

Article

A Solvable Algebra for Massless Fermions

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Abstract: We derive the stabiliser group of the four-vector, also known as Wigner’s little group, in case of massless particle states, as the maximal solvable subgroup of the proper orthochronous Lorentz group of dimension four, known as the Borel subgroup. In the absence of mass, particle states are disentangled into left- and right-handed chiral states, governed by the maximal solvable subgroups $\text{sol}_2^\pm$ of order two. Induced Lorentz transformations are constructed and applied to general representations of particle states. Finally, in our conclusions, it is argued how the spin-flip contribution might be closely related to the occurrence of nonphysical spin operators.

Keywords: solvable Lie group; Borel subgroup; massless particle states; chirality states

1. Introduction

As neutrino oscillations are observed in experiments, it seems obvious that all fermions carry a mass. Even though the mass spectrum reaches from small fractions of eV for neutrinos up to 175 GeV for the top quark, a hierarchy waiting still for an explanation, the fact that a fermion carries a mass allows going to the rest frame of the particle and observing both left-handed and right-handed states.

Therefore, the concept of massless fermions, moving with the speed of light, has to be considered as an approximation. This approximation holds true if some of the masses of fermions interacting in a perturbative calculation can be neglected compared to other, larger fermion masses. However, while assuming a fermion to be massless, one not only obtains an essential simplification of the calculation but also different symmetries, which are not given for fermions with small but finite mass. As an example, the breakdown of these symmetries can cause spin-flip effects where the result of the mass-zero limit differs from the result for massless fermions [1–11]. This effect can be understood as a discontinuity in freezing the spin of the fermion. However, to the best of our knowledge, a deeper understanding of these effects is still missing.

In this paper, we analyse the structure of Wigner’s little group for massless particles by adding a small but essential degree of freedom, given by the fact that the momentum vector of a massless particle defines a projective space. In doing so, we come to the conclusion that the stabiliser subgroup is not given by a semisimple group as for massive particles but by a solvable group. In Section 2, we give details on the Borel subgroup as the maximal solvable subgroup describing the stabiliser. In Section 3, we deal with the representation space in terms of common eigenvectors which, in a natural way, leads to the split-off of the representation space into left- and right-handed parts, described as a Kronecker sum in Section 4. The two-dimensional subspaces are governed by the solvable groups sol_2^- and sol_2^+ which are expressed in terms of the Chevalley basis in Section 5. Finally, in Section 6, we give our conclusions and present an outlook on how the Weyl equations for these massless states can be combined to a Dirac equation for fermions with mass.



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1.1. Analysis of Wigner's Little Group

In his paper “Sur la dynamique de l’électron” from July 1905 [12], Henri Poincaré formulated the “Principle of Relativity”, introduces the concepts of Lorentz transformation and the Lorentz group, postulating the covariance of the laws of nature under Lorentz transformations. The full Lorentz group is a six-dimensional, noncompact and non-abelian real Lie group which is not connected. The four connected components of this group are related to each other via discrete transformations (parity and time reversal). None of these components are simply connected. In describing physics, one usually considers the component connected to the identity, called the proper orthochronous Lorentz group $\text{Lor}(1, 3)$.

An important subgroup of $\text{Lor}(1, 3)$ that preserves a given four-vector p is Wigner’s little group. For p describing the momentum of a massive particle, the condition $\Lambda_p p = p$ for the elements Λ_p of the little group can be solved in the rest frame of the particle where the normalised momentum vector is given by $\hat{p} = (1; 0, 0, 0)^T$, leading to the block structure

$$\hat{\Lambda}_p = \begin{pmatrix} 1 & \vec{0}^T \\ \vec{0} & D \end{pmatrix} = R_p, \quad (1)$$

where $DD^T = \mathbb{1}_3 = D^T D$. Therefore, the little group of a massive particle is isomorphic to $\text{SO}(3)$. However, for a massless particle, the momentum vectors $p = (1; 0, 0, \varepsilon)$ with $\varepsilon = \pm 1$ for a movement along the z axis are projective vectors. Therefore, solving the generalised equations $\Lambda p = \lambda p$ and $\Lambda^T \eta p = \lambda^{-1} \eta p$ for $p = (1; 0, 0, \varepsilon)$ with a general value of ε and the Minkowskian metric $\eta = \text{diag}(1; -1, -1, -1)$ via the block ansatz

$$\Lambda = \begin{pmatrix} A & \vec{B}^T \\ \vec{C} & D \end{pmatrix} \quad (2)$$

leads to $\varepsilon B_3 = \lambda - A$, $\varepsilon C_3 = A - \lambda^{-1}$, $\varepsilon D_{33} = \varepsilon \lambda - C_3$, and $\varepsilon D_{33} = B_3 + \varepsilon \lambda^{-1}$. The two last conditions are in agreement if and only if

$$\varepsilon^2 \lambda - A + \lambda^{-1} = \lambda - A + \varepsilon^2 \lambda^{-1} \Leftrightarrow (1 - \varepsilon^2)(\lambda - \lambda^{-1}) = 0. \quad (3)$$

This equation marks the point where two different paths are possible to follow: for $\lambda = \lambda^{-1} = 1$ ($\lambda > 0$ for the proper orthochronous Lorentz group), one ends up again with Wigner’s little group $\text{SO}(3)$. For massless particles, however, one has $\varepsilon^2 = 1$ and, therefore, one can keep $\lambda > 0$ arbitrary, ending up with the Borel subgroup explained in the following.

1.2. Justification of the Extension

The introduction of an extension of Wigner’s little group needs justification. Wigner introduced the little group as a stabiliser group with respect to the momentum vector p . However, because the four-length of the momentum vector for a massless particle is zero and, therefore, the multiplication of this vector with an arbitrary scale does not change the physics of this particle, the physical situation is better described by a projective space. The existence of an invariant subspace is guaranteed by the Lie–Kolchin theorem,

Lie–Kolchin Theorem: *If G is a connected and solvable linear algebraic group defined over an algebraically closed field and $\rho : G \rightarrow \text{GL}(V)$ is a representation on a nonzero finite-dimensional vector space V , then there exists a one-dimensional linear subspace L of V such that*

$$\rho(G)(L) = L. \quad (4)$$

In 1956, Armand Borel generalised the Lie–Kolchin theorem as a fixed-point theorem for algebraic varieties [13] and, therefore, also for a projective space.

Borel Fixed-Point Theorem: *If G is a connected, solvable algebraic group acting regularly on a non-empty, complete algebraic variety V over an algebraically closed field, then there exists a fixed point of V .*

As expressed by Equation (3), the projectivity of the fixed-point is broken if $\varepsilon^2 < 1$, i.e., if the particle gains mass. In this case, we fall back to Wigner's little group. The extension can also be understood on the level of Lie algebras, as for massless particles the interchange of the space and time components of the momentum vector is an additional symmetry that is absent for massive particles. Note that in this context $E(2)$, as the little group for massless particles proposed by Wigner, is also a solvable group, though not maximal.

2. The Borel Subgroup $\text{Bor}(1, 3; p)$

From now on, we use ε only as the sign of the momentum 3-component. The fact that the momentum vector $p \sim (1; 0, 0, \varepsilon)$ for a massless particle is symmetric (up to the sign $\varepsilon = \pm 1$) under the interchange of the first and the last component gives an additional element of the algebra which is missing so far in Wigner's little group. In order to see this, one can find solutions for the character problem (summation over repeated indices is implied)

$$\Lambda^\mu_\nu(p)p^\nu = \lambda(\Lambda)p^\mu. \quad (5)$$

Solving this problem for the Lorentz matrix $\Lambda_p = (\Lambda^\mu_\nu(p))$ with $\Lambda_p^T \eta \Lambda_p = \eta$ (the matrix is transposed (indicated by the upper index T) for visualisation reasons only) one obtains

$$\Lambda_p = \begin{pmatrix} \cosh t + \frac{1}{2}e^{-t}(u^2 + v^2) & \varepsilon u & \varepsilon v & \varepsilon(\sinh t + \frac{1}{2}e^{-t}(u^2 + v^2)) \\ \varepsilon(u \cos w - v \sin w) & \cos w & -\sin w & u \cos w - v \sin w \\ \varepsilon(u \sin w + v \cos w) & \sin w & \cos w & u \sin w + v \cos w \\ \varepsilon(\sinh t - \frac{1}{2}e^{-t}(u^2 + v^2)) & -u & -v & \cosh t - \frac{1}{2}e^{-t}(u^2 + v^2) \end{pmatrix}^T, \quad (6)$$

