

Article

Estimates for Certain Rough Multiple Singular Integrals on Triebel–Lizorkin Space

Hussain Al-Qassem ¹  and Mohammed Ali ^{2,*} 
¹ Department of Mathematics and Statistics, Qatar University, Doha 2713, Qatar; husseink@qu.edu.qa

² Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid 22110, Jordan

* Correspondence: myali@just.edu.jo

Abstract: This paper focuses on studying the mapping properties of singular integral operators over product symmetric spaces. The boundedness of such operators is established on Triebel–Lizorkin spaces whenever their rough kernel functions belong to the Grafakos and Stefanov class. Our findings generalize, extend and improve some previously known results on singular integral operators.

Keywords: Triebel–Lizorkin space; singular integrals; rough kernels; product spaces

MSC: 42B20; 42B25; 42B35

1. Introduction and Main Results

Assume that $\mathbb{R}^s$ ($s = \kappa$ or η) is the $2 \leq s$ -Euclidean space and that $\mathbb{S}^{s-1}$ is the unit sphere in $\mathbb{R}^s$ equipped with the normalized Lebesgue surface measure $d\sigma_s(\cdot)$. Also assume that $w' = w/|w|$ for $w \in \mathbb{R}^s \setminus \{0\}$.

Let $\mathcal{U}$ be an integrable over $\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}$ and satisfy

$$\mathcal{U}(tu, rv) = \mathcal{U}(u, v), \quad \forall t, r > 0, \quad (1)$$

$$\int_{\mathbb{S}^{\kappa-1}} \mathcal{U}(u', v') d\sigma_{\kappa}(u') = \int_{\mathbb{S}^{\eta-1}} \mathcal{U}(u', v') d\sigma_{\eta}(v') = 0. \quad (2)$$

The singular integral operator $T_{\mathcal{U}}$ on symmetric spaces $\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta}$ is defined, initially for $h \in \mathcal{S}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})$, by

$$T_{\mathcal{U}}h(x, y) = \text{p.v.} \iint_{\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta}} h(x - u, y - v) \frac{\mathcal{U}(u', v')}{|u|^{\kappa}|v|^{\eta}} du dv.$$

In mathematical contexts, particularly in integral calculus and complex analysis, “p.v.” stands for principal value (short for the Cauchy principal value). It is often used when dealing with integrals that may have singularities (points where the function being integrated becomes infinite or undefined).

The theory of singular integrals is an important part of Fourier analysis due to its powerful role in dealing with many significant problems arising in such parts of analysis as partial differential equations and several complex variables. The study of the mapping properties for singular integral operators has attracted the attention of many mathematicians for a long time. Historically, the operator $T_{\mathcal{U}}$ was introduced by Fefferman and Stein in [1], in which they proved the L^p boundedness of $T_{\mathcal{U}}$ for all $p \in (1, \infty)$ if $\mathcal{U}$ satisfies certain Lipschitz conditions. Subsequently, the boundedness of $T_{\mathcal{U}}$ and some of its extensions have been investigated by many researchers. For example, Duoandikoetxea improved the above results in [2] by proving that $T_{\mathcal{U}}$ is bounded on $L^p(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})$ under the weaker condition $\mathcal{U} \in L^q(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$. Later on, the authors of [3] confirmed that $T_{\mathcal{U}}$ is



Citation: Al-Qassem, H.; Ali, M. Estimates for Certain Rough Multiple Singular Integrals on Triebel–Lizorkin Space. *Symmetry* **2024**, *16*, 1323. <https://doi.org/10.3390/sym16101323>

Academic Editor: Sergei Odintsov

Received: 20 August 2024

Revised: 3 October 2024

Accepted: 4 October 2024

Published: 8 October 2024



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bounded on $L^p(\mathbb{R}^\kappa \times \mathbb{R}^\eta)$ ($1 < p < \infty$) if $\mathcal{U} \in L(\log^+ L)^2(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$. In [4], the authors established the L^p boundedness of $\mathbf{T}_{\mathcal{U}}$ for $p \in (1, \infty)$, provided that $\mathcal{U}$ in the block space $B_q^{(0,1)}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$ for some $q > 1$. Thereafter, the discussion of the mapping properties of $\mathbf{T}_{\mathcal{U}}$ and its extensions under various conditions on $\mathcal{U}$ has received a large amount of attention by many authors (the readers are referred to [1–9]).

Our focus in this paper will be on studying the boundedness of $\mathbf{T}_{\mathcal{U}}$ whenever $\mathcal{U}$ belongs to a certain class of functions related to a class of functions introduced by Walsh in [10] and then developed by Grafakos and Stefanov in [11]. To clarify our purpose, we recall some definitions and some pertinent results related to our current study.

Definition 1. For $\alpha > 0$, we let $\mathbf{G}_\alpha(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$ be the class of all functions $\mathcal{U}$ which are integrable over $\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}$ and satisfy the condition on product spaces

$$\sup_{(\xi, \zeta) \in \mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}} \iint_{\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}} \log^{\alpha+1}(|\xi \cdot u'|^{-1}) \log^{\alpha+1}(|\zeta \cdot v'|^{-1}) \\ \times |\mathcal{U}(u', v')| d\sigma_\kappa(u') d\sigma_\eta(v') < \infty.$$

By following the same arguments as that employed in [11], we obtain the following:

$$\bigcup_{q>1} L^q(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}) \subsetneq \mathbf{G}_\alpha(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}) \text{ for any } \alpha > 0, \\ \bigcap_{\alpha>0} \mathbf{G}_\alpha(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}) \not\subseteq L(\log^+ L)^2(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}) \not\subseteq \bigcup_{\alpha>0} \mathbf{G}_\alpha(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}), \\ \bigcap_{\alpha>0} \mathbf{G}_\alpha(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}) \not\subseteq B_q^{(0,1)}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}) \not\subseteq \bigcup_{\alpha>0} \mathbf{G}_\alpha(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}).$$

