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Enhanced Oscillation Criteria for Non-Canonical Second-Order Advanced Dynamic Equations on Time Scales

Ahmed M. Hassan ^{1,*} , Osama Moaaz ^{2,3} , Sameh S. Askar ⁴ , Ahmad M. Alshamrani ⁴  and Samy E. Affan ¹ 

¹ Department of Mathematics, Faculty of Science, Benha University, Benha-Kalubia 13518, Egypt

² Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39, 00186 Roma, Italy

³ Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

⁴ Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

* Correspondence: ahmed.mohammed@fsc.bu.edu.eg

Abstract: This study aims to establish novel iterative oscillation criteria for second-order half-linear advanced dynamic equations in non-canonical form. The results extend and enhance recently established criteria for this type of equation by various authors and also encompass the classical criteria for related ordinary differential equations. Our methodology involves transforming the non-canonical equation into its corresponding canonical form. The inherent symmetry of these canonical forms plays a pivotal role in deriving our new criteria. By employing techniques from the theory of symmetric differential equations and utilizing symmetric functions, we establish precise conditions for oscillation. Several illustrative examples highlight the accuracy, applicability, and versatility of our results.

Keywords: oscillation; second order; advanced dynamic equations; non-canonical; canonical transformation



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1. Introduction

Dynamic equations on time scales, introduced by Stefan Hilger in 1988 [1], are a mathematical framework designed to provide a unified framework for the study of differential equations (continuous models) and difference equations (discrete models). A time-scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ that can represent discrete points, continuous intervals, or a combination of both.

The strength of this theory lies in its ability to generalize classical mathematical tools, such as calculus, to be applicable for various time domains. This unification allows for the analysis of differential and difference equations within a single framework, bridging the gap between discrete and continuous models.

Dynamic equations are applied in various fields, including biology, economics, and engineering, to model phenomena occurring over mixed time domains. This provides a powerful and flexible approach for analyzing dynamic systems comprehensively. To achieve a thorough understanding, it is essential to review some fundamental concepts of time scale theory. Key concepts in time scale theory include the forward and backward jump operators, denoted by σ and ρ , respectively. These operators are defined as follows:

$$\sigma(s) = \inf\{t \in \mathbb{T} \mid t > s\} \quad \text{and} \quad \rho(s) = \sup\{t \in \mathbb{T} \mid t < s\},$$

(supplemented by $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$).

In a time scale $\mathbb{T}$, a point s is classified as right-scattered, right-dense, left-scattered, or left-dense, depending on the relationships $\sigma(s) > s$, $\sigma(s) = s$, $\rho(s) < s$, or $\rho(s) = s$, respectively. The set $\mathbb{T}^{\kappa}$ is defined as $\mathbb{T}$ if it has no left-scattered maximum; otherwise, it is $\mathbb{T}$

without this maximum. The graininess function $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined as $\mu(s) = \sigma(s) - s$. $\mu(s)$ is 0 for continuous time ($\mathbb{T} = \mathbb{R}$) and s for discrete time ($\mathbb{T} = \mathbb{Z}$). In general, however, time scales may have variable graininess.

Within the framework of time scale theory, a function $h : \mathbb{T} \rightarrow \mathbb{R}$ is considered rd-continuous, denoted by $h \in C_{rd}(\mathbb{T}, \mathbb{R})$, if it is continuous at right-dense points and left-dense points in $\mathbb{T}$ and has finite left-hand limits. Differentiability of a function $h : \mathbb{T} \rightarrow \mathbb{R}$ at $\zeta \in \mathbb{T}$ is defined differently depending on whether $\sigma(s) = s$ or $\sigma(s) > s$. The derivative of h at ζ is denoted by h^Δ and defined as follows:

$$h^\Delta := \lim_{t \rightarrow s} \frac{h(s) - h(t)}{s - t}$$

when $\sigma(s) = s$ and

$$h^\Delta := \lim_{t \rightarrow s} \frac{h(\sigma(s)) - h(s)}{\mu(s)}$$

when h is continuous at s and $\sigma(s) > s$.

The product and quotient rules for two differentiable functions h and k are given by:

$$(hk)^\Delta(s) = h^\Delta(s)k(s) + h(\sigma(s))k^\Delta = h(s)k^\Delta(s) + h^\Delta(s)k(\sigma(s)), \quad (1)$$

$$\left(\frac{h}{k}\right)^\Delta(s) = \frac{h^\Delta(s)k(s) - h(s)k^\Delta(s)}{k(s)k(\sigma(s))}. \quad (2)$$

The chain rule states that

$$(h \circ k)^\Delta = \left\{ \int_0^1 h'(k + s\mu k^\Delta) ds \right\} g^\Delta. \quad (3)$$

where $h : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable and $k : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -differentiable.

For a rigorous treatment of the foundational concepts of time scale calculus, we direct the reader to the seminal works of Bohner and Peterson [2,3].

