ and $\overline{\mathbf{B}}$, we can conclude that the vector potential of $\overline{\mathbf{A}}$ is also the canonical coordinate (generator field) for the magnetic field strength.

Finally, in the electromagnetic field, the canonical momentum of $\overline{\mathbf{p}}_2$ to the coordinate of $\overline{\mathbf{q}}_2$

$$\overline{\mathbf{p}}_2 = \overline{\mathbf{E}} + c\overline{\mathbf{B}} \tag{64}$$

can be identified by the sum of the electric field strength of $\overline{\mathbf{E}}$ and the magnetic field strength of $\overline{\mathbf{B}}$. This is a combination of the electromagnetic variables, but the fields do not lose their identity from this mathematical step.

Without spoiling the generality, consider an electromagnetic wave for which the following choice of directions holds:

$$\mathbf{e}_{\mathrm{k}} = (1,0,0) \qquad \mathbf{E} = (0,\overline{E}_{\mathrm{y}},0) \qquad \mathbf{B} = (0,0,\overline{B}_{\mathrm{z}}). \tag{65}$$

The wave propagates in the direction of the x-axis as $\mathbf{e}_{\mathrm{k}}$ shows. Applying these, the related modified vector potential is

$$\overline{\mathbf{A}} = (0,\overline{A}_{\mathrm{y}},0). \tag{66}$$

The other canonical conjugated pair is $\overline{\mathbf{p}}_1$ and $\overline{\mathbf{q}}_1$. The $\overline{\mathbf{q}}_1$ is

$$\overline{\mathbf{q}}_1 = \overline{\mathbf{M}}; \qquad \dot{\overline{\mathbf{M}}} = -\overline{\mathbf{A}} \qquad \longrightarrow \qquad \overline{\mathbf{q}}_1 = \overline{\mathbf{M}} = \frac{1}{\mathrm{i}\omega}\overline{\mathbf{A}} \tag{67}$$

We need to express $\overline{\mathbf{p}}_1$. By applying Equations (7), (28), and (29), we obtain

$$\begin{aligned}\overline{\mathbf{p}}_1 \quad &\longrightarrow \quad \left(-\dddot{q} + 2\lambda\ddot{q} - \omega^2\dot{q}\right) + \left(-2\lambda\ddot{q} + 4\lambda^2\dot{q} - 2\lambda\omega^2 q\right) = -\dot{\mathbf{E}} - 2\lambda\mathbf{E} - c\dot{\mathbf{B}} - 2\lambda c\mathbf{B} \\ &\longrightarrow \quad -\frac{\omega}{\mathrm{i}}(\overline{\mathbf{E}} + c\overline{\mathbf{B}}). \end{aligned} \tag{68}$$

The Hamiltonian can be formulated by the related telegrapher equation's structure in Equation (10). To obtain the correct energy unit of the Hamiltonian, we multiply it by the permittivity of vacuum $\varepsilon_0$ as follows:

$$H \longrightarrow \overline{H}_{\mathrm{em}} = \varepsilon_0 H \tag{69}$$

We obtain the exact Hamiltonian by the following substitution of the calculated variables:

$$\begin{aligned} \overline{H}_{\mathrm{em}} &= \frac{1}{2}\varepsilon_0 p_2^2 - \varepsilon_0\omega^2 p_2 q_1 + \varepsilon_0 p_1 q_2 + 2\lambda\varepsilon_0 p_2 q_2 \\ &= \frac{1}{2}\varepsilon_0(\overline{E}+c\overline{B})^2 \underbrace{-\varepsilon_0\omega^2(\overline{E}+c\overline{B})\frac{\overline{A}}{\mathrm{i}\omega} + \varepsilon_0\frac{\omega}{\mathrm{i}}(\overline{E}+c\overline{B})\overline{A}}_{=0} - \sigma\overline{E}\,\overline{A} \\ &= \frac{1}{2}\varepsilon_0\overline{E}^2 + \frac{1}{2\mu_0}\overline{B}^2 - \sigma\overline{E}\,\overline{A}. \end{aligned} \tag{70}$$

