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The Connections Between Attribute-Induced and Object-Induced Decision Rules in Incomplete Formal Contexts

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Abstract: For a given incomplete context, object-induced approximate concepts have been defined, and this type of approximate concept can induce a type of decision rule. Based on the duality principle, another set of approximate concepts can be defined from the perspective of attributes, i.e., attribute-induced approximate concepts. Although object induced approximate concepts and attribute induced approximate concepts are symmetrical by duality principle, their induced decision rules exhibit different properties and the connections between attribute induced decision rules and object induced decision rules in incomplete formal contexts are not clear. To this end, a type of attribute-induced approximate concept and a method of extracting attribute-induced decision rules are presented. More importantly, it is revealed that given a type of decision rules, there must be corresponding decision rules of the other type, and both of them can provide some useful information, but they are not equivalent to each other. In other words, each type of decision rule can provide some unique and irreplaceable information.

Keywords: formal concept analysis; decision rule; incomplete formal context; approximate concept

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Citation: Wang, H.; Zhi, H.; Li, Y.; Zhu, D.; Xiahou, J. The Connections Between Attribute-Induced and Object-Induced Decision Rules in Incomplete Formal Contexts. *Symmetry* **2024**, *16*, 1403. https://doi.org/10.3390/sym16101403

Academic Editor: Mariano Torrisi

Received: 30 July 2024
Revised: 25 September 2024
Accepted: 11 October 2024
Published: 21 October 2024

1. Introduction

In daily life, due to various types of reasons, it may be very difficult to describe our target with complete certainty. Missing data are inevitable or even acceptable in many situations. At present, incomplete formal contexts have been widely used in characterizing incomplete information systems [1–3]. Intuitively, incomplete formal contexts extend classical formal contexts by introducing the uncertain parts. Burmeister and Holzer [1] interpreted the incompleteness of an original incomplete formal context by using a set of complete formal contexts, i.e., the so-called scaling technique. Besides the similarities shared by the incomplete context and the classical formal context, the former has its own prominent features. Specifically, for a given incomplete context, one can define object-induced approximate concepts [4] and attribute-induced approximate concepts, which can lead to two kinds of decision rules. The main idea of building an object-induced approximate concept is to find the possible attributes possessed by the objects as well the certain attributes possessed by the objects. Hence, the values of attributes form an interval, which can manage the incompleteness caused by the missing of some relations between objects and attributes [4]. To the best of our knowledge, only object-induced approximate concepts and approximate decision rules have been studied [4]. Therefore, how to define the attribute-induced approximate concept and attribute-induced decision rules, as well as the connections with the object-induced decision rules, still deserve further investigation.

Formal concept analysis (FCA) has been widely used in qualitative data analysis [5]. FCA provides an effective tool to assist data analysis. Each node of a concept lattice is a formal concept, consisting of two parts: extent and intent. Extent represents the instances covered by the concept, and intent is a description of a concept that covers the common features of the discussed instances. Moreover, a concept lattice graphically embodies the generalization and specialization relations between these concepts using Hasse diagrams. Therefore, concept lattices are considered to be a powerful tool for data analysis. The process of generating a concept lattice from a data set is essentially a conceptual clustering process. Concept lattices have been applied in many fields, such as information retrieval, digital libraries, software engineering, and knowledge discovery. To fulfill different needs, the classical formal concepts have been generalized to represent specific ideas of human thinking. Wang [6] presented a type of generalized concepts to relate one set of objects to the other set of objects in software development. Düntsch and Gediga [7] explored the uncertainties of information systems and put forward property-oriented concepts for qualitative data analysis. Combining the theory of FCA and rough set, Yao [8,9] presented object-oriented concepts and discussed the relationships between the necessity operator and the sufficiency operator. Furthermore, from the perspective of granule description, Yao [10] raised roughly definable concepts.

Generally speaking, given an incomplete information system, a set of objects may possess some attributes with certainty and possess some attributes with possibility. Dually, a set of attributes can be possessed by some objects with certainty and some objects with possibility. To this end, partially known formal concepts are defined based on interval sets [11]. Ren et al. [12] systematically enumerated the various types of partially known formal concepts by considering the uncertainties of extents and intents. Yao [13] interpreted interval sets by a family of sets bounded by a pair of sets and adopted this idea to investigate partially known data sets. Zhao et al. [14] discussed the transformation between concepts in dynamic incomplete information systems by building an approximate cognitive learning system. Wang et al. [15] designed an attribute reduction algorithm of SE-ISI concept lattices for incomplete formal contexts. Moreover, inspired by the *three-way decision* method [16,17], Zhi et al. generalized the existing approximate concepts and proposed approximate three-way concepts [18,19]. Wei et al. [20] discussed a rule mining technology based on three-way concepts.

As stated by Wille [21], FCA is effective in knowledge discovery from relational databases. For instance, Ganter et al. [22] and Valtchev et al. [23] presented their method of implication rule acquisition from formal contexts. Zaki [24] mined non-redundant association rules from relational databases. Quan et al. [25] presented a clustering-based approach to mine association rules. Stumme et al. [26] deleted redundant rules from original knowledge bases while keeping the deduction abilities of the knowledge bases. Wu et al. [27] explored the hierarchical structure of concept lattices to derive granular rules in decision making. Zhai et al. [28,29] built a framework to mine concise and optimal decision implications from a logical reasoning viewpoint. Furthermore, the results have also been extended to the fuzzy settings [30,31] and uncertain cases [32]. Zhang et al. [33] revealed the connections between concept rules and granular rules. Zhi et al. [34] proposed a way to mine association rules from complete formal contexts using a granule description technique.

In classical formal contexts, objects and attributes can be treated equally in a mathematical perspective. Nevertheless, if we think in threes or model a concept in an incomplete context, this thought does not hold in many circumstances. For instance, in *three-way concept analysis* (3WCA) [35–38], object-induced three-way concepts and attribute-induced three-way concepts cannot be directly derived from each other. In an incomplete formal context, Li et al. [4] defined a type of object-induced approximate concept by using interval sets. Following this work, Zhi et al. [18,19] proposed object-induced approximate three-way concepts and attribute-induced approximate three-way concepts.

With regard to knowledge discovery from incomplete contexts, there are already some important studies. For instance, Li et al. [4] defined a type of approximate rules based on kinds of approximate concepts. Furthermore, Zhi et al. [19] gave a solution to a granule description, based on which they described an approach to approximate decision rule extraction based on a type of approximate three-way object-oriented concepts. In fact, the existing methods are mainly based on object-induced concepts [4,19]. Then, defining the attribute-induced concepts and decision rules and discovering the connections between the attribute-induced decision rules and the object-induced decision rules become an open problem. To this end, we define attribute-induced approximate concepts and decision rules and discuss their connections to object-induced approximate concepts and decision rules.

To organize the study, Section 2 recalls some basic notions. Section 3 introduces the attribute-induced approximate concept lattice. Section 4 describes a method of mining attribute-induced decision rules from incomplete formal contexts. Section 5 explores the connections between the attribute-induced decision rules and object-induced decision rules. Finally, Section 6 summarizes the main results and discusses future research directions.

2. Related Theoretical Foundations

This section briefly recalls some basic notions in FCA, including incomplete formal contexts, object-induced approximate concepts, and decision rules.

In FCA, an information system is represented by a so-called formal context $K = (E, C, I)$, in which E and C are used to describe the object set and the attribute set, and I denotes the binary relation between E and C. Concretely, $I(e, c) = 1$ states that object e possesses the attribute c, while $I(e, c) = 0$ means the opposite. In this study, the formal context K is called a complete context to emphasize that it contains no undetermined values.

Definition 1 ([22])**.** *Let $K = (E, C, I)$ be a formal context. For $O \in 2^E$ and $P \in 2^C$, two operators are, respectively, defined as:*

$$O^* = \{c \in C \mid \forall o \in O, I(o, c) = 1\} \text{ and } P^* = \{e \in E \mid \forall p \in P, I(e, p) = 1\}.$$

Moreover, If $O^* = P$ and $P^* = O$, then we call the pair (O, P) a formal concept of K. By sorting all formal concepts of K into a hierarchical structure based on covering relations, one can derive a lattice, which is called the concept lattice of K, and it is denoted as $L(K)$.

For an information system with undetermined values, an incomplete formal context can be built to deal with the uncertainties. That is, binary relations of K are extended to three-valued ones. Formally, an incomplete formal context is a quadruple $\mathbb{K} = (E, C, \{0, ?, 1\}, R)$, where $R(e, c) =?$ means that the relation between e and c is not clear.

For a given incomplete context, which can be interpreted via possible world semantics [1,39,40], it is usually investigated by the so-called greatest and least completions as follows.

