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Exact Model of Gravitational Waves and Pure Radiation

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Abstract: An exact non-perturbative model of a gravitational wave with pure radiation is constructed. It is shown that the presence of dust matter in this model contradicts Einstein's field equations. The exact solution to Einstein's equations for gravitational wave and pure radiation is obtained. The trajectories of propagation and the characteristics of radiation are found. For the considered exact model of a gravitational wave, a retarded time equation for radiation is obtained. The obtained results are used to construct an exact model of gravitational wave and pure radiation for the Bianchi type IV universe.

Keywords: gravitational wave; pure radiation; light cone; retarded time of radiation; dust matter; Hamilton–Jacobi equation; Shapovalov spacetimes; Bianchi type IV universe

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Citation: Osetrin, K.E.; Epp, V.Y.; Filippov, A.E. Exact Model of Gravitational Waves and Pure Radiation. *Symmetry* **2024**, *16*, 1456. https://doi.org/10.3390/sym16111456

Academic Editors: Branko Dragovich and Ornella Juliana Piccinni

Received: 29 September 2024
Revised: 23 October 2024
Accepted: 31 October 2024
Published: 2 November 2024

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1. Introduction

The opening of the era of gravitational wave astronomy (Nobel Prize in Physics, 2017) provided new tools for obtaining information about astrophysical objects and the universe as a whole [1–3]. Studying the propagation of radiation and the motion of particles in a gravitational wave is an important task for gravitational wave astronomy since it can provide additional approaches and methods for recording and determining the characteristics of the gravitational wave background [4]. Recently, this direction has received an additional experimental base in connection with the release of a number of publications on observational data on time delays of signals from pulsars when their radiation passes through the stochastic gravitational wave cosmic background [5–7].

Usually, such models are considered in perturbative approximations for weak gravitational perturbations against the background of basic gravitational fields and are studied by numerical methods [8–12]. When constructing numerical models taking into account complex gravitational wave signals, the availability of exact models of gravitational waves is of great importance, providing an exact mathematical basis and a transparent physical interpretation and allowing one to exactly calculate the influence of a gravitational wave on the propagation of radiation. Such exact models allow us to clarify the behavioral features of solutions and serve as a basis for debugging more complex numerical models of gravitational waves.

In this paper, we will consider the most general exact model of a gravitational wave, whose metric in a privileged coordinate system depends only on one wave variable, along which the spacetime interval vanishes [13]. Along with the cosmological constant in Einstein's field equations, we will consider such sources as pure radiation and dust matter. In this case, the pure radiation model can have a physical interpretation as both high-frequency electromagnetic radiation and high-frequency gravitational radiation, as well as any other massless radiation against the background of a basic gravitational wave.

Exact models of gravitational waves are also of interest in the study of gravitational waves at the early stages of the universe's development since observational data on the microwave cosmological background indicate its anisotropy [14], and the generally accepted interpretation of this fact as a consequence of the kinematic effect during the motion of our galaxy also causes critical assessments [15–17]. If the universe had anisotropy at the early stages of its development, then gravitational wave models based on Bianchi's anisotropic universe models and their influence on the formation of the microwave electromagnetic background of the universe may be of interest for research. The model of gravitational waves considered by us allows for the existence of symmetries of Bianchi spaces [18], can have joint exact solutions with the electromagnetic field [19–26], and, thus, can also model gravitational and electromagnetic waves in the early anisotropic universe. We have also previously constructed a number of exact models of similar gravitational waves [27–29] for both Einstein's theory of gravity and modified theories of gravity [30–38], including by taking into account nonlinear terms in curvature in the field equations [39,40].

When studying the motion of test particles and the propagation of radiation in such exact models of gravitational waves, the Hamilton–Jacobi formalism is often used, which allows exactly integrating the equations of motion and obtaining the trajectories of particles and the trajectories of light propagation in a gravitational wave. The study of trajectories allows, among other things, for obtaining the ratios of the delay of light signals during their propagation in gravitational waves [41], which provides opportunities for experimental assessments of the characteristics of background gravitational waves.

From a general theoretical point of view, exact models of gravitational waves provide a general basis for deriving physical laws based on them, for example, for deriving Coulomb's law for the electric charge in a gravitational wave. From a technical point of view, non-perturbative exact models of gravitational waves are necessary to describe strong gravitational wave disturbances when perturbative methods do not work, for example, to describe gravitational waves at the early stages of the universe's development or near sources of strong gravitational wave disturbances. The delay of light signals in the stochastic gravitational wave background is currently being directly analyzed in the works of radio astronomers on long-term observations of pulsar signal delays.

2. The Gravitational Wave Exact Model

Let us consider the most general model of a gravitational wave, the metric of which in a privileged coordinate system depends on only one wave variable. The spacetime interval of the gravitational wave under consideration can be represented in the following general form [13]:

$$ds^2 = 2dx^0dx^1 + g_{pq}(x^0)\left(dx^p + f^p(x^0)dx^1\right)\left(dx^q + f^q(x^0)dx^1\right). \quad (1)$$

Here, s is the spacetime interval and the indices $p, q, r = 2, 3$. To write the metric (1), a "privileged" coordinate system with the wave variable x^0 is used (along the wave variable, the spacetime interval vanishes). The metric (1) in the privileged coordinate system allows for obtaining a complete integral for the Hamilton–Jacobi equation of test particles and for the eikonal equation [42–44].

The equation for test particles in a gravitational field has the following form in the Hamilton–Jacobi formalism [13]:

$$g^{\alpha\beta}\frac{\partial S}{\partial x^\alpha}\frac{\partial S}{\partial x^\beta} = m^2c^2, \qquad \alpha, \beta = 0, 1, 2, 3, \quad (2)$$

where m is the mass, S is the action function of the test particle (unlike the interval, the action is denoted by a capital letter S), and c is the speed of light, which we will set equal to unity in what follows.

As is known, the metric (1) belongs to the class of Shapovalov wave metrics [45] and allows for the existence of the so-called complete set of spacetime symmetries (a complete

set of Killing vector and tensor fields) that form a complete set of integrals of motion of the Hamilton–Jacobi equation for geodesics (2), allowing one to obtain a complete integral of this equation in a privileged coordinate system.

In a privileged coordinate system, the action function of a test particle can be represented in a separated form:

$$S(x^\alpha) = \phi_0(x^0) + \lambda_k\, x^k, \qquad i,j,k = 1,2,3. \tag{3}$$

where λ_k are constants (integrals of motion specified by initial or boundary conditions).

From the Hamilton–Jacobi Equation (2) for the metric (1), we can note the consequences that arise according to the method of the separation of variables when the action function of a test particle in the considered privileged coordinate system has the form:

$$S = \phi_0(x^0) + \sum_k \lambda_k x^k. \tag{4}$$

The Hamilton–Jacobi Equation (2) then gives the relation:

$$2\,\frac{d\phi_0}{dx^0}\left(\lambda_1 - \sum_q \lambda_q f^q(x^0)\right) = m^2 - \sum_{p,q} \lambda_p \lambda_q\, g^{pq}(x^0), \tag{5}$$

where λ_k are independent constants of motion of the test particle.

The action function of the test particle S in the gravitational wave under consideration takes the following form:

$$S = \sum_k \lambda_k x^k + \frac{1}{2}\int \frac{m^2 - \sum_{p,q} \lambda_p \lambda_q\, g^{pq}(x^0)}{\lambda_1 - \sum_q \lambda_q f^q(x^0)}\, dx^0. \tag{6}$$

The equations of the trajectory of the test particle in the Hamilton–Jacobi gravitational field formalism are written in the following form:

$$\frac{\partial S}{\partial \lambda_k} = \sigma_k, \tag{7}$$

where σ_k are additional independent constant parameters of the particle trajectory, determined by the initial or boundary conditions of the test particle motion.

Substituting the obtained form of the function S from (6) into relations (7), we obtain the following general form of the equations of the trajectory of test particles (i.e., the geodesic lines of spacetime) in the considered gravitational wave in the privileged coordinate system:

$$x^1 = \sigma_1 + \frac{1}{2}\int \frac{m^2 - \lambda_p \lambda_q\, g^{pq}(x^0)}{\left(\lambda_1 - \lambda_q f^q(x^0)\right)^2}\, dx^0, \tag{8}$$

$$x^p = \sigma_p - \frac{1}{2}\int \frac{f^p(x^0)\left(m^2 - \lambda_r \lambda_q\, g^{rq}(x^0)\right)}{\left(\lambda_1 - \lambda_q f^q(x^0)\right)^2}\, dx^0 + \int \frac{\lambda_q\, g^{pq}(x^0)}{\lambda_1 - \lambda_q f^q(x^0)}\, dx^0. \tag{9}$$

Here the wave variable x^0 plays the role of the parameter along the particle trajectories, determining the relationship between the variables x^1, x^2, and x^3 on the particle trajectories.

