

## Article

# Statistical Analysis and Several Estimation Methods of New Alpha Power-Transformed Pareto Model with Applications in Insurance

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**Abstract:** This article defines a new distribution using a novel alpha power-transformed method extension. The model obtained has three parameters and is quite effective in modeling skewed, complex, symmetric, and asymmetric datasets. The new approach has one additional parameter for the model. Certain distributional and mathematical properties are investigated, notably reliability, quartile, moments, skewness, kurtosis, and order statistics, and several approaches of estimation, notably the maximum likelihood, least square, weighted least square, maximum product spacing, Cramer-Von Mises, and Anderson Darling estimators of the model parameters were obtained. A Monte Carlo simulation study was conducted to evaluate the performance of the proposed techniques of estimation of the model parameters. The actuarial measures are computed for our recommended model. At the end of the paper, two insurance applications are illustrated to check the potential and utility of the suggested distribution. Evaluation using four selection criteria indicates that our recommended model is the most appropriate probability model for modeling insurance datasets.

**Keywords:** actuarial measures; asymmetric; insurance data; order statistics; parameter estimation; skewness; symmetric



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## 1. Introduction

Statistical models are commonly applied to describe real-world phenomena and are very much required for data modeling in different disciplines, to analyze and to know the pattern of the data generated by the stochastic nature of the system from the field of knowledge. Further, the baseline distribution is unsuitable for fitting several datasets. To solve this problem, a family of distribution is obtained using numerous techniques. These new techniques extend the classical model to obtain an approach distribution. The suggested tool is more efficient in fitting different real-world phenomena, likely engineering (see Wang et al. [1,2]), biomedical, actuarial science, medicine, insurance, and environmental. Notable examples of these generated distribution families include the beta-normal model, provided by Eugene et al. [3], and the inverse Weibull-G model, considered by Amal et al. [4]. Cordeiro et al. [5] introduced the Kumaraswamy-G family, Marshall, and

Olkin [6] and defined a new method for adding a parameter to a family of distributions, Zagrofos and Balakrishnan [7] defined the gamma-G class of distributions. In the same line, Morad Alizadeh et al. [8] investigated the Gompertz-G models and Brito et al. [9] proposed the Topp–Leone odd log-logistic family of distributions. Thomas et al. [10] generated the power generalized DUS family. Alzaatreh et al. [11] have introduced the transformed-transformer(T-X) class of distributions and Almetwaly and Meraou [12] have considered the sine Nadaraja Haghighi model.

Recently, Barry [13] defined Pareto (Par) distribution, which can be used in various applications, particularly in survival, medical science, hydrology, finance, and insurance. Numerous studies and generalized extensions of the Par distribution have been implemented in several fields to bring further flexibility to the Par distribution. One may refer to Abdul Majid et al. [14], Akinsete et al. [15], Bourguignon et al. [16], and Chhetri et al. [17]. The cumulative distribution function (cdf) and density probability function (pdf) of the parameter model can be investigated as

$$W(t; \eta, \delta) = 1 - \left(\frac{\eta}{t}\right)^\delta, \quad t > \eta, \quad \eta, \delta > 0, \quad (1)$$

and

$$w(t; \eta, \delta) = \frac{\delta \eta^\delta}{t^{\delta+1}}. \quad (2)$$

In the last couple of decades, a useful method for generating a flexible probability model, called a new class of distribution, was suggested by Mahdavi and Kundu [18], listed as the alpha power transformation (APT) family of distributions. The proposed class of probability models can be described as follows. Suppose  $X$  is a continuous RV. Then,  $X$  is said to have an APT class of distribution if its cdf and pdf are like

$$F(x) \begin{cases} \frac{\alpha^{K(x, \psi)} - 1}{\alpha - 1}, & \text{if } \alpha > 0; \alpha \neq 1, \\ K(x, \psi), & \text{if } \alpha = 1, \end{cases}$$

and

$$f(x) \begin{cases} \frac{\log \alpha}{\alpha - 1} \alpha^{K(x, \psi)} k(x, \psi), & \text{if } \alpha > 0; \alpha \neq 1, \\ k(x, \psi), & \text{if } \alpha = 1, \end{cases}$$

where  $\alpha$  is an additional parameter. Many authors applied the APT family frequently in numerous studies. For example, the APT Weibull (APTW) distribution (Nassar et al. [19]), APT generalized exponential (APTGE) model (Dey et al. [20]), APT Gompertz (APTGO) model (Eghwerido et al. [21]), and APT power Lindley (APTPL) distribution (Hassan et al. [22]). In the same context, APT Xgamma (APTXG) distribution (Shivanshi et al. [23]), APT extended exponential (APTEE) model (Hassan et al. [24]), APT Extended power Lindley (APT-EPL) (Fatehi and Chhaya [25]), and APT Dagum (APT-D) model (Reyad et al. [26]). In this work, we introduced an extension of the APT family, which is referred to as a new alpha power-transformed (NAPT) class of distributions. The cdf and pdf of the NAPT model are

$$G(t, \psi, \alpha) = \alpha^{-\log\left(\frac{1}{K(t, \psi)}\right)} = \alpha^{\log(K(t, \psi))}, \quad t \in \mathbb{R}, \alpha > 1,$$

and

$$g(t, \psi, \alpha) = \frac{1}{K(t, \psi)} \left( \log \alpha k(t, \psi) \alpha^{-\log\left(\frac{1}{K(t, \psi)}\right)} \right) = \frac{1}{K(t, \psi)} \left( \log \alpha k(t, \psi) \alpha^{\log(K(t, \psi))} \right).$$

This article focused on inventing a new model derived from Par distributions to explore more complex datasets and establish various properties. The obtained model was formed using the NAPT technique, and we listed it as the new alpha power-transformed Pareto (NAPT-Par) model. More advantages of the recommended NAPT-Par model can be considered. It can be applied to several sectors, notably insurance, finance, economics, environmental sciences, engineering, survival, and biology. Its applications extend to analyzing income and wealth inequality, firm size data, and Internet traffic datasets, among others.

Now, we can list certain motivations that motivated us to define the new KMIW model:

- The NAPT-Par distribution is adequate for fitting different types data due to its flexibility and utility.
- Another crucial motivation is that the proposed distribution offers mechanisms to introduce or manipulate skewness, allowing for the modeling of datasets where asymmetry is inherent or desired.
- A paramount goal of the NAPT-Par model is to provide several classical estimation procedures to estimate the unknown parameters of the recommended distribution, and a simulation analysis is performed to evaluate the estimator's performance. Moreover, the final aim is to explore four well-known risk measures of the proposed NAPT-Par model, notably the value at risk (VaR), Tail VaR (TVaR), tail variance (TV), and tail variance premium (TVP).
- the NAPT-Par model can capture various shapes and patterns in data, which often results in better goodness-of-fit compared to other distributions.

The order of the remaining sections is as follows. In Section 2, we provide the useful representations of cdf and pdf and certain important distributional properties of the newly recommended model. In Section 3, several mathematical properties of the NAPT-Par model are investigated. The estimation of model parameters using numerous estimation methods is presented in Section 4. Section 5 offers a brief Monte Carlo (MC) simulation study to evaluate the potential of the proposed estimators. Four risk measures of our distribution are calculated in Section 6, and one real dataset taken from the insurance area is analyzed to select the best-fitting models in Section 7. Finally, the conclusion is given in Section 8.

## 2. Model Formulation

In this section, the cdf, pdf, and corresponding reliability functions are developed.

A continuous random variable  $Z$  is said to have the recommended NAPT-Par distribution with parameters  $\eta, \delta$ , and  $\alpha$  if its cdf and pdf have the below forms

$$M(z, \eta, \delta, \alpha) = \alpha^{\log\left(1 - \left(\frac{\eta}{z}\right)^\delta\right)}, \quad z > \eta, \text{ and } \alpha > 1, \eta, \delta > 0, \quad (3)$$

and

$$m(z, \eta, \delta, \alpha) = \frac{1}{1 - \left(\frac{\eta}{z}\right)^\delta} \left\{ \log \alpha \frac{\delta \eta^\delta}{z^{\delta+1}} \alpha^{\log\left(1 - \left(\frac{\eta}{z}\right)^\delta\right)} \right\}. \quad (4)$$

For different values of the parameters  $\eta, \delta$ , and  $\alpha$ , several representations of the pdf of the NAPT-Par distribution are displayed in Figure 1. The pdf of the random variable  $Z$  confirms a variety of shapes. The proposed NAPT-Par model is decreasing, positively skewed, and unimodal.

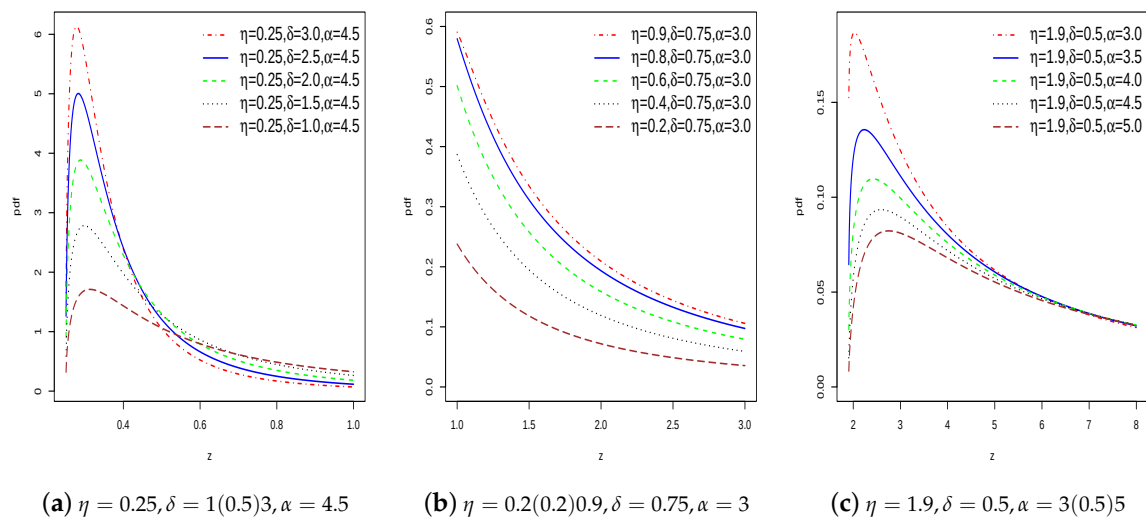


**Figure 1.** Various representations of pdf plots for NAPT-Par distribution under several parameter values.

Furthermore, the survival function (sf) and hazard rate function (hrf) of the random variable  $Z$  are obtained:

$$S(z, \eta, \delta, \alpha) = 1 - \alpha^{\log\left(1 - \left(\frac{\eta}{z}\right)^\delta\right)},$$

and

$$h(z, \eta, \delta, \alpha) = \frac{\log \alpha \frac{\delta \eta^\delta}{z^{\delta+1}} \alpha^{\log\left(1 - \left(\frac{\eta}{z}\right)^\delta\right)}}{\left[1 - \left(\frac{\eta}{z}\right)^\delta\right] \left[1 - \alpha^{\log\left(1 - \left(\frac{\eta}{z}\right)^\delta\right)}\right]}.$$

The plot of hrf for the NAPT-Par is summarized in Figure 2. The plots considering the hrf of the suggested model are obtained by applying numerous selected parameter values. The hrf of  $Z$  decreases, and the unimodal and reverse J-shaped functions are also decreasing. Consequently, the correct behavior of the hrf confirms that the NAPT-Par is a valid and reliable model for analyzing various datasets.