Definition 2 ([13])**.** *Let $\mathbb{K} = (E, C, \{0, ?, 1\}, R)$. (E, C, J) be called a completion of $\mathbb{K}$ provided that*

$$R(e, c) = 1 \Rightarrow J(e, c) = 1,$$
$$R(e, c) = 0 \Rightarrow J(e, c) = 0.$$

In fact, a completion is a modification of the initial incomplete one by changing each ? into either 1 or 0.

Definition 3 ([13])**.** *Let $\mathbb{K} = (E, C, \{0, ?, 1\}, R)$. We call*

$$\overline{\mathbb{K}} = (E, C, \overline{J}),\ \overline{J} = \{(e, c) \mid R(e, c) = 1\} \cup \{(e, c) \mid R(e, c) =?\},$$
$$\underline{\mathbb{K}} = (E, C, \underline{J}),\ \underline{J} = \{(e, c) \mid R(e, c) = 1\},$$

the greatest completion and the least completion of $\mathbb{K}$, respectively.

Definition 4. *Let $\mathbb{K} = (E, C, \{0, ?, 1\}, R)$. For $P \in 2^C$, two derivation operators $\underline{R}*, \overline{R}* : 2^C \to 2^E$ are, respectively, defined as:*

$$P^{\underline{R}*} = \{\, e \in E \mid \forall c \in P, R(e,c) = 1\},$$
$$P^{\overline{R}*} = \{\, e \in E \mid \forall c \in P, R(e,c) = 1 \text{ or } R(e,c) = ?\}.$$

Dually, for $O \in 2^E$*, the lower and upper derivation operators* $\underline{R}*, \overline{R}* : 2^E \to 2^C$ *are, respectively, defined as:*

$$O^{\underline{R}*} = \{\, c \in C \mid \forall e \in O, R(e,c) = 1\},$$
$$O^{\overline{R}*} = \{\, c \in C \mid \forall e \in O, R(e,c) = 1 \text{ or } R(e,c) = ?\}.$$

It is obvious that the operators $\underline{R}*, \overline{R}* :\ 2^C \to 2^E$ *with their associated operators* $\underline{R}*, \overline{R}* : 2^E \to 2^C$ *are actually the operator* $* :\ 2^C \to 2^E$ *with its associated operator* $* :\ 2^E \to 2^C$ *defined on* $\underline{\mathbb{K}}$ *and* $\overline{\mathbb{K}}$*, respectively.*

Definition 5 ([4])**.** *Let* $\mathbb{K} = (E, C, \{0, ?, 1\}, R)$*. For* $O \in 2^E$ *and* $(P, Q) \in 2^C \times 2^C$*, two operators* $\downarrow: 2^E \to 2^C \times 2^C$ *and* $\uparrow: 2^C \times 2^C \to 2^E$ *are, respectively, defined as:*

$$O^{\downarrow} = (O^{\underline{R}*}, O^{\overline{R}*}) \text{ and } (P,Q)^{\uparrow} = \{e \in E \mid (P,Q) \leq (\{e\}^{\underline{R}*}, \{e\}^{\overline{R}*})\}.$$

Moreover, if $O^{\downarrow} = (P, Q)$ and $(P, Q)^{\uparrow} = O$, we call $(O,(P, Q))$ an object-induced approximate concept of $\mathbb{K}$.

Then, by ordering all of the approximate concepts with set inclusion relations, a complete lattice is formed, which is known as an object-induced approximate concept lattice and is denoted by $OL(\mathbb{K})$.

By horizontally merging an incomplete formal context $\mathbb{K} = (E, C, \{0, ?, 1\}, R)$ and a formal context $F = (E, D, J)$ that meets $C \cap D = \emptyset$, we can derive an incomplete formal decision context $\mathbb{D} = (E, C, R, \{0, ?, 1\}, D, J)$. Here, we call C the conditional attribute set and D the decision attribute set of $\mathbb{D}$.

Moreover, for $(O,(P, Q)) \in OL(\mathbb{K})$ and $(U, S) \in L(F)$, if $O \subseteq U$, then we call $r_O : (P, Q) \to S$ a type-O decision rule of $\mathbb{D}$. By the way, type-O is adopted to emphasize that the decision rules are derived based on object-induced approximate concepts.

3. Attribute-Induced Approximate Concepts

In this section, in order to manage the uncertainties of incomplete formal contexts, we adopt intervals to show incompleteness and propose attribute-induced approximate concepts. Initially, we define two derivation operations in the incomplete context in preparation for modeling attribute-induced approximate concepts.

Definition 6. *Let* $\mathbb{K} = (E, C, \{0, ?, 1\}, R)$*. For* $(U, V) \in 2^E \times 2^E$ *and* $P \in 2^C$*, two operators* $\rhd : 2^E \times 2^E \to 2^C$ *and* $\lhd : 2^C \to 2^C \times 2^E$ *are, respectively, defined as:*

$$(U,V)^{\rhd} = U^{\underline{R}*} \cap V^{\overline{R}*} \text{ and } P^{\lhd} = (P^{\underline{R}*}, P^{\overline{R}*}).$$

From the semantic perspective, $\lhd$ maps a set of attributes to two sets of objects, i.e., $P^{\underline{R}*}$ and $P^{\overline{R}*}$. Concretely, $P^{\underline{R}*}$ returns the objects that possess all of the attributes in P with certainty, and $P^{\overline{R}*}$ returns the objects that possess all of the attributes in P with possibility. Furthermore, $P^{\underline{R}*}$ and $P^{\overline{R}*}$ form an interval, i.e., $[P^{\underline{R}*}, P^{\overline{R}*}]$. Moreover, $(U,U)^{\rhd}$ contains the attributes that are possessed by U with certainty and by $(U - V)$ with possibility.

Proposition 1. *Let* $\mathbb{K} = (E, C, \{0, ?, 1\}, R)$*. For* (U, V)*,* (U_1, V_1)*,* $(U_2, V_2) \in 2^E \times 2^E$ *and* P*,* P_1*,* $P_2 \in 2^C$*, the following properties hold.*

(i) $(U_1, V_1) \leq (U_2, V_2) \Rightarrow (U_1, V_1)^{\rhd} \supseteq (U_2, V_2)^{\rhd}$*;* $P_1 \subseteq P_2 \Rightarrow P_1^{\lhd} \geq P_2^{\lhd}$*.*

(ii) $(U, V) \leq (U, V)^{\rhd\lhd}$*;* $P \subseteq P^{\lhd\rhd}$*.*

(iii) $\rhd : 2^E \times 2^E \to 2^C$ *and* $\lhd : 2^C \to 2^E \times 2^E$ *form a Galois connection between* $(2^E \times 2^E, \leq)$ *and* $(2^C, \subseteq)$*.*

Proof. (i) If $(U_1, V_1) \leq (U_2, V_2)$, it follows that $U_1^{\underline{R}*} \supseteq U_2^{\underline{R}*}$ and $V_1^{\overline{R}*} \supseteq V_2^{\overline{R}*}$. Then, we can obtain $U_1^{\underline{R}*} \cap V_1^{\overline{R}*} \supseteq U_2^{\underline{R}*} \cap V_2^{\overline{R}*}$, i.e., $(U_1, V_1)^{\triangleright} \supseteq (U_2, V_2)^{\triangleright}$.

By $P_1 \subseteq P_2$, we can derive $P_1^{\underline{R}*} \supseteq P_2^{\underline{R}*}$ and $P_1^{\overline{R}*} \supseteq P_2^{\overline{R}*}$. Then, it follows that $(P_1^{\underline{R}*}, P_1^{\overline{R}*}) \geq (P_2^{\underline{R}*}, P_2^{\overline{R}*})$, i.e., $P_1^{\triangleleft} \supseteq P_2^{\triangleleft}$.

(ii) As $U^{\underline{R}*} \supseteq U^{\underline{R}*} \cap V^{\overline{R}*}$ and $V^{\overline{R}*} \supseteq U^{\underline{R}*} \cap V^{\overline{R}*}$, it follows that $U^{\underline{R}*\underline{R}*} \subseteq (U^{\underline{R}*} \cap V^{\overline{R}*})^{\underline{R}*}$ and $V^{\overline{R}*\overline{R}*} \subseteq (U^{\underline{R}*} \cap V^{\overline{R}*})^{\overline{R}*}$. Furthermore, due to the fact that $U \subseteq U^{\underline{R}*\underline{R}*}$ and $V \subseteq V^{\overline{R}*\overline{R}*}$, we can conclude that $U \subseteq (U^{\underline{R}*} \cap V^{\overline{R}*})^{\underline{R}*}$ and $V \subseteq (U^{\underline{R}*} \cap V^{\overline{R}*})^{\overline{R}*}$. Then, $(U, V) \leq ((U^{\underline{R}*} \cap V^{\overline{R}*})^{\underline{R}*}, (U^{\underline{R}*} \cap V^{\overline{R}*})^{\overline{R}*})$ is at hand. Therefore, $(U, V) \leq (U, V)^{\triangleright\triangleleft}$ follows due to $(U, V)^{\triangleright\triangleleft} = ((U^{\underline{R}*} \cap V^{\overline{R}*})^{\underline{R}*}, (U^{\underline{R}*} \cap V^{\overline{R}*})^{\overline{R}*})$.