In this work, we investigate the oscillatory characteristics of solutions to the non-canonical second-order dynamic equations

$$[r(s)u^\Delta(s)]^\Delta + p(s)u(\delta(s)) = 0, \quad (4)$$

on an above-unbounded arbitrary time-scale $\mathbb{T}$, and define the time-scale interval $[s_0, \infty)_{\mathbb{T}}$ by $[s_0, \infty)_{\mathbb{T}} := [s_0, \infty) \cap \mathbb{T}$. The following assumptions will be required throughout the paper:

H1. $r, p \in C_{rd}([s_0, \infty)_{\mathbb{T}}, (0, \infty))$ such that $p \not\equiv 0$.

H2. $\delta \in C_{rd}^1(\mathbb{T}, \mathbb{T})$, $\delta^\Delta(s) > 0$ and $\delta(s) \geq s$ for $s \geq s_0$.

According to Trench [4], (4) is in non-canonical form, that is,

$$\Psi(s_0) = \int_{s_0}^{\infty} \frac{\Delta\theta}{r(\theta)} < \infty. \quad (5)$$

If (5) does not hold, then (4) is said to be in canonical form.

An oscillatory solution $u(s)$ of Equation (4) is characterized by the property that it is neither eventually positive nor eventually negative. Conversely, a solution is considered non-oscillatory if it exhibits eventual positivity or eventual negativity. Equation (4) itself is deemed oscillatory when all of its solutions exhibit oscillatory behavior.

Oscillation theory plays a vital role in the study of dynamic equations as it allows us to determine whether solutions exhibit oscillatory behavior. In the context of advanced dynamic equations, which combine elements of continuous and discrete dynamics, the investigation of oscillation criteria becomes particularly important. The study of half-linear second-order advanced dynamic equations on time scales has gained significant attention

due to its broad applicability in various fields. These equations can model a wide range of phenomena, including mechanical systems, electrical circuits, population dynamics, and biological processes. As a special case of (4), when $\mathbb{T} = \mathbb{Z}$, Chatzarakis et al. [5] introduced an improved approach for determining the oscillatory behavior of the second-order advanced difference equation

$$\Delta(r(s)\Delta(u(s))) + p(s)u(\delta(s)) = 0, \quad (6)$$

where $r(s) > 0$, $\sum_{s=s_0}^{\infty} \frac{1}{r(s)} < \infty$ and $\delta(s) \geq s + 1$, and $\{\delta(s)\}$ represents a monotonically increasing sequence of integers. Recently, Chatzarakis et al. [6] established some new oscillation criteria for the second-order nonlinear difference equation

$$\Delta(r(s)\Delta(u(s))) + p(s)u^{\beta}(\delta(s)) = 0, \quad (7)$$

where $\beta \in (0, 1]$ are quotients of odd integers subject to $\sum_{s=s_0}^{\infty} \frac{1}{r(s)} < \infty$ in the both cases $\delta(s) \leq s$ and $\delta(s) \geq s$.

On the other hand, for $\mathbb{T} = \mathbb{R}$, Džurina [7] obtained the oscillation of the advanced differential equation

$$(r(s)(u(s))')' + p(s)u(\delta(s)) = 0, \quad (8)$$

in the non-canonical form. Contrary to most existing results, Baculikova [8] examined the oscillatory behavior of solutions of the delay and advanced differential Equation (8) via only one condition.

In more recent times, Hassan et al. [9] established improved oscillation results of the half linear advanced differential equations

$$(r(s)(\phi(u'(s)))' + p(s)\phi(u(\delta(s))) = 0, \quad (9)$$

where $\phi = |u|^{\gamma-1}u$, $\gamma > 0$ in the canonical form.

Newly, Grace [10] has explored novel criteria for the oscillation of nonlinear second-order delay dynamic equations of the form

$$\left(r(s)u^{\Delta}(s)\right)^{\Delta} + p(s)u^{\beta}(\delta(s)) = 0, \quad (10)$$

where $\beta \in (0, 1]$ is a ratio of odd integers in the non-canonical form and the following condition

$$\int_{s_0}^{\infty} \Psi(\theta)p(\theta)\Delta\theta = \infty. \quad (11)$$

In the work by Karpuz [11], a Hille–Nehari test was introduced to determine the non-oscillation/oscillation behavior of second-order dynamic equations.

$$(r(s)(u(s))^{\Delta})^{\Delta} + p(s)u(s) = 0, \quad (12)$$

and

$$(r(s)(u(s))^{\Delta})^{\Delta} + p(s)u(\sigma(s)) = 0. \quad (13)$$

Furthermore, it was determined that the critical constant for Equation (13) is $\frac{1}{4}$, mirroring the well-established cases of $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

In recent times, there has been a surge of interest among researchers in investigating the oscillatory behavior of specific instances of Equation (4); see [12–15]. As spatial case $\mathbb{T} = \mathbb{R}$; see [16–22] and for $\mathbb{T} = \mathbb{Z}$ [6,23,24].