where we have chosen $\lambda = e^t$ and introduced three additional parameters u , v , and w . Expanding in these parameters one obtains $\Lambda_p \approx \mathbb{1}_4 + Tt + Uu + Vv + Ww$, where

$$\begin{aligned} T = \frac{\partial \Lambda}{\partial t} \Big|_0 &= \begin{pmatrix} 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & 0 \end{pmatrix}, & U = \frac{\partial \Lambda}{\partial u} \Big|_0 &= \begin{pmatrix} 0 & \varepsilon & 0 & 0 \\ \varepsilon & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ V = \frac{\partial \Lambda}{\partial v} \Big|_0 &= \begin{pmatrix} 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & W = \frac{\partial \Lambda}{\partial w} \Big|_0 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (7)$$

and the lower index "0" symbolises the initial value $t = u = v = w = 0$. T , U , V , and W are generators of the maximal solvable Lie subgroup of $\text{Lor}(1, 3)$, i.e., the Borel subgroup $\text{Bor}(1, 3; p) \subset \text{Lor}(1, 3)$. For the corresponding Lie algebra $\mathfrak{g} = \text{span}_{\mathbb{R}}\{T, U, V, W\} = \text{bor}(1, 3; p)$ one easily obtains

$$[T, U] = U, \quad [T, V] = V, \quad [W, U] = -V, \quad [W, V] = U, \quad (8)$$

with all other commutators being zero. Accordingly, one has $[\mathfrak{g}, \mathfrak{g}] = \text{span}_{\mathbb{R}}\{U, V\}$ and $[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = 0$ such that $\mathfrak{g}$ is solvable. Note that the element (6) of the Borel subgroup $\text{Bor}(1, 3; p)$ is given by a polar decomposition, i.e., it can be restored by calculating

$$\Lambda_p = \exp(Uu + Vv) \exp(Tt + Ww). \quad (9)$$

Because $[T, W] = 0$, one has $\exp(Tt) \exp(Ww) = \exp(Tt + Ww) = \exp(Ww) \exp(Tt)$. The two parts of the second exponential factor commute with each other. They constitute

the maximal torus $\text{Tor}(1, 1; p)$ describing the transformations that leave the direction of the momentum vector p invariant: a boost directed along the z axis described by $\exp(Tt)$ and a rotation about the z axis described by $\exp(Ww)$. However, these two factors do not commute with the first exponential factor $\Lambda_{u,v} := \exp(Uu + Vv)$ which constitutes the physically nontrivial part $\mathcal{T}(2; p)$ of the Borel subgroup (translations),

$$\Lambda_{u,v} = \exp \begin{pmatrix} 0 & \varepsilon u & \varepsilon v & 0 \\ \varepsilon u & 0 & 0 & -u \\ \varepsilon v & 0 & 0 & -v \\ 0 & u & v & 0 \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{2}(u^2 + v^2) & \varepsilon u & \varepsilon v & -\frac{1}{2}\varepsilon(u^2 + v^2) \\ \varepsilon u & 1 & 0 & -u \\ \varepsilon v & 0 & 1 & -v \\ \frac{1}{2}\varepsilon(u^2 + v^2) & u & v & 1 - \frac{1}{2}(u^2 + v^2) \end{pmatrix}. \quad (10)$$

Note that due to the solvability, the series expansion breaks at the second order. Together, these two parts of the polar decomposition of Λ_p represent the Borel subgroup as a semidirect product,

$$\text{Bor}(1, 3; p) = \mathcal{T}(2; p) \rtimes \text{Tor}(1, 1; p). \quad (11)$$

A Bridge from Massive to Massless

Even though the main emphasis of this paper is on the independent treatment of the little group of massless particles as the maximal noncompact solvable subgroup of the proper orthochronous Lorentz group, there is still a way to find a bridge connecting this part of the Lorentz group to the maximal compact simple subgroup, which is quite remarkable. Starting with a massive particle, in the rest frame of this particle, a proper orthochronous Lorentz transformation $\hat{\Lambda}_p = B_{r,s,t} R_{u,v,w}$ can be written as a polar decomposition of the Wigner rotation matrix $R_{u,v,w}$ followed by a boost $B_{r,s,t}$, where

$$B_{r,s,t} = \exp \begin{pmatrix} 0 & r & s & t \\ r & 0 & 0 & 0 \\ s & 0 & 0 & 0 \\ t & 0 & 0 & 0 \end{pmatrix}, \quad R_{u,v,w} = \exp \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & w & -u \\ 0 & -w & 0 & -v \\ 0 & u & v & 0 \end{pmatrix}. \quad (12)$$

The transformation to the laboratory frame where the momentum vector of the particle is given by p is performed with the help of the boost matrix $B_p = B_{0,0,\varepsilon\tilde{\zeta}_p}$ parametrised by the momentum vector p ,

$$B_p = \exp \begin{pmatrix} 0 & 0 & 0 & \varepsilon\tilde{\zeta}_p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \varepsilon\tilde{\zeta}_p & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} c_p & 0 & 0 & \varepsilon s_p \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \varepsilon s_p & 0 & 0 & c_p \end{pmatrix}, \quad (13)$$

where $c_p = \cosh \tilde{\zeta}_p$ and $s_p = \sinh \tilde{\zeta}_p$ with a rapidity of $\tilde{\zeta}_p$. Accordingly, the proper orthochronous Lorentz transformation in the laboratory frame is given by

$$\Lambda_p = B_p B_{r,s,t} R_{u,v,w} B_p^{-1} = B_p B_{r,s,t} B_p^{-1} B_p R_{u,v,w} B_p^{-1}. \quad (14)$$

For the generic Lie algebra element generating the boost $B_{r,s,t}$ one obtains

$$B_p \begin{pmatrix} 0 & r & s & t \\ r & 0 & 0 & 0 \\ s & 0 & 0 & 0 \\ t & 0 & 0 & 0 \end{pmatrix} B_p^{-1} = \begin{pmatrix} 0 & c_p r & c_p s & t \\ c_p r & 0 & 0 & -\varepsilon s_p r \\ c_p s & 0 & 0 & -\varepsilon s_p s \\ t & \varepsilon s_p r & \varepsilon s_p s & 0 \end{pmatrix}. \quad (15)$$

Because $c_p, s_p \rightarrow \infty$ in the massless limit, r and s (but not t) have to be renormalised in order to obtain a finite matrix $B_p B_{r,s,t} B_p^{-1}$. This can be performed by replacing r by xr and s by xs where $xc_p = xs_p \rightarrow 1$ in the massless limit $x \rightarrow 0$. Raising the generic element in Equation (15) to the exponent, one obtains $B_{xr, xs, t} = B_{xr} B_{xs} B_t$, where the exponential factors B_{xr} and B_{xs} factorise and commute with each other and with the remaining factor B_t due to the smallness of the renormalised parameters xr and xs . The factor B_t describes a boost

along the z axis. Compared to the boost B_p in the same direction, the former is negligible in the massless limit. Therefore, one can replace B_p with $B_p B_t^{-1} = B_t^{-1} B_p$ and obtain

$$\Lambda_p = B_p B_{\varepsilon x r} B_{\varepsilon x s} B_p^{-1} B_p R_{u,v,w} B_p^{-1} B_t \rightarrow \Lambda_{\varepsilon r, \varepsilon s} B_p R_{u,v,w} B_p^{-1} B_t. \quad (16)$$

Because of the renormalisation, $B_p B_{x r} B_{x s} B_p^{-1}$ is finite in the massless limit and gives $\Lambda_{\varepsilon r, \varepsilon s}$, which can be seen by comparing the result of the exponentiation with Equation (10).

Looking at the second main factor in Λ_p , a similar consideration can be made for $B_p R_{u,v,w} B_p^{-1} B_t$. Starting from

$$B_p \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & w & -u \\ 0 & -w & 0 & -v \\ 0 & u & v & 0 \end{pmatrix} B_p^{-1} = \begin{pmatrix} 0 & \varepsilon_s p u & \varepsilon_s p v & 0 \\ \varepsilon_s p u & 0 & w & -c_p u \\ \varepsilon_s p v & -w & 0 & -c_p v \\ 0 & c_p u & c_p v & 0 \end{pmatrix}, \quad (17)$$

u and v (but not w) have to be renormalised, again using x with $x c_p = x s_p \rightarrow 1$. Raising the generic element in Equation (17) to the exponent, one obtains $R_{xu, xv, w} = R_{xu} R_{xv} R_w$ where all three factors again commute with each other. As R_w also commutes with B_p^{-1} , this factor can be pulled out, and the remaining product $B_p R_{xu} R_{xv} B_p^{-1}$ gives $\Lambda_{u,v}$ in the massless limit. Therefore, in this limit, Λ_p will decay into

$$\Lambda_p \rightarrow \Lambda_{\varepsilon r, \varepsilon s} \Lambda_{u,v} R_w B_t, \quad R_w = \exp(Ww), \quad B_t = \exp(Tt). \quad (18)$$

In this product, $\Lambda_{u,v} R_w B_t$ constitutes the generic element of the Borel subgroup $\text{Bor}(1, 3; p)$ and $\Lambda_{\varepsilon r, \varepsilon s}$ constitutes the rest class $\text{Lor}(1, 3) / \text{Bor}(1, 3; p)$. To conclude, the little groups of massive and massless particles are connected by a singular transformation, induced by an infinitesimal boost, interpreted as contraction in the sense of Inonu and Wigner [14].