Let us recall the definition of the homogeneous Triebel–Lizorkin space $\dot{F}_p^{\varepsilon, \vec{\gamma}}(\mathbb{R}^\kappa \times \mathbb{R}^\eta)$. For $p, \varepsilon \in (1, \infty)$ and $\vec{\gamma} = (\gamma_1, \gamma_2) \in \mathbb{R} \times \mathbb{R}$, the homogeneous Triebel–Lizorkin space $\dot{F}_p^{\varepsilon, \vec{\gamma}}(\mathbb{R}^\kappa \times \mathbb{R}^\eta)$ is the class of all tempered distributions h on $\mathbb{R}^\kappa \times \mathbb{R}^\eta$ that satisfy

$$\|h\|_{\dot{F}_p^{\varepsilon, \vec{\gamma}}(\mathbb{R}^\kappa \times \mathbb{R}^\eta)} = \left\| \left(\sum_{j,k \in \mathbb{Z}} 2^{j\gamma_1 \varepsilon} 2^{k\gamma_2 \varepsilon} |(\mathcal{A}_j \otimes \mathcal{B}_k) * h|^\varepsilon \right)^{1/\varepsilon} \right\|_{L^p(\mathbb{R}^\kappa \times \mathbb{R}^\eta)} < \infty,$$

where $\hat{\mathcal{A}}_j(u) = 2^{-j\kappa} \mathcal{A}(2^{-j}u)$ for $j \in \mathbb{Z}$, $\hat{\mathcal{B}}_k(v) = 2^{-k\eta} \mathcal{B}(2^{-k}v)$ for $k \in \mathbb{Z}$ and the radial functions $\mathcal{A} \in \mathcal{S}(\mathbb{R}^\kappa)$, $\mathcal{B} \in \mathcal{S}(\mathbb{R}^\eta)$ satisfy the following:

- (1) $0 \leq \mathcal{A} \leq 1$, $0 \leq \mathcal{B} \leq 1$;
- (2) $\text{supp}(\mathcal{A}) \subset \{u : \frac{1}{2} \leq |u| \leq 2\}$, $\text{supp}(\mathcal{B}) \subset \{v : \frac{1}{2} \leq |v| \leq 2\}$;
- (3) There exists $M > 0$ such that $\mathcal{A}(u), \mathcal{B}(v) \geq M$ for all $|u|, |v| \in [\frac{3}{5}, \frac{5}{3}]$;
- (4) $\sum_{j \in \mathbb{Z}} \mathcal{A}(2^{-j}u) = 1$ with $u \neq 0$ and $\sum_{k \in \mathbb{Z}} \mathcal{B}(2^{-k}v) = 1$ with $v \neq 0$.

The authors of [12] proved the following properties:

- (i) The Schwartz space $\mathcal{S}(\mathbb{R}^\kappa \times \mathbb{R}^\eta)$ is dense in $\dot{F}_p^{\varepsilon, \vec{\gamma}}(\mathbb{R}^\kappa \times \mathbb{R}^\eta)$;
- (ii) $\dot{F}_p^{2, \vec{0}}(\mathbb{R}^\kappa \times \mathbb{R}^\eta) = L^p(\mathbb{R}^\kappa \times \mathbb{R}^\eta)$ for $1 < p < \infty$;
- (iii) $\dot{F}_p^{\varepsilon_1, \vec{\gamma}}(\mathbb{R}^\kappa \times \mathbb{R}^\eta) \subseteq \dot{F}_p^{\varepsilon_2, \vec{\gamma}}(\mathbb{R}^\kappa \times \mathbb{R}^\eta)$ if $\varepsilon_1 \leq \varepsilon_2$.

In [13], Ying showed that if $\mathcal{U} \in \mathbf{G}_\alpha(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$ for some $\alpha > 0$, then $\mathbf{T}_{\mathcal{U}}$ is bounded on $L^p(\mathbb{R}^\kappa \times \mathbb{R}^\eta)$ for all $p \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$.

In the one parameter setting, the singular operator related to $\mathbf{T}_{\mathcal{U}}$ is given by

$$\mathbf{H}_{\mathcal{U}}h(x) = \text{p.v.} \int_{\mathbb{R}^{\kappa}} h(x-u) \frac{\mathcal{U}(u')}{|u|^{\kappa}} du.$$

For $\alpha > 0$, the class $\mathbf{G}_{\alpha}(\mathbb{S}^{\kappa-1})$ is the collection of all functions $\mathcal{U} \in L^1(\mathbb{S}^{\kappa-1})$ that satisfy the Grafakos–Stefanov condition

$$\sup_{\xi \in \mathbb{S}^{\kappa-1}} \int_{\mathbb{S}^{\kappa-1}} |\mathcal{U}(u')| \log^{\alpha+1}(|\xi \cdot u'|^{-1}) d\sigma_{\kappa}(u') < \infty.$$

In [14], the authors proved that the integral operator $\mathbf{H}_{\mathcal{U}}$ is bounded on $\dot{F}_p^{\varepsilon, \gamma_1}(\mathbb{R}^{\kappa})$ for $p \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$, $\varepsilon \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$ and $\gamma_1 \in \mathbb{R}$.

It is worth mentioning that the Triebel–Lizorkin space $\dot{F}_p^{\varepsilon, \gamma_1}(\mathbb{R}^{\kappa})$ covers several classes of many well-known function spaces including Lebesgue spaces $L^p(\mathbb{R}^{\kappa})$, the Hardy spaces $H^p(\mathbb{R}^{\kappa})$ and the Sobolev spaces $L_p^{\alpha}(\mathbb{R}^{\kappa})$. So, it is understood that the work on these spaces is more intricate than $L^p(\mathbb{R}^{\kappa})$. This clearly has instigated many authors to investigate the boundedness of $\mathbf{H}_{\mathcal{U}}$ and some of its extensions; see, for instance, [15–27].