Henceforth, the cumulative hrf with the reversed hrf of the random variable  $Z$  are written as follows:

$$H(z, \eta, \delta, \alpha) = -\log\left\{1 - \alpha^{\log\left(1 - \left(\frac{\eta}{z}\right)^\delta\right)}\right\},$$

and

$$R(z, \eta, \delta, \alpha) = \frac{\log \alpha \frac{\delta \eta^\delta}{z^{\delta+1}}}{1 - \left(\frac{\eta}{z}\right)^\delta}.$$

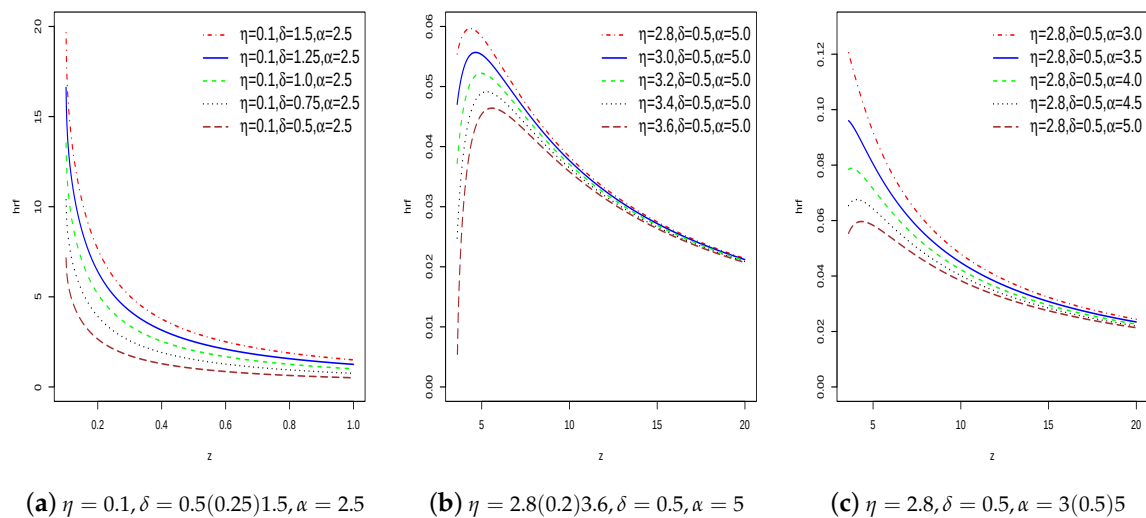


**Figure 2.** Various representations of hrf plots for NAPT-Par distribution under several parameter values.

### 3. Some Statistical Properties of NAPT-Par Model

We established certain mathematical properties of the NAPT-Par distribution. These properties covered the quantile function,  $k^{th}$  moment, mean, variance, moment generating function (MGF), and the distribution of the order statistics.

#### 3.1. Quantile Function

Let  $Z$  follows the NAPT-Par distribution. The corresponding quantile function ( $q$ ) is obtained by the inverse of Equation (3). That is,

$$q(u) = \eta \left( 1 - e^{\frac{\log u}{\log \alpha}} \right)^{-1/\delta}, \quad 0 < u < 1, \quad (5)$$

So, a random sample from the random variable  $Z$  may be generated using the above equation with  $u$  having a uniform random number (0,1).

The median of  $Z$  is given by

$$z_{0.5} = -\eta \left( 1 - e^{\frac{\log 0.5}{\log \alpha}} \right)^{-1/\delta}.$$

Next, both coefficients of skewness ( $SK$ ) and the kurtosis ( $KR$ ) of the recommended NAPT-Par model can be derived as

$$SK = \frac{q(0.25) + q(0.75) - 2q(0.5)}{q(0.75) - q(0.25)},$$

and

$$KR = \frac{q(0.875) - q(0.625) + q(0.375) - q(0.125)}{q(0.75) - q(0.25)}.$$

#### 3.2. Useful Expansion

First, the generalized binomial linear representation can be written as

$$\alpha^t = \sum_{i=0}^{\infty} \frac{(\log \alpha)^i}{i!} t^i.$$

As a result, we can obtain the cdf and pdf of NAPT-Par distribution, and they are given by

$$\begin{aligned}
 M(z, \eta, \delta, \alpha) &= \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{i!} \left[ \log \left( 1 - \left( \frac{\eta}{z} \right)^{\delta} \right) \right]^i \\
 &= \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{i!} [\log K(z, \eta, \delta)]^i,
 \end{aligned}$$

and

$$\begin{aligned}
 m(z, \eta, \delta, \alpha) &= \frac{\delta \eta^{\delta}}{z^{\delta+1} \left( 1 - \left( \frac{\eta}{z} \right)^{\delta} \right)} \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{i!} \left[ \log \left( 1 - \left( \frac{\eta}{z} \right)^{\delta} \right) \right]^i \\
 &= \frac{k(z, \eta, \delta)}{K(z, \eta, \delta)} \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{i!} [\log K(z, \eta, \delta)]^i,
 \end{aligned}$$

where,  $K(z, \eta, \delta)$  and  $k(z, \eta, \delta)$  are given in (1) and (2).

### 3.3. The $k^{\text{th}}$ Moment of the NAPT-Par

Now, by applying the new expression pdf of the RV  $Z$ , which follows the NAPT-Par model, we can express the  $k^{\text{th}}$ -moment of  $Z$ , and it is written as

$$u'_k = \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{i!} Y_{i,k}(z, \eta, \delta), \quad (6)$$

with  $Y_{i,k}(z, \eta, \delta) = \int_0^{\infty} z^k \frac{k(z, \eta, \delta)}{K(z, \eta, \delta)} [\log K(z, \eta, \delta)]^i dz$ .

Consequently, the expected value (mean) and variance of  $Z$  are

$$u'_1 = \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{i!} Y_{i,1}(z, \eta, \delta),$$

and

$$V = \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{i!} \left\{ Y_{i,2}(z, \eta, \delta) - \sum_{i=0}^{\infty} \frac{(\log \alpha)^i}{i!} Y_{i,1}^2(z, \eta, \delta) \right\}.$$

The index of dispersion of the RV  $Z$  can thus be obtained using the formula

$$ID = \frac{V}{u'_1}.$$

Finally, the MGR of  $Z$  is

$$M(t) = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{t^k (\log \alpha)^{i+1}}{k! i!} Y_{i,k}(z, \eta, \delta). \quad (7)$$

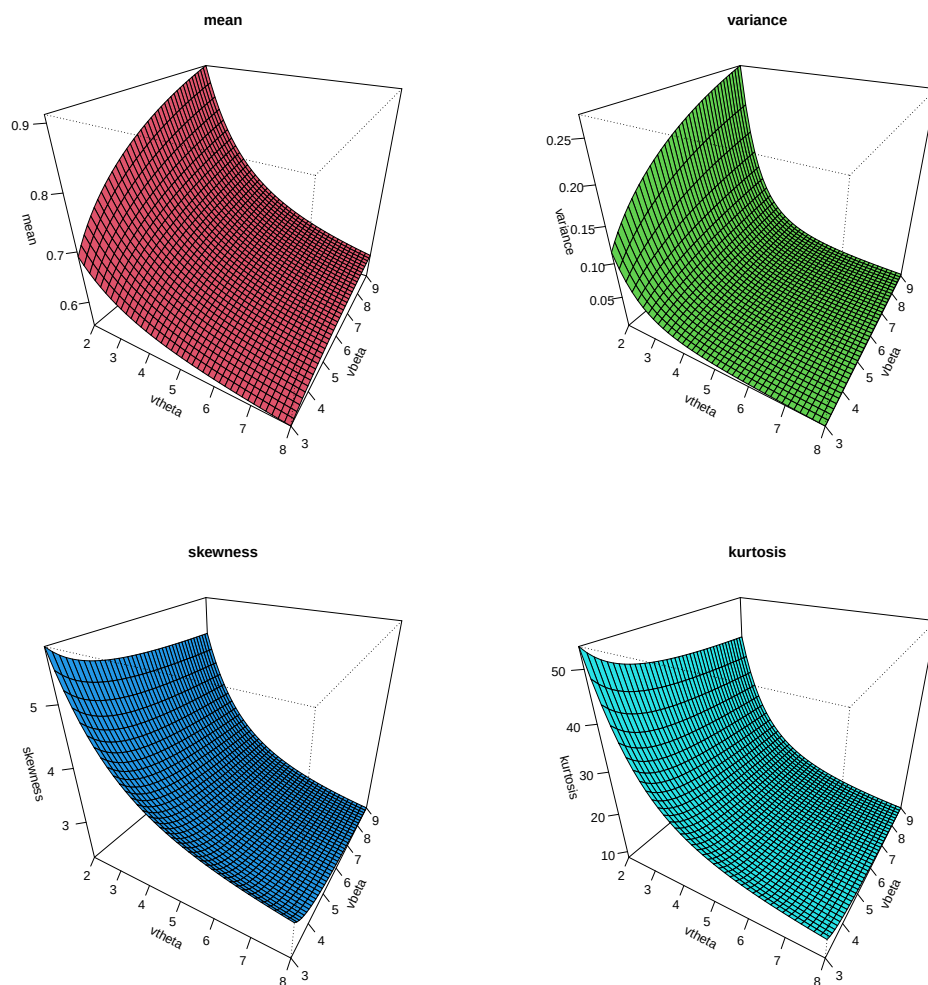
Tables 1 and 2 record numerous recommended mathematical properties as discussed previously, and Figures 3 and 4 demonstrate the 3d plots of these statistical measures. All these obtained values and representations show that our NAPT-Par distribution is a more suitable model for modeling several types of datasets.

**Table 1.** Several characteristic properties for the NAPT-Par model at  $\alpha = 1.75$ .

| $\delta$   |    | $\mu'_1$ | $V$     | ID     | SK     | KR      |
|------------|----|----------|---------|--------|--------|---------|
| $\eta = 2$ | 3  | 2.6433   | 1.6734  | 0.6331 | 9.8944 | 307.896 |
|            | 6  | 2.2631   | 0.165   | 0.0729 | 4.0128 | 30.177  |
|            | 9  | 2.1655   | 0.0574  | 0.0265 | 3.3245 | 18.362  |
|            | 12 | 2.1207   | 0.0288  | 0.0136 | 3.0614 | 14.827  |
| $\eta = 4$ | 3  | 5.2866   | 6.6935  | 1.2661 | 9.8944 | 307.896 |
|            | 6  | 4.5262   | 0.6602  | 0.1459 | 4.0128 | 30.177  |
|            | 9  | 4.3309   | 0.2298  | 0.0531 | 3.3245 | 18.362  |
|            | 12 | 4.2414   | 0.1154  | 0.0272 | 3.0614 | 14.827  |
| $\eta = 6$ | 3  | 7.9299   | 15.0603 | 1.8992 | 9.8944 | 307.896 |
|            | 6  | 6.7892   | 1.4854  | 0.2188 | 4.0128 | 30.177  |
|            | 9  | 6.4964   | 0.5170  | 0.0796 | 3.3245 | 18.362  |
|            | 12 | 6.3621   | 0.2596  | 0.0408 | 3.0614 | 14.827  |

**Table 2.** Several characteristic properties for the NAPT-Par model at  $\alpha = 3$ .

|            | $\delta$ | $\mu'_1$ | $V$    | ID     | SK     | KR      |
|------------|----------|----------|--------|--------|--------|---------|
| $\eta = 2$ | 3        | 3.0702   | 2.889  | 0.9410 | 8.8719 | 258.264 |
|            | 6        | 2.4267   | 0.2512 | 0.1035 | 3.3636 | 22.196  |
|            | 9        | 2.2664   | 0.0843 | 0.0372 | 2.7186 | 12.867  |
|            | 12       | 2.1936   | 0.0416 | 0.019  | 2.4707 | 10.110  |
| $\eta = 4$ | 3        | 6.1403   | 11.556 | 1.8820 | 8.8719 | 258.264 |
|            | 6        | 4.8535   | 1.0049 | 0.2070 | 3.3636 | 22.196  |
|            | 9        | 4.5328   | 0.3371 | 0.0744 | 2.7186 | 12.867  |
|            | 12       | 4.3872   | 0.1663 | 0.0379 | 2.4707 | 10.110  |
| $\eta = 6$ | 3        | 9.2105   | 26.001 | 2.8230 | 8.8719 | 258.264 |
|            | 6        | 7.2802   | 2.2610 | 0.3106 | 3.3636 | 22.196  |
|            | 9        | 6.7991   | 0.7584 | 0.1115 | 2.7186 | 12.867  |
|            | 12       | 6.5808   | 0.3742 | 0.0569 | 2.4707 | 10.110  |