The proper time of a test particle τ can be defined via the particle action function by the relation (see [13]):

$$\tau = S/m = \sum_k \tilde{\lambda}_k x^k + \frac{1}{2}\int \frac{1 - \tilde{\lambda}_p \tilde{\lambda}_q\, g^{pq}(x^0)}{\tilde{\lambda}_1 - \tilde{\lambda}_q f^q(x^0)}\, dx^0, \qquad \tilde{\lambda}_k = \lambda_k/m. \tag{10}$$

Then, we obtain, from (10), via the relations for x^1, x^2, and x^2 from trajectory Equations (8) and (9), the connection of the wave variable x^0 on the particle trajectory with the particle's proper time τ.

Thus, via Equations (8)–(10), we obtain the trajectories of test particles in the usual notation $x^\alpha = x^\alpha(\tau)$, although the functions $x^\alpha(\tau)$ are generally specified not explicitly but in parametric form.

The refinement of the form of the functions $g^{pq}(x^0)$ and $f^q(x^0)$, included in the metric of the wave model of spacetime, arises from the field equations of the physical model and the theory of gravity under consideration.

Note here that substituting the gravitational wave metric (1) into Einstein's vacuum equations

$$R_{\alpha\beta} = 0, \tag{11}$$

where $R_{\alpha\beta}$ is the Ricci tensor, leads to the following necessary conditions for the gravitational wave model under consideration (see [13]):

$$f^p(x^0) = 0. \tag{12}$$

3. Pure Radiation and Dust Matter in Gravitational Waves

Let us consider the gravitational wave model for the case of Einstein's theory of gravity with field equations of the following form:

$$R_{\alpha\beta} - \frac{1}{2} R\, g_{\alpha\beta} = \Lambda\, g_{\alpha\beta} + \epsilon\, l_\alpha l_\beta + \rho\, u_\alpha u_\beta, \tag{13}$$

where $g_{\alpha\beta}$ is the spacetime metric for a gravitational wave of signature $(+,-,-,-)$, $R_{\alpha\beta}$ is the Ricci tensor, R is the scalar curvature, Λ is the cosmological constant, ϵ is the radiation energy density function (null dust), ρ is the dust matter mass density function, l_α is the wave vector of pure radiation, and u_α is the four-velocity vector field of dust matter.

In this case, it is assumed that the following normalization conditions are satisfied:

$$g^{\alpha\beta} l_\alpha l_\beta = 0, \qquad g^{\alpha\beta} u_\alpha u_\beta = 1. \tag{14}$$

The convolution of the field Equation (13) then yields the following relation for the mass density of dust matter:

$$\rho = R - 4\Lambda. \tag{15}$$

Thus, if, for the model under consideration and Einstein's field equations, the scalar curvature R becomes constant, then the mass density of dust matter ρ will also be constant.

Calculating the scalar curvature R for the gravitational wave metric (1) yields:

$$R = -\frac{g}{2}\left(g^{33}\,{(b^2)'}^2 + (b^3)'\left(g^{22}\,(b^3)' - 2\,g^{23}\,(b^2)'\right)\right). \tag{16}$$

The components of the Ricci tensor $R_{\alpha\beta}$ for the gravitational wave metric (1) are obtained in the following form:

$$R_{00} = \frac{1}{4\,g^2}\left({g'}^2 + 2\,g^3\left({(g^{23})'}^2 - (g^{22})'\,(g^{33})'\right) - 2\,g\,g''\right), \tag{17}$$

$$R_{01} = \frac{1}{4}\Bigg[- 2g\left(g^{33}{(b^2)'}^2 + b^{3\prime}\left(-2g^{23}(b^2)' + g^{22}(b^3)'\right)\right)$$

$$+ b^3\left(2g\left((b^2)'(g^{23})' - (b^3)'(g^{22})'\right) + g^{23}\left(3g'(b^2)' + 2g(b^2)''\right)\right.$$

$$\left. - g^{22}\left(3g'(b^3)' + 2g(b^3)''\right)\right) + b^2\left(2g\left((b^3)'(g^{23})' - (b^2)'(g^{33})'\right)\right.$$

$$- g^{33}\left(3g'(b^2)' + 2g(b^2)''\right) + g^{23}\left(3g'(b^3)' + 2g(b^3)''\right)\Big)\Big], \tag{18}$$

$$R_{02} = \frac{g}{2}\Big((b^2)'(g^{33})' - (b^3)'(g^{23})' + g^{33}(b^2)'' - g^{23}(b^3)''\Big) + \frac{3g'}{4}\Big(g^{33}(b^2)' - g^{23}(b^3)'\Big), \tag{19}$$

$$R_{03} = \frac{g}{2}\Big((b^3)'(g^{22})' - (b^2)'(g^{23})' - g^{23}(b^2)'' + g^{22}(b^3)''\Big) + \frac{3g'}{4}\Big(g^{22}(b^3)' - g^{23}g'(b^2)'\Big), \tag{20}$$

$$R_{11} = \frac{-g^2\Big(b^3\big(g^{23}(b^2)' - g^{22}(b^3)'\big) + b^2\big(g^{23}(b^3)' - g^{33}(b^2)'\big)\Big)^2}{2}, \tag{21}$$

$$R_{12} = \frac{g^2}{2}\Big(g^{33}(b^2)' - g^{23}(b^3)'\Big)\times \times \Big(b^3\Big(g^{22}(b^3)' - g^{23}(b^2)'\Big) + b^2\Big(g^{33}(b^2)' - g^{23}(b^3)'\Big)\Big), \tag{22}$$

$$R_{13} = \frac{g^2}{2}\Big(g^{23}(b^2)' - g^{22}(b^3)'\Big)\times \times \Big(b^3\Big(g^{23}(b^2)' - g^{22}(b^3)'\Big) + b^2\Big(g^{23}(b^3)' - g^{33}(b^2)'\Big)\Big), \tag{23}$$

$$R_{22} = \frac{-g^2\big(g^{33}(b^2)' - g^{23}(b^3)'\big)^2}{2}, \tag{24}$$

$$R_{23} = \frac{g^2\big(g^{23}(b^2)' - g^{22}(b^3)'\big)\big(g^{33}(b^2)' - g^{23}(b^3)'\big)}{2}, \tag{25}$$

$$R_{33} = \frac{-g^2\big(g^{23}(b^2)' - g^{22}(b^3)'\big)^2}{2}. \tag{26}$$

Here, g is the determinant of the gravitational wave metric:

$$g = \det g_{\alpha\beta} = \frac{-1}{g^{22}g^{33} - (g^{23})^2}. \tag{27}$$

We choose a test particle as the base one, setting its mass m equal to unity and choosing its proper time τ as a parameter along the trajectory. Then, the components of the four-velocity of the particles in the used privileged coordinate system, according to the equations of trajectories (8)–(10), can be written as follows:

$$u_\alpha = g_{\alpha\beta}\,u^\beta = g_{\alpha\beta}\,\frac{dx^\beta}{d\tau} = \Big\{u_0(x^0), \lambda_1, \lambda_2, \lambda_3\Big\}. \tag{28}$$

$$u_0 = \frac{1 - \sum_{p,q}\lambda_p\lambda_q\,g^{pq}(x^0)}{2\Big(\lambda_1 - \sum_q \lambda_q f^q(x^0)\Big)}, \qquad p,q,r = 2,3. \tag{29}$$

Using relation (15) from the field equations with the components of the Ricci tensors R_{22}, R_{23}, and R_{33}, we obtain the following form of these equations:

$$(b^3)'^2 = \frac{-2\left(\varepsilon\, {l_2}^2 + \rho\, {\lambda_2}^2\right)}{g} - (10\,\Lambda + 3\,\rho)\,g^{33}, \tag{30}$$

$$(b^2)'\,(b^3)' = \frac{2\,(\varepsilon\, l_2\, l_3 + \rho\,\lambda_2\,\lambda_3)}{g} - (10\,\Lambda + 3\,\rho)\,g^{23}, \tag{31}$$

$$(b^2)'^2 = \frac{-2\left(\varepsilon\, {l_3}^2 + \rho\,{\lambda_3}^2\right)}{g} - (10\,\Lambda + 3\,\rho)\,g^{22}. \tag{32}$$

We obtain the compatibility condition for Equations (30)–(32) in the following form:

$$(2\,\Lambda + \rho)\,(10\,\Lambda + 3\,\rho)\,g + 4\,\varepsilon\,\rho\,(l_3\lambda_2 - l_2\lambda_3)^2 = 0. \tag{33}$$

From the field equations with components R_{11}, R_{12}, and R_{13}, we obtain the following relations:

$$\varepsilon\, l_1\,(l_1 + l_p\, b^p) + \rho\,\lambda_1\,(\lambda_1 + \lambda_p\, b^p) = 0, \tag{34}$$

$$\varepsilon\, l_2\,(l_1 + l_p\, b^p) + \rho\,\lambda_2\,(\lambda_1 + \lambda_p\, b^p) = 0, \tag{35}$$

$$\varepsilon\, l_3\,(l_1 + l_p\, b^p) + \rho\,\lambda_3\,(\lambda_1 + \lambda_p\, b^p) = 0. \tag{36}$$