Hassan et al. in [25] present iterative Hille-type oscillation criteria for second-order canonical half-linear advanced dynamic equations

$$\left[r(s)\left|u^{\Delta}(s)\right|^{\beta-1}u^{\Delta}(s)\right]^{\Delta} + q(s)|u(\delta(s))|^{\beta-1}u(\delta(s)) = 0, \quad (14)$$

where $\beta > 0$.

Also, the authors in [26] obtained Hille and Ohriska oscillation criteria for the canonical dynamic equations

$$\left[r(\tau)u^\Delta(s)\right]^\Delta + q(s)u^\gamma(\delta(s)) = 0, \quad (15)$$

where γ is a quotient of two odd positive integers.

These researchers have employed various techniques like generalized Riccati transformations, integral averaging, Kneser-type, and Hille-type oscillation to derive oscillation criteria for both canonical and non-canonical cases. In canonical equations, non-oscillatory solutions (positive at some point) exhibit a consistent sign throughout, with their derivative $u^\Delta(s)$ eventually becoming positive. In contrast, non-canonical equations allow for the derivative $u^\Delta(s)$ to eventually become positive or negative.

A prevalent approach employed in the literature is to utilize existing results from canonical equations to analyze such scenarios. However, this method has certain limitations, including the need to impose additional conditions (see [10]) and the lack of assurance regarding the oscillatory behavior of all solutions (for more detailed information, see [27]).

In light of these considerations, our aim is to present new rigorous criteria ensuring oscillation of all solutions to Equation (4). By identifying a suitable transformation capable of converting the non-canonical form of Equation (4) into a canonical form, we can proceed with our analysis. This transformation enables us to remove the condition (11). The results of this study extend and complement prior findings, including the special cases of $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

2. Preliminary Results

Without loss of generality, we will focus on non-oscillatory solutions of (4) and limit our attention to the positive case. This is because the negative case is analogous. For simplicity, we define

$$a(s) := r(s)\Psi(\sigma(s))\Psi(s), \quad \chi(s) := \frac{u(s)}{\Psi(s)}, \quad \text{and } q(s) := \Psi(\sigma(s))\Psi(\delta(s))p(s).$$

Lemma 1. Let $u(s)$ be an positive solution of (4) and (5) holds. Then, for all sufficiently large s , $u(s)$ fulfills either of the following conditions cases:

- (i) $u(s) > 0$, $r(s)u^\Delta(s) > 0$ and $(r(s)u^\Delta(s))^\Delta \leq 0$;
- (ii) $u(s) > 0$, $r(s)u^\Delta(s) < 0$ and $(r(s)u^\Delta(s))^\Delta \leq 0$.

Theorem 1. Assuming that (H1) and (H2) hold, then the non-canonical dynamic Equation (4) has a solution $u(s)$ if and only if it is a canonical representation

$$\left(a(s)\chi^\Delta(s)\right)^\Delta + q(s)\chi(\delta(s)) = 0, \quad (16)$$

has a solution $\chi(s)$.

Proof. By the same methodology as in ([28], Lemma 1), we can obtain

$$\left(r(s)u^\Delta(s)\right)^\Delta = \frac{1}{\Psi(\sigma(s))} \left(r(s)\Psi(\sigma(s))\Psi(s) \left(\frac{u(s)}{\Psi(s)}\right)^\Delta\right)^\Delta. \quad (17)$$

This leads to Equation (4), which can be expressed in the equivalent form (16). Moreover, it is clear that (16) in the canonical form such that

$$\int_{s_0}^{\infty} \frac{\Delta\theta}{a(\theta)} = \int_{s_0}^{\infty} \frac{\Delta\theta}{r(\theta)\Psi(\sigma(\theta))\Psi(\theta)} = \lim_{t \rightarrow \infty} \left(\frac{1}{\Psi(s)} - \frac{1}{\Psi(s_0)} \right) = \infty.$$

□

Corollary 1 ([28]). *An eventually positive solution for the non-canonical dynamic Equation (4) exists if and only if the canonical Equation (16) has an eventually positive solution.*

Corollary 1 shows that analyzing the oscillation of (4) is equivalent to analyzing the oscillation of (16). Hence, we can focus on one class of eventually positive solutions, specifically:

$$\chi^\Delta(s) > 0, \quad a(s)\chi^\Delta(s) > 0 \quad \text{and} \quad (a(s)\chi^\Delta(s))^\Delta \leq 0 \quad \text{for} \quad s \geq s_1 \geq s_0. \quad (18)$$

To keep things simple and direct, let us define

$$\Omega(s) = \int_{s_0}^s \frac{\Delta\theta}{a(\theta)}. \quad (19)$$

This definition highlights the symmetrical structure of the integral, consistent with the transformation of the equation into its canonical form.