**Figure 3.** Three-dimensional curves for the statistical properties of the NAPT-Par model at  $\eta = 2$ .

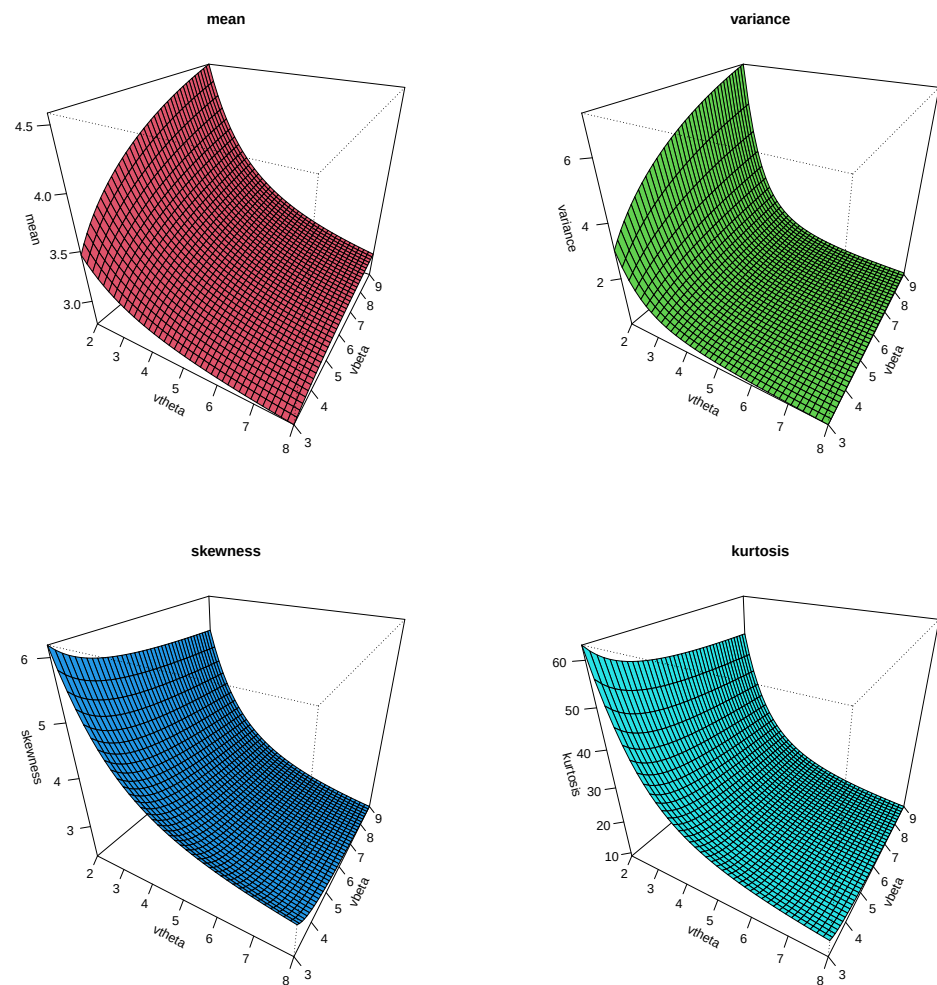


**Figure 4.** Three-dimensional curves for the statistical properties of the NAPT-Par model at  $\eta = 2.5$ .

### 3.4. Order Statistics

Let  $Y_{(1:n)}, Y_{(2:n)}, \dots, Y_{(n:n)}$  be an order random sample (RS) drawn from our suggested model. The pdf of the  $j^{\text{th}}$  order statistic of the RV  $Z$  is

$$\begin{aligned}
 f_{(j:n)}(z) &= \frac{n!}{(j-1)!(n-j)!} m(z, \eta, \delta, \alpha) [M(z, \eta, \delta, \alpha)]^{j-1} [1 - M(z, \eta, \delta, \alpha)]^{n-j} \\
 &= \frac{n! \left(1 - \left(\frac{\eta}{z}\right)^\delta\right)^{-1}}{(j-1)!(n-j)!} \left\{ \log \alpha \frac{\delta \eta^\delta}{z^{\delta+1}} \alpha^{\log\left(1 - \left(\frac{\eta}{z}\right)^\delta\right)} \right\} \left( \alpha^{\log\left(1 - \left(\frac{\eta}{z}\right)^\delta\right)} \right)^{j-1} \\
 &\quad \times \left( 1 - \alpha^{\log\left(1 - \left(\frac{\eta}{z}\right)^\delta\right)} \right)^{n-j}.
 \end{aligned}$$

This gives the pdf of  $Z_{(n:n)} = \max\{z_1, z_2, \dots, z_n\}$  as

$$f_{(n:n)}(z) = n \left(1 - \left(\frac{\eta}{z}\right)^\delta\right)^{-1} \left\{ \log \alpha \frac{\delta \eta^\delta}{z^{\delta+1}} \alpha^{\log\left(1 - \left(\frac{\eta}{z}\right)^\delta\right)} \right\} \left( \alpha^{\log\left(1 - \left(\frac{\eta}{z}\right)^\delta\right)} \right)^{n-1},$$

Also, we have the pdf of  $Z_{(1)} = \min\{z_1, z_2, \dots, z_n\}$  as

$$f_{(1:n)}(z) = n \left(1 - \left(\frac{\eta}{z}\right)^\delta\right)^{-1} \left\{ \log \alpha \frac{\delta \eta^\delta}{z^{\delta+1}} \alpha^{\log\left(1 - \left(\frac{\eta}{z}\right)^\delta\right)} \right\} \left(1 - \alpha^{\log\left(1 - \left(\frac{\eta}{z}\right)^\delta\right)}\right)^{n-1}.$$

Now, the cdf of the  $j^{th}$  order statistic of  $Z$  is obtained to be

$$\begin{aligned} F_{(j:n)}(z) &= \sum_{l=j}^n M^l(z, \eta, \delta, \alpha) [1 - M(z, \eta, \delta, \alpha)]^{n-l} \\ &= \sum_{l=j}^n \left( \alpha^{\log\left(1 - \left(\frac{\eta}{z}\right)^\delta\right)} \right)^l \left(1 - \alpha^{\log\left(1 - \left(\frac{\eta}{z}\right)^\delta\right)}\right)^{n-l}. \end{aligned}$$

#### 4. Parameter Estimation Procedures

In this part of the study, we discuss six parameter estimation processes for estimating the unknown parameters of the proposed model.

##### 4.1. Maximum Likelihood Estimation (MLE)

Suppose  $(z_1, z_2, \dots, z_n)$  is a set of samples drawn from the suggested NAPT-Par model, and  $\Theta = (\eta, \delta, \alpha)$  represents the parameter vector. Corresponding to Equation (4), the log-likelihood function, say  $\mathcal{LL}(z; \Theta)$ , is

$$\begin{aligned} \mathcal{LL}(z; \Theta) &= \sum_{i=1}^n \log m(z; \Theta) \\ &= n \log(\log(\alpha)) + n \log(\delta) + n \delta \log \eta - (\delta + 1) \sum_{i=1}^n \log z_i + (\log \alpha - 1) \sum_{i=1}^n \log \left(1 - \left(\frac{\eta}{z_i}\right)^\delta\right). \end{aligned} \quad (8)$$

Corresponding to Equation (8), the partial derivatives with respect parameters are obtained to be

$$\frac{\partial \mathcal{LL}(z; \Theta)}{\partial \eta} = \frac{n \delta}{\eta} + (1 - \log \alpha) \delta \eta^{\delta-1} \sum_{i=1}^n \left[ z_i^\delta \left(1 - \left(\frac{\eta}{z_i}\right)^\delta\right) \right]^{-1}, \quad (9)$$

$$\frac{\partial \mathcal{LL}(z; \Theta)}{\partial \delta} = \frac{n}{\delta} + n \log \eta - \sum_{i=1}^n \log z_i + (1 - \log \alpha) \sum_{i=1}^n \frac{\log\left(\frac{\eta}{z_i}\right) \left(\frac{\eta}{z_i}\right)^\delta}{1 - \left(\frac{\eta}{z_i}\right)^\delta}, \quad (10)$$

and

$$\frac{\partial \mathcal{LL}(z; \Theta)}{\partial \alpha} = \frac{n}{\alpha \log \alpha} + \frac{1}{\alpha} \sum_{i=1}^n \log \left(1 - \left(\frac{\eta}{z_i}\right)^\delta\right). \quad (11)$$

Based on Equations (9)–(11), it is difficult to compute them directly. To overcome this issue, we can use several iterative non-linear procedures, likely Newton–Raphson, secant, bisection, and fixed point techniques, to find the MLEs, say  $\omega_1$  of  $\Theta = (\eta, \delta, \alpha)$ .

##### 4.2. Least Squares Estimator (LSE)

Swain et al. [27] introduced an extensive tool to determine the estimation of unknown parameters for any distribution. Assume that an RS  $z_1, \dots, z_n$  of size  $n$  is drawn from a distribution function  $M(z; \eta, \delta, \alpha)$ , and  $z_{(1:n)}, \dots, z_{(n:n)}$  represents its order statistics. The proposed LSEs technique, say  $\omega_2$  of the  $\Theta = (\eta, \delta, \alpha)$ , is computed by minimizing

$$\omega_2 = \sum_{i=1}^n \left[ M(z_{(i:n)}; \Theta) - A_1 \right]^2,$$

where,  $A_1 = \frac{i}{n+1}$ . Consequently, we obtain the final estimates of  $\Theta$ , and they are expressed as follows:

$$\sum_{i=1}^n \left[ M(z_{(i:n)}; \Theta) - \frac{i}{n+1} \right] \Omega_1(z_{(i:n)}; \Theta) = 0,$$

$$\sum_{i=1}^n \left[ M(z_{(i:n)}; \Theta) - \frac{i}{n+1} \right] \Omega_2(z_{(i:n)}; \Theta) = 0,$$

and

$$\sum_{i=1}^n \left[ M(z_{(i:n)}; \Theta) - \frac{i}{n+1} \right] \Omega_3(z_{(i:n)}; \Theta) = 0,$$

with

$$\Omega_1(z_{(i:n)}; \Theta) = \frac{\partial}{\partial \eta} M(z_{(i:n)}; \Theta), \quad (12)$$

$$\Omega_2(z_{(i:n)}; \Theta) = \frac{\partial}{\partial \delta} M(z_{(i:n)}; \Theta), \quad (13)$$

and

$$\Omega_3(z_{(i:n)}; \Theta) = \frac{\partial}{\partial \alpha} M(z_{(i:n)}; \Theta), \quad (14)$$

#### 4.3. Weighted Least Squares Estimator (WLSE)

Similarly, by minimizing the below equation, we obtained the WLSE estimators, say  $\omega_3$  of  $\Theta$ . For more details, see

$$\omega_3 = \sum_{i=1}^n A_2 \left[ M(z_{(i:n)}; \Theta) - \frac{i}{n+1} \right]^2,$$

with  $A_2 = \frac{(n+1)^2(n+2)}{i(n-i+1)}$ . Consequently, we obtain the final estimates of  $\Theta$ , and they can be written as

$$\sum_{i=1}^n A_2 \left[ M(z_{(i:n)}; \Theta) - \frac{i}{n+1} \right] \Omega_1(z_{(i:n)}; \Theta) = 0,$$

$$\sum_{i=1}^n A_2 \left[ M(z_{(i:n)}; \Theta) - \frac{i}{n+1} \right] \Omega_2(z_{(i:n)}; \Theta) = 0,$$

and

$$\sum_{i=1}^n A_2 \left[ M(z_{(i:n)}; \Theta) - \frac{i}{n+1} \right] \Omega_3(z_{(i:n)}; \Theta) = 0.$$

#### 4.4. Maximum Product of Spacings (MPS)

To obtain the estimates of  $\Theta$  of the proposed NAPT-Par distribution using the MPS technique, we described this tool with the following steps: First, let

$$B_i = M(z_{(i:n)}; \Theta) - M(z_{(i-1:n)}; \Theta); \quad i = 1, \dots, n+1,$$

where

$$M(z_{(0)}; \Theta) = 0, \quad \text{and} \quad M(z_{(n+1)}; \Theta) = 1.$$

Evidently  $\sum_{i=1}^{n+1} B_i = 1$ . Secondly, by minimizing the below equation, the MPS estimators, say  $\omega_4$  of  $\Theta$ , are obtained to be

$$\omega_4 = \frac{1}{n+1} \sum_{i=1}^{n+1} \log \mathcal{MPS}.$$

By taking the derivative of the preceding equation, we obtain the final estimates of  $\Theta$ , and they are

$$\frac{\partial \omega_4}{\partial \eta} = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{\omega_4} \left\{ \Omega_1(z_{(i:n)}; \Theta) - \Omega_1(z_{(i:n)}; \Theta) \right\} = 0,$$

$$\frac{\partial \omega_4}{\partial \delta} = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{\omega_4} \left\{ \Omega_2(z_{(i:n)}; \Theta) - \Omega_2(z_{(i:n)}; \Theta) \right\} = 0,$$

and

$$\frac{\partial \omega_4}{\partial \alpha} = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{\omega_4} \left\{ \Omega_3(z_{(i:n)}; \Theta) - \Omega_3(z_{(i:n)}; \Theta) \right\} = 0,$$

#### 4.5. Cramer-Von Mises Method of Estimation (CVE)

Macdonald [28] defined empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators. This proposed estimator is based on the Cramer-von-Mises statistics, say  $\omega_5$ , provided by minimizing

$$\omega_5 = \frac{1}{12n} + \sum_{i=1}^n \left[ M(z_{(i:n)}; \Theta) - \frac{2i-1}{2n} \right]^2.$$