For any $p \in P$, it follows that $P^{\underline{R}*} \subseteq \{p\}^{\underline{R}*}$ and $P^{\overline{R}*} \subseteq \{p\}^{\overline{R}*}$. Furthermore, we can deduce that $P^{\underline{R}*\underline{R}*} \supseteq \{p\}^{\underline{R}*\underline{R}*} \supseteq \{p\}$ and $P^{\overline{R}*\overline{R}*} \supseteq \{p\}^{\overline{R}*\overline{R}*} \supseteq \{p\}$. Then, we can conclude that $p \in P^{\underline{R}*\underline{R}*} \cap P^{\overline{R}*\overline{R}*}$, i.e., $p \in P^{\triangleleft\triangleright}$. Therefore, $P \subseteq P^{\triangleleft\triangleright}$ is proved.

(iii) Using (i) and (ii), it is obvious. □

In the subsequent discussions, by using the operators $\triangleright$ and $\triangleleft$, we introduce the attribute-induced approximate concept and attribute-induced approximate concept lattice.

Definition 7. *Let $\mathbb{K} = (E, C, \{0, ?, 1\}, R)$, $(U, V) \in 2^E \times 2^E$, and $P \in 2^C$. If $(U, V)^{\triangleright} = P$ and $P^{\triangleleft} = (U, V)$, then we call $((U, V), P)$ an attribute-induced approximate concept.*

Furthermore, relation $\leq$ among attribute-induced approximate concepts is defined as:

$$((U_1, V_1), P_1) \leq ((U_2, V_2), P_2) \Leftrightarrow U_1 \subseteq U_2 \text{ and } V_1 \supseteq V_2 \Leftrightarrow P_1 \supseteq P_2.$$

Then, all of the attribute-induced approximate concepts contained in $\mathbb{K}$ ordered by $\leq$ form a complete lattice. In this case, we call the complete lattice an attribute-induced approximate concept lattice and denote it by $AL(\mathbb{K})$.

In fact, attribute-induced approximate concepts $((U, V), A)$ are a kind of ISI-SE concept [13]. In essence, for a given incomplete context, although attribute-induced approximate concepts and object-induced approximate concepts are symmetrical in a mathematical standpoint, they can characterize the target set from different perspectives. More importantly, in the subsequent discussions, we show that these two types of approximate concepts can give rise to different types of decision rules.

In essence, all of the existing models of concept lattices are in full compliance with regard to their incremental construction [41–43]. By using the existing framework, Algorithm 1 can be used to build attribute-induced approximate concept lattices.

The main idea of Algorithm 1 is to use a progressive method to generate a concept lattice. Concretely, the concept lattice is initialized to be empty, each attribute is processed one by one, and the concept lattice is dynamically adjusted until all attributes are processed, and the final result is obtained. Lines 6 to 11 describe the process of updating the current concept lattice in detail when dealing with a new attribute. It can be realized in the following two steps.

Step 1: Sort all attribute-induced approximate concepts in increasing order with respect to the cardinalities of intents.

Step 2: Visit each attribute-induced approximate concept and make necessary changes as follows.

(i) If $(\{c_i\}^{\underline{R}*}, \{c_i\}^{\overline{R}*}) \cap (U_j, V_j) = (U_j, V_j)$, then change $((U_j, V_j), P_j))$ to be $((U_j, V_j), P_j \cup \{c_i\})$.

(ii) If $(\{c_i\}^{\underline{R}*}, \{c_i\}^{\overline{R}*}) \cap (U_j, V_j) \neq (U_j, V_j)$, then add $((\{c_i\}^{\underline{R}*} \cap U_j, \{c_i\}^{\overline{R}*} \cap V_j), P_j \cup \{c_i\})$ to $AL(\mathbb{K})$.

The time complexity of Algorithm 1 is $O(|C|^2|E||AL(\mathbb{K})|)$. Here, $|AL(\mathbb{K})|$ is the number of attribute-induced approximate concepts of $\mathbb{K}$.

Algorithm 1 Building an attribute-induced approximate concept lattice

Require: $\mathbb{K} = (E, C, \{0, ?, 1\}, R)$, where $C = \{c_1, c_2, \cdots, c_{|C|}\}$.
Ensure: Attribute-induced approximate concept lattice $AL(\mathbb{K})$.
Initialize $AL(\mathbb{K}) = \emptyset$.
$AL(\mathbb{K}) \leftarrow ((\{c_1\}^{\underline{R}*}, \{c_1\}^{\overline{R}*}), \{c_1\})$.
If $(\{c_1\}^{\underline{R}*}, \{c_1\}^{\overline{R}*}) \neq (E, E)$, then add $((E, E), \emptyset)$ into $AL(\mathbb{K})$.
$i \leftarrow 2$.
While $i \leq |V|$ // sort c_i
    Sort $AL(\mathbb{K})$ into a sequence $((U_1, V_1), P_1), \cdots, ((U_{|AL(\mathbb{K})|}, V_{|AL(\mathbb{K})|}), P_{|AL(\mathbb{K})|})$ such that $|P_m| \leq |P_n|$ for $m < n$;
    $j \leftarrow 1$;
    While $j \leq |AL(\mathbb{K})|$ do // the current visited concept is $((U_j, V_j), P_j))$
        If $(\{c_i\}^{\underline{R}*}, \{c_i\}^{\overline{R}*}) \cap (U_j, V_j) = (U_j, V_j)$
            Then update $((U_j, V_j), P_j))$ to $((U_j, V_j), P_j \cup \{c_i\})$
            Else add $((\{c_i\}^{\underline{R}*} \cap U_j, \{c_i\}^{\overline{R}*} \cap V_j), P_j \cup \{c_i\})$ to $AL(\mathbb{K})$
        End If
        j++.
    End While
    i++.
End While
Return $AL(\mathbb{K})$.

Proposition 2. *Let $\mathbb{K} = (E, C, \{0, ?, 1\}, R)$. Then, the following properties hold:*

(i) If $(O_1, P) \in L(\underline{\mathbb{K}})$ and $(O_2, P) \in L(\overline{\mathbb{K}})$, then $((O_1, O_2), P) \in AL(\mathbb{K})$.

(ii) If $(O_1, P) \in L(\underline{\mathbb{K}})$, P is not an intent of any concept in $L(\overline{\mathbb{K}})$, and (O_3, Q) is the minimal one in $L(\overline{\mathbb{K}})$ such that $Q \supset P$, then $((O_1, O_3), P) \in AL(K)$.

(iii) If $(O_2, P) \in L(\overline{\mathbb{K}})$, P is not an intent of any concept in $L(\underline{\mathbb{K}})$, and (O_3, Q) is the minimal one in $L(\underline{\mathbb{K}})$ such that $Q \supset P$, then $((O_3, O_2), P) \in AL(K)$.

Proof. (i) If $(O_1, P) \in L(\underline{\mathbb{K}})$ and $(O_2, P) \in L(\overline{\mathbb{K}})$, it follows that $O_1 = P^{\underline{R}*}$, $O_1^{\underline{R}*} = P$, $O_2 = P^{\overline{R}*}$, and $O_2^{\overline{R}*} = P$. Then, $(O_1, O_2)^{\triangleright} = O_1^{\underline{R}*} \cap O_2^{\overline{R}*} = P$ and $P^{\triangleleft} = (O_1, O_2)$, which means $((O_1, O_2), P) \in AL(\mathbb{K})$.

(ii) If $(O_1, P) \in L(\underline{\mathbb{K}})$ and $(O_3, Q) \in L(\overline{\mathbb{K}})$, it follows that $O_1 = P^{\underline{R}*}$, $O_1^{\underline{R}*} = P$, $O_3 = Q^{\overline{R}*}$, and $O_3^{\overline{R}*} = Q$.