3. Oscillation Results

First, we introduce a non-decreasing sequence $\{\lambda_j\}_{j \in \mathbb{N}_0}$ define by

$$\lambda_j := \liminf_{\ell \rightarrow \infty} \left\{ \Omega(s) \int_s^\infty \left(\frac{\Omega(\delta(\theta))}{\Omega(\theta)} \right)^{\lambda_{j-1}} q(\theta) \Delta\theta \right\}, \quad (20)$$

where $\lambda_0 = 0$ and $0 < \lambda_j < 1$ for $j = 1, 2, \dots, n$. Then, from (20), we have

$$\Omega(s) \int_s^\infty \left(\frac{\Omega(\delta(\theta))}{\Omega(\theta)} \right)^{\lambda_{j-1}} q(\theta) \Delta\theta \geq \lambda_j.$$

This sequence plays a major role throughout our results.

Lemma 2. *Assume that $\chi(s)$ is a positive solution of equation. If there exist $n \in \mathbb{N}$ and $\lambda_j > 0$, $j = 1, 2, \dots, n$, such that (20) holds. Then,*

$$\left(\frac{\chi(s)}{\Omega^{\lambda_j}(s)} \right)^\Delta \geq 0,$$

eventually.

Proof. We can prove this using induction. As $\chi^\Delta(s) > 0$ eventually, it follows from Equation (16) that, for a large s ,

$$a(s)\chi^\Delta(s) \geq \int_s^\infty q(\theta)\chi(\delta(\theta))\Delta\theta \geq \int_s^\infty q(\theta)\chi(\theta)\Delta\theta \geq \chi(s) \int_s^\infty q(\theta)\Delta\theta,$$

Thus,

$$\begin{aligned} \left(\frac{\chi(s)}{\Omega^{\lambda_1}(s)} \right)^\Delta &= \frac{\Omega^{\lambda_1}(s)\chi^\Delta(s) - \chi(s)(\Omega^{\lambda_1}(s))^\Delta}{\Omega^{\lambda_1}(s)\Omega^{\lambda_1}(\sigma(s))} \\ &= \frac{\chi^\Delta(s)}{\Omega^{\lambda_1}(\sigma(s))} - \frac{\chi(s)(\Omega^{\lambda_1}(s))^\Delta}{\Omega^{\lambda_1}(s)\Omega^{\lambda_1}(\sigma(s))} \\ &\geq \frac{\chi(s)}{a(s)\Omega^{\lambda_1}(\sigma(s))} \int_s^\infty q(\theta)\Delta\theta - \frac{\chi(s)(\Omega^{\lambda_1}(s))^\Delta}{\Omega^{\lambda_1}(s)\Omega^{\lambda_1}(\sigma(s))}. \end{aligned} \quad (21)$$

Applying Pötzsche's chain rule ([2], Theorem 1.87), we find that

$$\begin{aligned}(\Omega^{\lambda_1}(s))^{\Delta} &= \lambda_1 \left\{ \int_0^1 [\Omega(s) + h\mu[\Omega(s)]^{\Delta}]^{\lambda_1-1} dh \right\} \Omega^{\Delta}(s) \\ &= \lambda_1 \left\{ \int_0^1 [(1-h)\Omega(s) + h\mu\Omega(\sigma(s))]^{\lambda_1-1} dh \right\} \Omega^{\Delta}(s) \\ &\leq \lambda_1 \Omega^{\lambda_1-1}(s) \Omega^{\Delta}(s).\end{aligned}\quad (22)$$

Therefore,

$$\frac{(\Omega^{\lambda_1}(s))^{\Delta}}{\Omega^{\lambda_1}(s)} \leq \lambda_1 \frac{\Omega^{\Delta}(s)}{\Omega(s)}.\quad (23)$$

This with (21), we have

$$\begin{aligned}\left(\frac{\chi(s)}{\Omega^{\lambda_1}(s)}\right)^{\Delta} &\geq \frac{\chi(s)}{a(s)\Omega^{\lambda_1}(\sigma(s))} \int_s^{\infty} q(\theta)\Delta\theta - \lambda_1 \chi(s) \frac{\Omega^{\Delta}(s)}{\Omega^{\lambda_1}(\sigma(s))\Omega(s)} \\ &\geq \frac{\chi(s)}{a(s)\Omega^{\lambda_1}(\sigma(s))\Omega(s)} \left[\Omega(s) \int_s^{\infty} q(\theta)\Delta\theta - \lambda_1 \right] \geq 0.\end{aligned}\quad (24)$$

Then, (2) holds for $j = 1$. Assume (2) holds for $j = k \in \mathbb{N}$, i.e., $\left(\frac{\chi(s)}{\Omega^{\lambda_k}(s)}\right)^{\Delta} \geq 0$ eventually.