Now, the final estimates of  $\Theta$  can be obtained to be

$$\sum_{i=1}^n \left[ M(z_{(i:n)}; \Theta) - \frac{2i-1}{2n} \right] \Omega_1(z_{(i:n)}; \Theta) = 0,$$

$$\sum_{i=1}^n \left[ M(z_{(i:n)}; \Theta) - \frac{2i-1}{2n} \right] \Omega_2(z_{(i:n)}; \Theta) = 0,$$

and

$$\sum_{i=1}^n \left[ M(z_{(i:n)}; \Theta) - \frac{2i-1}{2n} \right] \Omega_3(z_{(i:n)}; \Theta) = 0.$$

#### 4.6. Anderson–Darling Method of Estimation (ADE)

This proposed estimator is based on Anderson–Darling statistics, say  $\omega_6$ , obtained by minimizing

$$\omega_6 = -n - \frac{1}{n} \sum_{i=1}^n (2i-1) \left\{ \log M(z_{(i:n)}; \Theta) + \log(1 - M(z_{(i:n)}; \Theta)) \right\}.$$

By taking the derivative of the preceding equation, we obtain the final estimates of  $\Theta$ , and they are formulated as

$$\sum_{i=1}^n (2i-1) \left[ \frac{\Omega_1(z_{(i:n)}; \Theta)}{M(z_{(i:n)}; \Theta)} - \frac{\Omega_1(z_{(i:n)}; \Theta)}{1 - M(z_{(i:n)}; \Theta)} \right] = 0,$$

$$\sum_{i=1}^n (2i-1) \left[ \frac{\Omega_2(z_{(i:n)}; \Theta)}{M(z_{(i:n)}; \Theta)} - \frac{\Omega_2(z_{(i:n)}; \Theta)}{1 - M(z_{(i:n)}; \Theta)} \right] = 0,$$

and

$$\sum_{i=1}^n (2i-1) \left[ \frac{\Omega_3(z_{(i:n)}; \Theta)}{M(z_{(i:n)}; \Theta)} - \frac{\Omega_3(z_{(i:n)}; \Theta)}{1 - M(z_{(i:n)}; \Theta)} \right] = 0.$$

### 5. Simulation Experiment Study

A Monte Carlo simulation analysis is conducted over time using the R programming language to assess the performance of methods discussed in this study.

The samples are obtained through the application of the quantile function, given by

$$y_u = -\eta \left( 1 - e^{\frac{\log u}{\log \alpha}} \right)^{-1/\delta}, \quad 0 < u < 1, \quad (15)$$

The simulation study is undertaken for an arbitrarily chosen set of parameter values—Set 1:  $\Theta = (0.85, 0.5, 2.0)$  or Set 2:  $\Theta = (1.0, 0.75, 2.3)$ —and different selecting random samples, such as  $n = 25, 50, 70, 75, 100$ . Several criteria are applied to assess the simulation results, such as mean estimates (AE), mean bias (ABs), and mean square error (MSE). These criteria are detailed below

$$\text{Mean} = \frac{1}{1000} \sum_{i=1}^{1000} \hat{\Theta}_i,$$

$$\text{MSE} = \frac{1}{1000} \sum_{i=1}^{1000} | \hat{\Theta}_i - \Theta |,$$

$$\text{MSE} = \frac{1}{1000} \sum_{i=1}^{1000} (\hat{\Theta}_i - \Theta)^2.$$

The comparison is made using several metrics of deviations, notably the mean estimates (AE), mean bias (AB), and mean square errors (MSE) based on various sample sizes,  $n = \{25, 50, 75, 100\}$ . By taking  $N = 1000$  replications of the process, A random sample from the basic NAPT-Par model is generated using Equation (15). Tables 3 and 4 display the obtained results. The findings from the simulation study of the NAPT-Par distribution, as detailed in Tables 3 and 4, demonstrate that:

- As  $n$  continues to increase, the AE is observed to approach the initial value of parameters for all estimates procedures, which shows that all techniques are asymptotically unbiased.
- As  $n$  continues to increase, the MSE values decline, showing that all techniques are consistent.
- Since the MSE values of the MLE method are smaller, the MLE technique outperforms other proposed tools.

**Table 3.** The numerical representation of the simulation study of the NAPT-Par distribution is provided for Set 1 based on classical methods.

| $n$ | Method     | $\hat{\eta}$ |        |        | $\hat{\delta}$ |         |        | $\hat{\alpha}$ |        |        |
|-----|------------|--------------|--------|--------|----------------|---------|--------|----------------|--------|--------|
|     |            | AE           | AB     | MSE    | AE             | AB      | MSE    | AE             | AB     | MSE    |
| 25  | $\omega_1$ | 0.8722       | 0.0222 | 0.0007 | 0.5475         | 0.0475  | 0.0157 | 1.9871         | 0.0128 | 0.0807 |
|     | $\omega_2$ | 0.8244       | 0.0255 | 0.0044 | 0.5389         | 0.0389  | 0.0311 | 2.1824         | 0.1824 | 0.2619 |
|     | $\omega_3$ | 0.8319       | 0.0180 | 0.0059 | 0.4876         | 0.0123  | 0.0475 | 2.0850         | 0.0850 | 0.2802 |
|     | $\omega_4$ | 0.2603       | 0.0103 | 0.0020 | 0.4957         | 0.0042  | 0.0281 | 2.0863         | 0.0863 | 0.2018 |
|     | $\omega_5$ | 0.8655       | 0.0155 | 0.0023 | 0.4347         | 0.0652  | 0.0199 | 1.7637         | 0.2362 | 0.1360 |
|     | $\omega_6$ | 0.2457       | 0.6042 | 0.4925 | 0.4061         | 0.0938  | 0.0753 | 1.8016         | 0.1983 | 0.3972 |
| 50  | $\omega_1$ | 0.8614       | 0.0114 | 0.0005 | 0.5192         | 0.0192  | 0.0052 | 1.9940         | 0.0059 | 0.0317 |
|     | $\omega_2$ | 0.8381       | 0.0118 | 0.0039 | 0.4483         | 0.0516  | 0.0223 | 2.0253         | 0.0253 | 0.2328 |
|     | $\omega_3$ | 0.8423       | 0.0076 | 0.0056 | 0.4844         | 0.0155  | 0.0324 | 2.035          | 0.035  | 0.2445 |
|     | $\omega_4$ | 0.8683       | 0.0183 | 0.0016 | 0.4938         | 0.0061  | 0.0086 | 1.8139         | 0.1860 | 0.0590 |
|     | $\omega_5$ | 0.8747       | 0.0247 | 0.0015 | 0.5050         | 0.0050  | 0.0159 | 1.9911         | 0.0088 | 0.0539 |
|     | $\omega_6$ | 0.3910       | 0.2589 | 0.2504 | 0.3195         | 0.1804  | 0.0355 | 1.8938         | 0.0061 | 0.3516 |
| 75  | $\omega_1$ | 0.8619       | 0.0119 | 0.0003 | 0.5478         | 0.0478  | 0.0042 | 2.0187         | 0.0187 | 0.0176 |
|     | $\omega_2$ | 0.8482       | 0.0017 | 0.0032 | 0.4762         | 0.02378 | 0.0069 | 2.0539         | 0.0539 | 0.1208 |
|     | $\omega_3$ | 0.8461       | 0.0038 | 0.0042 | 0.4603         | 0.0396  | 0.0079 | 1.9661         | 0.0338 | 0.1500 |
|     | $\omega_4$ | 0.8544       | 0.0044 | 0.0008 | 0.4842         | 0.0157  | 0.0048 | 1.9414         | 0.0585 | 0.0196 |
|     | $\omega_5$ | 0.8651       | 0.0151 | 0.0006 | 0.4683         | 0.0316  | 0.0157 | 1.8970         | 0.1029 | 0.0111 |
|     | $\omega_6$ | 0.7289       | 0.1211 | 0.1529 | 0.4342         | 0.0618  | 0.0321 | 1.9277         | 0.0723 | 0.2356 |
| 100 | $\omega_1$ | 0.8506       | 0.0006 | 0.0001 | 0.5180         | 0.0180  | 0.0007 | 2.0282         | 0.0282 | 0.0091 |
|     | $\omega_2$ | 0.8382       | 0.0117 | 0.0007 | 0.5191         | 0.0191  | 0.0037 | 2.1477         | 0.1477 | 0.0674 |
|     | $\omega_3$ | 0.8691       | 0.0191 | 0.0032 | 0.5123         | 0.0123  | 0.0062 | 0.5212         | 0.0212 | 0.0041 |
|     | $\omega_4$ | 0.8571       | 0.0071 | 0.0005 | 0.4784         | 0.0215  | 0.0013 | 1.9583         | 0.0416 | 0.0173 |
|     | $\omega_5$ | 0.8545       | 0.0045 | 0.0004 | 0.5330         | 0.0330  | 0.0011 | 2.0888         | 0.0888 | 0.0106 |
|     | $\omega_6$ | 0.7723       | 0.0773 | 0.0925 | 0.4841         | 0.0159  | 0.0294 | 1.9729         | 0.0271 | 0.1241 |

**Table 4.** The numerical representation of the simulation study of the NAPT-Par distribution is provided for Set 2 based on classical methods.

| $n$ | Method     | $\hat{\eta}$ |        |        | $\hat{\delta}$ |         |        | $\hat{\alpha}$ |         |        |
|-----|------------|--------------|--------|--------|----------------|---------|--------|----------------|---------|--------|
|     |            | AE           | AB     | MSE    | AE             | AB      | MSE    | AE             | AB      | MSE    |
| 25  | $\omega_1$ | 1.0311       | 0.0311 | 0.0073 | 0.9074         | 0.1574  | 0.0711 | 1.9452         | 0.3547  | 0.2026 |
|     | $\omega_2$ | 1.0066       | 0.0066 | 0.0128 | 0.8026         | 0.0526  | 0.1216 | 2.7011         | 0.4011  | 1.7191 |
|     | $\omega_3$ | 0.9701       | 0.0298 | 0.0145 | 0.9141         | 0.1641  | 0.1482 | 2.8453         | 0.5453  | 2.4655 |
|     | $\omega_4$ | 1.0830       | 0.0830 | 0.0119 | 0.6733         | 0.0766  | 0.0879 | 2.0292         | 0.2707  | 0.2537 |
|     | $\omega_5$ | 0.9979       | 0.0020 | 0.0123 | 0.8638         | 0.1138  | 0.0882 | 2.8908         | 0.5908  | 0.2950 |
|     | $\omega_6$ | 0.2062       | 0.7937 | 0.7833 | 0.4988         | 0.2511  | 0.2767 | 2.9385         | 0.6385  | 3.4141 |
| 50  | $\omega_1$ | 1.0185       | 0.0185 | 0.0014 | 0.7173         | 0.0326  | 0.0154 | 2.0668         | 0.2331  | 0.1474 |
|     | $\omega_2$ | 0.9716       | 0.0283 | 0.0070 | 0.7329         | 0.0170  | 0.0559 | 2.6513         | 0.3513  | 0.8264 |
|     | $\omega_3$ | 0.9785       | 0.0214 | 0.0059 | 0.8049         | 0.0549  | 0.0440 | 2.7088         | 0.4088  | 0.9355 |
|     | $\omega_4$ | 1.0302       | 0.0302 | 0.0023 | 0.6311         | 0.1188  | 0.0206 | 1.9985         | 0.3014  | 0.1986 |
|     | $\omega_5$ | 0.9876       | 0.0123 | 0.0033 | 0.7889         | 0.03897 | 0.0429 | 2.5659         | 0.2659  | 0.2838 |
|     | $\omega_6$ | 0.1741       | 0.8258 | 0.6884 | 0.5328         | 0.2171  | 0.1588 | 1.9327         | 0.3672  | 1.4878 |
| 75  | $\omega_1$ | 1.0113       | 0.0113 | 0.0002 | 0.7419         | 0.0080  | 0.0061 | 2.0891         | 0.2108  | 0.0895 |
|     | $\omega_2$ | 0.9790       | 0.020  | 0.0059 | 0.7274         | 0.0225  | 0.0327 | 2.3480         | 0.0480  | 0.4795 |
|     | $\omega_3$ | 0.9899       | 0.010  | 0.0017 | 0.7705         | 0.0205  | 0.0310 | 2.6016         | 0.3016  | 0.4383 |
|     | $\omega_4$ | 1.0152       | 0.0152 | 0.0006 | 0.6836         | 0.0663  | 0.0238 | 2.0259         | 0.2740  | 0.1118 |
|     | $\omega_5$ | 0.9851       | 0.0148 | 0.0008 | 0.7930         | 0.0430  | 0.0300 | 2.4752         | 0.1752  | 0.2152 |
|     | $\omega_6$ | 0.3222       | 0.6777 | 0.6590 | 0.6799         | 0.0401  | 0.1425 | 2.8221         | 0.5221  | 1.1142 |
| 100 | $\omega_1$ | 1.0076       | 0.0076 | 0.0001 | 0.7367         | 0.0132  | 0.0034 | 2.3036         | 0.0036  | 0.0397 |
|     | $\omega_2$ | 1.0135       | 0.0135 | 0.0025 | 0.7358         | 0.0141  | 0.0132 | 2.2810         | 0.0189  | 0.3020 |
|     | $\omega_3$ | 0.9998       | 0.0001 | 0.0011 | 0.7593         | 0.0093  | 0.0234 | 2.2676         | 0.0323  | 0.0717 |
|     | $\omega_4$ | 1.0085       | 0.0085 | 0.0003 | 0.7421         | 0.0078  | 0.0045 | 2.2052         | 0.0947  | 0.0548 |
|     | $\omega_5$ | 0.9901       | 0.0098 | 0.0007 | 0.7785         | 0.0285  | 0.0063 | 2.3779         | 0.07796 | 0.1223 |
|     | $\omega_6$ | 0.6203       | 0.3797 | 0.3995 | 0.7091         | 0.0408  | 0.1204 | 2.8369         | 0.5369  | 0.4652 |

## 6. Actuarial Metrics

In this part of the work, we evaluated several metric measures for the NAPT-Par model, including VaR, TVaR, TV, and TVP. In the literature, many authors applied these actuarial measures, for example, the studies of Meraou et al. [29,30], Affify et al. [31], and Teamah et al. [32].