In light of the results obtained in [14] regarding the $\dot{F}_p^{\varepsilon, \gamma_1}$ boundedness of the singular integral $\mathbf{H}_{\mathcal{U}}$ in the one-parameter setting whenever $\mathcal{U} \in \mathbf{G}_{\alpha}(\mathbb{S}^{\kappa-1})$, and the work carried out in [13] regarding the L^p boundedness of the singular integral $\mathbf{T}_{\mathcal{U}}$ in the product domains whenever $\mathcal{U} \in \mathbf{G}_{\alpha}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$, we are motivated to investigate the boundedness of $\mathbf{T}_{\mathcal{U}}$ on $\dot{F}_p^{\varepsilon, \vec{\gamma}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})$ whenever $\mathcal{U}$ satisfies the Grafakos–Stefanov condition.

The main result of this paper is the following:

Theorem 1. Suppose that $\mathcal{U} \in \mathbf{G}_{\alpha}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$ for some $\alpha > 0$. Then, $\mathbf{T}_{\mathcal{U}}$ is bounded on $\dot{F}_p^{\varepsilon, \vec{\gamma}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})$ for $p \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$, $\varepsilon \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$ and $\vec{\gamma} \in \mathbb{R} \times \mathbb{R}$.

Remark 1.

- (1) The L^p boundedness of $\mathbf{T}_{\mathcal{U}}$ was proved in [1], provided that $\mathcal{U}$ satisfies certain Lipschitz conditions; however, in this work, the space was extended to be the Grafakos–Stefanov $\mathbf{G}_{\alpha}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$.
- (2) The author of [2] showed that $\mathbf{T}_{\mathcal{U}}$ is bounded on $L^p(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})$ whenever $\mathcal{U} \in L^q(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}) \subsetneq \mathbf{G}_{\alpha}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$. Hence, our result extends the result in [2].
- (3) If $\mathcal{U} \in \mathbf{G}_{\alpha}(\mathbb{S}^{\kappa-1})$, the authors of [13] confirmed the boundedness of the singular operator $\mathbf{T}_{\mathcal{U}}$ on $L^p(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta}) = \dot{F}_p^{2, \vec{0}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})$; however, we obtained the boundedness of $\mathbf{T}_{\mathcal{U}}$ not only for the case $\vec{\gamma} = \vec{0}$ but for all $\vec{\gamma} \in \mathbb{R} \times \mathbb{R}$. Therefore, our results generalize and improve the result in [13].
- (4) The authors of [3,4] establish the boundedness of $\mathbf{T}_{\mathcal{U}}$ whenever $\mathcal{U}$ belongs to $L(\log^+ L)^2(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$ or $B_q^{(0,1)}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$, respectively, which are totally different to the space $\mathbf{G}_{\alpha}(\mathbb{S}^{\kappa-1})$ that we are working on.
- (5) The methodology adapted in the current work is a combination of ideas and arguments from [14,26,28]. Precisely, to prove the main result, we shall first make an appropriate decomposition to $\mathbf{T}_{\mathcal{U}}$, and then keep tracking certain constants.

2. Auxiliary Lemmas

We devote this section to establishing some preliminary lemmas. For $\mathcal{U} \in L^1(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$, we consider the sequence of measures $\{Y_{t,r} : t, r \in \mathbb{R}\}$ and its corresponding maximal operator Y^* on $\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta}$ by

$$\iint_{\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta}} h dY_{t,r} = \iint_{I_{t,r}} h(u,v) \frac{\mathcal{U}(u',v')}{|u|^{\kappa}|v|^{\eta}} dudv$$

and

$$Y^*(h) = \sup_{t,r \in \mathbb{R}} |Y_{t,r}| * |h|,$$

where $I_{t,r} = \{(u, v) \in \mathbb{R}^k \times \mathbb{R}^\eta : 2^t \leq |u| < 2^{t+1}, 2^r \leq |v| < 2^{r+1}\}$.

By adapting the same argument used in [11] to the product case, it is easy to obtain the following:

Lemma 1. Let $\mathcal{U} \in G_\alpha(\mathbb{S}^{k-1} \times \mathbb{S}^{\eta-1})$ for some $\alpha > 0$ and satisfy the conditions (1) and (2). Then, there is a constant $C > 0$, such that the estimates

$$|\hat{Y}_{t,r}(\xi, \zeta)| \leq C, \quad (3)$$

$$|\hat{Y}_{t,r}(\xi, \zeta)| \leq C \min\{|2^t \xi|, (\log^+ |2^t \xi|)^{-(\alpha+1)}\}, \quad (4)$$

$$|\hat{Y}_{t,r}(\xi, \zeta)| \leq C \min\{|2^r \zeta|, (\log^+ |2^r \zeta|)^{-(\alpha+1)}\} \quad (5)$$

hold for all $t, r \in \mathbb{R}$ and $(\xi, \zeta) \in \mathbb{R}^k \times \mathbb{R}^\eta$.