The electromagnetic energy decreases by the term $-\sigma\overline{E}\,\overline{A}$ and turns into thermal energy. In this formulation, the shape of the calculated Hamiltonian is the usual one. This is important because, unlike the one used in the field theory of electrodynamics, here, the conducting current $\sigma\overline{\mathbf{E}}$ is an unknown function of time. This is worth emphasizing because this term is usually expressed with the electrical current density, i.e., the expression of $\mathbf{jA}$ is used, in which $\mathbf{j}$ is a known function of time [27]. This is convenient because it is not necessary to consider the current density as a variable quantity. It is easy to see that this otherwise seriously restricts the results and their interpretation. This is what we alluded to in Section 3. At the same time, this also proves that the Lagrangian potential description we use contains a generalization possibility in addition to equivalence.

## 7. Developing the Joule Hamiltonian and the Total Hamiltonian

It can be seen from the above electrodynamic relationships that it will be more appropriate to use the well-known vector potential instead of the $\mathbf{q}$ field used so far. This way, we have a description that is not only simpler but also more in line with the usual. Thus, by the relation in Equation (63), we can simplify the expression of the Joule part of the Hamiltonian:

$$\begin{aligned} \overline{H}_{Joule} &= 2C_{\mathrm{emT}}\,\sigma\lambda\overline{\mathbf{E}}\,\overline{T}\,\overline{\mathbf{A}} + C_{\mathrm{emT}}\,\sigma\frac{d}{dt}(\overline{\mathbf{E}}\,\overline{T})\overline{\mathbf{A}} - C_{\mathrm{emT}}\,\sigma\overline{\mathbf{E}}\,\overline{T}\,\dot{\overline{\mathbf{A}}} \\ &- \frac{1}{2}C_{\mathrm{emT}}\,\sigma\overline{\mathbf{E}}^2\overline{T} - \frac{1}{2}C_{\mathrm{emT}}\,\sigma c^2\overline{\mathbf{B}}^2\overline{T}. \end{aligned} \tag{71}$$

With this, we obtained the Hamiltonian of all three subprocesses in a suitable form in Equations (44), (70), and (71). Thus, the complete form of the Hamiltonian of the electromagnetic radiation absorption/emission thermal propagation phenomenon is as follows:

$$\overline{H} = \overline{H}_{\mathrm{th}} + \overline{H}_{\mathrm{em}} + \overline{H}_{\mathrm{Joule}}. \tag{72}$$

The Hamiltonian is explicitly independent of time; consequently, it is a conserved quantity. It reflects the conservation of energy during the transport. This is one of the most important consequences of the presented mathematical description.

A method often used in the Lagrangian description of dissipative processes, such as the damping oscillator, is to introduce a time-dependent exponential multiplication factor [28]. A direct consequence of this is that the Hamiltonian function expressing the energy of the system will also decrease exponentially. This means that although it describes the energy of the system well, it does not say anything about the energy lost to the system.

The calculation leading to the Equation (72) does not contain an explicit time dependence anywhere. Thus, the energy sum of the system remains constant, while continuous energy exchanges take place among the different internal processes. So, we can follow the energy transfer and flow at any time. It can be assumed that since the present procedure can be generalized to other physical systems, it can also help in exploiting their processes.

## 8. Asymmetry in the Radiation Processes, Dissipation, and Irreversibility

The phenomenon of absorption and emission is described by the equations established in the previous sections, namely the dissipated electrodynamic energy in Equation (22), and following that, the coupled equations, the evolution of the thermal field via the Joule heat source in Equation (51), and, reversely, the electric field strength regenerated by the thermal interaction in Equation (58), the parallel magnetic field strength in Equation (59), and the entire Hamiltonian expressing the conservation of the energy during the processes throughout the duration in Equation (72). At the same time, the general solution of the equations is presumably impossible. Since we want to point out an interesting and important consequence from a thermodynamic point of view, we will deal with a simplified solution.