On the one hand, as $Q \supset P$, it follows that $O_1^{\underline{R}*} \cap O_3^{\overline{R}*} = P \cap Q = P$, i.e., $(O_1, O_3)^{\triangleright} = P$. On the other hand, due to the fact that P is not an intent of any concept in $L(\overline{\mathbb{K}})$, and (O_3, Q) is the minimal one in $L(\overline{\mathbb{K}})$ such that $Q \supset P$, we can obtain that $P^{\overline{R}*} = O_3$. Then, we have that $P^{\triangleleft} = (P^{\underline{R}*}, P^{\overline{R}*}) = (O_1, O_3)$. To sum up, this item is at hand.

(iii) If $(O_2, P) \in L(\overline{\mathbb{K}})$ and $(O_3, Q) \in L(\underline{\mathbb{K}})$, then $O_2 = P^{\overline{R}*}$, $O_2^{\overline{R}*} = P$, $O_3 = Q^{\underline{R}*}$, and $O_3^{\underline{R}*} = Q$.

On the one hand, as $Q \supset P$, it follows that $O_3^{\underline{R}*} \cap O_2^{\overline{R}*} = Q \cap P = P$, i.e., $(O_2, O_3)^{\triangleright} = P$.

On the other hand, if P is not an intent of any concept in $L(\underline{\mathbb{K}})$ and (O_3, Q) is the minimal one in $L(\underline{\mathbb{K}})$ such that $Q \supset P$, we can obtain $P^{\underline{R}*} = O_3$, which implies that $P^{\triangleleft} = (P^{\underline{R}*}, P^{\overline{R}*}) = (O_3, O_2)$. In summary, this item is proved. □

Proposition 3. *Let $\mathbb{K} = (E, C, \{0, ?, 1\}, R)$. If $((U, V), P) \in AL(\mathbb{K})$, then $(U, P) \in L(\underline{\mathbb{K}})$ or $(V, P) \in L(\overline{\mathbb{K}})$.*

Proof. By $((U, V), P) \in AL(\mathbb{K})$, we can obtain that $U = P^{\underline{R}*}$ and $V = P^{\overline{R}*}$. On the one hand, we have that $P^{\underline{R}*\underline{R}*} \supseteq P$ and $P^{\overline{R}*\overline{R}*} \supseteq P$, which leads to $P^{\underline{R}*\underline{R}*} \cap P^{\overline{R}*\overline{R}*} \supseteq P$. On the other hand, as $((U, V), P)$ is an attribute-induced approximate concept, it implies that

$U^{\underline{R}*} \cap V^{\overline{R}*} = P^{\underline{R}*\underline{R}*} \cap P^{\overline{R}*\overline{R}*} = P$. In summary, we can obtain that if $((U, V), P) \in AL(\mathbb{K})$, then $P^{\underline{R}*\underline{R}*} = P$ or $P^{\overline{R}*\overline{R}*} = P$, i.e., $(U, P) \in L(\underline{\mathbb{K}})$ or $(V, P) \in L(\overline{\mathbb{K}})$. $\square$

In the sequel, we present an illustrative example to verify Propositions 2 and 3.

Example 1. *An incomplete context* $\mathbb{K} = (E, C, \{0, ?, 1\}, R)$ *is shown in Table 1.*

Table 1. $\mathbb{K}$ of example 1.

	a	***b***	***c***	***d***	***e***
1	0	1	1	1	0
2	1	1	0	1	0
3	0	1	1	?	0
4	?	0	1	1	1
5	1	1	0	0	1

Hereinafter, the braces and commas are omitted for convenience in some occasions when referring to a concept. For example, $(5, 45, ae)$ is the shorthand for $((\{5\}, \{4, 5\}), \{a, e\}))$. $L(\underline{\mathbb{K}})$, $L(\overline{\mathbb{K}})$, and $AL(\mathbb{K})$ are, respectively, shown in Figure 1, Figure 2, and Figure 3.

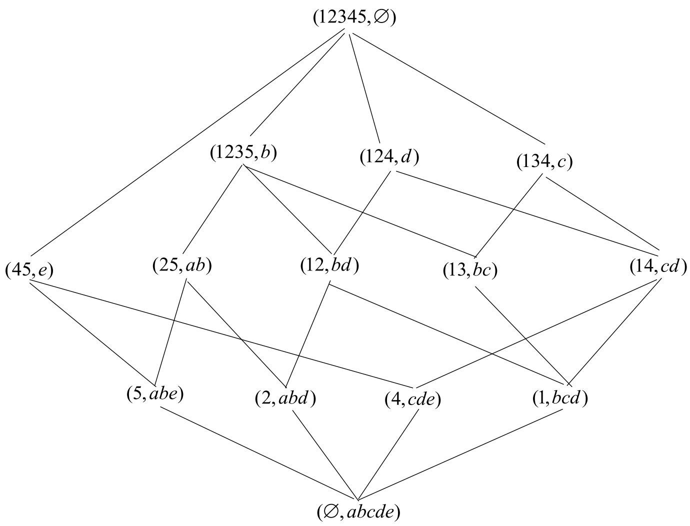

Figure 1. Concept lattice $L(\underline{\mathbb{K}})$.

The following instances can be observed to verify Propositions 2 and 3.

(i) On the one hand, we have $(1, bcd) \in L(\underline{\mathbb{K}})$ and $(13, bcd) \in L(\overline{\mathbb{K}})$. On the other hand, we have $(1, 13, bcd) \in AL(\mathbb{K})$. Therefore, Proposition 2 (i) is verified.

(ii) On the one hand, we have $(134, c) \in L(\underline{\mathbb{K}})$, and $(134, cd)$ is the smallest concept of $L(\overline{\mathbb{K}})$, with its intent containing the intent of $(134, c) \in L(\underline{\mathbb{K}})$. On the other hand, we have $(134, 13, c) \in AL(\mathbb{K})$. Then, Proposition 2 (ii) is verified.

(iii) On the one hand, we have $(24, ad) \in L(\overline{\mathbb{K}})$, and $(2, ade)$ is the smallest concept of $L(\underline{\mathbb{K}})$, with its intent containing the intent of $(24, ad) \in L(\overline{\mathbb{K}})$. On the other hand, we have $(2, 24, ad) \in AL(\mathbb{K})$. In summary, Proposition 2 (iii) is verified.

(iv) For any approximate concept $((U, V), P) \in AL(\mathbb{K})$, we can verify that $(U, P) \in L(\underline{\mathbb{K}})$ or $(V, A) \in L(\overline{\mathbb{K}})$. Thus, Proposition 3 can be verified.

Based on the duality principle, the following results can be immediately derived to display the connection between classical concept lattices and object-induced approximate concept lattices.

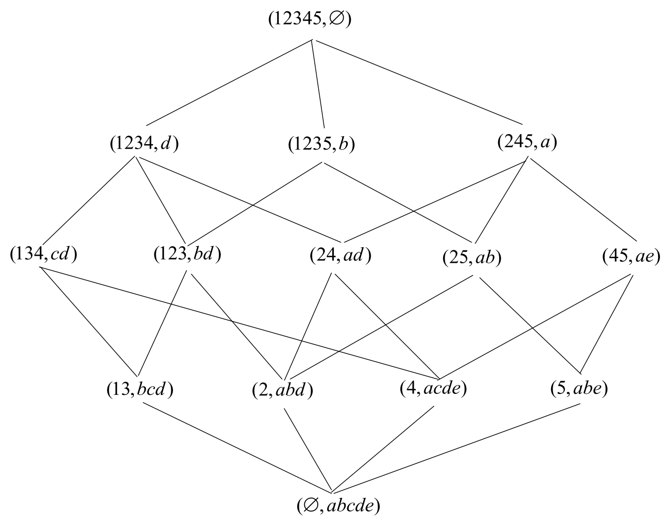

Figure 2. Concept lattice $L(\overline{\mathbb{K}})$.

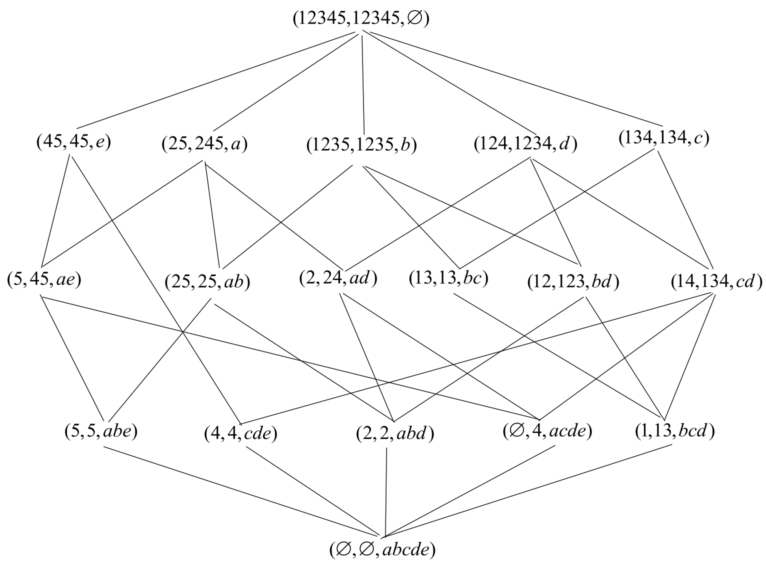

Figure 3. Approximate concept lattice $AL(\mathbb{K})$.