Assuming that $\rho \neq 0$, from Equations (34)–(36), we obtain corollaries of the form:

$$l_i\,\lambda_j - l_j\,\lambda_i = 0, \qquad (2\,\Lambda + \rho)\,(10\,\Lambda + 3\,\rho) = 0. \tag{37}$$

Equations (34)–(36), taking into account the relation $l_k = \kappa\lambda_k$ following from (37) (here, κ is an arbitrary constant), lead to the following condition:

$$\lambda_k\,(\lambda_1 + \lambda_p\, b^p)(\varepsilon\kappa^2 + \rho) = 0. \tag{38}$$

Since all λ_k cannot vanish simultaneously, we obtain a corollary of the form:

$$\varepsilon\kappa^2 + \rho = 0. \tag{39}$$

Substituting condition (39) into Equations (30)–(32), we obtain:

$$(b^3)'^2 = -(10\,\Lambda + 3\,\rho)\,g^{33}, \tag{40}$$

$$(b^2)'\,(b^3)' = -(10\,\Lambda + 3\,\rho)\,g^{23}, \tag{41}$$

$$(b^2)'^2 = -(10\,\Lambda + 3\,\rho)\,g^{22}, \tag{42}$$

Then, as a result of $g^{22}g^{33} - {g^{23}}^2 = \det g^{\alpha\beta} \neq 0$, we obtain equations of the form:

$$10\,\Lambda + 3\,\rho = 0 \quad \rightarrow \quad (b^p)' = 0. \tag{43}$$

The obtained conditions of compatibility of the field equations lead to a contradiction with Equation (15). Thus, we have arrived at a contradiction, from which it follows that in the gravitational wave model under consideration, dust matter in the given form cannot exist and $\rho = 0$.

4. Exact Solution to Einstein's Equations with Pure Radiation

Let us now consider the case when, in the considered model of gravitational wave, dust matter is absent and $\rho = 0$, but pure radiation with the wave vector l^α and energy density ε is preserved. Then, the reduction of the field equations, taking into account the presence of pure radiation, yields:

$$\rho = \Lambda = b^p = 0, \tag{44}$$

$$ds^2 = 2\,dx^0dx^1 + g_{pq}(x^0)\,dx^p dx^q, \qquad g = \det g_{\alpha\beta} = -\det g_{pq}, \quad p,q = 2,3, \tag{45}$$

$$l_\alpha = (l_0, 0, 0, 0), \qquad l^\alpha = (0, l_0, 0, 0), \tag{46}$$

$$\varepsilon\, {l_0}^2 = \frac{g'^2 - 2gg'' + 2g^3\left((g^{23})'^2 - (g^{22})'(g^{33})'\right)}{4g^2}. \tag{47}$$

In this case, the scalar curvature of the spacetime of the gravitational wave R and the cosmological constant Λ vanish.

This exact solution for the gravitational wave and pure radiation (44)–(47) can have several variants of physical interpretation depending on the formulation of the problem.

First, pure radiation can be considered an external source of energy (for example, external electromagnetic radiation or a flow of other massless particles). In this case, the external pure radiation is assumed to be specified through the wave vector l_α and the radiation energy density ε. Then, its presence gives an additional constraint on the three functions g^{22}, g^{33}, and g^{23}—the remaining components of the gravitational wave metric in the form of Equation (47). In this case, the obtained solution exists under the condition that the external radiation has a wave vector of the form (46). Thus, the characteristics of the external pure radiation (l_α and ε) determine both the choice of the used privileged coordinate system and the gravitational wave itself.

Secondly, pure radiation can be considered high-frequency gravitational radiation against the background of a slowly changing basic metric of the gravitational wave, determined by the components of the metric remaining after reducing g^{22}, g^{33}, and g^{23}, which, in this formulation of the problem, are considered arbitrary functions. Equation (47) then determines the intensity of the high-frequency part of the gravitational wave, and the wave vector (46) determines the direction of this high-frequency gravitational radiation against the background of the slowly changing part of the gravitational wave with the metric functions $g^{pq}(x^0)$.

Thirdly, pure radiation can be considered a flow of massless particles generated by a gravitational wave due to the energy of the wave (for example, the generation of an electromagnetic wave), i.e., the gravitational wave, in this case, will "glow". The direction of the radiation generated by the gravitational wave will be determined by the wave vector (46), and the intensity of this radiation will be determined by Equation (47), according to the metric of the gravitational wave.

5. Light Cone and Radiation Delay in Gravitational Waves

The propagation of light signals in a gravitational wave is determined by the eikonal equation

$$g^{\alpha\beta}\frac{\partial\Psi}{\partial x^\alpha}\frac{\partial\Psi}{\partial x^\beta} = 0, \tag{48}$$

where Ψ is the eikonal function that determines the propagation front of the light signal.

By analogy with the previous calculations for the trajectories of test particles, the solution to the eikonal Equation (48), by the method of the separation of variables, leads to the following form of equations for the trajectories of light rays in a gravitational wave with metric (1):

$$x^1 = \gamma_1 - \frac{1}{2}\int \frac{\sum_{p,q} k_p k_q\, g^{pq}(x^0)}{\left(k_1 - \sum_q k_q f^q(x^0)\right)^2}\, dx^0, \qquad p,q,r,s = 2,3. \tag{49}$$

$$x^p = \gamma_p + \frac{1}{2}\int \frac{f^p(x^0)\sum_{r,q} k_r k_q\, g^{rq}(x^0)}{\left(k_1 - \sum_q k_q f^q(x^0)\right)^2}\, dx^0 + \int \frac{\sum_q k_q\, g^{pq}(x^0)}{k_1 - \sum_q k_q f^q(x^0)}\, dx^0, \tag{50}$$

where the independent parameters k_i and γ_i are determined by the initial or boundary conditions of radiation propagation in a gravitational wave and the wave variable x^0 plays the role of a parameter along the trajectory of light ray propagation.

By selecting and fixing one of the world points $x^\alpha_{(D)}$, through which all possible trajectories of light rays (49) and (50) pass, we obtain the equation of the light "cone" for

this world point, along which light signals in a gravitational wave propagate, which can be detected by an observer at this point.

If we now additionally fix another world point $x^{\alpha}_{(S)}$, in which the radiation source is located, the light signal of which is detected by an observer at the point $x^{\alpha}_{(D)}$, then we obtain light trajectories that connect the world points of the source and the signal detector. These relations define boundary conditions that allow us to express the parameters of the light trajectory γ_k and k_q/k_1 through the coordinates of the world points of the source and detector of the signal $x^{\alpha}_{(S)}$ and $x^{\alpha}_{(D)}$, and also to find the equation of the delay of the light signal, which relates the coordinates of the world points of the source and detector. For calculation details, see Appendix A. Such a "delay" relation in reference systems with an explicitly distinguished time variable gives the connection between the moment of emission and the moment of detection of the light signal during its propagation in a gravitational wave (see [41]).

Solving the boundary condition equations for the parameters of the light signal trajectory in the exact gravitational wave model with metric (45) and conditions (44)–(47), we obtain expressions for the parameters γ_p of the form:

$$\gamma_p = \sum_q \left[\left(I - G_D G_S^{-1} \right)^{-1} \right]_{pq} \left(x_D^q - \sum_r \left[G_D G_S^{-1} \right]_{qr} x_S^r \right), \tag{51}$$

$$G^{pq}(x^0) = \int g^{pq}(x^0)\, dx^0, \qquad G_D = G^{pq}(x_D^0), \qquad G_S = G^{pq}(x_S^0). \tag{52}$$

where $G(x^0)$ is the matrix from relations (52) and I is the identity matrix (inverse matrices and matrix products are used in expressions). The subscript D means that the quantity refers to the detector, and the subscript S means that the quantity refers to the signal source. The parameters of the radiation trajectory k_p/k_1 we obtain from the boundary conditions can be expressed in the following form:

$$\frac{k_p}{k_1} = \sum_q \left[\left(G_D - G_S^{-1} \right)^{-1} \right]_{pq} x_D^q + \sum_q \left(\left[G_S^{-1} \right]_{pq} - \left[\left(G_S - G_S G_D^{-1} G_S^{-1} \right)^{-1} \right]_{pq} \right) x_S^q. \tag{53}$$

For the light ray trajectory parameter γ_1, we obtain the following expression:

$$\begin{aligned} \gamma_1 = {} & x_S^1 + \frac{1}{2} \sum_{p,q} x_S^p \left[G_S^{-1} \right]_{pq} x_S^q \\ & - \frac{1}{2} \sum_{p,q} \left(x_S^p - \sum_r \left[G_D G_S^{-1} \right]_{pr} x_S^r \right) \sum_r \left[(G_S - G_D)^{-1} \right]_{rp} x_S^r \\ & - \frac{1}{2} \sum_{p,q} x_S^p \left[(G_S - G_D)^{-1} \right]_{pq} \left(x_D^q - \sum_r \left[G_D G_S^{-1} \right]_{qr} x_S^r \right) \\ & - \frac{1}{2} \sum_{p,q} \left(x_S^p + \sum_r \left[G_D G_S^{-1} \right]_{pr} x_S^r \right) \left[\left(I - G_D G_S^{-1} \right)^{-1} \right]_{pq} \left(\sum_r \left[(G_S - G_D)^{-1} \right]_{pr} x_D^r \right. \\ & \left. - \sum_{s,r} \left[(G_S - G_D)^{-1} \right]_{ps} \left[G_D G_S^{-1} \right]_{sr} x_S^r \right). \end{aligned} \tag{54}$$

Note that the expressions obtained above for the parameters of the trajectory of a light signal propagating in a gravitational wave (51)–(54) can be formally redefined using the further obtained delay relation for a light beam. In this case, the numerical values of the parameters will of course not change.