From Equation (16) and using the increase in $\chi(s)$, this demonstrates that

$$\begin{aligned}a(s)\chi^{\Delta}(s) &\geq \int_s^{\infty} q(\theta)\chi(\delta(\theta))\Delta\theta \geq \int_s^{\infty} \left(\frac{\Omega(\delta(\theta))}{\Omega(\theta)}\right)^{\lambda_k} q(\theta)\chi(\theta)\Delta\theta \\ &\geq \chi(s) \int_s^{\infty} \left(\frac{\Omega(\delta(\theta))}{\Omega(\theta)}\right)^{\lambda_k} q(\theta)\Delta\theta.\end{aligned}\quad (25)$$

Using the increasing fact of $\Omega(s)$ and Pötzsche's chain rule, we have

$$\begin{aligned}\left(\frac{\chi(s)}{\Omega^{\lambda_{k+1}}(s)}\right)^{\Delta} &= \frac{\chi^{\Delta}(s)}{\Omega^{\lambda_{k+1}}(\sigma(s))} - \frac{(\Omega^{\lambda_{k+1}}(s))^{\Delta}\chi(s)}{\Omega^{\lambda_{k+1}}(s)\Omega^{\lambda_{k+1}}(\sigma(s))} \\ &\geq \frac{\chi^{\Delta}(s)}{\Omega^{\lambda_{k+1}}(\sigma(s))} - \lambda_{k+1}\chi(s) \frac{\Omega^{\Delta}(s)}{\Omega^{\lambda_{k+1}}(\sigma(s))\Omega(s)} \\ &\geq \frac{1}{a(s)\Omega^{\lambda_{k+1}}(\sigma(s))\Omega(s)} \left[\Omega(s)a(s)\chi^{\Delta}(s) - \lambda_{k+1}\chi(s) \right] \\ &\geq \frac{\chi(s)}{a(s)\Omega^{\lambda_{k+1}}(\sigma(s))\Omega(s)} \left[\Omega(s) \int_s^{\infty} \left(\frac{\Omega(\delta(\theta))}{\Omega(\theta)}\right)^{\lambda_k} q(\theta)\Delta\theta - \lambda_{k+1} \right].\end{aligned}\quad (26)$$

This indicates that (2) is valid when $j = k + 1$. As a result, (2) holds for all values of $j = 1, \dots, n$. $\square$

Theorem 2. If $\lambda_0 = 0$ and there exist $n \in \mathbb{N}$ and $\lambda_j > 0$ for $j = 1, 2, \dots, n$ such that (20) holds. If

$$\limsup_{s \rightarrow \infty} \int_{s_0}^s \left(\frac{\Omega^{\lambda_n}(\delta(t))}{\Omega^{\lambda_n-1}(t)} q(t) - \frac{1}{4a(t)\Omega(t)} \right) \Delta t = \infty,\quad (27)$$

then (4) is oscillatory.

Proof. Suppose, contrary to our assumption, that (4) is non-oscillatory. In accordance with Theorem 1, it can be deduced that (16) is likewise non-oscillatory. Assuming that $\chi(s)$

be an eventually positive solution of (16), there exists $s_1 \geq s_0$ such that $\chi(s) > 0$, for all $s \geq s_1$. Define

$$w(s) = \Omega(s) \frac{a(s)\chi^\Delta(s)}{\chi(s)}, \quad s \geq s_1. \quad (28)$$

Therefore,

$$\begin{aligned} w^\Delta(s) &= \left(a(s)\chi^\Delta(s) \frac{\Omega(s)}{\chi(s)} \right)^\Delta \\ &= \left(a(s)\chi^\Delta(s) \right)^\Delta \frac{\Omega(s)}{\chi(s)} + \left(a(s)\chi^\Delta(s) \right)^\sigma \left(\frac{\Omega(s)}{\chi(s)} \right)^\Delta \\ &= -\Omega(s)q(s) \frac{\chi(\delta(s))}{\chi(s)} + \left(a(s)\chi^\Delta(s) \right)^\sigma \left(\frac{1}{a(s)\chi^\sigma(s)} - \frac{\Omega(s)\chi^\Delta(s)}{\chi(s)\chi^\sigma(s)} \right) \\ &= -\Omega(s)q(s) \frac{\chi(\delta(s))}{\chi(s)} + \frac{1}{a(s)} \left(\frac{w(s)}{\Omega(s)} \right)^\sigma - \Omega(s) \frac{\chi^\Delta(s)}{\chi(s)} \left(\frac{w(s)}{\Omega(s)} \right)^\sigma. \end{aligned} \quad (29)$$

In view of the increasing fact of $\frac{\chi(s)}{\Omega^{\lambda_j}(s)}$ and $\chi(s)$, we infer that

$$\frac{\chi(\delta(s))}{\chi(s)} \geq \left(\frac{\Omega(\delta(s))}{\Omega(s)} \right)^{\lambda_j}. \quad (30)$$