Proposition 4. *Let* $\mathbb{K} = (E, C, \{0, ?, 1\}, R)$. *Then, it follows that:*

(i) If $(O, P) \in L(\underline{\mathbb{K}})$ *and* $(O, Q) \in L(\overline{\mathbb{K}})$, *then* $(O, (P, Q)) \in OL(\mathbb{K})$.

(ii) If $(U, P) \in L(\underline{\mathbb{K}})$, U *is not an extent of any concept in* $L(\overline{\mathbb{K}})$, *and* (V, Q) *is the minimal one in* $L(\overline{\mathbb{K}})$ *such that* $V \supset U$, *then* $(U, (P, Q)) \in OL(K)$.

(iii) If $(U, Q) \in L(\overline{\mathbb{K}})$, U *is not an intent of any concept in* $L(\underline{\mathbb{K}})$, *and* (V, P) *is the minimal one in* $L(\underline{\mathbb{K}})$ *such that* $V \supset U$, *then* $(U, (P, Q)) \in OL(K)$.

Proof. (i) If $(O, P) \in L(\underline{\mathbb{K}})$ and $(O, Q) \in L(\overline{\mathbb{K}})$, it follows that $P = O^{\underline{R}*}$, $O = P^{\underline{R}*}$, $Q = O^{\overline{R}*}$, and $O = Q^{\overline{R}*}$. Then, $O^{\downarrow} = (P, Q)$ and $(P, Q)^{\uparrow} = P^{\underline{R}*} \cap Q^{\overline{R}*} = O$, which means $(O, (P, Q)) \in OL(\mathbb{K})$.

(ii) If $(U, P) \in L(\underline{\mathbb{K}})$ and $(V, Q) \in L(\overline{\mathbb{K}})$, it follows that $U = P^{\underline{R}*}$, $U^{\underline{R}*} = P$, $V = Q^{\overline{R}*}$, and $V^{\overline{R}*} = Q$.

On the one hand, as $V \supset U$, it follows that $P^{\underline{R}*} \cap Q^{\overline{R}*} = U \cap V = U$, i.e., $(P, Q)^{\uparrow} = U$. On the other hand, due to the fact that U is not an extent of any concept in $L(\overline{\mathbb{K}})$, and (V, Q)

is the minimal one in $L(\overline{\mathbb{K}})$ such that $V \supset U$, we can obtain that $U^{\overline{R}*} = Q$. Then, we have $U^{\downarrow} = (U^{\underline{R}*}, U^{\overline{R}*}) = (P, Q)$. In summary, this item is at hand.

(iii) If $(U, Q) \in L(\overline{\mathbb{K}})$ and $(V, P) \in L(\underline{\mathbb{K}})$, then $U = Q^{\overline{R}*}$, $U^{\overline{R}*} = Q$, $V = P^{\underline{R}*}$, and $V^{\underline{R}*} = P$.

On the one hand, as $V \supset U$, it follows that $P^{\underline{R}*} \cap Q^{\overline{R}*} = V \cap U = U$, i.e., $(P, Q)^{\uparrow} = U$. On the other hand, if U is not an intent of any concept in $L(\underline{\mathbb{K}})$ and (V, P) is the minimal one in $L(\underline{\mathbb{K}})$ such that $V \supset U$, we can obtain $U^{\underline{R}*} = P$, which implies that $U^{\downarrow} = (U^{\underline{R}*}, U^{\overline{R}*}) = (P, Q)$. In summary, this item is proved. □

Proposition 5. *Let* $\mathbb{K} = (E, C, \{0, ?, 1\}, R)$*. If* $(O, (P, Q)) \in AL(\mathbb{K})$*, then* $(O, P) \in L(\underline{\mathbb{K}})$ *or* $(O, Q) \in L(\overline{\mathbb{K}})$*.*

Proof. Through $(O, (P, Q)) \in AL(\mathbb{K})$, we can obtain that $P = O^{\underline{R}*}$ and $Q = O^{\overline{R}*}$. On the one hand, we have $O^{\underline{R}*\underline{R}*} \supseteq O$ and $O^{\overline{R}*\overline{R}*} \supseteq O$, which leads to $O^{\underline{R}*\underline{R}*} \cap O^{\overline{R}*\overline{R}*} \supseteq O$. On the other hand, as $(O, (P, Q))$ is an object-induced approximate concept, it implies that $P^{\underline{R}*} \cap Q^{\overline{R}*} = O^{\underline{R}*\underline{R}*} \cap O^{\overline{R}*\overline{R}*} = O$. To sum up, if $(O, (P, Q)) \in AL(\mathbb{K})$, then $O^{\underline{R}*\underline{R}*} = O$ or $O^{\overline{R}*\overline{R}*} = O$, i.e., $(O, P) \in L(\underline{\mathbb{K}})$ or $(O, Q) \in L(\overline{\mathbb{K}})$. □

As attribute-induced approximate concepts and object-induced approximate concepts differ in semantics, further research on the decision rules induced by these two types of approximate concepts is still an open problem, which is discussed in the next section.

At the end of this section, we carry out experiments to show the factors that determine the sizes of object-induced approximate concept lattices and attribute-induced approximate concept lattices. At the same time, object-induced approximate concept lattices and attribute-induced approximate concept lattices can be compared in terms of the number of concepts.

Data sets, i.e., incomplete formal contexts, are randomly generated by controlling the number of objects ($|E|$), the number of attributes ($|C|$), and the ratios r_1:r_2:r_3, where r_1, r_2, and r_3 denote the number of $+$, $-$, and ? filled in the incomplete formal contexts.

First, we test the influences on the sizes of lattices caused by the variations of objects and attributes. Concretely, we fix the ratios r_1:r_2:r_3 to be 3:3:1 and change the number of objects $|E|$ and attributes $|C|$. For simplicity, we set $|C|/|E| = 1$, $|C|/|E| = 0.5$, and $|C|/|E| = 2$ in three rounds of experiments. The experimental results are, respectively, shown in Figures 4–6. It can be observed that: (1) for a given incomplete formal context, the number of object-induced approximate concepts does not equal that of attribute-induced approximate concepts; (2) when $|E| > |C|$, the number of object-induced approximate concepts is greater than that of attribute-induced approximate concepts, and when $|C| > |E|$, the number of attribute-induced approximate concepts is greater than that of object-induced approximate concepts; and (3) when $|E| = |C|$, the number of object-induced approximate concepts approximately equals that of attribute-induced approximate concepts.

Secondly, we test the influences on the sizes of lattices caused by the variations of fill ratios. Hence, we stipulate $|E| = |C| = 30$ and change the fill ratio r_1:r_2:r_3. The experimental results are shown in Figure 7. It can be concluded that when $+$, $-$, and ? take equal proportions, the sizes of object-induced approximate concept lattices and attribute-induced approximate concept lattices take the biggest values, and when ? tends to take a smaller proportion, the sizes of object-induced approximate concept lattices and attribute-induced approximate concept lattices gradually decrease.

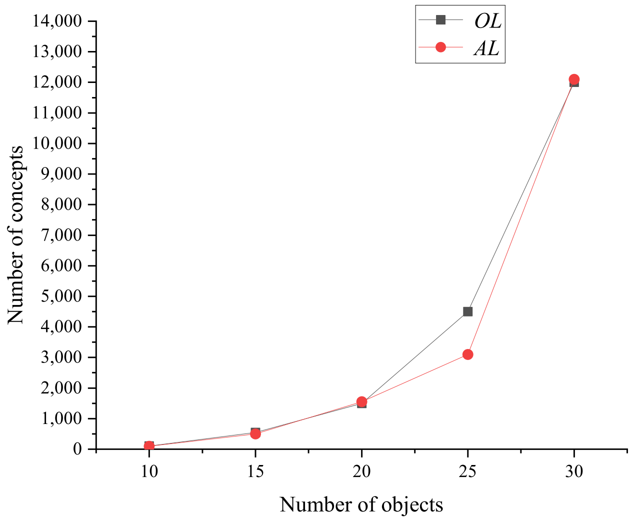

Figure 4. Experimental results caused by variations of objects ($|E|/|C| = 1$).