It is also well known from simple everyday experience that there is a certain asymmetry between the absorption of the incoming electromagnetic wave, the heating of the body, the cooling of the warm body, and then the emission of the electromagnetic wave. This statement about asymmetry can be expressed in the language of physics as if an electromagnetic wave with circular frequency $\omega_{\text{in}}$ arrives at the surface of the conductor and its energy is then thermalized; then, during the reverse process, another angular frequency $\omega_{\text{out}}$ appears, i.e., in case of the same amount of energy reduction, it does not result in the generation of an incoming circular frequency wave. This implies the irreversibility of the process, along with the fact that there is no loss of energy. After the absorption of the electromagnetic wave, the energy dissipates, and, during the thermal emission, a spectrum differs from the original.

In the following, we perform a simple, but real in terms of physical content, calculation to go through the individual steps of the solution. Despite the great simplification, the obtained results are realistic. When the electromagnetic wave arrives at the surface of the conductor, the wave begins to be absorbed precisely because of the electrical conduction. When the electromagnetic wave arrives at the surface of the conductor, the wave begins to be absorbed precisely because of the electrical conduction. We can describe this process by Equations (24) and (25) or (28) and (29). Let us assume that an electric wave of amplitude of $E_0$ arrives at the conductor's surface. The time evolution of the incoming Gaussian electric field strength is

$$\overline{E}_{\text{in}}(t) = E_0\, \mathrm{e}^{-a(t-b)^2} \tag{73}$$

where we later use the optional probe parameters $a = 100\ 1/\mathrm{s}^2$ and $b = 0.5$ s. During the absorption, the energy of electromagnetic waves turns into Joule heat, which is the source of the thermal effects. In the calculation, we only consider the change in amplitude over time, which simplifies the problem to be solved, but allows us to keep the essence. This means that we do not deal with the frequency of the incoming electromagnetic wave.

To study the processes in a bulk material, it seems sufficient, for the sake of simplicity—instead of the Lorentz invariant description—to consider the Fourier heat conduction:

$$c_{\mathrm{v}} \frac{\partial T}{\partial t} - \kappa \frac{\partial^2 T}{\partial x^2} = \sigma E^2. \tag{74}$$

We apply the half-Fourier form of this equation substituting the electric field strength from Equation (73):

$$c_{\mathrm{v}} \frac{\partial \overline{T}}{\partial t} + \kappa k^2 \overline{T} = \sigma \overline{E}_{\text{in}}^2 = \sigma E_0^2\, \mathrm{e}^{-2a(t-b)^2}. \tag{75}$$

Here, the wave number of $k$ relates to the thermal conduction; moreover, we substituted the Joule heat expression by the time dependence of the electric field strength in Equation (73). To determine the time dependence of the temperature, this equation must be solved by fixing the initial temperature. Readability can be increased by reducing the number of parameters in the equation. We introduce the following new parameters: $p = \kappa k^2/c_\mathrm{v}$, $q = \sigma E_0^2/c_\mathrm{v}$, and $r = 2\lambda = \sigma/\varepsilon_0$, by which we obtain

$$\frac{\partial \overline{T}}{\partial t} + p\overline{T} = q\,\mathrm{e}^{-2a(t-b)^2}. \tag{76}$$

Here, we do not forget that the variable $k$ is not a material parameter. The calculation suggests that the entire volume of $V_0$ takes part in the processes. However, only a thin skin of the surface has a role in the transfers. This is the reason why we reduced the formula for volume $V$. On the other hand, the solution of the homogeneous part is not interesting from our viewpoint. This relates to pure heat conduction, which contributes to the transports, but, in this first approximation, we neglect it due to its exponentially decreasing time dependence.