### 6.1. VaR Metric

The VaR, listed as  $(\mathcal{X}_1)$ , of our NAPT-Par model, is obtained by finding the inverse of the Equation (3). The final expression of  $\mathcal{X}_1$  can be expressed as

$$\mathcal{X}_1 = \eta \left( 1 - e^{\frac{\log q}{\log \alpha}} \right)^{-1/\delta}, \quad 0 < q < 1.$$

### 6.2. TVaR Metric

Another crucial metric measure is named the TVaR metric  $(\mathcal{X}_2)$ . It represents the amount utilized to perform the financial impact of a loss. For each random variable,  $Z$  follows the NAPT-Par model, and its  $\mathcal{X}_2$  is denoted by

$$\begin{aligned} \mathcal{X}_2 &= \frac{1}{(1-q)} \int_{\mathcal{V}_1}^{\infty} z m(z, \eta, \delta, \alpha) dz \\ &= \frac{\delta \eta^\delta}{(1-q)} \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{i!} \Phi_i(z, \eta, \delta), \end{aligned}$$

$$\text{where } \Phi_i(z, \eta, \delta) = \int_{\mathcal{X}_1}^{\infty} \left[ z^\delta \left( 1 - \left( \frac{\eta}{z} \right)^\delta \right) \right]^{-1} \left[ \log \left( 1 - \left( \frac{\eta}{z} \right)^\delta \right) \right]^i dz.$$

### 6.3. TV Metric

The TV metric  $(\mathcal{X}_3)$  for our recommended NAPT-Par model can be defined by the formula below

$$\begin{aligned} \mathcal{X}_3 &= \frac{1}{(1-p)} \int_{\mathcal{X}_1}^{\infty} z^2 m(z, \eta, \delta, \alpha) dy - (\mathcal{X}_2)^2 \\ &= \frac{\delta \eta^\delta}{(1-q)} \sum_{i=0}^{\infty} \frac{(\log \alpha)^{i+1}}{i!} \Phi'_i(z, \eta, \delta) - (\mathcal{X}_2)^2, \end{aligned}$$

$$\text{where } \Phi'_i(z, \eta, \delta) = \int_{\mathcal{X}_1}^{\infty} \left[ z^{\delta+1} \left( 1 - \left( \frac{\eta}{z} \right)^\delta \right) \right]^{-1} \left[ \log \left( 1 - \left( \frac{\eta}{z} \right)^\delta \right) \right]^i dz$$

### 6.4. TVP Metric

Another type of metric measure is referred to as the TVP metric, which is denoted by  $(\mathcal{X}_4)$ . It is a criterion used by the insurance industry. The expression of the TVP metric is defined as follows.

$$\mathcal{X}_4 = \mathcal{X}_2 + q\mathcal{X}_3,$$

### 6.5. Simulation Results of the Risk Measures for the NAPT-Par Model

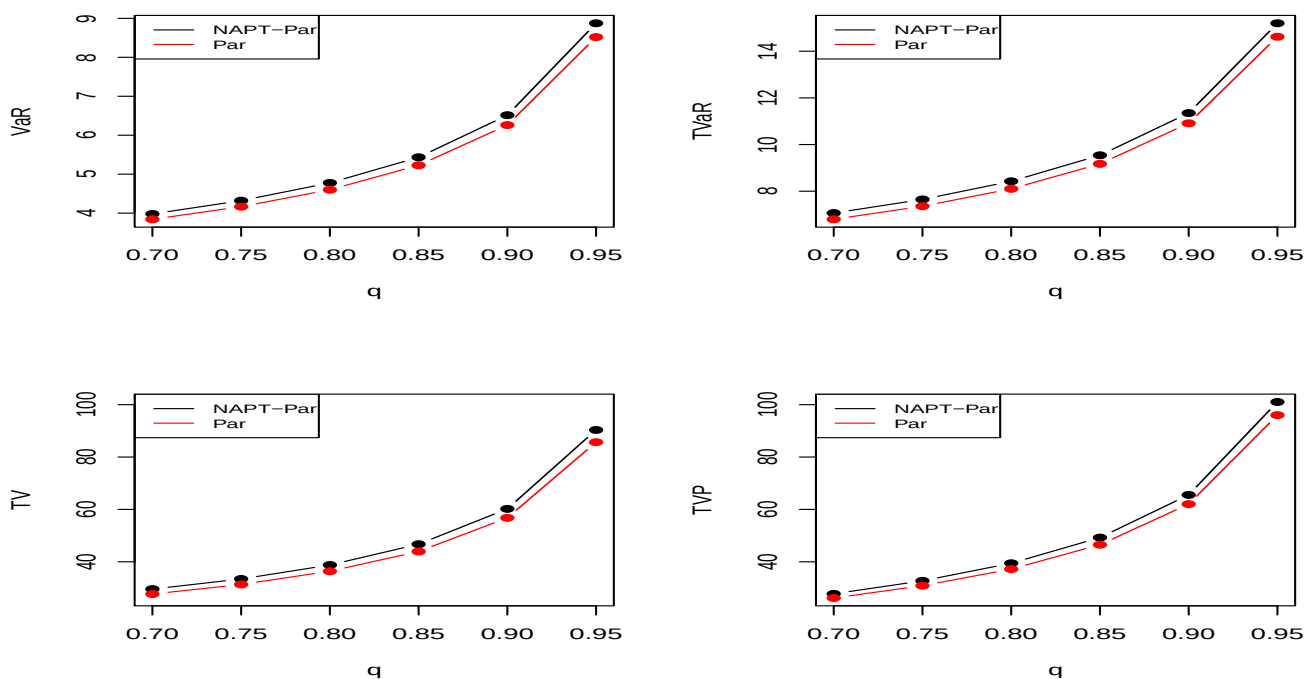
In this part, the results for  $\mathcal{X}_1$ ,  $\mathcal{X}_2$ ,  $\mathcal{X}_3$ , and  $\mathcal{X}_4$  using our proposed Par models based on two cases of the numerical values of the unknown parameters are performed. Tables 5 and 6 reported the obtained results. Furthermore, Figures 5 and 6 demonstrated the visual comparisons between the NAPT-Par and Par distributions. It is clear from the numerical experiments and Figures 5 and 6 that the NAPT-Par model is better than the other distribution since our recommended model has higher values of the actuarial measures than the Par distribution. This ensures that the NAPT-Par model is more efficient in heavy-tailed datasets.

**Table 5.** The numerical representation of the simulation study of  $\mathcal{X}_1$ ,  $\mathcal{X}_2$ ,  $\mathcal{X}_3$ , and  $\mathcal{X}_4$  for the NAPT-Par and Par models.

| Model    | Parameters                                 | $q$  | $\mathcal{X}_1$ | $\mathcal{X}_2$ | $\mathcal{X}_3$ | $\mathcal{X}_4$ |
|----------|--------------------------------------------|------|-----------------|-----------------|-----------------|-----------------|
| NAPT-Par | $\eta = 2.25, \delta = 2.25, \alpha = 3.0$ | 0.70 | 3.979           | 7.063           | 29.579          | 27.769          |
|          |                                            | 0.75 | 4.3205          | 7.6479          | 33.445          | 32.730          |
|          |                                            | 0.80 | 4.7766          | 8.4258          | 38.777          | 39.447          |
|          |                                            | 0.85 | 5.4343          | 9.5401          | 46.724          | 49.256          |
|          |                                            | 0.90 | 6.5145          | 11.349          | 60.220          | 65.547          |
|          |                                            | 0.95 | 8.8742          | 15.198          | 90.356          | 101.037         |
| Par      | $\eta = 2.25, \delta = 2.25$               | 0.70 | 3.8421          | 6.7967          | 27.693          | 26.182          |
|          |                                            | 0.75 | 4.1664          | 7.3566          | 31.350          | 30.869          |
|          |                                            | 0.80 | 4.6008          | 8.1027          | 36.399          | 37.222          |
|          |                                            | 0.85 | 5.2283          | 9.1727          | 43.942          | 46.524          |
|          |                                            | 0.90 | 6.260           | 10.911          | 56.795          | 62.028          |
|          |                                            | 0.95 | 8.5195          | 14.620          | 85.677          | 96.013          |

**Table 6.** The numerical representation of the simulation study of  $\mathcal{X}_1$ ,  $\mathcal{X}_2$ ,  $\mathcal{X}_3$ , and  $\mathcal{X}_4$  for the NAPT-Par and Par models.

| Model    | Parameters                               | $q$  | $\mathcal{X}_1$ | $\mathcal{X}_2$ | $\mathcal{X}_3$ | $\mathcal{X}_4$ |
|----------|------------------------------------------|------|-----------------|-----------------|-----------------|-----------------|
| NAPT-Par | $\eta = 3.0, \delta = 1.5, \alpha = 4.0$ | 0.70 | 8.0653          | 17.548          | 195.701         | 154.539         |
|          |                                          | 0.75 | 9.1609          | 19.341          | 215.542         | 180.997         |
|          |                                          | 0.80 | 10.689          | 21.706          | 241.398         | 214.825         |
|          |                                          | 0.85 | 13.018          | 25.021          | 277.758         | 261.116         |
|          |                                          | 0.90 | 17.146          | 30.106          | 338.372         | 334.642         |
|          |                                          | 0.95 | 27.352          | 38.932          | 512.671         | 525.971         |
| Par      | $\eta = 3.0, \delta = 1.5$               | 0.70 | 6.6943          | 14.860          | 161.729         | 128.071         |
|          |                                          | 0.75 | 7.5595          | 16.411          | 179.626         | 151.131         |
|          |                                          | 0.80 | 8.7720          | 18.482          | 203.060         | 180.930         |
|          |                                          | 0.85 | 10.626          | 21.435          | 235.784         | 221.852         |
|          |                                          | 0.90 | 13.924          | 26.107          | 287.742         | 285.075         |
|          |                                          | 0.95 | 22.104          | 34.978          | 412.776         | 427.116         |

**Figure 5.** Illustration graphics of  $\mathcal{X}_1$ ,  $\mathcal{X}_2$ ,  $\mathcal{X}_3$ , and  $\mathcal{X}_4$  for Table 5.