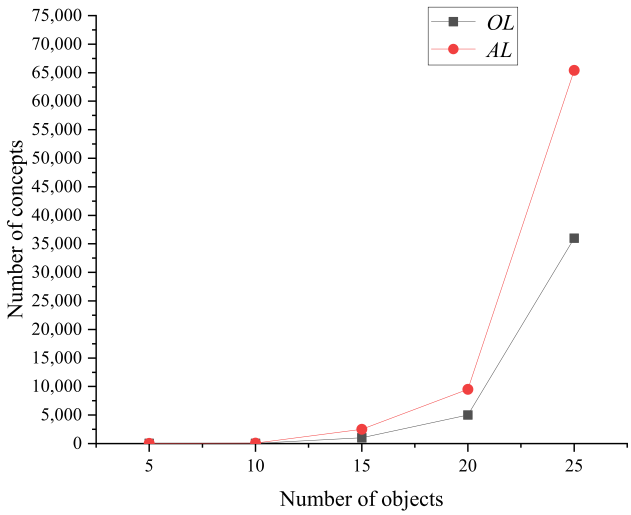

Figure 5. Experimental results caused by variations of objects ($|E|/|C| = 0.5$).

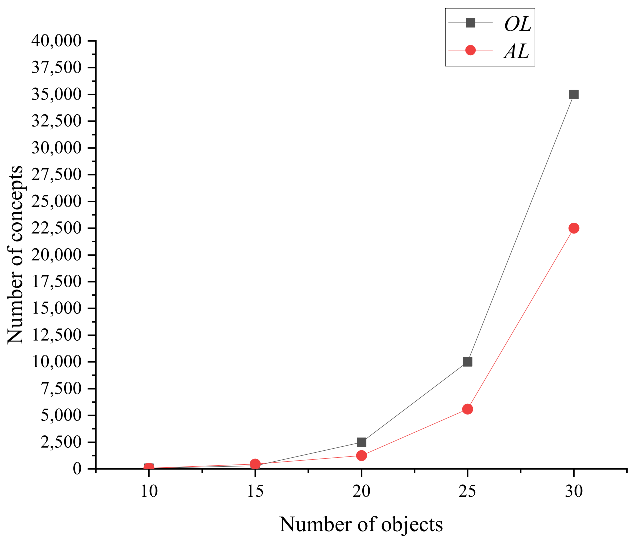

Figure 6. Experimental results caused by variations of objects ($|E|/|C| = 2$).

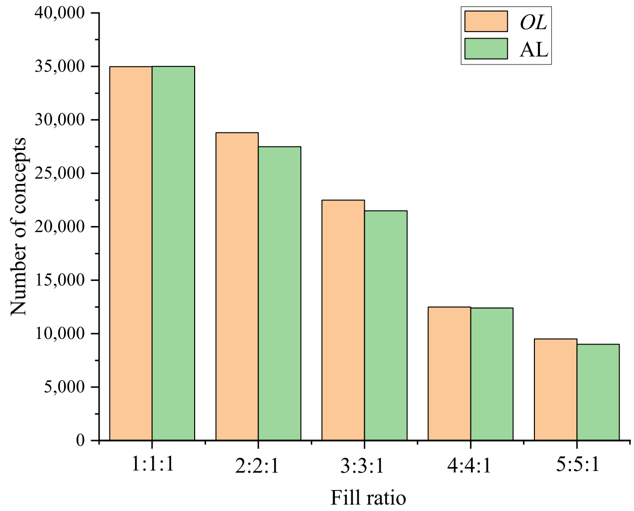

Figure 7. Experimental results caused by variations of fill ratios.

4. Decision Rules Acquisition from Incomplete Formal Contexts

This section presents a novel type of decision rules with incomplete formal contexts.

Definition 8. *Let $\mathbb{D} = (E, C, \{0, ?, 1\}, R, D, J)$, $((U, V), P) \in AL(E, C, \{0, ?, 1\}, R)$, and $(O, Q) \in L(E, D, J)$. If $V \subseteq O$, then $P \to Q$ is called a type-A decision rule of $\mathbb{D}$, which is supported by (U, V). Moreover, the set of type-A decision rules of $\mathbb{D}$ is denoted by $I_A(\mathbb{D})$.*

A type-A decision rule $r : P \to Q$ supported by (U, V) implies that if an object $u \in U$ has all the attributes in P, then u has all the attributes in Q. What is more, if $U \neq \emptyset$, then r is supported by the objects of U with certainty. Moreover, if $U - V \neq \emptyset$, then r is supported by the objects of $(V - U)$ with possibility.

As a one type-A decision rule may be induced by another type-A decision rule, it is necessary to keep $I_A(\mathbb{D})$ from redundancy.

Definition 9. *Let $\mathbb{D} = (E, C, \{0, ?, 1\}, R, D, J)$, $r_1 : P_1 \to Q_1 \in I_A(\mathbb{D})$, and $r_2 : P_2 \to Q_2 \in I_A(\mathbb{D})$. If $P_1 \subseteq P_2$ and $Q_1 \supseteq Q_2$, then it is said that r_1 implies r_2, which is denoted by $r_1 \Rightarrow r_2$, and r_2 is redundant in $I_A(\mathbb{D})$.*

Definition 10. *Let $\mathbb{D} = (E, C, \{0, ?, 1\}, R, D, J)$, $\Pi = \{V \mid ((U, V), P) \in AL(E, C, \{0, ?, 1\}, R)\}$, and $\Omega = \{O \mid (O, Q) \in L(E, D, J)\}$. For $V \in \Pi$ and $O \in \Omega$, we define*

$$f(V, O) = \begin{cases} 1, & f_p(V, O) = 1 \text{ and } f_q(V, O) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

where

$$f_p(V, O) = \begin{cases} 1, & V \subseteq O \text{ and for } \forall V' \in \Pi, V \subset V' \Rightarrow V' \nsubseteq O, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$f_q(V, O) = \begin{cases} 1, & V \subseteq O \text{ and for } \forall O' \in \Omega, O' \subset O \Rightarrow V \nsubseteq O', \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 1 shows the necessary and sufficient conditions of a non-redundant and a type-*A* decision rule.

Theorem 1. *Let $\mathbb{D} = (E, C, \{0, ?, 1\}, R, D, J)$ and $r : P \to Q \in I_A(\mathbb{D})$. Then, r is non-redundant in $I_A(\mathbb{D})$ if and only if $f(P^{\triangleleft}, Q^*) = 1$.*

Proof. $f(P^{\triangleleft}, Q^*) = 1$ holds if and only if there exist $U, V, O \in 2^E$ with $((U,V),P) \in AL(E,C,\{0,?,1\},R)$ and $(O,Q) \in L(E,D,J)$ such that $f(V,O) = 1$. Moreover, $f(V,O) = 1$ holds if and only if $f_p(V,O) = 1$ and $f_q(V,O) = 1$.

On the one hand, $f_p(V,O) = 1$ holds if and only if there is not $((U',V'),P') \in AL(E,C,\{0,?,1\},R)$ such that $V' \subseteq O$ and $V \subset V'$. As a result, it follows there is not $P' \to Q \in I_A(\mathbb{D}) - \{r\}$ with $P' \subset P$.

On the other hand, we can similarly prove that $f_q(V,O) = 1$ holds if and only if there is not $P \to Q' \in I_A(\mathbb{D}) - \{r\}$ with $Q \subset Q'$.

In summary, this theorem is proved. □

Algorithm 2 describes a method of mining all non-redundant type-A decision rules from $\mathbb{D}$.

Algorithm 2 Mining non-redundant type-A decision rules from $\mathbb{D}$

Require: $\mathbb{D} = (E,C,\{0,?,1\},R,D,J)$.
Ensure: Non-redundant type-A decision rules $I_A(\mathbb{D})$.

```
Compute L(E, D, J).
Compute AL(E, C, {0, ?, 1}, R).
Initialize I_A(D) ← ∅.
For each ((U, V), P) ∈ AL(E, C, {+, ?, −}, R) and (O, Q) ∈ L(E, D, J)
    If V ≠ ∅, O ≠ ∅, P ≠ ∅, Q ≠ ∅, and f(V, Z) = O
        Then r : P → Q is a non-redundant type-A decision rule; // supported by (U, V)
          I_A(D) ← I_A(D) ∪ {r};
    End If
End For
Return I_A(D).
```

Note that the building of concept lattices shares a similar process [42] and takes up the major time consumption, and the time complexity of Algorithm 2 is $O(|C|^2|E||AL_p| + |E|^2|D||L_d| + |C|(|AL_p| + |L_d|)|AL_p||L_d|$. Here, $|AL_p|$ is the number of attribute-induced approximate concepts of $(E,C,\{0,?,1\},R)$ and $|L_d|$ is the number of classical concepts of (E,D,J).