Proof. By the definition of $\hat{Y}_{t,r}(\xi, \zeta)$, it is easy to see that

$$|\hat{Y}_{t,r}(\xi, \zeta)| \leq (\log 2)^2 \|\mathcal{U}\|_{L^1(\mathbb{S}^{k-1} \times \mathbb{S}^{\eta-1})}, \quad (6)$$

which proves (3). By a change in variable, we deduce that

$$|\hat{Y}_{t,r}(\xi, \zeta)| \leq \iint_{\mathbb{S}^{k-1} \times \mathbb{S}^{\eta-1}} |\mathcal{U}(u, v)| \int_{2^r}^{2^{r+1}} |J_t(\xi, u, l)| \frac{d\tau}{\tau} d\sigma_k(u) d\sigma_\eta(v), \quad (7)$$

where

$$J_t(\xi, u, l) = \int_1^2 e^{-i(l2^t \xi \cdot u)} \frac{dl}{l}$$

which leads to

$$|J_t(\xi, u, l)| \leq C |2^t \xi \cdot u|^{-1/2}.$$

Hence, by the last estimate and the trivial estimate $|J_t(\xi, u, l)| \leq (\log 2)$, along with the fact that $t/(\log t)^\alpha$ is increasing on $(2^\alpha, \infty)$, we obtain

$$|J_t(\xi, u, l)| \leq C \frac{(\log 2 |\xi' \cdot u|^{-1})^{\alpha+1}}{(\log |2^t \xi|)^{\alpha+1}} \text{ if } |2^t \xi| > 2^\alpha. \quad (8)$$

Thus, the inequalities (7) and (8) give

$$\begin{aligned} & |\hat{Y}_{t,r}(\xi, \zeta)| \\ & \leq C (\log |2^t \xi|)^{-(\alpha+1)} \iint_{\mathbb{S}^{k-1} \times \mathbb{S}^{\eta-1}} (\log (2 |\xi' \cdot u|^{-1}))^{\alpha+1} |\mathcal{U}(u, v)| d\sigma_k(u) d\sigma_\eta(v), \end{aligned}$$

which in turn implies that

$$|\hat{Y}_{t,r}(\xi, \zeta)| \leq C (\log |2^t \xi|)^{-(\alpha+1)} \text{ if } |2^t \xi| > 2^\alpha. \quad (9)$$

Similarly, we derive that

$$|\hat{Y}_{t,r}(\xi, \zeta)| \leq C (\log |2^r \zeta|)^{-(\alpha+1)} \text{ if } |2^r \zeta| > 2^\alpha. \quad (10)$$

Now, by the cancellation property (1), we have

$$\begin{aligned} |\hat{Y}_{t,r}(\xi, \zeta)| &\leq C \iint_{\mathbb{S}^{k-1} \times \mathbb{S}^{\eta-1}} |\hat{U}(u, v)| \int_{2^r}^{2^{r+1}} \int_1^2 \left| e^{-il2^t \xi \cdot u} - 1 \right| \frac{dld\tau}{l\tau} d\sigma_\kappa(u) d\sigma_\eta(v) \\ &\leq C |2^t \xi|. \end{aligned} \quad (11)$$

In the same manner, we obtain

$$|\hat{Y}_{t,r}(\xi, \zeta)| \leq C |2^r \zeta|. \quad (12)$$

Therefore, by combining (9) with (11), we obtain (4), and by combining (10) with (12), we obtain (5). The lemma is proved. $\square$

The following lemma can be found in [4] (see also [2,3,6]).

Lemma 2. Let $\hat{U} \in L^1(\mathbb{S}^{k-1} \times \mathbb{S}^{\eta-1})$. Then, there exists a constant $C_p > 0$ such that

$$\|Y^*(f)\|_{L^p(\mathbb{R}^k \times \mathbb{R}^\eta)} \leq C_p \|h\|_{L^p(\mathbb{R}^k \times \mathbb{R}^\eta)} \|\hat{U}\|_{L^1(\mathbb{S}^{k-1} \times \mathbb{S}^{\eta-1})} \quad (13)$$

for all $1 < p < \infty$ and $h \in L^p(\mathbb{R}^k \times \mathbb{R}^\eta)$.

Let $\mathcal{A} \in \mathcal{S}(\mathbb{R}^k)$ and $\mathcal{B} \in \mathcal{S}(\mathbb{R}^\eta)$ be radial functions satisfying the following:

- (1) $0 \leq \mathcal{A}, \mathcal{B} \leq 1$;
- (2) $\text{supp}(\mathcal{A}) \subset \{u : \frac{1}{2} \leq |u| \leq 2\}$, $\text{supp}(\mathcal{B}) \subset \{v : \frac{1}{2} \leq |v| \leq 2\}$;
- (3) There is a constant $M > 0$ such that $\mathcal{A}(u), \mathcal{B}(v) \geq M$ for all $|u|, |v| \in [\frac{3}{5}, \frac{5}{3}]$;
- (4) $\int_{\mathbb{R}} |\hat{\mathcal{A}}(2^t u)|^2 = 1$ with $u \neq 0$ and $\int_{\mathbb{R}} |\hat{\mathcal{B}}(2^r v)|^2 = 1$ with $v \neq 0$.

For simplicity, we denote $\hat{\mathcal{A}}(tu)$ by $\hat{\mathcal{A}}_t(u)$ and $\hat{\mathcal{B}}(rv)$ by $\hat{\mathcal{B}}_r(v)$. Then, it is clear that $\mathcal{A}_{2^t}(u) = 2^{-tk} \mathcal{A}(u/2^t)$ and $\mathcal{B}_{2^r}(v) = 2^{-r\eta} \mathcal{B}(v/2^r)$. Let $\mathcal{W}_{2^t, 2^r}(h)(u, v) = (\mathcal{A}_{2^t} \otimes \mathcal{B}_{2^r}) * h(u, v)$. Hence, for any $h \in \mathcal{S}(\mathbb{R}^k \times \mathbb{R}^\eta)$, we have

$$\begin{aligned} \|h\|_{\dot{F}_p^{\varepsilon, \vec{0}}(\mathbb{R}^k \times \mathbb{R}^\eta)} &\sim \left\| \left(\iint_{\mathbb{R}^+ \times \mathbb{R}^+} |(\mathcal{A}_t \otimes \mathcal{B}_r) * h|^\varepsilon \frac{dtdr}{tr} \right)^{1/\varepsilon} \right\|_{L^p(\mathbb{R}^k \times \mathbb{R}^\eta)} \\ &\sim \left\| \left(\iint_{\mathbb{R} \times \mathbb{R}} |\mathcal{W}_{2^t, 2^r}(h)|^\varepsilon dtdr \right)^{1/\varepsilon} \right\|_{L^p(\mathbb{R}^k \times \mathbb{R}^\eta)}. \end{aligned}$$

Let us give the following result regarding the boundedness of the measures $|\mathbf{Y}_{t,r}| * |h|$ on $\dot{F}_p^{\varepsilon, \vec{0}}(\mathbb{R}^k \times \mathbb{R}^\eta)$.