Applying the time dependence of the thermal temperature, we can use Equation (60) to formulate the electric field strength appearing in the emitted wave:

$$\ddot{\overline{E}}_\mathrm{out} - r\dot{\overline{E}}_\mathrm{out} + \omega_\mathrm{out}^2\,\overline{E}_\mathrm{out} = -rC_\mathrm{emT}\left(-2\dot{\overline{E}}_\mathrm{out}\,\dot{\overline{T}} + \overline{E}_\mathrm{out}\,\ddot{\overline{T}} + r\overline{E}_\mathrm{out}\,\dot{\overline{T}}\right). \tag{77}$$

where $\omega_\mathrm{out}$ is the angular frequency of the outgoing electromagnetic wave. Applying the calculated surface temperature solution, we attempt to solve this equation for the electric field strength. However, from this point on, the analytical solution does not work. Therefore, we perform an illustrative calculation for the case of copper.

The applied material data is as follows: mass density, $\varrho = 8930$ kg/m$^3$; specific heat, $C_\mathrm{v} = 382$ J/kgK by which the factor $c_\mathrm{v} = \varrho C_\mathrm{v}$; heat conductivity, $\kappa = 400$ W/m K; and electrical conductivity, $\sigma = 6 \cdot 10^7$ S/m. We need the value of the permittivity of the vacuum: $\varepsilon_0 = 8.854 \cdot 10^{-12}$ F/m.

The amplitude of the electric field strength can be taken as $E_0 = 900$ V/m, which relates to an intensity of about 1000 W/m$^2$. The thermal wavelength of $k$ may be a variable of the solution. We can now answer what happens in the case of phonons with a wave number of $k$ and an angular frequency of $\omega$. The speed of sound (phonon velocity) in copper is $c_\mathrm{ph} = 4500$ m/s. We assume a wave number of $k = 100$ 1/m, then $\omega_\mathrm{ph} = 4.5 \cdot 10^5$ 1/s. The temperature increase can be calculated using Equation (76). The wave number $k$ is in the parameter $p$. The initial temperature is zero.

The time dependence of the electric field strength amplitude can be plotted using the formulation of Equation (73), taking into account the above physical data. We can read the characteristic absorption time from Figure 1. The heating interval falls between 0.3 and 0.7 s values, i.e., the total length is at most 0.4 s.

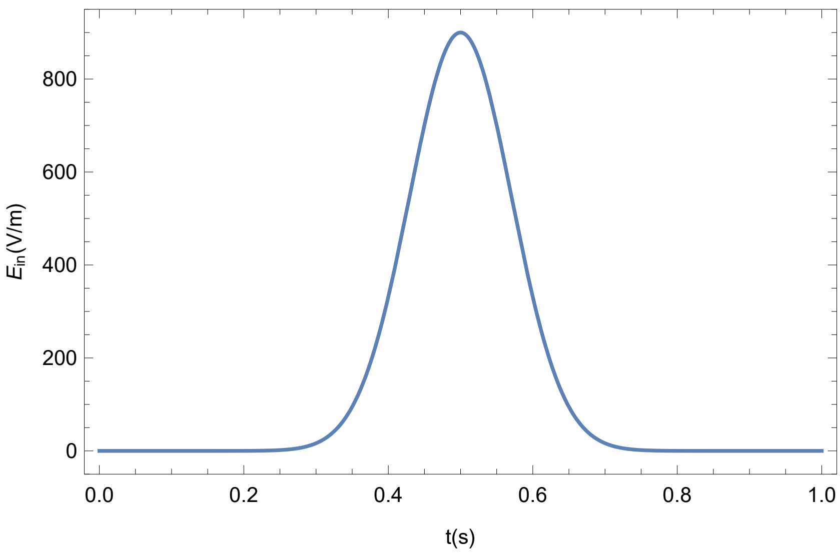


**Figure 1.** The electric field strength of $E$ as a function of time during the absorption.

Taking into account the relatively short irradiation time, we can roughly estimate a thermal penetration depth of $10^{-5}$ m. This also means that the volume ratio of $V/V_0$ is approximately $10^5$, taking into account that the range around the peak of the Gaussian curve is considered to be effective.