Consequently,

$$w^\Delta(s) \leq -\frac{\Omega^{\lambda_j}(\delta(s))}{\Omega^{\lambda_j-1}(s)} q(s) + \frac{1}{a(s)} \left(\frac{w(s)}{\Omega(s)} \right)^\sigma - \Omega(s) \frac{\chi^\Delta(s)}{\chi(s)} \left(\frac{w(s)}{\Omega(s)} \right)^\sigma. \quad (31)$$

By using the facts that $\chi(s)$ is increasing and $a(s)\chi^\Delta(s)$ is decreasing, we obtain

$$\chi^\sigma(s) \geq \chi(s),$$

and

$$a(s)\chi^\Delta(s) \geq \left(a(\zeta)\chi^\Delta(\zeta) \right)^\sigma,$$

which yields that

$$\frac{\chi^\Delta(s)}{\chi(s)} \geq \frac{1}{a(s)} \left(\frac{a(s)\chi^\Delta(s)}{\chi(s)} \right)^\sigma = \frac{1}{a(s)} \left(\frac{w(s)}{\Omega(s)} \right)^\sigma. \quad (32)$$

Therefore, from (31) and (32), we have

$$w^\Delta(s) \leq -\frac{\Omega^{\lambda_j}(\delta(s))}{\Omega^{\lambda_j-1}(s)} q(s) + \frac{1}{a(s)} \left(\frac{w(s)}{\Omega(s)} \right)^\sigma - \frac{\Omega(s)}{a(s)} \left[\left(\frac{w(s)}{\Omega(s)} \right)^\sigma \right]^2.$$

It is simple to verify that

$$\begin{aligned} w^\Delta(s) &\leq -\frac{\Omega^{\lambda_j}(\delta(s))}{\Omega^{\lambda_j-1}(s)} q(s) - \frac{\Omega(s)}{a(s)} \left[\frac{1}{2\Omega(s)} - \left(\frac{w(s)}{\Omega(s)} \right)^\sigma \right]^2 \\ &\quad + \frac{1}{4a(s)\Omega(s)} \\ &\leq -\frac{\Omega^{\lambda_j}(\delta(s))}{\Omega^{\lambda_j-1}(s)} q(s) + \frac{1}{4a(s)\Omega(s)}. \end{aligned} \quad (33)$$

Integrating (33) from s_1 to s , we have

$$w(s) \leq w(s_1) - \int_{s_1}^s \left(\frac{\Omega^{\lambda_j}(\delta(\theta))}{\Omega^{\lambda_j-1}(\theta)} q(\theta) - \frac{1}{4a(\theta)\Omega(t)} \right) \Delta\theta.$$

Letting $s \rightarrow \infty$, we obtain a contradiction from (27). This completes the proof. $\square$

Theorem 3. Assume that $\lambda_0 = 0$. Given that there exists $n \in \mathbb{N}$ and $\lambda_j > 0$ for $j = 1, 2, \dots, n$ such that (20) holds. If any of the following dynamic equations

$$\left(a(s)\chi^\Delta(s) \right)^\Delta + \left(\frac{\Omega(\delta(s))}{\Omega(s)} \right)^{\lambda_j} q(s)\chi(s) = 0, \quad j = 1, 2, \dots, n, \quad (34)$$

is oscillatory, then Equation (4) is oscillatory.

Proof. Suppose, contrary to our assumption, that (4) is non-oscillatory. In accordance with Theorem 1, it can be deduced that (16) is likewise non-oscillatory. Let $\chi(s) > 0$, in view of (H2) and the increasing fact of $\frac{\chi(s)}{\Omega^{\lambda_j}(s)}$, we infer that

$$\chi(\delta(s)) \geq \left(\frac{\Omega(\delta(s))}{\Omega(s)} \right)^{\lambda_j} \chi(s). \quad (35)$$

Consequently, it follows from (16) that

$$\left(a(s)\chi^\Delta(s) \right)^\Delta + \left(\frac{\Omega(\delta(s))}{\Omega(s)} \right)^{\lambda_j} q(s)\chi(s) \leq 0. \quad (36)$$

Since $\chi(s)$ is increasing, then by integrating (36) on $[s, \tau]_{\mathbb{T}}$ and letting $\tau \rightarrow \infty$, we have

$$\chi^\Delta(s) \geq \frac{1}{a(s)} \int_s^\infty \left(\frac{\Omega(\delta(\theta))}{\Omega(\theta)} \right)^{\lambda_j} q(\theta)\chi(\theta)\Delta\theta. \quad (37)$$

Integrating from s_0 to s , we obtain

$$\chi(s) \geq \chi(s_0) + \int_{s_0}^s \left(\frac{1}{a(\theta)} \int_\theta^\infty \left(\frac{\Omega(\delta(\theta_1))}{\Omega(\theta_1)} \right)^{\lambda_j} q(\theta_1)\chi(\theta_1)\Delta\theta_1 \right) \Delta\theta. \quad (38)$$