Lemma 3. Let $\hat{U} \in L^1(\mathbb{S}^{k-1} \times \mathbb{S}^{\eta-1})$. Then, the estimate

$$\| |\mathbf{Y}_{t,r}| * |h| \|_{\dot{F}_p^{\varepsilon, \vec{0}}(\mathbb{R}^k \times \mathbb{R}^\eta)} \leq C_p \|h\|_{\dot{F}_p^{\varepsilon, \vec{0}}(\mathbb{R}^k \times \mathbb{R}^\eta)} \|\hat{U}\|_{L^1(\mathbb{S}^{k-1} \times \mathbb{S}^{\eta-1})} \quad (14)$$

holds for all $1 < p, \varepsilon < \infty$.

Proof. Let $h \in \dot{F}_p^{\varepsilon, \vec{0}}(\mathbb{R}^K \times \mathbb{R}^\eta)$. Then, for any function $f \in \dot{F}_{p'}^{\varepsilon', \vec{0}}(\mathbb{R}^K \times \mathbb{R}^\eta)$ such that $\|f\|_{\dot{F}_{p'}^{\varepsilon', \vec{0}}(\mathbb{R}^K \times \mathbb{R}^\eta)} \leq 1$, Holder's inequality leads to

$$\begin{aligned} & |\langle |Y_{t,r}| * |h|, f \rangle| \\ & \leq \left| \iint_{\mathbb{R}^K \times \mathbb{R}^\eta} \iint_{\mathbb{R} \times \mathbb{R}} |Y_{t,r}| * \mathcal{W}_{2^{t+n}, 2^{r+m}}(|h|) \mathcal{W}_{2^{t+n}, 2^{r+m}}^*(f)(u, v) dndmdudv \right| \\ & \leq \left\| \left(\iint_{\mathbb{R} \times \mathbb{R}} |Y_{t,r}| * \mathcal{W}_{2^{t+n}, 2^{r+m}}(|h|)^\varepsilon dndm \right)^{1/\varepsilon} \right\|_p \\ & \quad \times \left\| \left(\iint_{\mathbb{R} \times \mathbb{R}} |\mathcal{W}_{2^{t+n}, 2^{r+m}}^*(f)|^{\varepsilon'} dndm \right)^{1/\varepsilon'} \right\|_{p'} \end{aligned}$$

which in turn implies

$$\| |Y_{t,r}| * |h| \|_{\dot{F}_p^{\varepsilon, \vec{0}}(\mathbb{R}^K \times \mathbb{R}^\eta)} \leq C \left\| \left(\iint_{\mathbb{R} \times \mathbb{R}} |Y_{t,r}| * \mathcal{W}_{2^{t+n}, 2^{r+m}}(|h|)^\varepsilon dndm \right)^{1/\varepsilon} \right\|_p. \quad (15)$$

Let us now estimate the L^p -norm of $\left(\iint_{\mathbb{R} \times \mathbb{R}} |Y_{t,r}| * \mathcal{W}_{2^{t+n}, 2^{r+m}}(|h|)^\varepsilon dndm \right)^{1/\varepsilon}$. Since $p > 1$, by duality there exists the function $g \in L^{p'}(\mathbb{R}^K \times \mathbb{R}^\eta)$ such that $\|g\|_{L^{p'}(\mathbb{R}^K \times \mathbb{R}^\eta)} = 1$ and

$$\begin{aligned} & \left\| \iint_{\mathbb{R} \times \mathbb{R}} |Y_{t,r}| * \mathcal{W}_{2^{t+n}, 2^{r+m}}(|h|)^\varepsilon dndm \right\|_p = \iint_{\mathbb{R} \times \mathbb{R}} \langle |Y_{t,r}| * \mathcal{W}_{2^{t+n}, 2^{r+m}}(|h|)^\varepsilon, g \rangle dndm \\ & \leq \iint_{\mathbb{R} \times \mathbb{R}} \langle |\mathcal{W}_{2^{t+n}, 2^{r+m}}(|h|)(u, v)|, Y^*(\bar{g})(u, v) \rangle dndm \\ & \leq \left\| \iint_{\mathbb{R} \times \mathbb{R}} \mathcal{W}_{2^{t+n}, 2^{r+m}}(|h|) dndm \right\|_p \|Y^*(\bar{g})\|_{p'} \\ & \leq \left\| \iint_{\mathbb{R} \times \mathbb{R}} \mathcal{W}_{2^{t+n}, 2^{r+m}}(|h|) dndm \right\|_p \|\bar{U}\|_{L^1(\mathbb{S}^{K-1} \times \mathbb{S}^{\eta-1})}, \end{aligned} \quad (16)$$

where $\bar{g}(u, v) = g(-u, -v)$ and the last inequality is obtained by Lemma 2.