The absorbed electromagnetic energy is transformed into the energy of phonons, which manifests itself as a temperature increase in the bulk. The model cannot currently tell in which distribution the individual phonon energies appear. We take the phonon wave number as $k = 100$ 1/m, and we assume that this is the wave number for all phonons. Solving Equation (76), we obtain the time dependence of the surface temperature. We can see its graph in Figure 2.

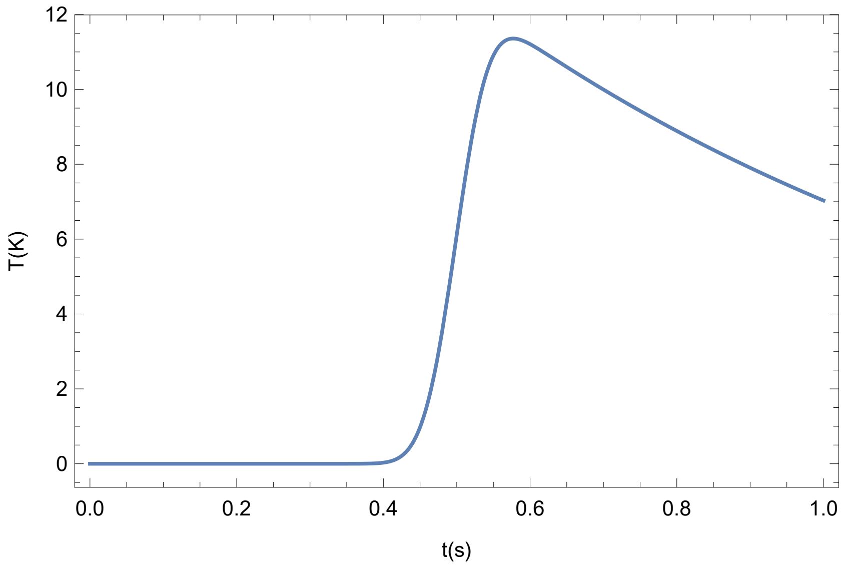


**Figure 2.** The temperature increase in $T$ as a function of time during the absorption.

As a result of the irradiation, the temperature increases, while thermal conduction starts. The right side of the curve nicely shows the temperature drop due to thermal conduction. In terms of energy balance, this is a loss in the emitted radiation.

The emission of the electromagnetic field can be generated from the surface temperature using Equation (77). The equation is quite complicated, so we use suitable simplifications that are physically justifiable. The first and third terms on the left side of the equation, $\ddot{\overline{E}}_{\text{out}} + \omega_{\text{out}}^2 \overline{E}_{\text{out}}$, describe the outgoing electromagnetic wave. It can be said that at the moment of leaving the surface, it is already a free wave, separated from the surface. Therefore, the formula of $0 = \ddot{\overline{E}}_{\text{out}} + \omega_{\text{out}}^2 \overline{E}_{\text{out}}$ is valid for these two terms that describe a freely spreading space. That is, these two terms are eliminated from the equation. In the third term within the bracket on the right-hand side of the equation, $r\overline{E}_{\text{out}} \dot{\overline{T}}$, there is a multiplication by r. Here, $r = \sigma/\varepsilon_0 = 6.77 \cdot 10^{18}$ 1/s, which gives an extremely large value to this term. For this reason, the other two terms within the bracket, $-2\dot{\overline{E}}_{\text{out}} \dot{\overline{T}} + \overline{E}_{\text{out}} \ddot{\overline{T}}$, are negligible. Finally, after simplifications, we obtain a reduced equation for the time evolution of the electric field strength as follows:

$$\dot{\overline{E}}_{\text{out}} = rC_{\text{emT}}\overline{E}_{\text{out}} \dot{\overline{T}}. \tag{78}$$

The solution of this equation is

$$\overline{E}_{\text{out}} \sim e^{rC_{\text{emT}} \overline{T}}. \tag{79}$$

The coefficient $C_{\text{emT}}$ is undefined, and there is no physical explanation for its value. During the plotting, we found a value of $1.2 \cdot 10^{-20}$ s/K, by which the obtained curve seemed realistic. Moreover, we must consider that there is no radiation at zero temperature, so we have to subtract one from the expression within the bracket as follows:

$$\overline{E}_{\text{out}} = E_{0,\text{out}}\left(e^{rC_{\text{emT}}\,\overline{T}} - 1\right). \tag{80}$$

The factor of $E_{0,\text{out}}$ is the amplitude of the outgoing wave. This can be approximated from the energy balance: incoming wave energy vs. outgoing wave + thermal conduction. In our present example, its value is about 580 V/m. We plot the amplitude of the electric field strength in the outgoing wave in Figure 3.