For the exact model of the gravitational wave and pure radiation (44)–(47), we present, omitting for brevity the calculations described above, the form of the resulting **equation of the delay of the light signal**, connecting the coordinates of the world points of the source $x^{\alpha}_{(S)}$ and the detector of the light signal $x^{\alpha}_{(D)}$ in the gravitational wave in the privileged wave coordinate system:

$$0 = 2\left(x^1_D - x^1_S\right) + \sum_{p,q=2}^{3}\left(x^p_D - x^p_S\right)\left[\left(G_D - G_S\right)^{-1}\right]_{pq}\left(x^q_D - x^q_S\right). \tag{55}$$

In square brackets is the inverse matrix of the difference between the matrices $G_D = G^{pq}(x^0_D)$ and $G_S = G^{pq}(x^0_S)$ at the world points of the detector and the source (see (52)). We obtained the equation in "finite differences"; in fact, it is not "local" but integral. The obtained relation (55) for the gravitational wave under consideration is a certain analog of the interval along the trajectory of light propagation in flat Minkowski spacetime.

The equation of the delay of radiation propagating in a gravitational wave in a simple exact analytical form (55) has been obtained for the first time for a non-perturbative exact model of a gravitational wave of arbitrary intensity and can be the basis for calculating various physical phenomena occurring with the participation of gravitational waves, including the calculation of the delay time of electromagnetic signals from pulsars during the passage of gravitational waves between pulsars and observers [5–7].

6. Synchronous Frame of Reference

The advantage of the approach considered in the paper using the Hamilton–Jacobi formalism also includes the possibility of analytically constructing a synchronous frame of reference, which is associated with an observer freely falling in a gravitational field, with an explicitly distinguished time variable (the observer's proper time). Such a construction is based on the fact that we can analytically construct a complete integral for the action function of test particles in a privileged wave coordinate system and construct the trajectories of particle motion. By choosing a complete set of integrals of motion and the proper time of a particle for new independent variables, we obtain a transformation from a privileged wave coordinate system to a synchronous frame of reference (see [13]).

Such a synchronous frame of reference is a significant advantage for astronomical observations and allows one to analytically represent the equation for the time delay of radiation in a gravitational field, calculate the time delay of radiation in a gravitational wave, and reconstruct the characteristics of a gravitational wave from the time delay of radiation propagating against the background of a gravitational wave.

By choosing the constants λ_1, λ_2, and λ_3 in the equations of the trajectory of the test particles (8) and (9) as new spatial variables, and the proper time of the particle τ in (10) as the time variable of the new reference frame, we will construct a synchronous reference frame.

For the gravitational wave metric (45), the transformation from the privileged wave coordinate system $\{x^{\alpha}\}$ to the synchronous reference frame $\{y^{\alpha}\} = \{t, y^1, y^2, y^3\}$ can be written from (8)–(10) in the following form:

$$x^0 \to t y^1, \tag{56}$$

$$x^1 \to \frac{t}{2y^1} - \frac{y^p y^q}{2\left(y^1\right)^2} G^{pq}\left(t y^1\right), \tag{57}$$

$$x^p \to \frac{y^q}{2y^1} G^{pq}\left(t y^1\right), \qquad p, q = 2, 3, \tag{58}$$

$$G^{pq}(x^0) = \int g^{pq}(x^0)\, dx^0, \qquad G^{pq}\left(t y^1\right) = G^{qp}(x^0)\Big|_{x^0 \to t y^1}. \tag{59}$$

The gravitational wave metric $\tilde{g}_{\alpha\beta}$ in the synchronous reference frame y^α takes the following form:

$$ds^2 = dt^2 - dl^2 = dt^2 + \tilde{g}_{ik}(t, y^1, y^2, y^3)\, dy^i dy^k, \qquad i, j, k = 1, 2, 3, \tag{60}$$

where t is the time variable and y^k are the spatial variables of the synchronous reference frame.

For the equation of radiation delay in a gravitational wave (55), we can obtain, in a synchronous reference system, an analytical relationship between the time of signal emission at the point of the source and its detection at the point of the observer during the propagation of the signal in a gravitational wave (see, e.g., [41]).

7. Exact Model of Gravitational Waves and Pure Radiation for Bianchi Type IV Universe

As an example of using the approach proposed in this paper, we will consider the application of the results obtained above to a specific model of a gravitational wave with pure radiation in a cosmological problem.

The metric for a gravitational wave (45) with symmetries of the Bianchi type IV space can be represented as follows

$$ds^2 = 2\, dx^0\, dx^1 + \left(x^0\right)^{(1-\nu)} \left[\gamma^2 (\sin\phi)^2 \left(dx^2\right)^2 + \left(\left(\log x^0 - \gamma \cos\phi\right) dx^2 + dx^3 \right)^2 \right], \tag{61}$$

$$g = \det g_{\alpha\beta} = -\gamma^2 \sin^2(\phi) \left(x^0\right)^{2(1-\nu)}, \qquad \gamma \neq 0, \qquad 0 < \phi < \pi, \tag{62}$$

where x^0 is the wave variable along which the spacetime interval vanishes and ν, γ, and ϕ are constant parameters of the gravitational wave model for the Bianchi type IV universe.

This model was obtained [18,27] by imposing symmetries of Bianchi type IV space on the general gravitational wave model (45). Depending on the value of the parameter ν, we obtain an expanding or collapsing model of the Bianchi type IV universe with a gravitational wave.

This gravitational wave model of spacetime (61) admits a three-parameter subgroup of motions, forming a homogeneity group with Killing vectors $X_{(1)}$, $X_{(2)}$, and $X_{(3)}$ with a positive-definite metric on the orbits of the group:

$$X^k_{(1)} = (0, 0, 1, 0), \qquad X^k_{(2)} = (0, 0, 0, 1), \qquad X^k_{(3)} = \left(-x^0, x^1, \omega x^2, \omega x^3 - x^2\right), \tag{63}$$

where the following notation is used $\omega = (1 - \nu)/2$.

The commutation relations for the homogeneity group Killing vectors $X_{(1)}$, $X_{(2)}$, and $X_{(3)}$ correspond to type IV according to the Bianchi classification:

$$\left[X_{(1)}, X_{(2)}\right] = 0, \qquad \left[X_{(1)}, X_{(3)}\right] = \omega X_{(1)} - X_{(2)}, \qquad \left[X_{(2)}, X_{(3)}\right] = \omega X_{(2)}. \tag{64}$$

Solutions (46) and (47) to Einstein's equations for pure radiation, with the energy density ε and wave vector l^α for the metric (61), have the form:

$$l_\alpha = \left\{ l_0, 0, 0, 0 \right\}, \qquad l^\alpha = \left\{ 0, l_0, 0, 0 \right\}, \tag{65}$$

$$\varepsilon\, {l_0}^2 = \frac{E}{\left(x^0\right)^2}, \qquad E = \frac{-1 + (1 - \nu^2)\gamma^2 \sin^2\phi}{2\gamma^2 \sin^2\phi} = \text{const}. \tag{66}$$

For cases of positive energy density of pure radiation $\varepsilon \geq 0$, it is convenient to introduce the angular parameter ψ instead of the parameter ν:

$$\nu = \cos\psi, \qquad 0 < \psi < \pi, \qquad -1 < \nu < 1. \tag{67}$$

Then, we obtain

$$E = \frac{(\gamma \sin\phi \sin\psi)^2 - 1}{2(\gamma \sin\phi)^2}, \qquad \gamma \neq 0, \qquad 0 < \phi < \pi, \qquad 0 < \psi < \pi. \tag{68}$$

The maximum value of the constant E is achieved at values of angular parameters $\phi = \psi = \pi/2$ (at $\nu = 0$) and equal

$$E_{max} = \frac{\gamma^2 - 1}{2\gamma^2} = \frac{1}{2}\left(1 - \frac{1}{\gamma^2}\right), \qquad E_{max} < \frac{1}{2}. \tag{69}$$

From Einstein's equations with pure radiation and the sign of the radiation energy density ε, we obtain restrictions on the gravitational wave parameters ν and γ.

1. The energy density of pure radiation ε becomes zero at the following values of the gravitational wave parameters:

$$|\gamma| = \frac{1}{\sin(\phi)\,\sin(\psi)}, \quad \nu = \cos\psi, \quad 0 < \phi < \pi, \quad 0 < \psi < \pi. \tag{70}$$

This is the case of an expanding Bianchi IV universe.