3. Common (Pseudo)eigenvectors

The exponential representation (9) is a special case of the representation

$$\Lambda(\omega) = \exp\left(-\frac{1}{2} \omega_{\alpha\beta} e^{\alpha\beta}\right) \quad (19)$$

of the full Lorentz group where $(e^{\alpha\beta})^\mu{}_\nu = \eta^{\alpha\mu} \eta^{\beta\nu} - \eta^{\alpha\nu} \eta^{\beta\mu}$, $\eta = \text{diag}(1, -1, -1, -1)$. Of course, the generators T, U, V , and W can then be expressed in terms of $e^{\alpha\beta}$,

$$T = -\varepsilon e^{03}, \quad U = e^{31} - \varepsilon e^{01}, \quad V = e^{32} - \varepsilon e^{02}, \quad W = e^{12} \quad (20)$$

with the non-vanishing parameters $\varepsilon \omega_{03} = \varepsilon \omega_{30} = t$, $\varepsilon \omega_{01} = \varepsilon \omega_{10} = \omega_{13} = -\omega_{31} = u$, $\varepsilon \omega_{02} = \varepsilon \omega_{20} = \omega_{23} = -\omega_{32} = v$, and $\omega_{21} = -\omega_{12} = w$. For technical reasons, instead of $\{T, U, V, W\}$ we may use the notation $\{T_0^\varepsilon, T_1^\varepsilon, T_2^\varepsilon, T_3^\varepsilon\} = \{T_i^\varepsilon\}_0^3$ in the following. The upper index ε indicates the dependence on ε , where $T_0^{-\varepsilon} = -T_0^\varepsilon$ and $T_3^{-\varepsilon} = T_3^\varepsilon$. Because $T_3^\varepsilon = W$ does not depend on ε , one can skip the index in this case.

According to Lie's theorem, a solvable algebra has a single common eigenvector. Solving the equations $T_i^\varepsilon \ell_0 = \lambda_i^{(0)} \ell_0$ ($i = 0, 1, 2, 3$), one obtains

$$\ell_0 = (1; 0, 0, \varepsilon)^T / \sqrt{2}, \quad \lambda_0^{(0)} = +1, \quad \lambda_1^{(0)} = 0, \quad \lambda_2^{(0)} = 0, \quad \lambda_3^{(0)} = 0. \quad (21)$$

Not very surprisingly, the common eigenvector is given simply as p . In order to specify the defective matrices $\hat{T}_i^\varepsilon$ of the solvable algebra, the equations

$$\begin{aligned}
T_i^\varepsilon \ell_1 &= \lambda_i^{(1)} \ell_1 + \gamma_{1i}^0 \ell_0 \\
T_i^\varepsilon \ell_2 &= \lambda_i^{(2)} \ell_2 + \gamma_{2i}^1 \ell_1 + \gamma_{2i}^0 \ell_0 \\
T_i^\varepsilon \ell_3 &= \lambda_i^{(3)} \ell_3 + \gamma_{3i}^2 \ell_2 + \gamma_{2i}^1 \ell_1 + \gamma_{2i}^0 \ell_0
\end{aligned} \tag{22}$$

are solved in a step-wise manner to obtain a system of pseudo-eigenvectors and -eigenvalues. Collecting all these equations in a single equation, after some normalisation one obtains

$$T_i^\varepsilon P = P \begin{pmatrix} \lambda_i^{(0)} & \gamma_{1i}^0 & \gamma_{2i}^0 & \gamma_{3i}^0 \\ 0 & \lambda_i^{(1)} & \gamma_{2i}^1 & \gamma_{3i}^1 \\ 0 & 0 & \lambda_i^{(2)} & \gamma_{3i}^2 \\ 0 & 0 & 0 & \lambda_i^{(3)} \end{pmatrix}, \quad P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & -\varepsilon \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ \varepsilon & 0 & 0 & 1 \end{pmatrix}, \tag{23}$$

where $P = (\ell_0, \ell_1, \ell_2, \ell_3)$ is rearranged in order to be unitary, $P^{-1} = P^\dagger$. Turning back to the original notation, one obtains $TP = P\hat{T}$, $UP = P\hat{U}$, $VP = P\hat{V}$, and $WP = P\hat{W}$, where

$$\begin{aligned}
\hat{T} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, & \hat{U} &= \begin{pmatrix} 0 & \varepsilon & i\varepsilon & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\hat{V} &= \begin{pmatrix} 0 & i\varepsilon & \varepsilon & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \hat{W} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{24}$$

are upper triangular forms of the four generators.

Generating the (Pseudo)eigenvectors

Even though the four generators have only a single common eigenvector, this is not the case for the generic element $\Lambda_p \in \text{Bor}(1,3;p)$ in Equation (6). Solving the fourth-order equation $\det(\Lambda_p - \lambda \mathbb{1}) = 0$ for λ leads to $\lambda \in \{e^t, e^{i\omega}, e^{-i\omega}, e^{-t}\}$. The corresponding system of eigenvectors can be calculated. However, here we give a more elegant method to calculate this system of eigenvectors. Using the exponential representation (9) and

$$(Uu + Vv)P = P(\hat{U}u + \hat{V}v), \quad (Tt + Ww)P = P(\hat{T}t + \hat{W}w), \tag{25}$$

one obtains $\Lambda_p = PK_{u,v}K_{t,w}P^{-1}$ with unipotent $K_{u,v} = \exp(\hat{U}u + \hat{V}v)$ and semisimple $K_{t,w} = \exp(\hat{T}t + \hat{W}w)$,

$$K_{u,v} = \begin{pmatrix} 1 & \varepsilon(u + iv) & \varepsilon(v + iu) & -\varepsilon(u^2 + v^2) \\ 0 & 1 & 0 & -u + iv \\ 0 & 0 & 1 & -v + iu \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad K_{t,w} = \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{i\omega} & 0 & 0 \\ 0 & 0 & e^{-i\omega} & 0 \\ 0 & 0 & 0 & e^{-t} \end{pmatrix}. \tag{26}$$

Because $K_{t,w}$ is a diagonal matrix containing the four eigenvectors, the system of eigenvectors is given by the matrix Q obeying $\Lambda_p Q = QK_{t,w}$. Inserting $\Lambda_p = PK_{u,v}K_{t,w}P^{-1}$ into this eigenvalue equation, after some rearrangements, one obtains

$$P^{-1}Q = K_{u,v}K_{t,w}P^{-1}QK_{t,w}^{-1}. \tag{27}$$

This equation for the unknown quantity $P^{-1}Q$ can be solved iteratively, starting with $P^{-1}Q = \mathbb{1}$, i.e., $Q = P$. The iterative solution can be shown to converge to

$$P^{-1}Q = \begin{pmatrix} 1 & \frac{\varepsilon(u+iv)}{1-e^{t-iw}} & \frac{\varepsilon(v+iu)}{1-e^{t+iw}} & \frac{-\varepsilon(u^2+v^2)}{(1-e^{t-iw})(1-e^{t+iw})} \\ 0 & 1 & 0 & \frac{-u+iv}{1-e^{t+iw}} \\ 0 & 0 & 1 & \frac{-v+iu}{1-e^{t-iw}} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (28)$$

Multiplying with P from the left, one finally obtains the system of eigenvectors

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \frac{\varepsilon(u+iv)}{1-e^{t-iw}} & \frac{\varepsilon(v+iu)}{1-e^{t+iw}} & -\varepsilon - \frac{\varepsilon(u^2+v^2)}{(1-e^{t-iw})(1-e^{t+iw})} \\ 0 & 1 & i & \frac{-u+iv}{1-e^{t+iw}} + i \frac{-v+iu}{1-e^{t-iw}} \\ 0 & i & 1 & i \frac{-u+iv}{1-e^{t+iw}} + \frac{-v+iu}{1-e^{t-iw}} \\ \varepsilon & \frac{u+iv}{1-e^{t-iw}} & \frac{v+iu}{1-e^{t+iw}} & 1 - \frac{\varepsilon(u^2+v^2)}{(1-e^{t-iw})(1-e^{t+iw})} \end{pmatrix}. \quad (29)$$

Expressed in a slightly philosophical manner, one can say that starting from the very sparse boundary of four defective matrices, the Lie algebra (in this case, the Borel subalgebra) knits the sweater Q for the Lie group in a straightforward, iterative way.