Example 2. $\mathbb{D} = (E,C,\{0,?,1\},R,D,J)$ *shown in Table 2 depicts a workgroup, where U contains five participants, V denotes five software development abilities, and* $D = \{d_1, d_2, d_3\}$ *stands for three types of research jobs. Specially, a, b, c, d, and e denote advanced mathematics, discrete mathematics, Java programming, computer vision, and C programming, respectively;* d_1, d_2, *and* d_3 *represent experimental calculation design, algorithm design, and graphic user interface design, respectively.*

Table 2. An incomplete decision context $\mathbb{K}$.

	a	b	c	d	e	d_1	d_2	d_3
x_1	0	1	0	1	1	1	1	0
x_2	1	1	0	?	0	0	0	1
x_3	1	1	1	0	0	0	1	0
x_4	?	0	1	1	1	0	1	1
x_5	1	?	0	0	1	1	1	0

Initially, we can obtain

$$AL(E,C,\{+,?,-\},R)=\left\{\begin{array}{l}((E,E),\emptyset),\\ ((\{x_1,x_2,x_3\},\{x_1,x_2,x_3,x_5\}),\{b\}),\\ ((\{x_1,x_4\},\{x_1,x_2,x_4\}),\{d\}),\\ ((\{x_2,x_3,x_5\},\{x_2,x_3,x_4,x_5\}),\{a\}),\\ ((\{x_1,x_4,x_5\},\{x_1,x_4,x_5\}),\{e\}),\\ ((\{x_3,x_4\},\{x_3,x_4\}),\{c\}),\\ ((\{x_1\},\{x_1,x_2\}),\{b,d\}),\\ ((\{x_2,x_3\},\{x_2,x_3,x_5\}),\{a,b\}),\\ ((\{x_1\},\{x_1,x_5\}),\{b,e\}),\\ ((\{x_1,x_4\},\{x_1,x_4\}),\{d,e\}),\\ ((\emptyset,\{x_2,x_4\}),\{a,d\}),\\ ((\{x_3\},\{x_3,x_4\}),\{a,c\}),\\ ((\{x_5\},\{x_4,x_5\}),\{a,e\}),\\ ((\{x_1\},\{x_1\}),\{b,d,e\}),\\ ((\emptyset,\{x_2\}),\{a,b,d\}),\\ ((\{x_3\},\{x_3\}),\{a,b,c\}),\\ ((\{x_4\},\{x_4\}),\{c,d,e\}),\\ ((\emptyset,\{x_5\}),\{a,b,e\}),\\ ((\emptyset,\{x_4\}),\{a,c,d,e\}),\\ ((\emptyset,\emptyset),C)\end{array}\right\},$$

and

$$L(E,D,J)=\{(E,\emptyset),(\{x_1,x_3,x_4,x_5\},\{d_2\}),(\{x_2,x_4\},$$
$$\{d_3\}),(\{x_1,x_5\},\{d_1,d_2\}),(\{x_4\},\{d_2,d_3\}),(\emptyset,D)\}.$$

Using Algorithm 2, the following non-redundant type-A decision rules can be mined:

$$\begin{array}{ll} r_{A1}: e\to d_2 & //\text{supported by } (\{x_1,x_4,x_5\},\{x_1,x_4,x_5\}).\\ r_{A2}: c\to d_2 & //\text{supported by } (\{x_3,x_4\},\{x_3,x_4\}).\\ r_{A3}: ad\to d_3 & //\text{supported by } (\emptyset,\{x_2,x_4\}).\\ r_{A4}: be\to d_1d_2 & //\text{supported by } (\{x_1\},\{x_1,x_5\}).\\ r_{A5}: cde\to d_2d_3 & //\text{supported by } (\{x_4\},\{x_4\}).\end{array}$$

The obtained decision rules describe the correlations between the software development abilities and the software development activities. As an instance, r_{A4} states that if a person is good at discrete mathematics and C programming, then this person is qualified for experimental calculation design and algorithm design with certainty. Moreover, x_4 supports this statement with certainty and x_5 supports this statement with possibility.

5. The Connections Between Type-O Decision Rules and Type-A Decision Rules

This section explores the connections between type-O decision rules and type-A decision rules. The following Theorems 2 and 3 indicate the related type-O decision rules of a given type-A decision rule.

Theorem 2. *Let $\mathbb{D}=(E,C,\{0,?,1\},R,D,J)$, $r_A: P\to Q$ be a type-A decision rule supported by U. Then, there must be a type-O decision rule $r_O: (P,P')\to Q$ supported by U.*

Proof. As $r_A: P\to Q$ is a decision rule of $\mathbb{D}$ supported by U, it follows that $((U,U),P)$ is an attribute-induced approximate concept of $(E,C,\{0,?,1\},R)$, i.e., $P^{\underline{R}*}=P^{\overline{R}*}=U$, and $U^{\underline{R}*}\cap U^{\overline{R}*}=P$. Moreover, due to the fact that $U^{\underline{R}*}\subseteq U^{\overline{R}*}$, it is easy to obtain $U^{\underline{R}*}=P$. By combining $U^{\underline{R}*}=P$ and $P^{\underline{R}*}=U$, we come to the fact that (U,P) is a concept of the least completion of $(E,C,\{0,?,1\},R)$. Then, in light of Proposition 4, we can deduce that there must be an object-induced approximate concept $(U,(P,P'))$ of $(E,C,\{0,?,1\},R)$. Moreover, through the condition that i_A is supported by U, then there must be a concept $(O,Q)\in L(U,N,J)$ such that $U\subseteq O$. Therefore, by the definition of an approximate decision rule, we can conclude that there must be an approximate decision rule $r_O : (P, P') \to Q$ supported by U. □

Theorem 2 has twofold meanings. On the one hand, for any $e \in E$, if e has all the attributes in P with certainty, then e has all the attributes in Q with certainty. On the other hand, r_A embodies the fact that for any $e \in E$, if e has all attributes in P with certainty, then e has all attributes in $(P' - P)$ with possibility.

Theorem 3. *Let $\mathbb{D} = (E, C, \{0, ?, 1\}, R, D, J)$, $r_A : P \to Q$ be a type-A decision rule supported by (U, V) and $U \subset V$. Then, there must be a type-O decision rule $r_O : (P', P) \to Q$ supported by V.*

Proof. As $r_A : P \to Q$ is a type-A decision rule $\mathbb{D}$ supported by (U, V) and $U \subset V$, it follows that $((U, V), P)$ is an attribute-induced approximate concept of $(E, C, \{0, ?, 1\}, R)$, i.e., $P^{\underline{R}*} = U$, $P^{\overline{R}*} = V$, and $U^{\underline{R}*} \cap V^{\overline{R}*} = P$. Moreover, due to the fact that $U^{\underline{R}*} = P^{\underline{R}*\underline{R}*} \supseteq P$, it follows that $V^{\overline{R}*} = P$. By considering $V^{\overline{R}*} = P$ and $P^{\overline{R}*} = V$, we come to the fact that (V, P) is a concept of the greatest completion of $(E, C, \{0, ?, 1\}, R)$. Then, through Proposition 4, we can obtain that there must be an object-induced approximate concept $(V, (P', P))$ of $(E, C, \{0, ?, 1\}, R)$. Moreover, through the condition that r_A is supported by (U, V), there must be a concept $(O, Q) \in L(E, D, J)$ such that $V \subseteq O$. Then, we can conclude that there must be a type-O decision rule $r_O : (P', P) \to Q$ supported by V. □

In short, Theorem 3 states that for a given type-A decision rule supported by a specific granule U with certainty and supported by another specific granule V with possibility, there must be an approximate decision rule supported by granule V.

In what follows, it points out the related type-A decision rules of the given type-O decision rule.

Theorem 4. *Let $\mathbb{D} = (E, C, \{0, ?, 1\}, R, D, J)$, $r_O : (P, P) \to Q$ be a type-O decision rule supported by U. Then, there must be a type-O decision rule $r_A : P \to Q$ supported by U.*

Proof. As $r_O : (P, P) \to Q$ is a type-O decision rule supported by U, it follows that $(U, (P, P))$ is an object-induced approximate concept of $(E, C, \{0, ?, 1\}, R)$, which implies that $U^{\underline{R}*} = U^{\overline{R}*} = P$ and $P^{\underline{R}*} = P^{\overline{R}*} = U$. Then, there must be an attribute-induced approximate concept $((U, U), P)$ of $(E, C, \{0, ?, 1\}, R)$. Moreover, using the condition that r_O is supported by U, there must be a concept $(O, Q) \in L(E, D, J)$ such that $U \subseteq O$. Therefore, we can conclude that there must be a type-A $i_A : P \to Q$ supported by U. □

Theorem 4 points out that for a type-O decision rule, there must be a type-A decision rule supported by the same granule. Furthermore, by considering Theorems 2 and 4, it can be concluded that type-A and type-O decision rules have equal abilities in dealing with certain data sets.