By following similar arguments as that employed in the proof of Lemma 2 in [4], we obtain

$$\left\| \sup_{t,r \in \mathbb{R}} |Y_{t,r}| * \mathcal{W}_{2^{t+n}, 2^{r+m}}(|h|)^\varepsilon \right\|_p \leq C_p \left\| \sup_{t,r \in \mathbb{R}} |\mathcal{W}_{2^{t+n}, 2^{r+m}}(|h|)| \right\|_p \|\bar{U}\|_{L^1(\mathbb{S}^{K-1} \times \mathbb{S}^{\eta-1})}. \quad (17)$$

By interpolating between (16) and (17), we obtain

$$\left\| \left(\iint_{\mathbb{R} \times \mathbb{R}} |Y_{t,r}| * \mathcal{W}_{2^{t+n}, 2^{r+m}}(|h|)^{\varepsilon} dndm \right)^{1/\varepsilon} \right\|_p \leq \|\mathcal{U}\|_{L^1(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})} \|h\|_{\dot{F}_p^{\varepsilon, \vec{0}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})}.$$

Consequently, by the last inequality and (15), we obtain (14). $\square$

3. Proof of Theorem 1

Let $\mathcal{U} \in \mathbf{G}_{\alpha}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$ for some $\alpha > 0$. By the translation invariance of $\mathbf{T}_{\mathcal{U}}$, it suffices to prove the boundedness of $\mathbf{T}_{\mathcal{U}}$ on $\dot{F}_p^{\varepsilon, \vec{\gamma}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})$, only whenever $\vec{\gamma} = \vec{0}$. It is clear that

$$\begin{aligned} \mathbf{T}_{\mathcal{U}}h(x, y) &= \iint_{\mathbb{R} \times \mathbb{R}} Y_{t,r} * h(x, y) dt dr \\ &= \iint_{\mathbb{R} \times \mathbb{R}} \mathcal{Q}_{n,m}(h) dndm, \end{aligned} \quad (18)$$

where

$$\mathcal{Q}_{n,m}(h) = \iint_{\mathbb{R} \times \mathbb{R}} \mathcal{W}_{2^{t+n}, 2^{r+m}}(Y_{t,r} * \mathcal{W}_{2^{t+n}, 2^{r+m}}(h)) dt dr.$$

Let us estimate the $\|\mathcal{Q}_{n,m}\|_{\dot{F}_p^{\varepsilon, \vec{0}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})}$. By following the same steps in proving (15), we obtain

$$\|\mathcal{Q}_{n,m}(h)\|_{\dot{F}_p^{\varepsilon, \vec{0}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})} \leq C \left\| \left(\iint_{\mathbb{R} \times \mathbb{R}} |Y_{t,r} * \mathcal{W}_{2^{t+n}, 2^{r+m}}(h)|^{\varepsilon} dt dr \right)^{1/\varepsilon} \right\|_p. \quad (19)$$

We need to now consider three cases:

Case 1. $p = 2 = \varepsilon$. In this case, we have $\dot{F}_2^{2, \vec{0}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta}) = L^2(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})$. So, by invoking Plancherel's theorem, we obtain

$$\begin{aligned} &\|\mathcal{Q}_{n,m}(h)\|_{\dot{F}_2^{2, \vec{0}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})}^2 \\ &\leq C \iint_{\mathbb{R} \times \mathbb{R}} \iint_{\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta}} \left| \left(\widehat{\mathcal{A}}(2^{t+n}\xi) \otimes \widehat{\mathcal{B}}(2^{r+m}\zeta) \right) \widehat{Y}_{t,r}(\xi, \zeta) \widehat{f}(\xi, \zeta) \right|^2 d\xi d\zeta dr dt \\ &\leq C \iint_{\mathbb{R} \times \mathbb{R}} \iint_{\Delta_{t+n, r+m}} \left| \left(\widehat{\mathcal{A}}(2^{t+n}\xi) \otimes \widehat{\mathcal{B}}(2^{r+m}\zeta) \right) \widehat{Y}_{t,r}(\xi, \zeta) \widehat{h}(\xi, \zeta) \right|^2 d\xi d\zeta dr dt \\ &\leq C((1+|n|)(1+|m|))^{-\alpha-1} \|h\|_{L^p(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})}, \end{aligned} \quad (20)$$

where $\Delta_{t,r} = \left\{ (\xi, \zeta) \in \mathbb{R}^{\kappa} \times \mathbb{R}^{\eta} : \frac{1}{2} \leq \mathcal{A}(2^t \xi) \leq 2 \text{ and } \frac{1}{2} \leq \mathcal{B}(2^r \zeta) \leq 2 \right\}$.

Case 2. $p = \varepsilon$. By (15), we obtain

$$\|\mathcal{Q}_{n,m}(h)\|_{\dot{F}_p^{p, \vec{0}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})} \quad (21)$$

$$\begin{aligned}
&\leq C \left(\iint_{\mathbb{R} \times \mathbb{R}} \iint_{\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta}} \left(\iint_{\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}} \mathbf{M}_{u,v}(\mathcal{W}_{2^{t+n}, 2^{r+m}}(h)(u, v)) \right. \right. \\
&\quad \times \left. \left. |\mathcal{U}(u, v)| \sigma_{\kappa}(u) d\sigma_{\eta}(v) \right)^{\varepsilon} dx dy dt dr \right)^{1/\varepsilon} \\
&\leq C \left(\iint_{\mathbb{R} \times \mathbb{R}} \left(\iint_{\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}} |\mathcal{U}(u, v)| \right. \right. \\
&\quad \times \left. \left. \|\mathbf{M}_{u,v}(\mathcal{W}_{2^{t+n}, 2^{r+m}}(h))\|_p \sigma_{\kappa}(u) d\sigma_{\eta}(v) \right)^p dt dr \right)^{1/p},
\end{aligned}$$

where

$$\mathbf{M}_{u,v}(h)(u, v) = \sup_{k_1, k_2 \in \mathbb{R}} \frac{1}{k_1 k_2} \int_0^{k_2} \int_0^{k_1} h(x - tu, y - rv) dt dr$$

which is bounded on $L^p(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})$ for $1 < p < \infty$. Therefore,

$$\|\mathcal{Q}_{n,m}(h)\|_{\dot{F}_p^{\varepsilon, \vec{0}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})} \leq C \|\mathcal{U}\|_{L^1(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})} \|h\|_{\dot{F}_p^{\varepsilon, \vec{0}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})}. \quad (22)$$