4. Kronecker Sum of Solvable Algebras

Although we were able to analyse the solvable algebra $\text{bor}(1,3;p)$, the representation in terms of the generators T , U , V , and W is not the best one to see the structure of this algebra. Therefore, we use a second one, namely

$$\begin{aligned} J_3^\varepsilon &= \frac{1}{2}(-T - iW), & J_-^\varepsilon &= \frac{1}{2}(-U + iV), \\ K_3^\varepsilon &= \frac{1}{2}(T - iW), & K_+^\varepsilon &= \frac{1}{2}(U + iV), \end{aligned} \quad (30)$$

obeying

$$\begin{aligned} [J_3^\varepsilon, J_3^\varepsilon] &= 0, & [J_3^\varepsilon, J_-^\varepsilon] &= -J_-^\varepsilon, & [J_-^\varepsilon, J_-^\varepsilon] &= 0, \\ [K_3^\varepsilon, K_3^\varepsilon] &= 0, & [K_3^\varepsilon, K_+^\varepsilon] &= K_+^\varepsilon, & [K_+^\varepsilon, K_+^\varepsilon] &= 0 \end{aligned} \quad (31)$$

and $[J_3^\varepsilon, K_3^\varepsilon] = [J_3^\varepsilon, K_+^\varepsilon] = [J_-^\varepsilon, K_3^\varepsilon] = [J_-^\varepsilon, K_+^\varepsilon] = 0$. The first justification for the sign notations for J_-^ε and K_+^ε is given by the commutator relations (31). In terms of the pairs $\{J_3^\varepsilon, J_-^\varepsilon\}$ and $\{K_3^\varepsilon, K_+^\varepsilon\}$ of generators, $\text{bor}(1,3;p)$ can be rewritten as a Kronecker sum $\text{sol}_2^- \boxplus \text{sol}_2^+$ of two two-dimensional solvable algebras, as will be detailed in the following.

4.1. Weyl's Unitary Trick

A deeper look at this change of representation unveils that this change is actually a composition of several steps. In order to illustrate these steps, one can start again with the proper orthochronous Lorentz group $\text{Lor}(1,3) \subset \text{SO}(1,3)$, the elements of which are given by the exponential representation (19) where $(e^{\alpha\beta})^\mu{}_\nu = \eta^\alpha{}_\nu \eta^{\beta\mu} - \eta^{\alpha\mu} \eta^\beta{}_\nu$, $\eta = \text{diag}(1, -1, -1, -1)$. This representation can be written in a different form as

$$\Lambda_p = \exp(\vec{\tau} \cdot \vec{E} + \vec{\omega} \cdot \vec{B}), \quad (32)$$

using an analogy to the electromagnetic field strength tensor $F^{\mu\nu}$ to write

$$-\frac{1}{2}\omega_{\alpha\beta}e^{\alpha\beta} = \begin{pmatrix} 0 & \omega_{01} & \omega_{02} & \omega_{03} \\ \omega_{01} & 0 & \omega_{12} & -\omega_{31} \\ \omega_{02} & -\omega_{12} & 0 & \omega_{23} \\ \omega_{03} & \omega_{31} & -\omega_{23} & 0 \end{pmatrix} = \vec{\tau} \cdot \vec{E} + \vec{\omega} \cdot \vec{B}, \quad (33)$$

where $\vec{\tau} = (\omega_{01}, \omega_{02}, \omega_{03})$, $\vec{\omega} = (\omega_{23}, \omega_{31}, \omega_{12})$, and $\vec{E} = -(e^{01}, e^{02}, e^{03})$, $\vec{B} = -(e^{23}, e^{31}, e^{12})$, or $e^{0i} = -E_i$, $e^{ij} = -\epsilon_{ijk}B_k$ with the convention of lower indices for $\vec{E}$ and $\vec{B}$ and related three-vectors, $\epsilon_{123} = 1$. The $3 + 3$ generators of $\text{Lor}(1, 3)$ obey the commutation relations

$$[B_i, B_j] = \epsilon_{ijk}B_k, \quad [B_i, E_j] = \epsilon_{ijk}E_k, \quad [E_i, E_j] = -\epsilon_{ijk}B_k. \quad (34)$$

Obviously, the algebra $\text{lor}(1, 3)$ is a real algebra. It contains a compact subalgebra $\mathfrak{k}$ related to the B^i which is isomorphic to the compact algebra $\text{so}(3)$. Actually, $\text{lor}(1, 3)$ is in the shape of a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ characterised by the values $\phi(\mathfrak{k}) = \mathfrak{k}$, $\phi(\mathfrak{p}) = -\mathfrak{p}$ of an involution ϕ . As vector spaces, $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal, because given a scalar product $(\mathfrak{k}, \mathfrak{p})$ invariant under this involution, one obtains

$$(\mathfrak{k}, \mathfrak{p}) = (\phi(\mathfrak{k}), \phi(\mathfrak{p})) = (\mathfrak{k}, -\mathfrak{p}) = -(\mathfrak{k}, \mathfrak{p}) \Rightarrow (\mathfrak{k}, \mathfrak{p}) = 0 \quad (35)$$

However, $[\mathfrak{k}, \mathfrak{p}] \neq 0$. Therefore, we used the symbol $+$ instead of the symbol $\oplus$ for the direct sum. The algebra $\mathfrak{g}$ can be transformed to a compact form by using Weyl's unitary trick. The result is an algebra $\mathfrak{g}^* = \mathfrak{k} + i\mathfrak{p}$, where the implications for introducing an imaginary factor will be explained later. In case of $\text{lor}(1, 3)$, the involution is given by

$$\phi : e^{\mu\nu} \rightarrow \eta(e^{\mu\nu}) := \eta e^{\mu\nu} \eta = -e^{\mu\nu T} \quad (36)$$

(matrix indices are suppressed) or $\phi(\vec{B}) = \vec{B}$, $\phi(\vec{E}) = -\vec{E}$. Therefore, the compact form of $\text{lor}(1, 3)$ is given by the generators B^i and iE^i obeying the commutation relations

$$[B_i, B_j] = \epsilon_{ijk}B_k, \quad [B_i, (iE_j)] = \epsilon_{ijk}(iE_k), \quad [(iE_i), (iE_j)] = \epsilon_{ijk}B_k. \quad (37)$$

Considered as a real algebra, this algebra is isomorphic to $\text{so}(4)$. However, the generators are antihermitean, $B_i^\dagger = -B_i$, and $(iE_i)^\dagger = -iE_i$ and, therefore, the group is unitary. In general, Weyl's unitary trick can be seen to lead always to unitary Lie groups.

4.2. Duplication and Complexification

The addition of an imaginary factor i turns the real algebra into a complex algebra, at least for intermediate steps. In general, this process is called complexification and is denoted by a lower index $\mathbb{C}$ (or additional argument) to the algebra symbol. Given a real Lie algebra L , the duplication of this algebra is given by [15]

$$L + iL := \{x + iy \mid x, y \in L\}. \quad (38)$$

$L + iL$ is still a real vector space. In defining the multiplication of an element $x + iy \in L_{\mathbb{C}}$ with a complex number $\alpha = a + ib \in \mathbb{C}$ by $(a + ib)(x + iy) := (ax - by) + i(bx + ay)$ and the commutator of two elements $x + iy$ and $x' + iy'$ by

$$[x + iy, x' + iy'] := [x, x'] - [y, y'] + i([x, y'] + [y, x']), \quad (39)$$

$L + iL$ constitutes a complex form, denoted by $L_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} L$. This complex form again is a complex Lie algebra, which is called the complexification of L . Applied to the actual case, the complexification turns the real algebra $\text{so}(1, 3)$ into the complex algebra $\text{so}(4, \mathbb{C})$.

However, it is obvious that the algebra given by the commutator relations (37) is real, not complex. The final algebra, therefore, is a real form of this complex algebra, defined as follows: a subalgebra K of the duplicated algebra $L + iL$ is called real form if the complexification of this subalgebra is the same as the original algebra, $K_{\mathbb{C}} = L$. As the

duplication is not unique (for instance, a part of the basis elements can be duplicated with i , another part with $-i$), there are also different real forms to a given complex algebra.