Theorem 5. *Let $\mathbb{D} = (E, C, \{0, ?, 1\}, R, D, J)$, $r_O : (P, Q) \to S$ be a type-O decision rule supported by U and $P \subset Q$. If there exists another type-O decision rule $(P, P) \to T$ supported by V, then there must be a type-A decision rule $r_A : Q \to S$ supported by (U', U). Otherwise, there must be a type-A decision rule $r_A : P \to S$ supported by (U, U).*

Proof. As $r_O : (P, Q) \to S$ is a type-O decision rule supported by U, it can be deduced that $(U, (P, Q))$ is an object-induced approximate concept of $(E, C, \{0, ?, 1\}, R)$. Moreover, as $(P, P) \to T$ is another type-O decision rule supported by V, it follows that $(V, (P, P))$ is also an object-induced approximate concept of $(E, C, \{0, ?, 1\}, R)$. Then, $U^{\underline{R}*} = P$, $U^{\overline{R}*} = Q$, $P^{\underline{R}*} \cap Q^{\overline{R}*} = U$, and $P^{\underline{R}*} = P^{\overline{R}*} = V \supset U$, and a simple manipulation leads to $U^{\overline{R}*} = Q$. As a result, it can be concluded that (U, Q) is a formal concept of the greatest completion of $(E, C, \{0, ?, 1\}, R)$. Then, through Proposition 4, there must be an attribute-induced approximate concept $((U', U), Q)$ of $(E, C, \{0, ?, 1\}, R)$. Moreover, as r_O is supported by U, then there must be a formal concept $(O, Q) \in L(E, D, J)$ such that $U \subseteq O$. Therefore, we can conclude that there must be a type-A decision rule $r_A : Q \to S$ supported by (U', U). The remaining part can proved in a similar way. $\square$

Theorem 5 implies that for a type-O decision rule that has an uncertain part in the premise, if the certain part is also the premise of another type-O decision rule, then there must be a type-A decision rule supported by a granule with certainty and a rule supported by another granule with possibility. Otherwise, there must be a type-A decision rule supported by a specific granule with certainty.

The above discussion also manifests that in some cases, a type-A decision rule can show different details about the supported granules. That is, a type-A decision rule can show which object supports this rule with certainty and which object supports this rule with possibility, which cannot be obtained from a type-O decision rule.

Example 3. *Continued with Example 2. We can derive*

$$OL(E, C, \{0, ?, 1\}, R) = \left\{ \begin{array}{l} (E, (\emptyset, \emptyset)), \\ (\{x_1, x_2, x_3, x_5\}, (\emptyset, \{b\})), \\ (\{x_2, x_3, x_4, x_5\}, (\emptyset, \{a\})), \\ (\{x_1, x_2, x_3\}, (\{b\}, \{b\})), \\ (\{x_1, x_2, x_4\}, (\emptyset, \{d\})), \\ (\{x_1, x_4, x_5\}, (\{e\}, \{e\})), \\ (\{x_2, x_3, x_5\}, (\{a\}, \{a, b\})), \\ (\{x_3, x_4\}, (\{c\}, \{a, c\})), \\ (\{x_1, x_2\}, (\{b\}, \{b, d\})), \\ (\{x_2, x_3\}, (\{a, b\}, \{a, b\})), \\ (\{x_1, x_5\}, (\{e\}, \{b, e\})), \\ (\{x_1, x_4\}, (\{d, e\}, \{d, e\})), \\ (\{x_2, x_4\}, (\emptyset, \{a, d\})), \\ (\{x_4, x_5\}, (\{e\}, \{a, e\})), \\ (\{x_1\}, (\{b, d, e\}, \{b, d, e\})), \\ (\{x_2\}, (\{a, b\}, \{a, b, d\})), \\ (\{x_3\}, (\{a, b, c\}, \{a, b, c\})), \\ (\{x_4\}, (\{c, d, e\}, \{a, c, d, e\})), \\ (\{x_5\}, (\{a, e\}, \{a, b, e\})), \\ (\emptyset, (C, C)) \end{array} \right\},$$

by using the method in [4], and the obtained non-redundant type-O decision rules are as follows:

$$\begin{array}{lll} r_{O1} : & (e, e) \to d_2 & //\text{supported by } \{x_1, x_4, x_5\}. \\ r_{O2} : & (c, ac) \to d_2 & //\text{supported by } \{x_3, x_4\}. \\ r_{O3} : & (\emptyset, ad) \to d_3 & //\text{supported by } \{x_2, x_4\}. \\ r_{O4} : & (e, be) \to d_1 d_2 & //\text{supported by } \{x_1, x_5\}. \\ r_{O5} : & (cde, acde) \to d_2 d_3 & //\text{supported by } \{x_4\}. \end{array}$$

In what follows, we take several examples to illustrate the relationships between the type-A decision rules and the type-O decision rules.

(i) For a given type-A decision rule $r_{A1} : e \to d_2$ supported by $\{x_1, x_4, x_5\}$, there must be a type-O decision rule $(e, e) \to d_2$ supported by $\{x_1, x_4, x_5\}$ (i.e., r_{O1}). Therefore, Theorem 2 is verified.

(ii) For a given type-A decision rule $r_{A3} : ad \to d_3$ supported by $(\emptyset, \{x_2, x_4\})$, there must be a type-$O$ decision rule $(\emptyset, ad) \to d_3$ supported by $\{x_2, x_4\}$ (i.e., r_{O3}). Therefore, Theorem 3 is verified.

(iii) For a given type-O decision rule $r_{O1} : (e, e) \to d_2$ supported by $\{x_1, x_4, x_5\}$, there must be a type-A decision rule $e \to d_2$ supported by $\{x_1, x_4, x_5\}$ (i.e., r_{A1}). Therefore, Theorem 4 is verified.

(iv) For a given type-O decision rule $r_{O4} : (e, be) \to d_1d_2$ supported by $\{x_1, x_5\}$, as there exists an approximate decision rule $r_{O1} : (e, e) \to d_1$ supported by $\{x_1, x_4, x_5\}$, there must be a type-A decision rule $be \to d_1d_2$ supported by $(\{x_1\}, \{x_1, x_5\})$ (i.e., r_{A4}); thus, Theorem 5 is verified.

(v) For a given type-O decision rule $r_{O5} : (cde, acde) \to d_2d_3$ supported by $\{x_4\}$, as there is not an approximate decision rule with its premise equal to cde, there must be a type-A decision rule $cde \to d_2d_3$ supported by $(\{x_4\}, \{x_4\})$ (i.e., r_{A5}); thus, Theorem 5 is verified.

6. Conclusions

Rule acquisition is a major topic in the development of a knowledge-based system. Decision rules mining from the incomplete decision formal context still deserves further exploration. In this study, we define attribute-induced approximate concepts and decision rules based on attribute-induced approximate concepts, i.e., type-A decision rules. More importantly, we explore the connections between type-A decision rules and type-O decision rules, i.e., decision rules based on object-induced approximate concepts, and we obtain several important propositions.

Essentially, in this study, a target set is described by pointing out what attributes it possesses. However, sometimes a target set needs to be characterized by what attributes it does not possess [44–46]. In other words, we have to generalize the spirit of 3WCA and investigate target sets from multiple perspectives [47]. Moreover, how to model approximate concepts by combining both object-induced and attribute-induced manners, and how to derive decision rules based on the proposed type of approximate concepts are promising and interesting topics.Moreover, it is also possible to generalize the obtained results to more complicated cases, such as incomplete intuitionistic fuzzy information systems [48].

Author Contributions: Conceptualization, H.W. and H.Z.; methodology, H.Z.; software, H.Z.; validation, Y.L., D.Z. and J.X.; formal analysis, H.Z.; investigation, H.W.; resources, J.X.; data curation, H.Z.; writing—original draft preparation, H.W. and H.Z.; writing—review and editing, H.W. and H.Z.; visualization, H.Z.; supervision, Y.L.; project administration, H.Z.; funding acquisition, H.Z. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported in part by the Natural Science Foundation of Fujian Provincial Science and Technology Department under Grant No. 2023H6034 and No. 2021H6037; in part by the Key Project of Quanzhou Science and Technology Plan under Grant No. 2021C008R; and in part by the sixth batch of Quanzhou City's introduction of high-level talent team projects under Grant No. 2.

Data Availability Statement: Dataset available upon request from the authors.

Conflicts of Interest: The authors declare no conflicts of interest.

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