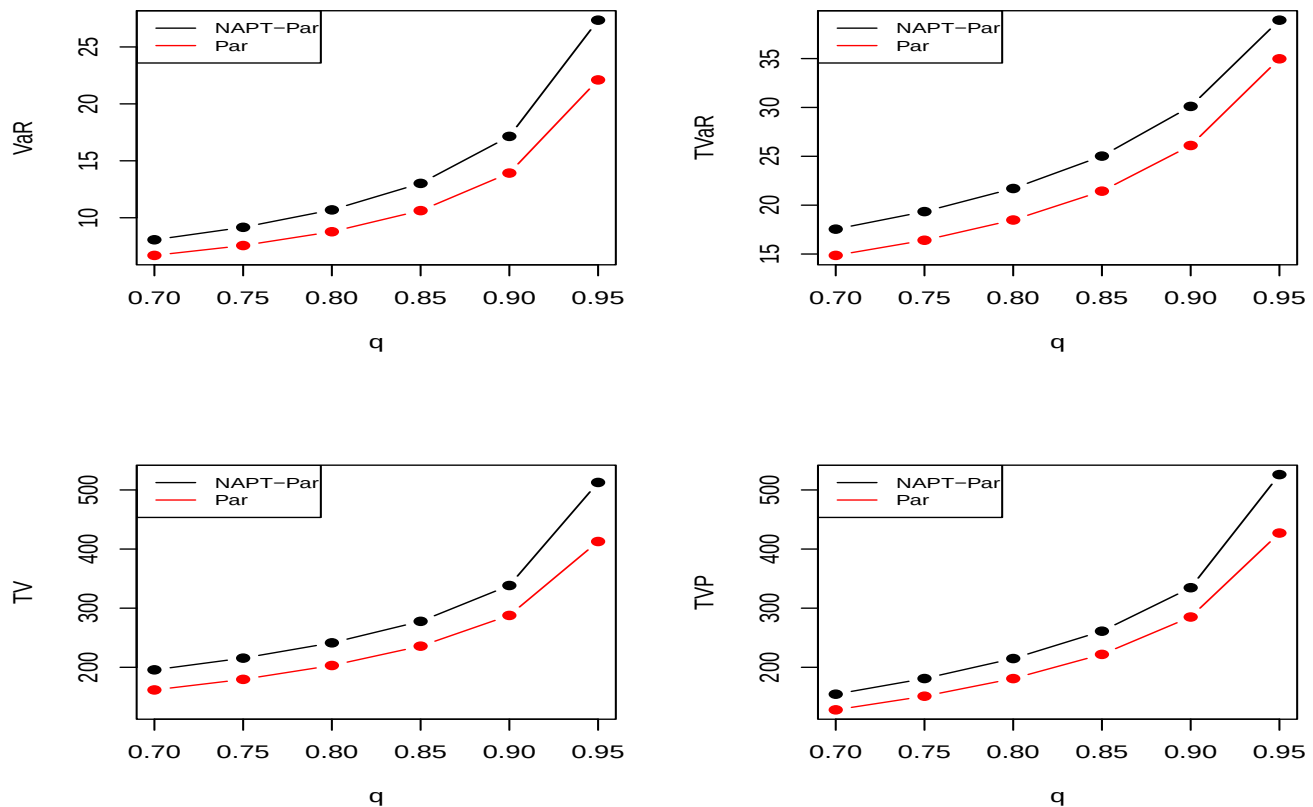


Figure 6. Illustration graphics of  $\mathcal{X}_1$ ,  $\mathcal{X}_2$ ,  $\mathcal{X}_3$ , and  $\mathcal{X}_4$  for Table 6.

## 7. Statistical Modeling

### 7.1. Dataset 1

A real-life application taken from the insurance sector was analyzed in this section to demonstrate the utility and performance of our NAPT-Par model. The considered dataset reported the monthly unemployment insurance metrics from 08 July 2008 to 13 April 2013, and it was drawn from <https://opendata.maryland.gov/w/3x6e-7i3k/gz96-f9ea?cur=aQL1UsyAvhe&from=oyc8IB21VJ5>. Henceforth, it was applied by Riad et al. [33]. The considered dataset can be tabulated in Table 7

Table 7. The monthly unemployment insurance values.

|      |      |      |      |      |      |       |      |          |      |      |       |      |      |
|------|------|------|------|------|------|-------|------|----------|------|------|-------|------|------|
| 48.2 | 48.2 | 55.6 | 49.4 | 54.7 | 84.4 | 85.6  | 92.1 | 115.7    | 92.7 | 87.4 | 104.1 | 86.6 | 96.7 |
| 74.1 | 70.4 | 84   | 80.3 | 86.5 | 86.9 | 104.7 | 74.3 | 66.7     | 79.1 | 65.7 | 78.6  | 59.9 | 59   |
| 72.6 | 66.6 | 89.9 | 75.2 | 72.1 | 63.2 | 70.3  | 58.9 | 59.7     | 72.6 | 55.3 | 62.6  | 55.5 | 60.4 |
| 84.2 | 68.9 | 65.2 | 70   | 56.6 | 56.3 | 71.9  | 60.3 | 54, 65.2 | 56.3 | 69.6 | 68.8  | 66   | 63.5 |
| 64.9 |      |      |      |      |      |       |      |          |      |      |       |      |      |

### 7.2. Dataset 2

These data define the values of the stream health cost (% of GDP) for Saudi Arabia (KSA). The dataset values are taken from the World Bank (accessed on 15 September 2021) as well from Emam [34] and Muqrin [35]. The records are tabulated in Table 8

Table 8. The current health expenditure values for KSA.

|            |            |            |            |            |
|------------|------------|------------|------------|------------|
| 42.1159410 | 44.6173573 | 42.4937630 | 39.7909737 | 35.8400607 |
| 34.1867280 | 36.1922503 | 35.6228733 | 29.7100425 | 42.9041958 |
| 36.4785600 | 37.1177721 | 40.1962376 | 44.6568298 | 52.2795486 |
| 59.9834490 | 58.3562946 | 62.6256323 | 57.4845695 | 56.8828773 |

### 7.3. Dataset 3

These data represent 43 observations of long patients with head and neck cancer surviving. The dataset is provided by Ceren et al. [36], and its records are tabulated in Table 9

**Table 9.** 43 observations of long patients with head and neck cancer survive.

|       |       |       |       |       |     |       |       |       |       |
|-------|-------|-------|-------|-------|-----|-------|-------|-------|-------|
| 12.20 | 23.56 | 23.74 | 25.87 | 31.98 | 37  | 41.35 | 47.38 | 55.46 | 58.36 |
| 63.47 | 68.46 | 78.26 | 74.47 | 81.43 | 84  | 92    | 94    | 110   | 112   |
| 119   | 127   | 130   | 133   | 140   | 146 | 155   | 159   | 173   | 179   |
| 194   | 195   | 209   | 249   | 281   | 319 | 339   | 432   | 469   | 519   |
| 633   | 725   | 817   |       |       |     |       |       |       |       |

A summary of statistics and some non-parametric plots of the proposed three datasets, notably the scaled total time on the test (TTT), Q-Q, and box plots, were depicted, respectively, in Table 10 and Figure 7. It is well known that the scaled TTT transform plot indicates the shape of the hazard function. For example, if the plot is a concave (convex) function, then it indicates that the hrf function is an increasing (decreasing) function. The TTT plot demonstrates that the hrfs for the three datasets are monotonically increasing. It also demonstrates that the NAPT-Par has a asymmetric bimodal density with a right tail.

**Table 10.** Summary statistics for the three proposed datasets.

| Dataset | $Q_1$  | Median | Mean   | $Q_3$  | ID     | $\mathcal{K}$ | $\mathcal{R}$ |
|---------|--------|--------|--------|--------|--------|---------------|---------------|
| 1       | 60.0   | 71.52  | 69.25  | 83.08  | 3.1232 | 0.7244        | 0.0749        |
| 2       | 36.104 | 41.345 | 42.477 | 53.430 | 4.4251 | 0.8688        | 1.4129        |
| 3       | 65.97  | 127.00 | 187.37 | 202.00 | 1.0235 | 1.7327        | 2.3258        |

Henceforth, we consider certain fitted distributions to select the more suitable model for analyzing the three real datasets. For this purpose, we consider power inverse Nadarajah Haghighi (PINH, [37]), Burr X (BurXD, [38]), Truncated Poisson Lindley (TPLind, [39]), Truncated Poisson log-normal (TPLN, [40]), and Truncated Poisson exponential (TPExp, [41]) models. The fitted cdfs of the comparison models can be formulated by

- BurXD:

$$F(z) = 1 - e^{-(\eta z^\delta)^2}, \quad z > 0, \eta, \delta > 0.$$

- TPLN:

$$F(z) = \frac{e^{\alpha \Phi\left(\frac{\ln z - \eta}{\delta}\right)} - 1}{e^\alpha - 1}, \quad z > 0, \eta \in \mathbb{R}, \delta, \alpha > 0.$$

- TPExp:

$$F(z) = \frac{e^{\eta(1-e^{-\delta z})} - 1}{e^\eta - 1}, \quad z, \delta, \eta > 0.$$

- TPLind:

$$F(z) = \frac{e^{\eta \left[ 1 - \left( \frac{\delta z}{\delta + 1} + 1 \right) e^{-\delta z} \right]} - 1}{e^\eta - 1}, \quad z > 0, \eta, \delta > 0,$$

- PINH:

$$F(y) = e^{1-(1+\eta/y^\delta)^\alpha}, \quad z, \eta; \delta, \alpha > 0.$$

Next, for validation purposes, we employ several statistical criteria: Kolmogorov–Smirnov ( $\mathcal{KS}$ ) statistics with their associated  $\mathcal{P}$ -values, Akaike Information Criterion ( $\mathcal{C}_1$ ), and Bayesian Information Criterion ( $\mathcal{C}_2$ ). The results are summarized in Table 11. From the numerical values of Table 11, our recommended NAPT-Par model is a suitable distribution for the three datasets. Additionally, the fitted pdf, cdf, and sf plots of the proposed model and other competitor distributions presented in Figures 8, 9, 10 and 11 ensure that our suggested model fits the three datasets well.