4.3. Compactified and Decompactified Real Forms

Most important real forms are the normal real form where the duplicates are again taken as separate elements, and the compact real form which exists for all (semi)simple complex Lie algebras. Because we will meet these forms in the lower-dimensional case, we postpone the discussion about the different real forms. In the actual case, one of the compact real forms is $\mathfrak{so}(4)$. However, another one is given by

$$A_i = \frac{1}{2}(B_i + iE_i), \quad \bar{A}_i = \frac{1}{2}(B_i - iE_i) \quad (40)$$

with the commutation rules

$$[A_i, A_j] = \epsilon_{ijk} A_k, \quad [A_i, \bar{A}_j] = 0, \quad [\bar{A}_i, \bar{A}_j] = \epsilon_{ijk} \bar{A}_k. \quad (41)$$

Therefore, the algebra decomposes into two separate algebras which are isomorphic to $\mathfrak{su}(2)$ ($\mathfrak{su}(2)$ is preferred instead of $\mathfrak{so}(3)$ because A_i and $\bar{A}_i$ are antihermitean, leading to unitary groups). Turning back to solvable groups, the decomposition into $\mathfrak{su}(2) = \text{span}_{\mathbb{R}}\{A_i\}$ and $\mathfrak{su}(2) = \text{span}_{\mathbb{R}}\{\bar{A}_i\}$ does not yet conform with the definitions given in Equation (30). Looking at the definitions of T_i^ϵ , on the one hand, and the definitions of E_i and B_i , on the other hand, one obtains

$$J_3^\epsilon = \frac{1}{2}(\epsilon e^{03} - ie^{12}) = \frac{1}{2}(-\epsilon E_3 + iB_3), \quad K_3^\epsilon = \frac{1}{2}(-\epsilon e^{03} - ie^{12}) = \frac{1}{2}(\epsilon E_3 + iB_3) \quad (42)$$

and generally, $J_i^\epsilon = \frac{1}{2}(-\epsilon E_i + iB_i) = i\bar{A}_i$, $K_i^\epsilon = \frac{1}{2}(\epsilon E_i + iB_i) = iA_i$. As A_i and $\bar{A}_i$ are antihermitean, J_i^ϵ and K_i^ϵ are hermitean and, therefore, constitute decompactified subgroups generated by $\exp(ij_i J_i^\epsilon)$ and $\exp(ik_i K_i^\epsilon)$. One obtains

$$\begin{aligned} J_1^\epsilon - iJ_2^\epsilon &= \frac{1}{2}(-\epsilon E_1 + B_2 + i(\epsilon E_2 + B_1)) = \frac{1}{2}(-T_1^\epsilon + iT_2^\epsilon) = J_-^\epsilon, \\ K_1^\epsilon + iK_2^\epsilon &= \frac{1}{2}(\epsilon E_1 - B_2 + i(\epsilon E_2 + B_1)) = \frac{1}{2}(T_1^\epsilon + iT_2^\epsilon) = K_+^\epsilon \end{aligned} \quad (43)$$

which is the other justification for the sign notations in J_-^ϵ and K_+^ϵ . Actually, the algebra looks like $\mathfrak{sl}(2, \mathbb{R})$ with one generator missing (J_+^ϵ or K_-^ϵ , respectively). In $\mathfrak{lor}(1, 3)$, these missing generators exist. In $\mathfrak{bor}(1, 3; p)$, however, the generators are found in the respective algebra with an opposite sign ϵ ,

$$\begin{aligned} J_+^\epsilon &= J_1^\epsilon + iJ_2^\epsilon = \frac{1}{2}(-\epsilon E_1 - B_2 + i(-\epsilon E_2 + B_1)) = \frac{1}{2}(T_1^{-\epsilon} + iT_2^{-\epsilon}) = K_+^{-\epsilon}, \\ K_-^\epsilon &= K_1^\epsilon - iK_2^\epsilon = \frac{1}{2}(\epsilon E_1 + B_2 + i(-\epsilon E_2 + B_1)) = \frac{1}{2}(-T_1^{-\epsilon} + iT_2^{-\epsilon}) = J_-^{-\epsilon}, \end{aligned} \quad (44)$$

while $J_3^{\pm\epsilon} = K_3^{\mp\epsilon}$. Therefore, $\mathfrak{bor}(1, 3; p)$ splits up into the subalgebras

$$\begin{aligned} \mathfrak{sol}_2^- &:= \text{span}_{\mathbb{R}}\{J_3^\epsilon, J_-^\epsilon\} = \text{span}_{\mathbb{R}}\{K_3^{-\epsilon}, K_-^{-\epsilon}\} \quad \text{and} \\ \mathfrak{sol}_2^+ &:= \text{span}_{\mathbb{R}}\{K_3^\epsilon, K_+^\epsilon\} = \text{span}_{\mathbb{R}}\{J_3^{-\epsilon}, J_+^{-\epsilon}\}. \end{aligned} \quad (45)$$

As there is a homomorphism between the two algebras $\mathfrak{sol}_2^+$ and $\mathfrak{sol}_2^-$, both solvable algebras are maximal and, therefore, are Borel subalgebras of the larger algebra $\mathfrak{sl}(2, \mathbb{R})$. For a free choice of ϵ one can represent the two Borel subalgebras as being generated by a solvable part of the set $\{J_3^{\pm\epsilon}, J_+^{\pm\epsilon}, J_-^{\pm\epsilon}\}$ of generators of $\mathfrak{sl}(2, \mathbb{R})$, thereby skipping the second (redundant) set $\{K_3^{\mp\epsilon}, K_+^{\mp\epsilon}, K_-^{\mp\epsilon}\}$. Alternatively, one can use the two sets and skip $\epsilon = +1$. Though the first choice is more intriguing, for this paper we stay with the clearer second one.

4.4. In Search of Left and Right

Searching for eigenvectors of the set $\{J_3, J_+, J_-\}$ one finds that these eigenvectors are disjoint, as is known for semisimple algebras. The same holds for the set $\{K_3, K_+, K_-\}$. However, for each of the solvable subalgebras sol_2^- and sol_2^+ one obtains only a single common eigenvector. In order to analyse the eigenvector structure, we return to the eigenvectors

$$\ell_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \varepsilon \end{pmatrix}, \quad \ell_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \quad \ell_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix} \text{ and } \ell_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -\varepsilon \end{pmatrix} \quad (46)$$

of Section 3 to which we apply the algebra elements, obtaining

$$\begin{aligned} J_3 \ell_0 &= -\frac{1}{2} \ell_0 & J_+ \ell_0 &= -\varepsilon \ell_1 & J_- \ell_0 &= 0 & K_3 \ell_0 &= +\frac{1}{2} \ell_0 & K_+ \ell_0 &= 0 & K_- \ell_0 &= \varepsilon \ell_1 \\ J_3 \ell_1 &= +\frac{1}{2} \ell_1 & J_+ \ell_1 &= 0 & J_- \ell_1 &= -\varepsilon \ell_0 & K_3 \ell_1 &= +\frac{1}{2} \ell_1 & K_+ \ell_1 &= 0 & K_- \ell_1 &= \varepsilon \ell_3 \\ J_3 \ell_2 &= -\frac{1}{2} \ell_2 & J_+ \ell_2 &= -\varepsilon \ell_3 & J_- \ell_2 &= 0 & K_3 \ell_2 &= -\frac{1}{2} \ell_2 & K_+ \ell_2 &= \varepsilon \ell_0 & K_- \ell_2 &= 0 \\ J_3 \ell_3 &= +\frac{1}{2} \ell_3 & J_+ \ell_3 &= 0 & J_- \ell_3 &= -\varepsilon \ell_2 & K_3 \ell_3 &= -\frac{1}{2} \ell_3 & K_+ \ell_3 &= \varepsilon \ell_1 & K_- \ell_3 &= 0 \end{aligned} \quad (47)$$