2. The energy density of pure radiation ε is greater than zero and is given by relations (68) for the following values of the gravitational wave parameters:

$$|\gamma| > \frac{1}{\sin(\phi)\,\sin(\psi)}, \quad \nu = \cos\psi, \quad 0 < \phi < \pi, \quad 0 < \psi < \pi. \tag{71}$$

This is the case of an expanding Bianchi IV universe.

3. The energy density of pure radiation ε is negative (interpreted as radiation generation due to gravitational wave energy) for the following two ranges of parameter values:

3.A. The value of the constant E is negative and is given by relations (68) and

$$0 < |\gamma| < \frac{1}{\sin(\phi)\,\sin(\psi)}, \quad \nu = \cos\psi, \quad 0 < \phi < \pi, \quad 0 < \psi < \pi. \tag{72}$$

This is the case of an expanding Bianchi IV universe.

3.B. The value of the constant E is negative and is given by the following relations

$$E = \frac{1}{2}\left(1 - \nu^2 - \frac{1}{\gamma^2(\sin\phi)^2}\right) < 0, \qquad |\nu| \geq 1, \qquad \gamma \neq 0, \qquad 0 < \phi < \pi. \tag{73}$$

For $\nu > 1$, this is the case of a collapsing Bianchi type IV universe, and for $\nu \leq 1$, this is the case of an expanding universe.

The integration of the Hamilton–Jacobi Equation (2) for the gravitational wave metric (61) has a special case for the parameter value $\nu = 0$, so we will consider two separate cases: when $\nu \neq 0$ and when $\nu = 0$.

7.1. Exact Solution for Gravitational Waves and Radiation in Bianchi Type IV Universe ($\nu \neq 0$)

In this subsection, we will use auxiliary notations:

$$A = 1 + \gamma^2\nu^2\cos^2\phi > 1, \qquad L\left(y^0 y^1\right) = 1 + \nu\left(\gamma\cos\phi - \log\left(y^0 y^1\right)\right). \tag{74}$$

In a privileged wave coordinate system, the equation of radiation signal delay for a gravitational wave with metric (61) takes the following form

$$\Delta_1 = \frac{-\gamma^2\nu^3\cos^2\phi}{2}\left(A\left(\left(x_D^0\right)^\nu-\left(x_S^0\right)^\nu\right)^2-\nu^2\left(\log(x_D^0)-\log(x_S^0)\right)^2\left(x_D^0\right)^\nu\left(x_S^0\right)^\nu\right)^{-1}\times$$

$$\times\left\{-2\left[\log(x_D^0)\left(x_D^0\right)^\nu-\log(x_S^0)\left(x_S^0\right)^\nu+\left(\frac{1}{\nu}+\gamma\cos\phi\right)\left(\left(x_S^0\right)^\nu-\left(x_D^0\right)^\nu\right)\right]\Delta_2\,\Delta_3\right.$$

$$-\left[\left(\log(x_D^0)-2\left(\frac{1}{\nu}+\gamma\cos\phi\right)\right)\left(x_D^0\right)^\nu\log(x_D^0)-\left(\log(x_S^0)-2\left(\frac{1}{\nu}+\gamma\cos\phi\right)\right)\left(x_S^0\right)^\nu\log(x_S^0)\right.$$

$$\left.\left.+\left(\frac{2}{\nu^2}+\frac{2\,\gamma\cos\phi}{\nu}+\gamma^2\right)\left(\left(x_D^0\right)^\nu-\left(x_S^0\right)^\nu\right)\right]\Delta_2{}^2+\left(\left(x_S^0\right)^\nu-\left(x_D^0\right)^\nu\right)\Delta_3{}^2\right\}, \tag{75}$$

where Δ_k denotes the difference between the coordinates of the detector and the source

$$\Delta_k = x_D^k - x_S^k. \tag{76}$$

The law for the transformation of variables from a privileged wave coordinate system $\{x^\alpha\}$ to a synchronous reference system $\{y^\alpha\}$ takes the form:

$$x^0 \to y^0\,y^1, \tag{77}$$

$$x^1 \to \frac{y^0}{2y^1}-\frac{\left(y^0y^1\right)^\nu}{2\gamma^2\nu^3\left(y^1\right)^2\cos^2\phi}\left(\left(\nu y^2+Ly^3\right)^2+A\left(y^3\right)^2\right), \tag{78}$$

$$x^2 \to \frac{\left(y^0y^1\right)^\nu}{\gamma^2\nu^2y^1\cos^2\phi}\left(\nu y^2+Ly^3\right), \tag{79}$$

$$x^3 \to \frac{\left(y^0y^1\right)^\nu}{\gamma^2\nu^3y^1\cos^2\phi}\left(L\left(\nu y^2+Ly^3\right)+Ay^3\right). \tag{80}$$

The metric of the gravitational wave (61) in the synchronous reference system $\{y^\alpha\}=\{t,y^1,y^2,y^3\}$ takes the following form:

$$\tilde{g}_{00}=1,\qquad \tilde{g}_{01}=\tilde{g}_{02}=\tilde{g}_{03}=0, \tag{81}$$

$$\tilde{g}_{11}=-\frac{t^2}{\left(y^1\right)^2}+\frac{A\left(ty^1\right)^\nu t}{\left(\gamma^2\cos^2\phi\right)^2\nu^6\left(y^1\right)^3}\left(\left(\nu y^2+(1+L)y^3\right)^2+\gamma^2\nu^2\left(y^3\right)^2\cos^2\phi\right), \tag{82}$$

$$\tilde{g}_{12}=\frac{A\,t\left(ty^1\right)^\nu}{\left(\gamma^2\cos^2\phi\right)^2\nu^5\left(y^1\right)^2}\left(\nu y^2+(1+L)y^3\right), \tag{83}$$

$$\tilde{g}_{13}=\frac{A\,t\left(ty^1\right)^\nu}{\left(\gamma^2\cos^2\phi\right)^2\nu^6\left(y^1\right)^2}\left((1+L)\,\nu\,y^2+\left((1+L)^2+\gamma^2\nu^2\cos^2\phi\right)y^3\right), \tag{84}$$

$$\tilde{g}_{22}=-\frac{A\;t\left(t\,y^1\right)^\nu}{\left(\gamma^2\cos^2\phi\right)^2\nu^4\,y^1}, \tag{85}$$

$$\tilde{g}_{23}=-\frac{A\,(1+L)\;t\left(t\,y^1\right)^\nu}{\left(\gamma^2\cos^2\phi\right)^2\nu^5\,y^1}, \tag{86}$$

$$\tilde{g}_{33}=-\frac{A\,t\left(t\,y^1\right)^\nu}{\left(\gamma^2\cos^2\phi\right)^2\nu^6\,y^1}\left((1+L)^2+\gamma^2\,\nu^2\cos^2\phi\right), \tag{87}$$

where the constants γ, ν, ϕ, and A are parameters of the gravitational wave, variables y^k are spatial coordinates of the synchronous reference system, t is a time variable, and the function L is defined by the relation

$$L(t, y^1) = 1 + \nu\Big(\gamma\cos\phi - \log(y^1) - \log t\Big). \tag{88}$$

The determinant of the metric in a synchronous reference system $\tilde{g}$ takes the following form:

$$\tilde{g} = \det\tilde{g}_{\alpha\beta} = -\frac{A^2\, t^4\, (t\, y^1)^{2\,\nu}}{(\gamma^2\cos^2\phi)^3\, \nu^8\, (y^1)^4}. \tag{89}$$

The equation of radiation delay in a gravitational wave (55), which relates the coordinates of the world points of the source and the detector of the signal, in a synchronous frame of reference, acquires a direct physical representation of the connection by separating a single time variable t. For a gravitational wave (61), the equation of radiation delay in a synchronous frame of reference can be reduced to the following form:

$$\Big[(\nu\Delta_{12} + L_D\Delta_{13})^2 + A{\Delta_{13}}^2\Big]\Big(t' y_S^1\Big)^{\nu} - \Big[(\nu\Delta_{12} + L_S\Delta_{13})^2 + A{\Delta_{13}}^2\Big]\Big(t y_D^1\Big)^{\nu}$$

$$+\,\gamma^2\nu^3\cos^2\phi\left[\Big((t y_D^1)^{-\nu} - (t' y_S^1)^{-\nu}\Big)^2 - \frac{\nu^2}{A}\Big(\log(t y_D^1) - \log(t' y_S^1)\Big)^2\right] y_D^1 y_S^1 \Delta_{01} = 0, \tag{90}$$

where

$$\Delta_{12} = y_D^2 y_S^1 - y_D^1 y_S^2, \quad \Delta_{13} = y_D^3 y_S^1 - y_D^1 y_S^3, \quad \Delta_{01} = y_D^1\, t' - y_S^1\, t, \tag{91}$$

$$L_D = 1 + \nu\Big(\gamma\cos\phi - \log(y_D^1) - \log t\Big), \quad L_S = 1 + \nu\Big(\gamma\cos\phi - \log(y_S^1) - \log t'\Big). \tag{92}$$

Here, t' is the time of signal emission by the source and t is the time of signal detection by the observer. The constants γ, ν, ϕ, and A are the parameters of the gravitational wave. The spatial coordinates y_S^i specify the position of the radiation source at the moment of emission t', and the coordinates y_D^i specify the position of the observer detecting the signal coming from the source.