Subsequently, we establish a sequence denoted as $\{\omega_m(s)\}_{m \in \mathbb{N}_0}$ by

$$\begin{aligned} \omega_0(s) &= \chi(s) \\ \omega_{m+1}(s) &= \chi(s_0) + \int_{s_0}^s \left(\frac{1}{a(\theta)} \int_\theta^\infty \left(\frac{\Omega(\delta(\theta_1))}{\Omega(\theta_1)} \right)^{\lambda_j} q(\theta_1)\omega_m(\theta_1)\Delta\theta_1 \right) \Delta\theta, \quad m \in \mathbb{N}_0. \end{aligned}$$

By employing induction, it can be readily verified that the sequence $\{\omega_m(s)\}$ is well defined and decreasing while fulfilling the following condition

$$\chi(s_0) \leq \omega_m(s) \leq \chi(s) \quad \text{for } s \geq s_0 \text{ and } m \in \mathbb{N}_0.$$

Consequently, there exists a function ω defined on the interval $[s_0, \infty)_{\mathbb{T}}$ such that

$$\lim_{m \rightarrow \infty} \omega_m(s) = \omega(s) \quad \text{and} \quad \chi(s_0) \leq \omega_m(s) \leq \chi(s).$$

By applying Lebesgue's convergence theorem, we conclude that

$$\omega(s) = \chi(s_0) + \int_{s_0}^s \left(\frac{1}{a(\theta)} \int_{\theta}^{\infty} \left(\frac{\Omega(\delta(\theta_1))}{\Omega(\theta_1)} \right)^{\lambda_j} q(\theta_1) \omega(\theta_1) \Delta\theta_1 \right) \Delta\theta. \quad (39)$$

It is clear that (39) is a positive solution of (34), contradicting the oscillatory assumption of (34), thereby completing the proof. $\square$

Theorem 4. If $\lambda_0 = 0$ and there exist $n \in \mathbb{N}$ and $\lambda_j > 0$ for $j = 1, 2, \dots, n$ such that (20) holds. If

$$\lambda_n > \frac{1}{4},$$

then Equation (4) is oscillatory.

Proof. To maintain generality, let us assume that $n \in \mathbb{N}$ represents the smallest number for which

$$\Omega(s) \int_s^{\infty} \left(\frac{\Omega(\delta(\theta))}{\Omega(\theta)} \right)^{\lambda_{j-1}} q(\theta) \Delta\theta \geq \lambda_j > \frac{1}{4} \quad \text{for } j = 1, 2, \dots, n. \quad (40)$$

In case there exists a smaller number satisfying (40), we will replace n with that smallest value. Consequently, by employing Theorem 3 of [11], substituting $q(s)$ with $\left(\frac{\Omega(\delta(\theta))}{\Omega(\theta)} \right)^{\lambda_j} q(s)$ in Equation (34), it becomes evident that Equation (34) exhibits oscillatory behavior with $j = n$. Consequently, according to Theorem 3, we conclude that Equation (4) is also oscillatory. $\square$

4. Examples

Example 1. For special time scales $\mathbb{T}$, such that $\delta(s) \geq s\sigma(s) \in \mathbb{T}$ for all $s \in \mathbb{T}$. Consider

$$\left(s\sigma(s)u^{\Delta}(s) \right)^{\Delta} + p_0 \frac{\delta(s)}{s} u(\delta(s)) = 0, \quad s \geq 1. \quad (41)$$

We note that $r(s) = s\sigma(s)$, $p(s) = p_0 \frac{\delta(s)}{s}$ and $\delta(s) \geq s\sigma(s)$. Here

$$\Psi(s) = \int_s^{\infty} \frac{\Delta\theta}{r(\theta)} = \int_s^{\infty} \frac{\Delta\theta}{\theta\sigma(\theta)} = \frac{1}{s} < \infty.$$

Consequently, $a(s) = 1$ and $q(s) = \frac{p_0}{s\sigma(s)}$. Thus, the transformed canonical equation is

$$\chi^{\Delta\Delta}(s) + \frac{p_0}{s\sigma(s)} \chi(\delta(s)) = 0, \quad (42)$$

and $\Omega(s) = s - 1$. Also, (20) take the form

$$\begin{aligned} \lambda_j &:= \liminf_{\ell \rightarrow \infty} \left\{ (s-1) \int_s^{\infty} \left(\frac{\delta(\theta)-1}{\theta-1} \right)^{\lambda_{j-1}} \frac{p_0}{\theta\sigma(\theta)} \Delta\theta \right\} \\ &\geq \liminf_{\ell \rightarrow \infty} \left\{ p_0(s-1) \int_s^{\infty} \frac{\Delta\theta}{\theta\sigma(\theta)} \right\} \\ &= p_0. \end{aligned}$$

Then, we can choose $\lambda_j = p_0$. Hence, using Theorem 4, Equation (41) is oscillatory for $p_0 > \frac{1}{4}$.