For $\{J_3, J_-\}$ the common eigenvector is given as a linear combination of ℓ_0 and ℓ_2 while for $\{K_3, K_+\}$ the common eigenvector is given by the linear combination of ℓ_0 and ℓ_1 . On the other hand, the common eigenvector for $\{J_3, J_+\}$ is a linear combination of ℓ_3 and ℓ_1 while the common eigenvector of $\{K_3, K_-\}$ is a linear combination of ℓ_3 and ℓ_2 . While ℓ_0 (ℓ_3) is proportional to the (space-inverted) momentum four-vector p , the interpretation of the eigenvectors ℓ_1 and ℓ_2 deserves more effort. For this, one can return to the circular polarisation [16,17]. The representation

$$\vec{E}(z, t) = E_0 \text{Re} \left((\vec{e}_x + i\vec{e}_y) e^{ikz - i\omega t} \right) = E_0 (\vec{e}_x \cos(kz - \omega t) - \vec{e}_y \sin(kz - \omega t)) \quad (48)$$

describes the right turn of the electric vector in the (x, y) plane, as can be seen by comparing the solution for $z = 0$ at $t = 0$ and after a short time $t = \Delta t$. Therefore, the vector ℓ_1 can be identified with a right turn. However, a turn can be identified with handedness or chirality only in combination with a direction of propagation as in case of the circular polarisation by the argument $kz - \omega t$. (In optics, this solution is called left polarised, as looked at from the direction the light comes from (passive direction). In our case, however, we consider the direction of propagation (active direction).) This direction is given by ℓ_0 (or ℓ_3). Therefore, one can interpret (in case of $\varepsilon = 1$)

$$\begin{aligned} \{J_3, J_-\} &\quad \text{as forward-propagating left-handed,} \\ \{K_3, K_+\} &\quad \text{as forward-propagating right-handed,} \\ \{J_3, J_+\} &\quad \text{as backward-propagating left-handed, and} \\ \{K_3, K_-\} &\quad \text{as backward-propagating right-handed.} \end{aligned} \quad (49)$$

4.5. The Irreducible Representation

In terms of 4×4 matrices, the generators J_i and K_i ($i = 3, \pm$) are, of course, not given in the irreducible representation. However, they can be related to irreducible representations in an easy way. In fact, there is a similarity transformation such that

$$J_i \rightarrow S^{-1} J_i S =: J_i^{\boxplus}, \quad K_i \rightarrow S^{-1} K_i S =: K_i^{\boxplus} \quad (50)$$

(a deeper understanding of the representation index $\boxplus$ will be given soon) where

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -i & 0 & 0 & -i \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad (51)$$

and $S^{-1} = S^\dagger$. In detail, one obtains

$$\begin{aligned} J_3^\boxplus &= \frac{1}{2}(\sigma_3 \otimes \mathbb{1}), & J_\pm^\boxplus &= \frac{1}{2}(\sigma_\pm \otimes \mathbb{1}), \\ K_3^\boxplus &= \frac{1}{2}(\mathbb{1} \otimes \sigma_3), & K_\pm^\boxplus &= \frac{1}{2}(\mathbb{1} \otimes \sigma_\pm), \end{aligned} \quad (52)$$

where the outer product is defined by $(A \otimes B)_{(ik)(jl)} := A_{ij}B_{kl}$, i.e., the first matrix sets the frame for the second one. The matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (53)$$

are the usual Pauli matrices, and $\sigma_\pm = \sigma_1 \pm i\sigma_2$. The same similarity transformation via S can be applied also to the generators E_i and B_i of the proper orthochronous Lorentz group $\text{Lor}(1,3)$. One obtains

$$\begin{aligned} E_i^\boxplus &:= S^{-1}E_iS = -\frac{1}{2}(\sigma_i \otimes \mathbb{1} - \mathbb{1} \otimes \sigma_i), \\ B_i^\boxplus &:= S^{-1}B_iS = -\frac{i}{2}(\sigma_i \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_i). \end{aligned} \quad (54)$$

These two results can be rewritten by employing the Kronecker sum

$$A \boxplus B := A \otimes \mathbb{1} + \mathbb{1} \otimes B. \quad (55)$$

Using this notation, one obtains

$$E_i^\boxplus = -\frac{1}{2}\sigma_i \boxplus \left(+\frac{1}{2}\sigma_i\right), \quad B_i^\boxplus = -\frac{i}{2}\sigma_i \boxplus \left(-\frac{i}{2}\sigma_i\right). \quad (56)$$

Therefore, the representation index $\boxplus$ indicates that in this representation obtained via the similarity transformation with S , the matrix can be written as a Kronecker sum. It is characteristic that

$$J_i^\boxplus = \frac{1}{2}\sigma_i \boxplus \mathbb{0}, \quad K_i^\boxplus = \mathbb{0} \boxplus \frac{1}{2}\sigma_i \quad (57)$$

contribute only to the first or second component of the Kronecker sum, respectively. Following the argumentation of Section 4.4, one can conclude that the first component of the Kronecker sum (and thereby $J_i^\boxplus$) is left-handed while the second component of the Kronecker sum (and thereby $K_i^\boxplus$) is right-handed. Finally, we conclude that via the same similarity transformation S , the maximal solvable algebra $\text{bor}(1,3;p)$ in the representation of this section can indeed be decomposed into the Kronecker sum $\text{sol}_2^- \boxplus \text{sol}_2^+$.

5. The Chevalley Basis

From Equation (57), it is obvious that sol_2^- and sol_2^+ are isomorphic to Borel subalgebras of the real algebra $\text{sl}(2, \mathbb{R})$ given in the Chevalley basis by the three generators

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}. \quad (58)$$

One can write $\text{sol}_2^\pm = \text{span}_{\mathbb{R}}\{\sigma_3, \sigma_\pm\}$. The algebra $\text{sl}(2, \mathbb{R}) = \text{span}_{\mathbb{R}}\{\sigma_3, \sigma_+, \sigma_-\}$ can be complexified to obtain $\text{sl}(2, \mathbb{C}) = \text{span}_{\mathbb{C}}\{\sigma_3, \sigma_+, \sigma_-\}$. Therefore, $\text{sl}(2, \mathbb{R})$ is a real form of $\text{sl}(2, \mathbb{C})$. The compact real form of $\text{sl}(2, \mathbb{C})$ is given by $\text{su}(2) = \text{span}_{\mathbb{R}}\{\sigma_1, \sigma_2, \sigma_3\}$, while

$\mathfrak{sl}(2, \mathbb{R})$ can be called the decompactified real form of $\mathfrak{sl}(2, \mathbb{C})$. Similar to how the complexified version of $\mathfrak{lor}(1, 3)$ is isomorphic to $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, the complexified version of the extended little algebra $\mathfrak{bor}(1, 3; p)$ is isomorphic to $\mathfrak{sol}_2^- \oplus \mathfrak{sol}_2^+$.

5.1. Common Eigenvectors

The concept of common eigenvectors introduced in Section 4.4 pulls through to the very core, i.e., to the irreducible representation. The set of generators $\{\sigma_3, \sigma_+\}$ of $\mathfrak{sol}_2^+$ have the common eigenvector $(1, 0)^T$ and the set $\{1, 0\}$ of eigenvalues, while for $\{\sigma_3, \sigma_-\}$ (i.e., $\mathfrak{sol}_2^-$) the common eigenvector is $(0, 1)^T$ with eigenvalues $\{-1, 0\}$. Reintroducing the sign ε , the two non-trivial eigenvalue equations can be cast into the form

$$\sigma_3 \psi_+ = \varepsilon \psi_+, \quad \psi_+ = \begin{pmatrix} (1 + \varepsilon) \psi^1 \\ (1 - \varepsilon) \psi^2 \end{pmatrix}. \quad (59)$$

This is the first quantisation step. Indeed, introducing

$$\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1} \quad (60)$$

one obtains the Weyl equation $((\sigma^\mu) := (\sigma_0, \sigma_1, \sigma_2, \sigma_3))$

$$0 = (\varepsilon \sigma_0 - \sigma_3) \psi_+ = \varepsilon p_\mu \sigma^\mu \psi_+ =: \varepsilon \sigma(p) \psi_+. \quad (61)$$

However, this is not the only possible quantisation. Equivalently, one may write

$$\sigma_3 \psi_- = -\varepsilon \psi_-, \quad \psi_- = \begin{pmatrix} (1 - \varepsilon) \psi^1 \\ (1 + \varepsilon) \psi^2 \end{pmatrix} \quad (62)$$

or

$$0 = (\varepsilon \sigma_0 + \sigma_3) \psi_- = \varepsilon p_\mu \tilde{\sigma}^\mu \psi_- =: \varepsilon \tilde{\sigma}(p) \psi_- \quad (63)$$

$((\tilde{\sigma}^\mu) = (\sigma_0, -\sigma_1, -\sigma_2, -\sigma_3))$ which is the dual Weyl equation. In using the tilde notation for $\tilde{\sigma}$, one avoids the breakdown of the covariant notation. Using Weyl's representation

$$\gamma_W^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma_W^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \quad (64)$$

of the Dirac matrices, for finite mass m , one ends up with the Dirac equation

$$(p_\mu \gamma_W^\mu - mc) \psi_W = 0, \quad \psi_W = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}. \quad (65)$$

ψ_+ is the right-handed spinor, and ψ_- is the left-handed spinor. This is in agreement with the usual definition $\psi_R = \frac{1}{2}(1 + \gamma^5) \psi_W = (0, \psi_+)^T$ and $\psi_L = \frac{1}{2}(1 - \gamma^5) \psi_W = (\psi_-, 0)^T$.