The resulting Equation (90) determines the relationship between the time of radiation emission t' and the time of signal detection by the observer t; it gives the retarded time of the signal as it passes in a gravitational wave (61).

The equations that determine the trajectory of a test particle in a gravitational wave (61) in a synchronous frame of reference make it possible to express the spatial coordinates of the trajectory y^1, y^2, and y^3 as functions of time t.

The equation that determines the dependence of the spatial coordinate y^1 of a test particle on the time variable t can be written in the following form:

$$A\,\Big(t\, y^1\Big)^{\nu}\Big(m^2\, t\,(y^1)^2 + 2\,\lambda_1\,(\lambda_1\,\sigma_1 + \lambda_2\,\sigma_2 + \lambda_3\,\sigma_3)\, y^1 - {\lambda_1}^2\, t\Big) +$$

$$+\, y^1\gamma^2\cos^2\phi\,\nu\,{\lambda_1}^2\left(A\,{\sigma_2}^2 + \Big[\sigma_2 - \nu\,\sigma_3 + \nu\sigma_2\,\Big(\gamma\cos\phi - \log y^1 - \log t\Big)\Big]^2\right) = 0. \tag{93}$$

Equation (93) defines, albeit implicitly, the spatial coordinate of the trajectory of a test particle y^1 as a function of time t. Then, the remaining spatial coordinates of the particle trajectory y^2 and y^3 are determined through $y^1(t)$ by the following trajectory equations:

$$y^2(t) = y^1\left(\frac{\lambda_2}{\lambda_1} + \frac{\gamma^2\,\nu\,\cos^2\phi\,\big((A + L^2)\,\sigma_2 - L\,\nu\,\sigma_3\big)}{A\,(t\, y^1)^{\nu}}\right), \tag{94}$$

$$y^3(t) = y^1\left(\frac{\lambda_3}{\lambda_1} - \frac{\gamma^2\,\nu^2\cos^2\phi\,(L\,\sigma_2 - \nu\,\sigma_3)}{A\,(t\, y^1)^{\nu}}\right). \tag{95}$$

where λ_i and σ_i are constant parameters of the trajectory of test particle, γ, ν, ϕ, A are constant parameters of the gravitational wave, and the function L is determined by relation (88).

The trajectories of a light beam in a gravitational wave in a synchronous frame of reference can be written in such a way as to single out the equation that relates the spatial coordinate of the trajectory y^1 to the time variable t. Then, the spatial coordinates of the light trajectory y^2 and y^3 will be determined through $y^1(t)$:

$$A\ \left(t\,y^1\right)^{\nu}\left(\kappa_1\,t-2\left(\kappa_1\,\gamma_1+\kappa_2\,\gamma_2+\kappa_3\,\gamma_3\right)y^1\right)$$

$$-\gamma^2\,\nu\,\kappa_1\,\cos^2\phi\left(A\,{\gamma_2}^2+\left(L\,\gamma_2-\nu\,\gamma_3\right)^2\right)y^1=0, \tag{96}$$

$$y^2(t)=y^1\left(\frac{\kappa_2}{\kappa_1}+\frac{\gamma^2\cos^2\phi\,\nu\left(\left(A+L^2\right)\gamma_2-L\,\nu\,\gamma_3\right)}{A\ \left(t\,y^1\right)^{\nu}}\right), \tag{97}$$

$$y^3(t)=y^1\left(\frac{\kappa_3}{\kappa_1}-\frac{\gamma^2\cos^2\phi\,\nu^2\left(L\,\gamma_2-\nu\,\gamma_3\right)}{A\ \left(t\,y^1\right)^{\nu}}\right), \tag{98}$$

where k_i and γ_i are constant parameters of the trajectory of the light beam; γ, ν, ϕ, and A are constant parameters of the gravitational wave; and the function L is determined by relation (88).

7.2. Exact Solution for Gravitational Waves and Radiation in Bianchi Type IV Universe ($\nu=0$)

The peculiarity that arises when integrating the Hamilton–Jacobi equation of test particles for the gravitational wave metric in a Bianchi type IV universe when the parameter ν becomes zero requires that this case be considered separately.

The metric of a gravitational wave in a privileged wave coordinate system in this case takes on the following special form:

$$ds^2=2\,dx^0\,dx^1+x^0\left[\gamma^2(\sin\phi)^2\left(dx^2\right)^2+\left(\left(\log x^0-\gamma\cos\phi\right)dx^2+dx^3\right)^2\right], \tag{99}$$

where x^0 is the wave variable and the constants γ and ϕ ($0<\phi<\pi$) are the parameters of the gravitational wave.

Solutions (46) and (47) to Einstein's equations for pure radiation, with the energy density ε and wave vector l^α for the metric (99), have the form:

$$l_\alpha=\left\{l_0,0,0,0\right\},\qquad l^\alpha=\left\{0,l_0,0,0\right\}, \tag{100}$$

$$\varepsilon\,{l_0}^2=\frac{E}{\left(x^0\right)^2},\qquad E=\frac{\gamma^2-\csc^2(\phi)}{2\gamma^2}=\text{const}. \tag{101}$$

The energy density of pure radiation ε has a positive value for the following range of parameter values:

$$|\gamma|>\frac{1}{\sin(\phi)},\qquad 0<\phi<\pi. \tag{102}$$

The result of integrating the Hamilton–Jacobi equation for test particles in the privileged wave coordinate system (8)–(10) for the considered case of the gravitational wave will take the following form:

$$x^0(\tau)=\lambda_1(\tau-\tau_0), \tag{103}$$

$$x^1(\tau)=\sigma_1-\frac{1}{4\gamma^2{\lambda_1}^2\sin^2(\phi)}\left[2\log(\lambda_1\tau)\left(\gamma^2{\lambda_3}^2+2\gamma\lambda_2\lambda_3\cos(\phi)+{\lambda_2}^2\right)\right.$$

$$\left.-2\gamma^2\lambda_1\tau\sin^2(\phi)-2\lambda_3\log^2(\lambda_1\tau)(\gamma\lambda_3\cos(\phi)+\lambda_2)+\frac{2}{3}{\lambda_3}^2\log^3(\lambda_1\tau)\right], \tag{104}$$

$$x^2(\tau) = \sigma_2 + \frac{\log(\lambda_1\tau)}{2\gamma^2\lambda_1\sin^2(\phi)}\Big(2\lambda_2 + 2\gamma\lambda_3\cos(\phi) - \lambda_3\log(\lambda_1\tau)\Big), \tag{105}$$

$$x^3(\tau) = \sigma_3 + \frac{\log(\lambda_1\tau)}{6\gamma^2\lambda_1\sin^2(\phi)}\Big(6\gamma^2\lambda_3 + 6\gamma\cos(\phi)(\lambda_2 - \lambda_3\log(\lambda_1\tau)) - 3\lambda_2\log(\lambda_1\tau) + 2\lambda_3\log^2(\lambda_1\tau)\Big), \tag{106}$$

where τ is the proper time of the particle and the constants λ_k, σ_k, and τ_0 are parameters determined by the initial or boundary conditions of the particle motion.

The delay time equation for the propagation of radiation in a gravitational wave (55), for the case under consideration in the privileged wave coordinate system, will take the following form:

$$0 = \Delta_1 + \frac{6\gamma^2\sin^2(\phi)\Big(\log(x_D^0x_S^0) - 2\gamma\cos(\phi)\Big)\Delta_2\Delta_3}{\log(x_D^0/x_S^0)\Big(6\gamma^2 - 6\gamma^2\cos(2\phi) + \log^2(x_D^0/x_S^0)\Big)}$$
$$+ \frac{2\gamma^2\sin^2(\phi)\Big(3\gamma^2 - 3\gamma\cos(\phi)\log(x_D^0x_S^0) + \log(x_D^0)\log(x_S^0) + \log^2(x_D^0) + \log^2(x_S^0)\Big)\Delta_2{}^2}{\log(x_D^0/x_S^0)\Big(6\gamma^2 - 6\gamma^2\cos(2\phi) + \log^2(x_D^0/x_S^0)\Big)}$$
$$+ \frac{6\gamma^2\sin^2(\phi)\Delta_3{}^2}{\log(x_D^0/x_S^0)\Big(6\gamma^2 - 6\gamma^2\cos(2\phi) + \log^2(x_D^0/x_S^0)\Big)}, \tag{107}$$

where x_D^α are the coordinates of the world point of the detector of signal, x_S^α are the coordinates of the world point of the signal source, x^0 is the wave variable, the constants γ and ϕ ($0 < \phi < \pi$) are the parameters of the gravitational wave, and

$$\Delta_k = x_D^k - x_S^k, \qquad i,j,k = 1,2,3.$$

In accordance with the solutions to the equations of test particle motion in the Hamilton–Jacobi formalism for this case (103)–(106), we obtain the following formulas for the transition from the privileged wave coordinate system $\{x^\alpha\}$ to the synchronous reference system $\{y^\alpha\} = (\tau, y^1, y^2, y^3)$:

$$x^0 \to y^1\tau,$$
$$x^1 \to -\frac{1}{4\gamma^2y^1{}^2\sin^2(\phi)}\Big(2\log(y^1\tau)\Big(\gamma^2y^3{}^2 + 2\gamma y^2y^3\cos(\phi) + y^2{}^2\Big) - 2\gamma^2y^1\tau\sin^2(\phi)$$
$$-2y^3\log^2(y^1\tau)(\gamma y^3\cos(\phi) + y^2) + \frac{2}{3}y^3{}^2\log^3(y^1\tau)\Big), \tag{108}$$

$$x^2 \to \frac{\log(y^1\tau)\big(2\gamma y^3\cos(\phi) - y^3\log(y^1\tau) + 2y^2\big)}{2\gamma^2y^1\sin^2(\phi)}, \tag{109}$$

$$x^3 \to \frac{\log(y^1\tau)}{6\gamma^2y^1\sin^2(\phi)}\Big(6\gamma^2y^3 + 6\gamma\cos(\phi)\Big(y^2 - y^3\log(y^1\tau)\Big) - 3y^2\log(y^1\tau) + 2y^3\log^2(y^1\tau)\Big). \tag{110}$$

Here, τ is a time variable and the variables y^k are spatial coordinates.