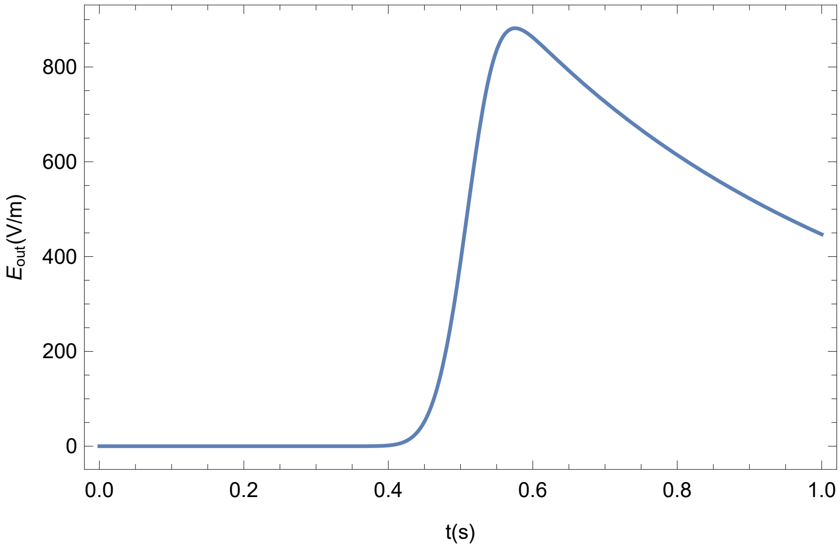


**Figure 3.** The emitted electric field strength as a function of time.

It can be seen that the outgoing wave follows the temperature changes. It seems physically relevant. We emphasize that the numerical calculation performed above is a test to see how it works. The general solutions from Equations (51), (58), and (59), together with the explicitly time-independent Hamiltonian that keeps the entire energy of the system constant in Equation (72) describing the temporal evolution of the spaces require further investigations.

We can conclude that the above-described physical phenomena are asymmetric. After the absorption of a monochromatic wave, the emitted waves have different frequencies and the thermal conducing process dissipates a part of the incoming energy. The irreversible behavior is obvious.

## 9. Conclusions

The goal stated in the article was the coupling of the electromagnetic field and the thermal field in an electrically conductive medium. This was completed within the framework of the Lagrangian–Hamiltonian formalism. This may seem a strong statement because the mathematical form of the equations describing these processes (Maxwell's and Fourier's equations) is very different. The equations of electrodynamics initially describe motion with a finite propagation speed. If a conductive medium is also present, then the equation is of the telegraph type. In this case, we needed to introduce potentials that are suitable for handling the first-time derivative. This may even differ from the usual vector and scalar potentials, but we have shown the relationship between the two. In addition to all of this, Lorentz transformation can be applied. This was the reason why it is advisable to use a Lorentz invariant form instead of the Fourier equation. After that, the complete Hamiltonian formalism (canonical quantities, Hamiltonian function) was constructed. The mathematical relationships include the absorption of the incoming electromagnetic wave, the resulting thermal conduction, and the emission of the electromagnetic wave by the heated surface. The correlations are interesting because they contain the material parameters. It is also important that, due to the lack of explicit time dependence, the conservation of total energy is fulfilled during the processes and can be formulated as a criterion. Finally, an extremely simple example was used to show how the connections can be used. Here, both the asymmetry and dissipative behavior of the phenomenon can be easily observed.

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