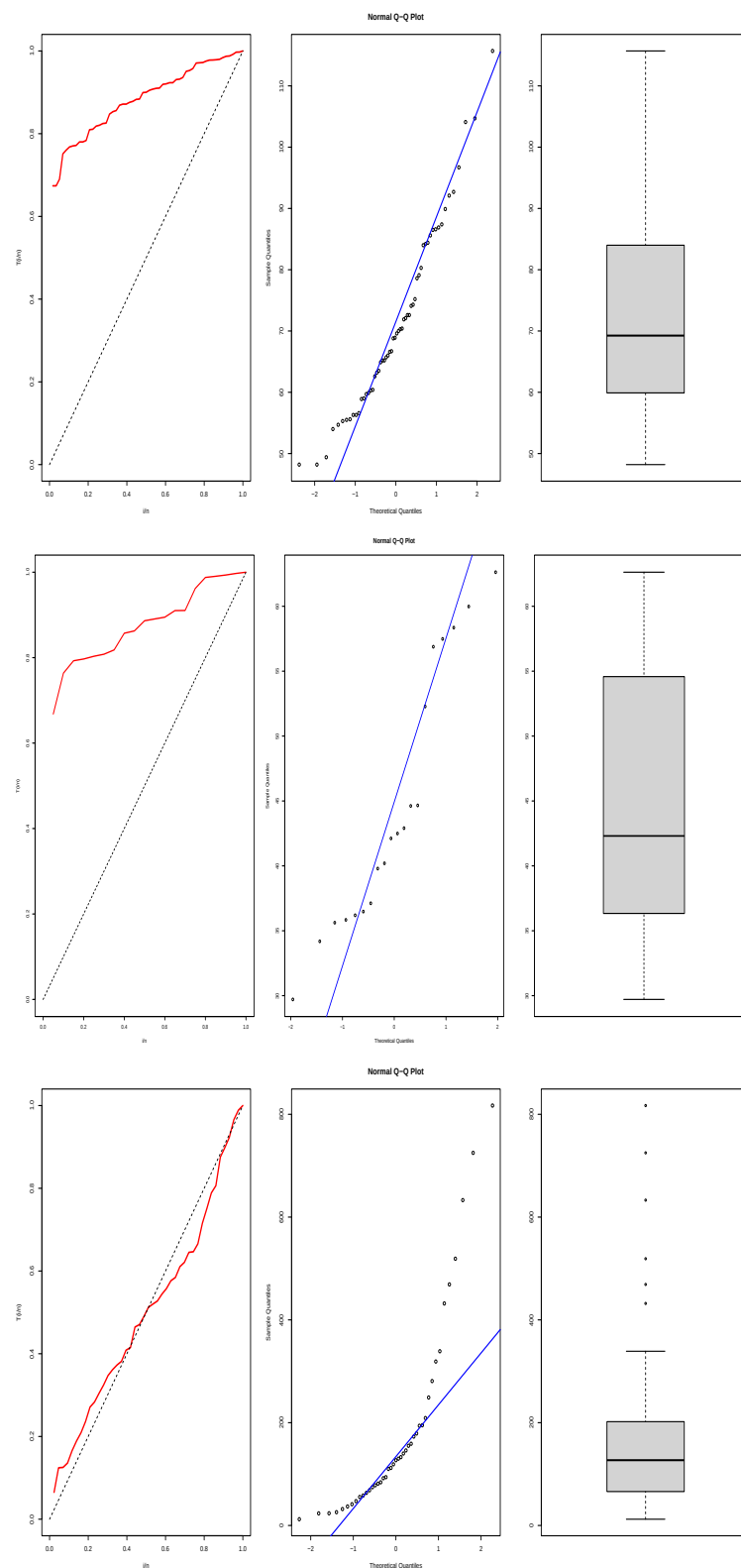


**Figure 7.** Theoretical-TTT, QQ, and box plots using the three proposed real datasets.

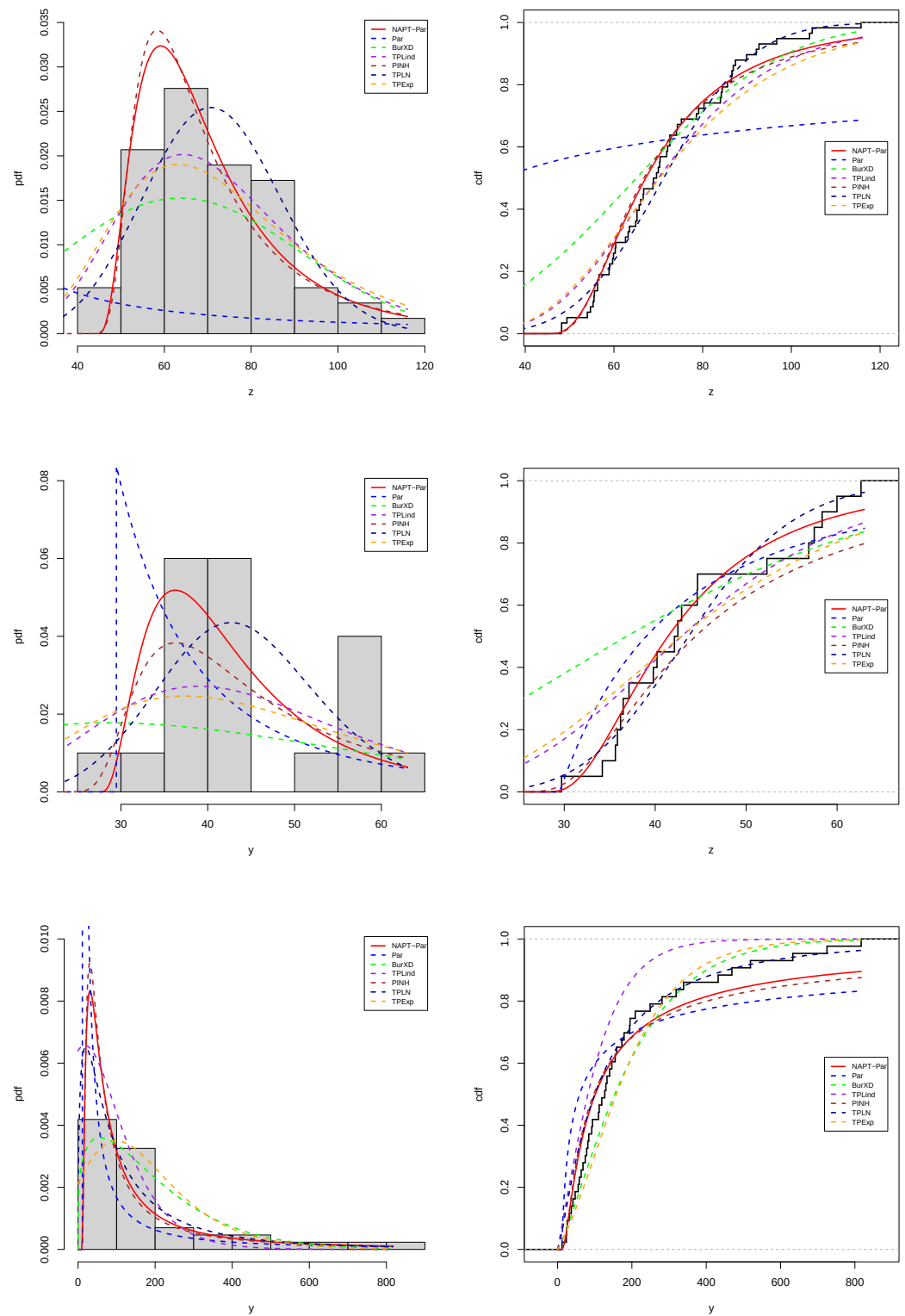


**Figure 8.** Fitted pdf and cdf plots compared to the histogram and empirical cdf for the three datasets.

**Table 11.** Estimation and values of the selection criterion of the fitted distributions.

| Data | Distribution | $\hat{\eta}$ | $\hat{\delta}$ | $\hat{\alpha}$ | $\mathcal{KS}$ | $\mathcal{P}$ -Value   | $\mathcal{C}_1$ | $\mathcal{C}_2$ |
|------|--------------|--------------|----------------|----------------|----------------|------------------------|-----------------|-----------------|
| 1    | NAPT-Par     | 44.544       | 4.6620         | 76.957         | 0.0934         | 0.6914                 | 481.028         | 487.209         |
|      | PINH         | 6061.75      | 933.85         | 3.8538         | 0.1049         | 0.5452                 | 484.328         | 490.509         |
|      | TPLN         | 5.2787       | 0.4761         | 41.589         | 0.1026         | 0.5744                 | 482.184         | 488.366         |
|      | TPExp        | 0.0517       | 26.273         |                | 0.1485         | 0.1548                 | 493.665         | 497.786         |
|      | Par          | 5.7369       | 0.3861         |                | 0.5603         | $3.33 \times 10^{-16}$ | 719.425         | 723.546         |
|      | BurXD        | 0.0021       | 1.4294         |                | 0.2824         | 0.0001                 | 510.927         | 515.047         |
|      | TPLind       | 0.0673       | 14.255         |                | 0.1393         | 0.2104                 | 489.537         | 493.658         |
| 2    | NAPT-Par     | 27.678       | 4.5090         | 49.720         | 0.1209         | 0.8981                 | 152.155         | 155.142         |
|      | PINH         | 135.703      | 120.570        | 2.7258         | 0.2054         | 0.3221                 | 155.896         | 158.883         |
|      | TPLN         | 3.6550       | 0.2314         | 1.8068         | 0.1580         | 0.6438                 | 152.623         | 155.610         |
|      | TPExp        | 0.0669       | 12.255         |                | 0.2388         | 0.1731                 | 160.912         | 162.903         |
|      | Par          | 29.475       | 2.4721         |                | 0.2739         | 0.0811                 | 157.124         | 159.115         |
|      | BurXD        | 0.0328       | 0.89592        |                | 0.4039         | 0.0018                 | 174.493         | 176.485         |
|      | TPLind       | 0.0937       | 8.2562         |                | 0.2177         | 0.2593                 | 157.876         | 159.868         |
| 3    | NAPT-Par     | 0.8572       | 7.9343         | 334.21         | 0.1186         | 0.5405                 | 537.577         | 542.861         |
|      | PINH         | 52.544       | 0.2261         | 0.6795         | 0.1357         | 0.3730                 | 552.007         | 557.291         |
|      | TPLN         | 0.3492       | 5.8578         | 10.247         | 0.1324         | 0.4023                 | 542.679         | 552.861         |
|      | TPExp        | 0.0086       | 2.3503         |                | 0.149          | 0.2678                 | 544.912         | 548.435         |
|      | Par          | 0.4162       | 11.214         |                | 0.2998         | 0.0006                 | 575.730         | 579.252         |
|      | BurXD        | 0.0356       | 0.6259         |                | 0.1374         | 0.3577                 | 541.156         | 544.678         |
|      | TPLind       | 0.0127       | 1.2503         |                | 0.2048         | 0.0464                 | 554.6856        | 558.208         |

Further, the suggested datasets were used to estimate the unknown parameters using the proposed estimation method. Table 12 reported the obtained results.

**Table 12.** Several proposed estimation techniques for the NAPT-Par model using the three proposed datasets.

| Data | Parameters | $\omega_2$ | $\omega_3$ | $\omega_4$ | $\omega_5$ | $\omega_6$ |
|------|------------|------------|------------|------------|------------|------------|
| 1    | $\eta$     | 47.497     | 45.301     | 44.297     | 48.568     | 41.038     |
|      | $\delta$   | 4.7525     | 5.1417     | 4.2351     | 4.8083     | 3.9132     |
|      | $\alpha$   | 33.8735    | 34.892     | 37.463     | 34.210     | 30.302     |
| 2    | $\eta$     | 24.743     | 22.389     | 21.694     | 26.138     | 25.851     |
|      | $\delta$   | 4.5329     | 4.5587     | 4.5438     | 4.5197     | 4.5637     |
|      | $\alpha$   | 49.684     | 49.781     | 49.718     | 49.704     | 49.735     |
| 3    | $\eta$     | 0.8327     | 0.8719     | 0.8368     | 0.8411     | 0.8479     |
|      | $\delta$   | 7.9237     | 7.9008     | 7.9164     | 7.9185     | 7.9209     |
|      | $\alpha$   | 332.97     | 331.52     | 330.87     | 333.49     | 333.51     |

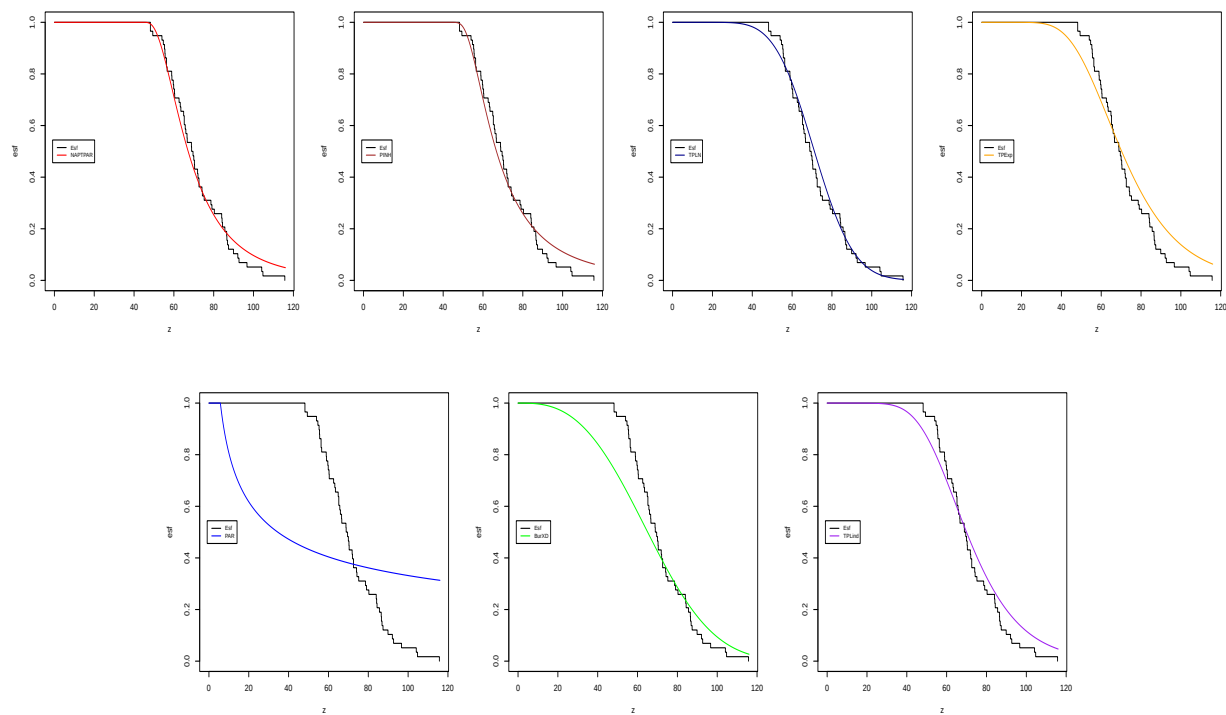


Figure 9. Fitted SF plot in comparison to the empirical SF and for the first dataset.