5.2. Induced Lorentz Transformations

The contractions of the momentum four-vector p with σ and $\tilde{\sigma}$ induces two (proper orthochronous) Lorentz transformations A_Λ and $\tilde{A}_\Lambda$ which make the diagram

$$\begin{array}{ccccc} A_\Lambda : & \sigma(p) & \longrightarrow & \sigma(\Lambda p) \\ \pi \uparrow & \uparrow \sigma & & \uparrow \sigma \\ \Lambda : & p & \longrightarrow & \Lambda p \\ \tilde{\pi} \downarrow & \downarrow \tilde{\sigma} & & \downarrow \tilde{\sigma} \\ \tilde{A}_\Lambda : & \tilde{\sigma}(p) & \longrightarrow & \tilde{\sigma}(\Lambda p) \end{array}$$

commutative. The induced Lorentz transformations are defined by

$$A_\Lambda \sigma(p) A_\Lambda^\dagger = \sigma(\Lambda p), \quad \tilde{A}_\Lambda \tilde{\sigma}(p) \tilde{A}_\Lambda^\dagger = \tilde{\sigma}(\Lambda p). \quad (66)$$

A long but straightforward calculation shows that

$$A_\Lambda = \frac{\sigma^\mu \Lambda_{\mu\nu} \tilde{\sigma}^\nu}{2 \operatorname{tr}(A_\Lambda^\dagger)}, \quad A_\Lambda^{-1} = \frac{\sigma^\mu \Lambda_{\nu\mu} \tilde{\sigma}^\nu}{2 \operatorname{tr}(A_\Lambda^\dagger)}, \quad \tilde{A}_\Lambda = \frac{\tilde{\sigma}^\mu \Lambda_{\mu\nu} \sigma^\nu}{2 \operatorname{tr}(\tilde{A}_\Lambda^\dagger)}, \quad \tilde{A}_\Lambda^{-1} = \frac{\tilde{\sigma}^\mu \Lambda_{\nu\mu} \sigma^\nu}{2 \operatorname{tr}(\tilde{A}_\Lambda^\dagger)}. \quad (67)$$

A_Λ and $\tilde{A}_\Lambda$ can be written in an exponential form similar to Equation (19),

$$A_\Lambda(\omega) = \exp\left(-\frac{1}{2}\omega_{\alpha\beta}a^{\alpha\beta}\right), \quad \tilde{A}_\Lambda(\omega) = \exp\left(-\frac{1}{2}\omega_{\alpha\beta}\tilde{a}^{\alpha\beta}\right). \quad (68)$$

For the exponential coefficients of A_Λ one obtains

$$a^{\alpha\beta} = \frac{1}{4}(e^{\alpha\beta})_{\mu\nu}\sigma^\mu\tilde{\sigma}^\nu = -\frac{1}{2}(\sigma^\alpha\tilde{\sigma}^\beta - \sigma^\beta\tilde{\sigma}^\alpha) \quad (69)$$

which can be detailed into $a^{ij} = \frac{i}{2}\epsilon_{ijk}\sigma_k =: -\epsilon_{ijk}b_k$, $a^{0i} = \frac{1}{2}\sigma_i =: -e_i$ with

$$b_i = -\frac{i}{2}\sigma_i, \quad e_i = -\frac{1}{2}\sigma_i, \quad (70)$$

where b_i and e_i obey the algebra $\mathfrak{lor}(1,3)$,

$$[b_i, b_j] = \epsilon_{ijk}b_k, \quad [b_i, e_j] = \epsilon_{ijk}e_k, \quad [e_i, e_j] = -\epsilon_{ijk}b_k. \quad (71)$$

For the exponential coefficient of $\tilde{A}_\Lambda$ one obtains

$$\tilde{a}^{\alpha\beta} = \frac{1}{4}(e^{\alpha\beta})_{\mu\nu}\tilde{\sigma}^\mu\sigma^\nu = -\frac{1}{2}(\tilde{\sigma}^\alpha\sigma^\beta - \tilde{\sigma}^\beta\sigma^\alpha) \quad (72)$$

which gives $\tilde{a}^{ij} = -\frac{i}{2}\epsilon_{ijk}\sigma_k =: \epsilon_{ijk}\tilde{b}_k$, $\tilde{a}^{0i} = \frac{1}{2}\tilde{\sigma}_i =: \tilde{e}_i$. The generators

$$\tilde{b}_i = -\frac{i}{2}\tilde{\sigma}_i, \quad \tilde{e}_i = \frac{1}{2}\tilde{\sigma}_i \quad (73)$$

(note the sign changes compared to b_i, e_i) obey again the algebra $\mathfrak{lor}(1,3)$,

$$[\tilde{b}_i, \tilde{b}_j] = \epsilon_{ijk}\tilde{b}_k, \quad [\tilde{b}_i, \tilde{e}_j] = \epsilon_{ijk}\tilde{e}_k, \quad [\tilde{e}_i, \tilde{e}_j] = -\epsilon_{ijk}\tilde{b}_k. \quad (74)$$

Formally, the transitions to the induced Lorentz transformations can be considered as mappings $\pi: \Lambda \rightarrow A_\Lambda$ with $\pi(e^{\alpha\beta}) = a^{\alpha\beta}$ and $\tilde{\pi}: \Lambda \rightarrow \tilde{A}_\Lambda$ with $\tilde{\pi}(e^{\alpha\beta}) = \tilde{a}^{\alpha\beta}$. Under these mappings, the generators J_i^ε and K_i^ε of $\mathfrak{sol}_2^\pm$ are mapped onto the Chevalley basis. Under π one obtains

$$\begin{aligned} J_3^\varepsilon &\rightarrow \frac{1}{4}(1+\varepsilon)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & K_3^\varepsilon &\rightarrow \frac{1}{4}(1-\varepsilon)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ J_-^\varepsilon &\rightarrow \frac{1}{4}(1+\varepsilon)\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, & K_+^\varepsilon &\rightarrow \frac{1}{4}(1-\varepsilon)\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \end{aligned} \quad (75)$$

while under $\tilde{\pi}$ one obtains

$$\begin{aligned} J_3^\varepsilon &\rightarrow -\frac{1}{4}(1-\varepsilon)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & K_3^\varepsilon &\rightarrow -\frac{1}{4}(1+\varepsilon)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ J_-^\varepsilon &\rightarrow -\frac{1}{4}(1-\varepsilon)\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, & K_+^\varepsilon &\rightarrow -\frac{1}{4}(1+\varepsilon)\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \end{aligned} \quad (76)$$

i.e., the same result as for $\varepsilon \leftrightarrow -\varepsilon$ and the total sign interchanged. Again, we are faced with the fact that half of the generators are mapped to zero. Taking into account the relations to $J_i^\boxplus$ and $K_i^\boxplus$, one can state that π maps to the first component of the Kronecker sum while $\tilde{\pi}$

maps to the second component of the Kronecker sum. Due to Section 4, π is the mapping to the left-handed sector, $\tilde{\pi}$ the mapping into the right-handed sector.

Using Equation (67) and performing a couple of simple conversions, one obtains the explicit shape for A_Λ for Λ of Equation (6) with dependence on ε ,

$$A_\Lambda^\varepsilon = \begin{pmatrix} e^{(-\varepsilon t + i w)/2} & \frac{1}{2}(1 - \varepsilon)(u - i v)e^{(-\varepsilon t - i w)/2} \\ -\frac{1}{2}(1 + \varepsilon)(u + i v)e^{(\varepsilon t + i w)/2} & e^{(\varepsilon t - i w)/2} \end{pmatrix} \quad (77)$$

($\tilde{A}_\Lambda^\varepsilon = A_\Lambda^{-\varepsilon}$), which can be rewritten as

$$A_\Lambda^\varepsilon = R_{-\varepsilon t} A_{u,v}^\varepsilon R_{i w}, \quad (78)$$

where

$$R_x := \begin{pmatrix} e^{x/2} & 0 \\ 0 & e^{-x/2} \end{pmatrix}, \quad A_{u,v}^\varepsilon := \begin{pmatrix} 1 & \frac{1}{2}(1 - \varepsilon)(u - i v) \\ -\frac{1}{2}(1 + \varepsilon)(u + i v) & 1 \end{pmatrix} \quad (79)$$

and $\det A_{u,v}^\varepsilon = \det R_x = 1$. Using $(A_\Lambda^{\varepsilon\dagger})^{-1} = (A_\Lambda^\varepsilon)^{-1\dagger} = A_\Lambda^{-\varepsilon} = \tilde{A}_\Lambda^\varepsilon$ and $(\tilde{A}_\Lambda^{\varepsilon\dagger})^{-1} = A_\Lambda^\varepsilon$ and Equation (66), one obtains the Lorentz transformation $\psi_W(\Lambda p) = U(\Lambda)\psi_W(p)$ of the Weyl spinor where

$$U(\Lambda) = \begin{pmatrix} A_\Lambda^\varepsilon & 0 \\ 0 & \tilde{A}_\Lambda^\varepsilon \end{pmatrix}, \quad \psi_W(p) = \begin{pmatrix} \psi_-(p) \\ \psi_+(p) \end{pmatrix}. \quad (80)$$