The components of the metric of the gravitational wave (99) in the synchronous reference system will take the following form:

$$\tilde{g}^{00} = 1, \quad \tilde{g}^{01} = 0, \quad \tilde{g}^{02} = 0, \quad \tilde{g}^{03} = 0,$$

$$\tilde{g}^{11} = -\frac{{y^1}^2}{\tau^2}, \qquad \tilde{g}^{12} = -\frac{y^1 y^2}{\tau^2}, \qquad \tilde{g}^{13} = -\frac{y^1 y^3}{\tau^2},$$

$$\tilde{g}^{22} = -\frac{{y^2}^2}{\tau^2} + \frac{4\gamma^2 y^1 \sin^2(\phi)}{\tau \log^2(y^1\tau)\left(6\gamma^2(1-\cos(2\phi)) + \log^2(y^1\tau)\right)^2}\left(3\gamma^2\log^2(y^1\tau)\right.$$

$$\left. + 18\gamma^4 - 6\gamma^2\cos(2\phi)\left(3\gamma^2 - \log^2(y^1\tau)\right) - 6\gamma\cos(\phi)\log^3(y^1\tau) + \log^4(y^1\tau)\right),$$

$$\tilde{g}^{23} = -\frac{y^2 y^3}{\tau^2} + \frac{12\gamma^2 y^1 \sin^2(\phi)\left(3\gamma^3\cos(3\phi) - 3\gamma\cos(\phi)\left(\gamma^2 + \log^2(y^1\tau)\right) + \log^3(y^1\tau)\right)}{\tau\log^2(y^1\tau)\left(6\gamma^2(1-\cos(2\phi)) + \log^2(y^1\tau)\right)^2},$$

$$\tilde{g}^{33} = -\frac{{y^3}^2}{\tau^2} + \frac{36\gamma^2 y^1 \sin^2(\phi)\left(\log^2(y^1\tau) - 2\gamma^2\cos(2\phi) + 2\gamma^2\right)}{\tau\log^2(y^1\tau)\left(6\gamma^2(1-\cos(2\phi)) + \log^2(y^1\tau)\right)^2},$$

where τ is the time variable, the variables y^k are spatial coordinates, and the constants γ and ϕ ($0 < \phi < \pi$) are the parameters of the gravitational wave.

The retarded time equation for the propagation of radiation (55) in a gravitational wave (99) can be written in a synchronous frame of reference with a time variable τ in the following form:

$$\begin{aligned}
0 = {} & \csc(\phi)\log^3(\tau' y^1_S)\left[6\gamma^2 y^1_D y^1_S \sin^2(\phi)(\tau y^1_S - \tau' y^1_D)\right.\\
& - 6\log(\tau y^1_D)\left(\gamma^2(y^3_D y^1_S - y^1_D y^3_S)^2 + (y^1_D)^2(y^2_S)^2 - 2y^1_D y^2_D y^1_S y^2_S + (y^2_D)^2(y^1_S)^2\right)\\
& + 2\log(\tau y^1_D)(y^3_D y^1_S - y^1_D y^3_S)\left(3\gamma\cos(\phi)\left(\log(\tau y^1_D)(y^3_D y^1_S - y^1_D y^3_S) + 2y^1_D y^2_S - 2y^2_D y^1_S\right)\right.\\
& \left.\left. + \log(\tau y^1_D)\left(\log(\tau y^1_D)(y^1_D y^3_S - y^3_D y^1_S) - 3y^1_D y^2_S + 3y^2_D y^1_S\right)\right)\right]\\
& + 3\csc(\phi)\log(\tau y^1_D)\log^2(\tau' y^1_S)\left[6\gamma^2 y^1_D y^1_S \sin^2(\phi)(\tau' y^1_D - \tau y^1_S)\right.\\
& + 2\log^2(\tau y^1_D)(y^3_D y^1_S - y^1_D y^3_S)\left(\gamma\cos(\phi)(y^3_D y^1_S - y^1_D y^3_S) - y^1_D y^2_S + y^2_D y^1_S\right)\\
& + 2\log(\tau y^1_D)\left(-2\gamma(y^3_D y^1_S - y^1_D y^3_S)\left(\gamma\cos(2\phi)(y^3_D y^1_S - y^1_D y^3_S) + \cos(\phi)(y^2_D y^1_S - y^1_D y^2_S)\right)\right.\\
& \left.\left. + (\gamma y^1_D y^3_S - \gamma y^3_D y^1_S + y^1_D y^2_S - y^2_D y^1_S)(\gamma y^1_D y^3_S - \gamma y^3_D y^1_S - y^1_D y^2_S + y^2_D y^1_S)\right)\right]\\
& + 6\gamma^2 y^1_D y^1_S \log(\tau y^1_D)(\tau' y^1_D - \tau y^1_S)\left(\sin(\phi)\left(9\gamma^2 + \log^2(\tau y^1_D)\right) - 3\gamma^2\sin(3\phi)\right)\\
& - 3\csc(\phi)\log(\tau' y^1_S)\left[24\gamma^4 y^1_D y^1_S \sin^4(\phi)(\tau' y^1_D - \tau y^1_S)\right.\\
& + 6\gamma^2\sin^2(\phi)\log(\tau y^1_D)\left(y^1_D y^1_S \log(\tau y^1_D)(\tau' y^1_D - \tau y^1_S)\right.\\
& + 4\left(\gamma^2(y^3_D y^1_S - y^1_D y^3_S)^2 + (y^1_D)^2(y^2_S)^2 - 2y^1_D y^2_D y^1_S y^2_S + (y^2_D)^2(y^1_S)^2\right)\Big)\\
& + 4\gamma\cos(\phi)\log(\tau y^1_D)(y^2_D y^1_S - y^1_D y^2_S)(y^3_D y^1_S - y^1_D y^3_S)\left(12\gamma^2\sin^2(\phi) + \log^2(\tau y^1_D)\right)\\
& \left. + 2\log^3(\tau y^1_D)\left(\gamma^2(y^3_D y^1_S - y^1_D y^3_S)^2 + (y^1_D)^2(y^2_S)^2 - 2y^1_D y^2_D y^1_S y^2_S + (y^2_D)^2(y^1_S)^2\right)\right]. \qquad (111)
\end{aligned}$$

The retarded time equation for the propagation of radiation (111) gives us the relationship between the signal detection time τ and the emission time of this signal τ' when the signal passes in a gravitational wave. The resulting equation contains the spatial coordinates of the source y^k_S and the spatial coordinates of the radiation detector y^k_D (observer).

8. Conclusions

This paper considers a general, exact model of a gravitational wave, with a spacetime metric depending, in a privileged coordinate system, on one wave variable and with sources in the form of pure radiation, dust matter, and a cosmological constant. It is shown that the compatibility conditions of Einstein's field equations for this model lead to a contradiction with the presence of dust matter in the model. An exact solution to the field equations for a gravitational wave with pure radiation is obtained. Light trajectories and radiation propagation characteristics are found. A retarded time equation of light signals in a gravitational wave is obtained. Based on the obtained general results, an exact model of a gravitational wave and pure radiation for the Bianchi type IV universe is constructed.

Author Contributions: Conceptualization, K.E.O.; methodology, K.E.O. and V.Y.E.; validation, V.Y.E., A.E.F. and K.E.O.; investigation, K.E.O., V.Y.E. and A.E.F.; writing—original draft preparation, K.E.O.; supervision, K.E.O.; project administration, V.Y.E.; funding acquisition, V.Y.E. All authors have read and agreed to the published version of the manuscript.

Funding: The study was supported by the Russian Science Foundation (grant No. 23-22-00343, accessed on 13 January 2023, https://rscf.ru/en/project/23-22-00343/).

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A. Parameters of Light Ray Trajectories in Exact Model of Gravitational Waves

This appendix presents a variant of calculating the parameters for the trajectory of radiation propagation (49) and (50) against the background of a gravitational wave (45) between the world point of the radiation source and the world point of signal detection by the observer.