Example 2. For special time scales $\mathbb{T}$, which satisfy $\zeta s \in \mathbb{T}$ for all $s \in \mathbb{T}$ and $\zeta \geq 1$ is a constant. Consider the dynamic equation

$$\left(s\sigma(s)u^\Delta(s)\right)^\Delta + p_0u(\zeta s) = 0, \quad s \geq 0. \quad (43)$$

We note that $r(s) = s\sigma(s)$, $p(s) = p_0$ and $\delta(s) = \zeta s$. Here,

$$\Psi(s) = \int_s^\infty \frac{\Delta\theta}{r(\theta)} = \int_s^\infty \frac{\Delta\theta}{\theta\sigma(\theta)} = \frac{1}{s} < \infty.$$

Consequently, $a(s) = 1$ and $q(s) = \frac{p_0}{\zeta s\sigma(s)}$. Thus, the transformed canonical equation is

$$\chi^{\Delta\Delta}(s) + \frac{p_0}{\zeta s\sigma(s)}\chi(\zeta s) = 0, \quad (44)$$

and $\Omega(s) = s$. Also, (20) take the form

$$\begin{aligned} \lambda_j &:= \liminf_{\ell \rightarrow \infty} \left\{ s \int_s^\infty \left(\frac{\zeta\theta}{\theta} \right)^{\lambda_{j-1}} \frac{p_0}{\zeta\theta\sigma(\theta)} \Delta\theta \right\} \\ &= \liminf_{\ell \rightarrow \infty} \left\{ p_0 \zeta^{\lambda_{j-1}-1} s \int_s^\infty \frac{\Delta\theta}{\theta\sigma(\theta)} \right\} \\ &= p_0 \zeta^{\lambda_{j-1}-1}. \end{aligned}$$

Then, we can choose $\lambda_j = p_0 \zeta^{\lambda_{j-1}-1}$, where $\lambda_1 = \frac{p_0}{\zeta}$. Hence, using Theorem 4 Equation (43) is oscillatory for $\frac{p_0}{\zeta} > \frac{1}{4}$. If not, then we try to find $\lambda_n > \frac{1}{4}$; in this case, Theorem 4 ensures that the oscillatory nature of (43) is guaranteed when $\lambda_n = p_0 \zeta^{\lambda_{n-1}-1} > \frac{1}{4}$. The importance of the methodology can be shown by the numerical verification of some special cases of Equation (43).

The first special cases of (43) is where $\zeta = 1$; therefore, Theorem 4 leads to equation

$$\left(s\sigma(s)u^\Delta(s)\right)^\Delta + p_0u(s) = 0, \quad s \geq 0,$$

is oscillatory for $p_0 > \frac{1}{4}$.

The second special cases of (43) is where $\zeta = 2$; therefore, Theorem 4 leads to equation

$$\left(s\sigma(s)u^\Delta(s)\right)^\Delta + p_0u(2s) = 0, \quad s \geq 0, \quad (45)$$

is oscillatory for $p_0 > \frac{1}{2}$. If $p_0 \leq \frac{1}{2}$, then we can find $\lambda_n = p_0 \zeta^{\lambda_{n-1}-1} > \frac{1}{4}$ for some values of q_0 , see Table 1.

Table 1. Relation between p_0 and λ_n .

p_0	$\lambda_n = p_0 \zeta^{\lambda_{n-1}-1}$
0.5	$\lambda_2 = 0.297302$
0.49	$\lambda_2 = 0.290348$
0.48	$\lambda_2 = 0.283438$
0.47	$\lambda_2 = 0.276573$
0.46	$\lambda_2 = 0.269752$
0.45	$\lambda_2 = 0.262975$
0.44	$\lambda_2 = 0.256241$
0.43	$\lambda_3 = 0.255600$
0.425	$\lambda_3 = 0.253065$
0.421	$\lambda_4 = 0.250192$

This, with Theorem 4, ensures that (45) is oscillatory for $p_0 \geq 0.421$.

5. Conclusions

This paper presents iterative Hille-type oscillation criteria for advanced dynamic equations in a non-canonical form. The adopted approach effectively combines the well-known Riccati transformation with an iterative framework, significantly improving the results, as demonstrated in Theorem 2. Additionally, our findings extend and complement some existing results. For instance, in [26], for the special case where $\gamma = 1$, Theorem 4 enhances ([26], Theorem 2, Theorem 4). Furthermore, our results are different from those in [25], where $\beta = 1$, as we focused on the canonical case in our study. Moreover, our presented findings are valid for all time scales, such as $\mathbb{T} = \mathbb{R}, \mathbb{T} = \mathbb{Z}, \mathbb{T} = q^{\mathbb{N}_0}$ where $q > 1$, etc., without enforcing any restrictive conditions.

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