5.3. Representations of the Proper Orthochronous Lorentz Group

Using the two mappings $\pi : \text{Lor}(1,3) \rightarrow \text{SL}(2, \mathbb{C})$, $\tilde{\pi} : \text{Lor}(1,3) \rightarrow \text{SL}(2, \mathbb{C})$ ($\pi : \Lambda \mapsto A_\Lambda$ and $\tilde{\pi} : \Lambda \mapsto \tilde{A}_\Lambda$) and the Kronecker sum, one can define the representation $(1/2, 1/2)$ by

$$\pi^{(1/2, 1/2)}(\Lambda) := (\pi \otimes \tilde{\pi})(\Lambda \boxplus \Lambda) = \pi(\Lambda) \boxplus \tilde{\pi}(\Lambda), \quad (81)$$

for which (and for the choice $\varepsilon = +1$)

$$\begin{aligned} \pi^{(1/2, 1/2)}(E_i) &= -\frac{1}{2}\sigma_i \boxplus \left(+\frac{1}{2}\sigma_i\right) = E_i^\boxplus, & \pi^{(1/2, 1/2)}(J_i) &= \frac{1}{2}\sigma_i \boxplus 0 = J_i^\boxplus, \\ \pi^{(1/2, 1/2)}(B_i) &= -\frac{i}{2}\sigma_i \boxplus \left(-\frac{i}{2}\sigma_i\right) = B_i^\boxplus, & \pi^{(1/2, 1/2)}(K_i) &= 0 \boxplus \frac{1}{2}\sigma_i = K_i^\boxplus. \end{aligned} \quad (82)$$

Therefore, the map $\pi^{(1/2, 1/2)} : \text{Lor}(1,3) \rightarrow \text{SL}(2, \mathbb{C}) \otimes \text{SL}(2, \mathbb{C})$ may replace the similarity transformation via S . The benefit of using this map instead of the similarity transformation is that such a construction can easily be generalised to a representation $\pi^{(k,l)}$ of the proper orthochronous Lorentz group.

In proceeding to these general (k, l) representations, the common eigenvectors $(1, 0)^T$ of the set $\{\sigma_3, \sigma_+\}$ and $(0, 1)^T$ of the set $\{\sigma_3, \sigma_-\}$ can be written as states $|l; m\rangle = |1/2; 1/2\rangle$ and $|l; m\rangle = |1/2; -1/2\rangle$, respectively, with

$$\sigma_3|l; m\rangle = 2m|l; m\rangle, \quad \sigma_\pm|l; m\rangle = 2\rho(l; \pm m)|l; m \pm 1\rangle, \quad (83)$$

where $\rho(l; m) = \sqrt{(l - m)(l + m + 1)}$. For the general (k, l) representation the states are given by $|k, l; m_k, m_l\rangle$, with

$$\begin{aligned} \pi^{(k,l)}(J_3)|k, l; m_k, m_l\rangle &= 2m_k|k, l; m_k, m_l\rangle, \\ \pi^{(k,l)}(J_\pm)|k, l; m_k, m_l\rangle &= 2\rho(k; \pm m_k)|k, l; m_k \pm 1, m_l\rangle, \\ \pi^{(k,l)}(K_3)|k, l; m_k, m_l\rangle &= 2m_l|k, l; m_k, m_l\rangle, \\ \pi^{(k,l)}(K_\pm)|k, l; m_k, m_l\rangle &= 2\rho(l; \pm m_l)|k, l; m_k, m_l \pm 1\rangle. \end{aligned} \quad (84)$$

Of these $(2k + 1) \times (2l + 1)$ states, only those for $\rho(k; -m_k) = 0$ or $\rho(l; m_l) = 0$, i.e., for $m_k = -k$ or $m_l = l$ are common eigenstates of $\text{sol}_2^- = \{J_3, J_-\}$ and $\text{sol}_2^+ = \{K_3, K_+\}$, respectively, with $|k, l; -k, l\rangle$ being the common eigenvector for both algebras [18].

5.4. Helicity

In order to define a helicity

$$H(\vec{p}) = \hat{p} \cdot \vec{s}, \quad \hat{p} = \frac{\vec{p}}{|\vec{p}|}, \quad (85)$$

one needs a spin vector $\vec{s}$. This vector can be defined by $s_i = i\hbar b_i$, because then the commutation relation $[b_i, b_j] = \epsilon_{ijk} b_k$ for the generators of A_Λ^ϵ leads to the usual commutation relation

$$[s_i, s_j] = i\hbar \epsilon_{ijk} s_k \quad (86)$$

of an angular momentum algebra. For the three-vector part $\vec{p} = (0, 0, 1)^T$ of the momentum vector p generating the Borel subgroup $\text{Bor}(1, 3; p)$, one obtains

$$H(\vec{p}) = s_3 = i\hbar b_3 = \frac{\hbar}{2} \sigma_3 = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (87)$$

Therefore, the common eigenvector $(1, 0)^T$ of sol_2^+ has a helicity $h = +\hbar/2$, and the common eigenvector $(0, 1)^T$ of sol_2^- has a helicity $h = -\hbar/2$, in agreement (for $\epsilon = +1$) with the previous understanding of left and right.

As b_i is the two-dimensional representation of B_i , the concept of helicity can be generalised to representations (k, l) ,

$$H^{(k,l)}(\vec{p}) = \pi^{(k,l)}(i\hbar B_3) = \frac{\hbar}{2} \pi^{(k,l)}(\sigma_3 \boxplus \sigma_3), \quad (88)$$

Applied to the state $|k, l; m_k, m_l\rangle$, one obtains

$$H^{(k,l)}(\vec{p})|k, l; m_k, m_l\rangle = \hbar(m_k + m_l)|k, l; m_k, m_l\rangle \quad (89)$$

which means that the helicity of this state is $h = \hbar(m_k + m_l)$.

6. Conclusions and Outlook

In this paper, we have calculated the stabiliser group of the proper orthochronous Lorentz group, which turns out to be the maximal solvable or Borel subgroup of dimension four. We have explained the continuous transition between the stabiliser groups of massive and massless particles that describes the massless limit but fails for exactly massless states. We have dealt with the system of eigenvectors of the Borel subgroup and shown that the Borel subgroup can be described by a Kronecker sum of two two-dimensional solvable groups $\text{sol}_2^\pm$ representing right- and left-handed chirality states. Finally, in the Chevalley basis we have derived the Weyl and Dirac equations for massless and massive particles and have defined the helicity of the massless states. Note that without the generator T , such a Kronecker sum of chiral states would not emerge. The Borel subgroup, as the maximal solvable subgroup of the proper orthochronous Lorentz group, provides exactly four eigenvectors describing these two chiral states, of which the left-handed state is populated by massless fermions and the right-handed is populated by antifermions. This is the physical content of our extension.

Even though the foundations for an explanation of the spin-flip effect are prepared by this, an exact formulation is not gained here but is aimed at in a future publication. The effect is closely related to the concept of mass which we want to understand in more detail. In our argumentation, we obtained unexpected help from a not yet published seminal work explaining in detail the construction of a spin operator by a linear combination

of components of the Pauli–Lubanski pseudovector [19]. Not unexpectedly, the authors end up with two spin (tensor) operators and corresponding chirality states that are interchanged under parity transformation. Parity eigenstates can be constructed as particle or antiparticle compound states. Applying the Lorentz transformation to the massive states of Ref. [19], the parity eigenstates are shown to evolve to solutions of the Dirac equation.

In Ref. [19] it is emphasised that the two spin operators are neither axial nor Hermitian, and the same holds for the spin operators in the $(1/2, 0) \oplus (0, 1/2)$ representation. However, both properties are restored if applied to particle and antiparticle states. On the other hand, as both properties are essential for physical states, we can conclude that massless left- and right-handed states are physical only in the total absence of mass. This “gap of (un)physicalness” as an explanation for the spin-flip effect has to be investigated in detail.

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