Let us denote the variables of the radiation source (light signal) as x_S^α and the variables of the observer (detector) as x_D^α; then, we obtain a system of equations for the trajectory of the light beam connecting these points (points x_S^α and x_D^α):

$$x_S^1 = \gamma_1 - \frac{k_p k_q}{2(k_1)^2} G^{pq}(x_S^0), \tag{A1}$$

$$x_S^p = \gamma_p + \frac{k_q}{k_1} G^{pq}(x_S^0), \tag{A2}$$

$$x_D^1 = \gamma_1 - \frac{k_p k_q}{2(k_1)^2} G^{pq}(x_D^0), \tag{A3}$$

$$x_D^p = \gamma_p + \frac{k_q}{k_1} G^{pq}(x_D^0), \tag{A4}$$

where

$$G^{pq}(x^0) = \int g^{pq}(x^0)\, dx^0, \qquad G_D = G^{pq}(x_D^0), \qquad G_S = G^{pq}(x_S^0).$$

To shorten the text, we will use the notation G_S for the matrix $G^{pq}(x_S^0)$ and the notation G_D for the matrix $G^{pq}(x_D^0)$:

$$G^{pq}(x_S^0) = [G_S]^{pq}, \quad G^{pq}(x_D^0) = [G_D]^{pq}, \tag{A5}$$

where $G(x^0)$ is a matrix constructed from the integrals of the metric components.

To obtain the "signal delay" relation, i.e., the relation linking the variables x_D^α and x_S^α, it is necessary to exclude the light signal parameters γ_1, γ_2, γ_3, k_2/k_1, and k_3/k_1 from Equations (A1)–(A4). As a result, there remains one relation linking the coordinates of the world points of the source and detector x_S^α and x_D^α.

Thus, from Equation (A2), we obtain an expression for k_p/k_1 through the parameters γ_p:

$$\frac{k_p}{k_1} = \sum_q \left[G_S^{-1}\right]_{pq} \left(x_S^q - \gamma_q\right). \tag{A6}$$

Then, from Equation (A1), taking into account (A6), we obtain the relation for γ_1 through the parameters γ_p:

$$\gamma_1 = x_S^1 + \frac{k_p k_q}{2(k_1)^2} G^{pq}(x_S^0) = x_S^1 + \frac{1}{2}\sum_{p,q}\left[G_S^{-1}\right]_{pq}\left(x_S^p - \gamma_p\right)\left(x_S^q - \gamma_q\right). \tag{A7}$$

Equation (A4), taking into account (A6), can be written as follows:

$$x_D^p = \gamma_p + \frac{k_q}{k_1} G^{pq}(x_D^0) = \gamma_p + \sum_{r,q} G_D^{pq}\left[G_S^{-1}\right]_{qr}(x_S^r - \gamma_r) =$$

$$= \sum_r \gamma_r \left(\delta_p^r - \sum_q G_D^{pq}\left[G_S^{-1}\right]_{qr}\right) + \sum_{r,q} G_D^{pq}\left[G_S^{-1}\right]_{qr} x_S^r. \tag{A8}$$

The obtained relation (A8) allows us to determine the parameters γ_p for the light signal through the coordinates of the source and the detector, using matrix notations to shorten the notation

$$\gamma_p = \sum_q \left[\left(I - G_D G_S^{-1}\right)^{-1}\right]_{pq} \left(x_D^q - \sum_r \left[G_D G_S^{-1}\right]_{qr} x_S^r\right), \tag{A9}$$

where $G(x^0)$ is the matrix of integrals of the metric components in relations (52), I is the identity matrix, for the designation of whose components the Kronecker symbols δ_p^q are also used.

Using the obtained relation (A9) for γ_p, we can find, from relation (A7), the required form of the parameter γ_1 for the light signal from the world point x_S^α of the source to the world point x_D^α of the detector. For γ_1, we have:

$$\gamma_1 = x_s^1 + \frac{1}{2}\sum_{p,q} x_s^p \left(\left[G_s^{-1}\right]_{pq} x_s^q - \left[G_s^{-1}\right]_{pq}\gamma_q\right)$$

$$-\frac{1}{2}\sum_{r,p}\left(x_s^p - \sum_q \left[G_D G_s^{-1}\right]_{pq} x_s^q\right)\left(\left[(G_s - G_D)^{-1}\right]_{rp} x_s^r - \left[(G_s - G_D)^{-1}\right]_{rp}\gamma_r\right)$$

$$= x_s^1 + \frac{1}{2}\sum_{p,q} x_s^p \left[G_s^{-1}\right]_{pq} x_s^q - \frac{1}{2}\sum_{r,p}\left(x_s^p - \sum_q \left[G_D G_s^{-1}\right]_{pq} x_s^q\right)\left[(G_s - G_D)^{-1}\right]_{rp} x_s^r$$

$$-\frac{1}{2}\sum_{p,q} x_s^p \left[G_s^{-1}\right]_{pq}\gamma_q - \frac{1}{2}\sum_p \left(x_s^p + \sum_q \left[G_D G_s^{-1}\right]_{pq} x_s^q\right)\sum_r \left[(G_s - G_D)^{-1}\right]_{rp}\gamma_r. \tag{A10}$$

Next, we will need to use the relations:

$$\sum_q \left[G_S^{-1}\right]_{pq}\gamma_q = \sum_q \left[(G_S - G_D)^{-1}\right]_{pq}\left(x_D^q - \sum_r \left[G_D G_S^{-1}\right]_{qr} x_S^r\right), \tag{A11}$$

$$\sum_q \left[(G_S - G_D)^{-1}\right]_{pq}\gamma_q$$

$$= \sum_q \left[\left(I - G_D G_S^{-1}\right)^{-1}\right]_{pq}\left(\sum_q \left[(G_S - G_D)^{-1}\right]_{pq} x_D^q - \sum_q \left[(G_S - G_D)^{-1}\right]_{pq}\sum_r \left[G_D G_S^{-1}\right]_{qr} x_S^r\right). \tag{A12}$$

As a result, we obtain the following expression for the parameter of the light beam γ_1:

$$\gamma_1 = x_S^1 + \frac{1}{2}\sum_{p,q} x_S^p \left[G_S^{-1}\right]_{pq} x_S^q - \frac{1}{2}\sum_{p,q}\left(x_S^p - \sum_r \left[G_D G_S^{-1}\right]_{pr} x_S^r\right)\sum_r \left[(G_S - G_D)^{-1}\right]_{rp} x_S^r$$
$$- \frac{1}{2}\sum_{p,q} x_S^p \left[(G_S - G_D)^{-1}\right]_{pq}\left(x_D^q - \sum_r \left[G_D G_S^{-1}\right]_{qr} x_S^r\right)$$
$$- \frac{1}{2}\sum_{p,q}\left(x_S^p + \sum_r \left[G_D G_S^{-1}\right]_{pr} x_S^r\right)\left[\left(I - G_D G_S^{-1}\right)^{-1}\right]_{pq}\left(\sum_r \left[(G_S - G_D)^{-1}\right]_{pr} x_D^r\right.$$
$$\left. - \sum_{s,r}\left[(G_S - G_D)^{-1}\right]_{ps}\left[G_D G_S^{-1}\right]_{sr} x_S^r\right). \quad \text{(A13)}$$

Similarly, from relations (A6), taking into account relations (A9) for γ_p, we can find the required form of parameters k_p/k_1:

$$\frac{k_p}{k_1} = \sum_q \left[G_S^{-1}\right]_{pq} x_S^q - \sum_{q,s}\left[G_S^{-1}\right]_{pq}\left[\left(I - G_D G_S^{-1}\right)^{-1}\right]_{qs}\left(x_D^s - \sum_r \left[G_D G_S^{-1}\right]_{sr} x_S^r\right)$$
$$= \sum_q \left[G_S^{-1}\right]_{pq} x_S^q + \sum_q \left[\left(G_D - G_S^{-1}\right)^{-1}\right]_{pq}\left(x_D^q - \sum_r \left[G_D G_S^{-1}\right]_{qr} x_S^r\right)$$
$$= \sum_q \left[\left(G_D - G_S^{-1}\right)^{-1}\right]_{pq} x_D^q + \sum_{q,r} x_S^r\left(\delta_r^q \left[G_S^{-1}\right]_{pq} - \left[\left(G_D - G_S^{-1}\right)^{-1}\right]_{pq}\left[G_D G_S^{-1}\right]_{qr}\right)$$
$$= \sum_q \left[\left(G_D - G_S^{-1}\right)^{-1}\right]_{pq} x_D^q + \sum_q \left(\left[G_S^{-1}\right]_{pq} - \left[\left(G_S - G_S G_D^{-1} G_S^{-1}\right)^{-1}\right]_{pq}\right) x_S^q. \quad \text{(A14)}$$

Note that the expressions obtained above for the parameters of the light signal can be formally redefined using the obtained retarded time equation for the light beam (55), which gives an additional equation linking the coordinates of the source and the detector. In this case, the numerical values of the parameters will not change.

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