Case 3. $p > \varepsilon$. By duality, there is a non-negative function, ϕ , that lies in the space $L^{(p/\varepsilon)'}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})$ such that $\|\phi\|_{L^{(p/\varepsilon)'}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})} = 1$ and

$$\begin{aligned}
&\|\mathcal{Q}_{n,m}(h)\|_{\dot{F}_p^{\varepsilon, \vec{0}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})}^{\varepsilon} \\
&\leq C \iint_{\mathbb{R} \times \mathbb{R}} \iint_{\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta}} \left| \iint_{I_{t,r}} \frac{\mathcal{U}(u', v')}{|u|^{\kappa} |v|^{\eta}} \mathcal{W}_{2^{t+n}, 2^{r+m}}(h)(x - u, y - v) du dv \right|^{\varepsilon} \phi(x, y) dx dy dt dr \\
&\leq C \|\mathcal{U}\|_{L^1(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})}^{\varepsilon/\varepsilon'} \iint_{\mathbb{R} \times \mathbb{R}} \iint_{\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta}} \iint_{I_{t,r}} \frac{|\mathcal{U}(u', v')|}{|u|^{\kappa} |v|^{\eta}} \\
&\quad \times |\mathcal{W}_{2^{t+n}, 2^{r+m}}(h)(x - u, y - v)|^{\varepsilon} du dv \phi(x, y) dx dy dt dr \\
&\leq C \|\mathcal{U}\|_{L^1(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})}^{\varepsilon/\varepsilon'} \iint_{\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta}} Y^*(\bar{\phi})(x, y) \left(\iint_{\mathbb{R} \times \mathbb{R}} |\mathcal{W}_{2^{t+n}, 2^{r+m}}(h)(x, y)|^{\varepsilon} dt dr \right) dx dy \\
&\leq C \|\mathcal{U}\|_{L^1(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})}^{\varepsilon/\varepsilon'} \left\| \iint_{\mathbb{R} \times \mathbb{R}} |\mathcal{W}_{2^{t+n}, 2^{r+m}}(h)(x, y)|^{\varepsilon} dt dr \right\|_{(p/q)} \|Y^*(\bar{\phi})\|_{(p/q)'} \\
&\leq C \|\mathcal{U}\|_{L^1(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})}^{\varepsilon/\varepsilon'+1} \|h\|_{\dot{F}_p^{\varepsilon, \vec{0}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})}^{\varepsilon}.
\end{aligned}$$

Thus, by the last inequality and (22), we obtain

$$\|\mathcal{Q}_{n,m}(h)\|_{\dot{F}_p^{\varepsilon, \vec{0}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})} \leq C \|\mathcal{U}\|_{L^1(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})} \|h\|_{\dot{F}_p^{\varepsilon, \vec{0}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})}. \quad (23)$$

for all $p \geq \varepsilon$. Therefore, by employing duality along with the interpolation, we conclude that the Inequality (23) holds for all $1 < p < \infty$ and $1 < \varepsilon < \infty$, which, when interpolated with (20), enables us to obtain

$$\|\mathcal{Q}_{n,m}(h)\|_{\dot{F}_p^{\varepsilon, \vec{0}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})} \leq C ((1 + |n|)(1 + |m|))^{-\theta(\alpha+1)} \|h\|_{\dot{F}_p^{\varepsilon, \vec{0}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})} \quad (24)$$

for all $\theta \in (0, 1)$, $\frac{\theta}{2} < \frac{1}{p} < 1 - \frac{\theta}{2}$ and $\frac{\theta}{2} < \frac{1}{\varepsilon} < 1 - \frac{\theta}{2}$. Since

$$\|\mathbf{T}_{\mathcal{U}}(h)\|_{\dot{F}_p^{\varepsilon, \vec{0}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})} \leq C \iint_{\mathbb{R} \times \mathbb{R}} \|\mathcal{Q}_{n,m}(h)\|_{\dot{F}_p^{\varepsilon, \vec{0}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})} dn dm,$$

then by invoking (24) and choosing $\theta > \frac{1}{\alpha+1}$, we end with

$$\|\mathbf{T}_{\mathcal{U}}(h)\|_{\dot{F}_p^{\varepsilon, \vec{\gamma}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})} \leq C \|h\|_{\dot{F}_p^{\varepsilon, \vec{\gamma}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})} \quad (25)$$

for all $p, \varepsilon \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$.

4. Conclusions

In this work, we proved the boundedness of the singular integrals $\mathbf{T}_{\mathcal{U}}$ on Triebel–Lizorkin spaces $\dot{F}_p^{\varepsilon, \vec{\gamma}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})$ for all $p, \varepsilon \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$ whenever the kernel function $\mathcal{U}$ in $\mathbf{G}_{\alpha}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$ for some $\alpha > 0$. The main result in this paper generalizes and improves the main results proved in [1,2,13]. In future work, we aim to prove the boundedness of $\mathbf{T}_{\mathcal{U}}$ on $\dot{F}_p^{\varepsilon, \vec{\gamma}}(\mathbb{R}^{\kappa} \times \mathbb{R}^{\eta})$ for a wider range of p , provided that $\mathcal{U} \in \mathbf{G}_{\alpha}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$.