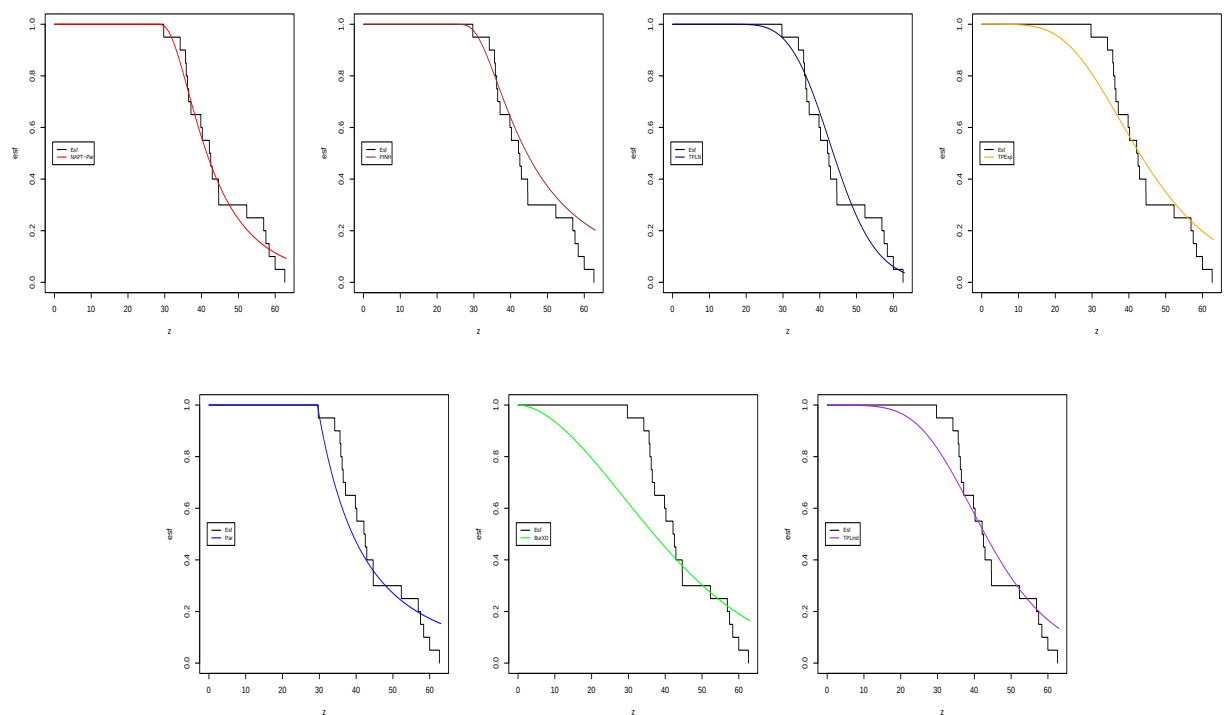


Figure 10. Fitted SF plot in comparison to the empirical SF and for the second dataset.

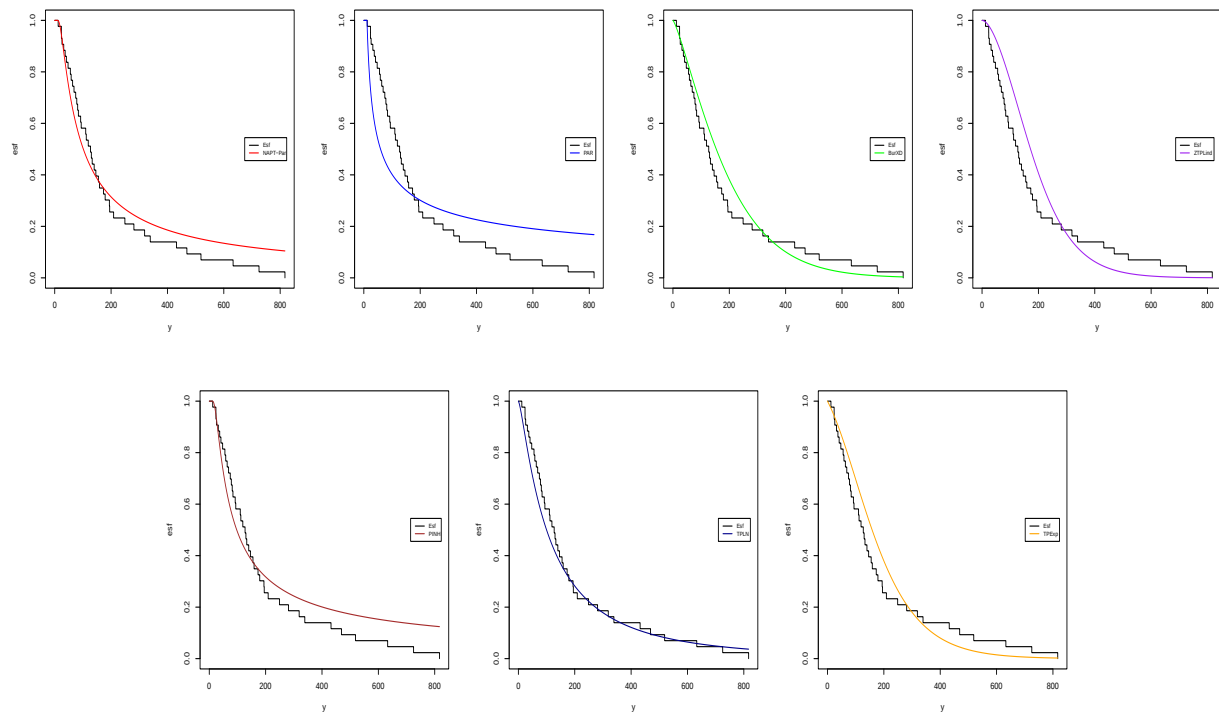


**Figure 11.** Fitted SF plot in comparison to the empirical SF and for the third dataset.

At the end, we performed the values of risk measures of the NAPT-Par and Par model by applying the three datasets. Table 13 reported the obtained results. From Table 13, the risk measure values of the NAPT-Par distribution are approaches to their associated empirical values for the three datasets. Thus, the proposed model is the best distribution for modeling the suggested datasets.

**Table 13.** Results of  $\mathcal{X}_1$ ,  $\mathcal{X}_2$ ,  $\mathcal{X}_3$ , and  $\mathcal{X}_4$  using the insurance dataset.

| Data | Model     | Parameters                                                            | $q$  | $\mathcal{X}_1$ | $\mathcal{X}_2$ | $\mathcal{X}_3$ | $\mathcal{X}_4$ |
|------|-----------|-----------------------------------------------------------------------|------|-----------------|-----------------|-----------------|-----------------|
| 1    | Empirical |                                                                       | 0.55 | 73.398          | 84.666          | 75.254          | 126.056         |
|      |           |                                                                       | 0.65 | 77.279          | 87.337          | 64.291          | 129.127         |
|      |           |                                                                       | 0.75 | 81.601          | 90.518          | 53.976          | 131.001         |
|      |           |                                                                       | 0.85 | 87.011          | 94.752          | 43.534          | 131.756         |
|      | NAPT-Par  | $\hat{\eta}=44.544$<br>$\hat{\delta}=4.6620$<br>$\hat{\alpha}=76.957$ | 0.55 | 69.161          | 90.523          | 721.410         | 487.298         |
|      |           |                                                                       | 0.65 | 73.898          | 95.977          | 793.115         | 611.502         |
|      |           |                                                                       | 0.75 | 80.303          | 103.605         | 905.375         | 782.636         |
|      |           |                                                                       | 0.85 | 90.489          | 116.070         | 1114.7          | 1063.645        |
|      | Par       | $\hat{\eta}=5.7369$<br>$\hat{\delta}=0.3861$                          | 0.55 | 45.378          | 160.839         | 44954.2         | 24885.6         |
|      |           |                                                                       | 0.65 | 87.002          | 188.769         | 54247.5         | 35449.6         |
|      |           |                                                                       | 0.75 | 207.968         | 210.083         | 73886.7         | 55625.1         |
|      |           |                                                                       | 0.85 | 780.872         | 77.016          | 62139.1         | 52895.2         |
|      | Empirical |                                                                       | 0.55 | 42.678          | 53.310          | 302.493         | 218.327         |
|      |           |                                                                       | 0.65 | 44.631          | 56.038          | 329.461         | 270.329         |
|      |           |                                                                       | 0.75 | 53.430          | 59.067          | 378.129         | 356.113         |
|      |           |                                                                       | 0.85 | 57.615          | 60.322          | 479.517         | 475.953         |
| 2    | NAPT-Par  | $\hat{\eta}=27.678$<br>$\hat{\delta}=4.5090$<br>$\hat{\alpha}=49.720$ | 0.55 | 42.678          | 56.384          | 305.956         | 224.660         |
|      |           |                                                                       | 0.65 | 45.682          | 59.889          | 337.883         | 279.513         |
|      |           |                                                                       | 0.75 | 49.762          | 64.805          | 387.897         | 355.728         |
|      |           |                                                                       | 0.85 | 56.283          | 72.871          | 481.522         | 482.165         |
|      | Par       | $\hat{\eta}=29.475$<br>$\hat{\delta}=2.4721$                          | 0.55 | 40.713          | 67.7463         | 2166.00         | 1259.04         |
|      |           |                                                                       | 0.65 | 45.069          | 74.884          | 2555.13         | 1735.72         |
|      |           |                                                                       | 0.75 | 51.641          | 85.599          | 3173.93         | 2466.04         |
|      |           |                                                                       | 0.85 | 63.494          | 104.756         | 4364.70         | 3814.75         |

Table 13. Cont.

| Data | Model     | Parameters                                                            | $q$  | $\mathcal{X}_1$ | $\mathcal{X}_2$ | $\mathcal{X}_3$ | $\mathcal{X}_4$ |
|------|-----------|-----------------------------------------------------------------------|------|-----------------|-----------------|-----------------|-----------------|
| 3    | Empirical |                                                                       | 0.55 | 133.70          | 333.315         | 187.371         | 436.370         |
|      |           |                                                                       | 0.65 | 163.2           | 382.200         | 191.542         | 506.702         |
|      |           |                                                                       | 0.75 | 202.0           | 453.818         | 191.542         | 597.475         |
|      |           |                                                                       | 0.85 | 333.0           | 562.00          | 191.542         | 724.811         |
|      | NAPT-Par  | $\hat{\eta}=0.8572$<br>$\hat{\delta}=7.9343$<br>$\hat{\alpha}=334.21$ | 0.55 | 131.10          | 311.89          | 185.269         | 434.219         |
|      |           |                                                                       | 0.65 | 161.47          | 379.579         | 188.374         | 503.981         |
|      |           |                                                                       | 0.75 | 197.268         | 449.197         | 189.067         | 592.741         |
|      |           |                                                                       | 0.85 | 329.197         | 558.341         | 189.643         | 719.358         |
|      | Par       | $\hat{\eta}=0.4162$<br>$\hat{\delta}=11.214$                          | 0.55 | 123.478         | 297.367         | 167.394         | 411.378         |
|      |           |                                                                       | 0.65 | 154.297         | 334.297         | 176.218         | 456.297         |
|      |           |                                                                       | 0.75 | 172.974         | 384.379         | 177.397         | 529.648         |
|      |           |                                                                       | 0.85 | 281.397         | 453.671         | 178.397         | 642.167         |

## 8. Conclusions

In light of the relevance of the Par distribution as well as its extraordinary flexibility in applications in the fields of dependability, engineering, and actuarial sciences, amongst others, a novel alpha power-transformed Pareto (NAPT-Par) model with three parameters has been investigated. We provided distributional and mathematical properties of this new distribution, notably density function, cumulative probability function, survival function, hazard function, quantile, mean, moment generating function, and distribution of order statistics. Furthermore, several estimation methods are considered to estimate the model parameters, and it is shown that the MLE procedure is the most suitable estimator by several simulation experiments for estimating the unknown parameters. Additionally, we established several indicator actuarial sciences of NAPT-Par distribution. Finally, two real insurance datasets have been considered for applications. By applying different statistical criteria, it is observed that the proposed NAPT-Par distribution was the best candidate model for analyzing insurance datasets.

The following are some future research directions:

1. We plan to explore and apply various goodness-of-fit (GOF) statistical tests for right-censored distributional validation.
2. Investigate the extension of the proposed novel probabilistic Pareto model to the multivariate setting. Explore the mathematical properties and implications of incorporating different copulas to model dependencies among multiple variables.
3. Researchers may compare the proposed NAPT-Par model with other asymmetric well-known distributions to determine its advantages and limitations in capturing the features of complex data.
4. Future research might explore Bayesian methods to estimate the model parameters, especially when dealing with limited data or incorporating prior information into the estimation process.

## 9. Limitations of Study

It is well documented that limitations accompany our findings. Though advantageous for engineering and insurance loss, the proposed mode might not universally apply across all types of datasets. This specificity raises questions about the model's versatility and adaptability to other complex datasets. Further, the effectiveness of the NAPT-Par distribution is deeply intertwined with the quality and comprehensiveness of the data it analyzes. In scenarios where data are sparse, incomplete, or biased, the model's performance could be significantly compromised, potentially affecting the reliability of survival time predictions.

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**Conflicts of Interest:** The authors declare no conflicts of interest.

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