Author Contributions: Formal analysis, H.A.-Q. and M.A.; writing—original draft preparation, M.A.; methodology, H.A.-Q. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

- Fefferman, R.; Stein, E. Singular integrals on product spaces. *Adv. Math.* **1982**, *45*, 117–143. [\[CrossRef\]](#)
- Duoandikoetxea, J. Multiple singular integrals and maximal functions along hypersurfaces. *Ann. Inst. Four.* **1986**, *36*, 185–206. [\[CrossRef\]](#)
- Al-Salman, A.; Al-Qassem, H.; Pan, Y. Singular integrals on product domains. *Indiana Univ. Math.* **2006**, *55*, 369–387. [\[CrossRef\]](#)
- Fan, D.; Guo, K.; Pan, Y. Singular integrals with rough kernels on product spaces. *Hokkaido Math. J.* **1999**, *28*, 435–460. [\[CrossRef\]](#)
- Fefferman, R. Singular integrals on product domains. *Bull. Am. Math. Soc.* **1981**, *4*, 195–201. [\[CrossRef\]](#)
- Al-Qassem, H.; Ali, M. L^p boundedness for singular integral operators with $L(\log^+ L)^2$ Kernels on Product Spaces. *Kyungpook Math. J.* **2008**, *46*, 377–387.
- Saker, S.; Rezk, H.; Abohela, I.; Baleanu, D. Refinement Multidimensional Dynamic Inequalities with General Kernels and Measures. *J. Ineq. Appl.* **2019**, *2019*, 306. [\[CrossRef\]](#)
- Ahmed, A.; Saker, S.; Kenawy, M.; Rezk, H. Lower Bounds on a Generalization of Cesaro Operator on Time Scales, Dynamics of Continuous, Discrete and Impulsive Systems Series A. *Math. Anal.* **2021**, *28*, 345–355.
- Saker, S.; Zakarya, M.; AlNemer, G.; Rezk, H. Structure of a generalized class of weights satisfy weighted reverse Hölder’s inequality. *J. Ineq. Appl.* **2023**, *2023*, 76. [\[CrossRef\]](#)
- Walsh, T. On the function of Marcinkiewicz. *Studia Math.* **1972**, *44*, 203–217. [\[CrossRef\]](#)
- Grafakos, L.; Stefanov, A. L^p bounds for singular integrals and maximal singular integrals with rough kernel. *Indiana J. Math.* **1998**, *47*, 455–469. [\[CrossRef\]](#)
- Fan, D.; Wu, H. On the generalized Marcinkiewicz integral operators with rough kernels. *Canad. Math. Bull.* **2011**, *54*, 100–112. [\[CrossRef\]](#)
- Ying, Y. Investigations on Some Operators with Rough Kernels in Harmonic Analysis. Ph.D. Thesis, Zhejiang University, Hangzhou, China, 2002.
- Al-Qassem, H.; Chen, L.; Pan, Y. Boundedness of rough integral operators on Triebel–Lizorkin space. *Publ. Mat.* **2012**, *56*, 261–277. [\[CrossRef\]](#)
- Chen, J.; Fan, D.; Pan, Y. Singular integral operators on function spaces. *J. Math. Anal. Appl.* **2002**, *276*, 691–708. [\[CrossRef\]](#)
- Chen, J.; Jia, H.; Jiand, L. Boundedness of rough oscillatory singular integral on Triebel–Lizorkin spaces. *J. Math. Anal. Appl.* **2005**, *306*, 385–397. [\[CrossRef\]](#)
- Chen, J.; Zhang, C. Boundedness of rough singular integral operators on the Triebel–Lizorkin spaces. *J. Math. Anal. Appl.* **2008**, *337*, 1048–1052. [\[CrossRef\]](#)
- Chen, Y.; Ding, Y. Rough singular integrals on Triebel–Lizorkin space and Besov space. *J. Math. Anal. Appl.* **2008**, *347*, 493–501. [\[CrossRef\]](#)
- Coifman, R.; Weiss, G. Extensions of Hardy spaces and their use in analysis. *Bull. Am. Math. Soc.* **1977**, *83*, 569–645. [\[CrossRef\]](#)
- Seeger, A. Singular integral operators with rough convolution kernels. *J. Am. Math. Soc.* **1996**, *9*, 95–105. [\[CrossRef\]](#)
- Seeger, A.; Tao, T. Sharp Lorentz space estimates for rough operators. *Math. Annal.* **2001**, *320*, 381–415. [\[CrossRef\]](#)
- Zhang, C.; Tao, T. Weighted estimates for certain rough singular integrals. *J. Korean Math. Soc.* **2008**, *45*, 1561–1576. [\[CrossRef\]](#)

23. Cheng, L.; Pan, Y. L^p bounds for singular integrals associated to surfaces of revolution. *J. Math. Anal. Appl.* **2002**, *265*, 163–169. [[CrossRef](#)]
24. Cho, Y.; Hong, S.; Kim, J.; Yang, C. Multiparameter singular integrals and maximal operators along flat surfaces. *Rev. Matem. Iber.* **2008**, *24*, 1047–1073. [[CrossRef](#)]
25. Duoandikoetxea, J.; Rubio de Francia, J. Maximal functions and singular integral operators via Fourier transform estimates. *Invent. Math.* **1986**, *84*, 541–561. [[CrossRef](#)]
26. Fan, D.; Pan, Y. Singular integral operators with rough kernels supported by subvarieties. *Am. J. Math.* **1997**, *119*, 799–839. [[CrossRef](#)]
27. Le, H. A note on singular integrals with dominating mixed smoothness in Triebel-Lizorkin spaces. *Acta Math. Sci.* **2014**, *34*, 1331–1344. [[CrossRef](#)]
28. Fan, D.; Gou, K.; Pan, Y. A note of a rough singular integral operator. *Math. Ineq. Appl.* **1999**, *2*, 73–81. [[CrossRef](#)]

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